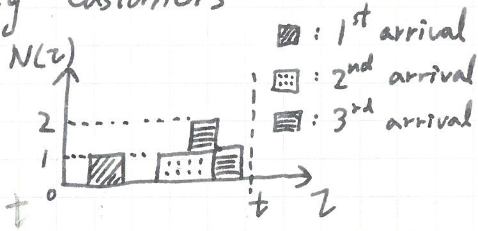
P2

Deriving Little's Theorem:

Let $N(t)$ = # of customers in the system at time t

$\alpha(t)$ = # of customers arrived in interval $[0, t]$

T_i = Time spent in the system by the i-th arriving customers



$$N_t = \frac{1}{t}\int_0^t N(\tau)\cdot d\tau$$

time average of $N(\tau)$ up to time t

at steady state,

N_t converges to $N = \lim_{t\to\infty} N_t$

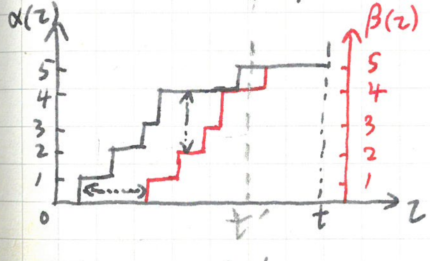
$$\lambda_t = \frac{\alpha(t)}{t}$$

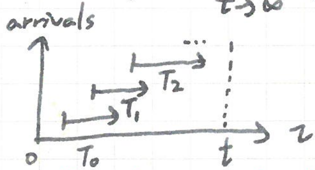
time average arrival rate over $[0, t]$

at steady state, $\lambda = \lim_{t\to\infty} \lambda_t$

customer delay

$$T_t = \frac{1}{\alpha(t)}\sum_{i=0}^{\alpha(t)} T_i \quad , \quad T = \lim_{t\to\infty} T_t$$





let $\beta(t)$ = # of customers departed in $[0, t]$

$\Rightarrow N(\tau) = \alpha(\tau) - \beta(\tau)$

The area between curves $\alpha(\tau)$ and $\beta(\tau)$ is $\int_0^t N(\tau)\,d\tau$.

The area is also equal to $\sum_{i=0}^{\alpha(\tau)} T_i$ if $N(t) = 0$

if $N(t') > 0$

$$\sum_{i=0}^{\beta(t)} T_i \le \int_0^t N(\tau)\,d\tau \le \sum_{i=0}^{\alpha(t)} T_i$$

departure rate

assuming $\frac{\beta(t)}{t} = \frac{\alpha(t)}{t}$ when $t\to\infty$

$= \lambda \quad = \lambda$

$$\Rightarrow \underbrace{\frac{1}{t}\cdot\int_0^t N(\tau)\,d\tau}_{N} = \underbrace{\frac{\alpha(t)}{t}}_{\lambda}\cdot\underbrace{\frac{\sum_{i=0}^{\alpha(t)} T_i}{\alpha(t)}}_{T}$$

at steady state

$$N = \lambda \cdot T$$