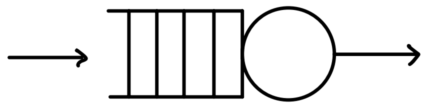
2.2 slowdown

(a) $\lambda = \frac{1}{2}$ job/sec

FCFS



$$S = \begin{cases} 1 & \text{with probability } \frac{3}{4} \\ 2 & \text{otherwise } 1-\frac{3}{4}=\frac{1}{4} \end{cases}$$

$$E[T] = \frac{29}{12}$$

$$E[\text{slowdown}] = ?$$

$$\text{slowdown}(j) = \frac{T(j)}{S(j)}$$

$T(j) \rightarrow$ resp. t.

$S(j) \rightarrow$ ser. t.

$$E[S] = 1\cdot\frac{3}{4} + 2\cdot\frac{1}{4} = \frac{3}{4} + \frac{2}{4} = \frac{5}{4}$$

$$\rho = \lambda \cdot E[S] = \frac{1}{2} \times \frac{5}{4} = \frac{5}{8}, \quad \frac{5}{8} < 1$$

$$E[\text{slowdown}] = \frac{E[T]}{E[S]} = \frac{\frac{29}{12}}{\frac{5}{4}} = \frac{29}{\cancel{12}_{3}} \times \frac{\cancel{4}^{1}}{5} = \frac{29}{15} \#$$

(b) If the service order were SJF, this methed wouldn't work directly. As SJF prioritizes shorter job, altering the response time distribution. Thus, using the simple ratio of mean response time to mean service time would be inaccurate.

6.2 Simplified Power Usage in Server Farms

We have 10 r.v. each representing the time for which a server is on for a specfic Job. $X_1, X_2, \cdots, X_{10}$.
The total time that all server are on is $X = X_1 + X_2 + \cdots + X_{10}$.

Each variable X_i is uniformly disturbuted between 1 and 9 sec.
PDF for X_i is:

$$p(x) = \frac{1}{8} \text{ for } 1 \leq x \leq 9$$

the rang of the distribution is 8 sec.
$p(x) = \frac{1}{b-a}$ for $a \leq x \leq b$

The expected value of uniform distributed r.v. X_i over an interval $[a, b]$ is given by $E(X_i) = \frac{a+b}{2}$.

$$E(X_i) = \frac{1+9}{2} = 5 \text{ seconds}$$

Since there are 10 jobs, each with an expected time of 5 seconds, the total time expected $E(X)$ for all 10 server is:

$$E(X) = 10 \times 5 = 50 \text{ seconds}$$

Each consumes power at a rate of $P = 240$ watts when it is on. The average power consumed over time can be calculated by

Average Power consumption $= 240 \times 50 = \underline{12000 \text{ watt}}$. #

8.1 Solving for Limiting Distributions

CPU intstruction (C)

Memory intstruction (M)

User interaction intstruction (U)

transition matrix

$$P = \begin{array}{c} \\ C \\ M \\ U \end{array} \begin{array}{c} \begin{array}{ccc} C & M & U \end{array} \\ \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.8 & 0.1 & 0.1 \\ 0.9 & 0.1 & 0 \end{pmatrix} \end{array}$$

The stationary distribution (π_C, π_M, π_U) satifies the equations:

$$\pi_C = \pi_C(0.7) + \pi_M(0.8) + \pi_U(0.9)$$
$$\pi_M = \pi_C(0.2) + \pi_M(0.1) + \pi_U(0.1)$$
$$\pi_U = \pi_C(0.1) + \pi_M(0.1) + \pi_U(0)$$

We have the normalization equation: $\boxed{1 = \pi_C + \pi_M + \pi_U}$

$$\pi_C = 0.7\pi_C + 0.8\pi_M + (0.1\pi_C + 0.1\pi_M)(0.9) = 0.79\pi_C + 0.89\pi_M$$
$$\Rightarrow 0.21\pi_C - 0.89\pi_M = 0 \Rightarrow \pi_M = \frac{0.21}{0.89}\pi_C$$
$$\pi_M = 0.2\pi_C + 0.1\pi_M + (0.1\pi_C + 0.1\pi_M)(0.1) = 0.21\pi_C + 0.11\pi_M$$
$$\Rightarrow 0.89\pi_M - 0.21\pi_C = 0 \Rightarrow \pi_M = \frac{0.21}{0.89}\pi_C$$

$$\pi_C + \frac{0.21}{0.89}\pi_C + \left(0.1\pi_C + 0.1 \cdot \frac{0.21}{0.89}\pi_C\right) = 1$$

$$\boxed{\pi_C = 0.74, \quad \pi_M = 0.17, \quad \pi_U = 0.09}$$ #

This mean the program speeds approximately 74% of the time in C, 17% in M, and 9% in U.

8.6 Threshold Queue

(a) transition matrix $\mathbf{P}$ given:

$$
\mathbf{P} = \begin{array}{c|ccccccc}
i \backslash j & 0 & 1 & 2 & \cdots & T & T+1 & \\
\hline
0 & 0.4 & 0.6 & & & & & \\
1 & 0.4 & & 0.6 & & & & \\
2 & & 0.4 & & & & & \\
 & & & 0.4 & & & & \\
\vdots & & & & \ddots & 0.6 & & \\
T & & & & & 0.2 & 0.4 & \\
T+1 & & & & & 0.6 & & 0.4 \\
 & & & & & & 0.6 & \\
 & & & & & & & 0.6
\end{array}
$$

$N < T$:

$P(N \to N+1) = 0.6$

$P(N \to N-1) = 0.4$

$N \geq T$:

$P(N \to N+1) = 0.4$

$P(N \to N-1) = 0.6$

transit probability

$\alpha_n = P(N=n \to N=n+1)$

$\beta_n = P(N=n \to N=n-1)$

$n < T$:

$\alpha_n = 0.6$

$\beta_n = 0.4$

$n \geq T$:

$\alpha_n = 0.4$

$\beta_n = 0.6$

$n \le T$, $\frac{0.6}{0.4} = \frac{3}{2}$, $\pi_n = (\frac{3}{2})^n \pi_0$ $\qquad \pi_T = \pi_0 (\frac{3}{2})^T$

$n > T$, $\frac{0.4}{0.6} = \frac{2}{3}$, $\pi_n = (\frac{2}{3})^{n-T} \pi_T = (\frac{2}{3})^{n-T} \pi_0 (\frac{3}{2})^T$

$\sum_{n=0}^{\infty} \pi_n = 1$, $S_0 : n = 0 \sim T \quad S_2 : n = T+1 \sim \infty$

$$S_1 = \sum_{n=0}^{T} \pi_n = \pi_0 \sum_{n=0}^{T} (\frac{3}{2})^n = \pi_0 \frac{(\frac{3}{2})^{T+1} - 1}{\frac{3}{2} - 1} = 2\pi_0 [(\frac{3}{2})^{T+1} - 1] \qquad \boxed{\frac{r^{n+1} - 1}{r - 1}}$$

$m = n - T$

$$S_2 = \sum_{n=T+1}^{\infty} \pi_n = \pi_0 (\frac{3}{2})^T \sum_{m=1}^{\infty} (\frac{2}{3})^m = \pi_0 (\frac{3}{2})^T (\frac{\frac{2}{3}}{1 - \frac{2}{3}}) = 2\pi_0 (\frac{3}{2})^T \qquad \boxed{\frac{r}{1 - r}}$$

$$S = S_1 + S_2 = 2\pi_0 [(\frac{3}{2})^{T+1} - 1 + (\frac{3}{2})^T]$$

$$2\pi_0 [(\frac{3}{2})^{T+1} - 1 + (\frac{3}{2})^T] = 1$$

$$2\pi_0 [\frac{3}{2} \cdot (\frac{3}{2})^T - 1 + (\frac{3}{2})^T] = 1$$

$$2\pi_0 [\frac{5}{2} \cdot (\frac{3}{2})^T - 1] = 1$$

$$\pi_0 = \frac{1}{2[\frac{5}{2} \cdot (\frac{3}{2})^T - 1]} = \frac{1}{5(\frac{3}{2})^T - 2} \quad \#$$

$n \le T$:

$$\pi_n = \frac{(\frac{3}{2})^n}{2[\frac{5}{2} \cdot (\frac{3}{2})^T - 1]}$$

$n > T$:

$$\pi_n = \frac{(\frac{3}{2})^T \cdot (\frac{2}{3})^{n-T}}{2[\frac{5}{2} \cdot (\frac{3}{2})^T - 1]}$$

(b) $E[N] = \sum_{n=0}^{\infty} n\pi_n = \sum_{n=0}^{T} n\pi_n + \sum_{n=T+1}^{\infty} n\pi_n = E_1 + E_2$

$$E_1 = \sum_{n=0}^{T} n\pi_n = \pi_0 \sum_{n=0}^{T} n\left(\frac{3}{2}\right)^n$$

$$S_1 = \sum_{n=0}^{\infty} n\left(\frac{3}{2}\right)^n = \frac{\frac{3}{2}\left[T\cdot\left(\frac{3}{2}\right)^{T+1} - (T+1)\cdot\left(\frac{3}{2}\right)^T + 1\right]}{\left(\frac{3}{2}-1\right)^2 = 0.25}$$

$$= 6\left[T\cdot\left(\frac{3}{2}\right)^{T+1} - (T+1)\cdot\left(\frac{3}{2}\right)^T + 1\right]$$

$$E_1 = \pi_0 \times S_1 = 6\pi_0\left[T\cdot\left(\frac{3}{2}\right)^{T+1} - (T+1)\cdot\left(\frac{3}{2}\right)^T + 1\right]$$

$n > T$, $k = n - T$

$$E_2 = \sum_{n=T+1}^{\infty} n\pi_n = \pi_0\left(\frac{3}{2}\right)^T \sum_{k=1}^{\infty}(k+T)\left(\frac{2}{3}\right)^k$$

$$= \pi_0\left(\frac{3}{2}\right)^T\left[\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k + T\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k\right]$$

$$\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k = \frac{\frac{2}{3}}{\left(1-\frac{2}{3}\right)^2} = \frac{\frac{2}{3}}{\left(\frac{1}{3}\right)^2} = \frac{\frac{2}{3}}{\frac{1}{9}} = 6$$

$$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2$$

$$E_2 = \pi_0\left(\frac{3}{2}\right)^T(6+2T)$$

$$E[N] = E_1 + E_2$$

$$= 6\pi_0\left[T\cdot\left(\tfrac{3}{2}\right)^{T+1} - (T+1)\cdot\left(\tfrac{3}{2}\right)^{T} + 1\right] + \pi_0\left(\tfrac{3}{2}\right)^{T}(6+2T)$$

$$6\pi_0\left[T\cdot\left(\tfrac{3}{2}\right)^{T+1} - (T+1)\cdot\left(\tfrac{3}{2}\right)^{T} + 1\right] = 6\pi_0\left[T\cdot\tfrac{3}{2}\cdot\left(\tfrac{3}{2}\right)^{T} - (T+1)\cdot\left(\tfrac{3}{2}\right)^{T} + 1\right]$$

$$= 6\pi_0\left(\tfrac{3}{2}\right)^{T}\left[\boxed{\tfrac{3}{2}T - (T+1)} + \frac{1}{\left(\frac{3}{2}\right)^{T}}\right]$$

$$\Downarrow$$

$$\tfrac{3}{2}T - T - 1 = \tfrac{1}{2}T - 1$$

$$E[N] = 6\pi_0\left(\tfrac{3}{2}\right)^{T}\left(\tfrac{1}{2}T - 1 + \frac{1}{\left(\frac{3}{2}\right)^{T}}\right) + \pi_0\left(\tfrac{3}{2}\right)^{T}(6+2T)$$

$$= \pi_0\left(\tfrac{3}{2}\right)^{T}\left[6\left(\tfrac{1}{2}T - 1 + \frac{1}{\left(\frac{3}{2}\right)^{T}}\right) + 6 + 2T\right]$$

$$6\left(\tfrac{1}{2}T - 1 + \frac{1}{\left(\frac{3}{2}\right)^{T}}\right) + 6 + 2T = 3T - 6 + \frac{6}{\left(\frac{3}{2}\right)^{T}} + 6 + 2T$$

$$= 5T + \frac{6}{\left(\frac{3}{2}\right)^{T}}$$

$$E[N] = \pi_0\left(\tfrac{3}{2}\right)^{T}\left(5T + \frac{6}{\left(\frac{3}{2}\right)^{T}}\right) = \pi_0\left(5T\left(\tfrac{3}{2}\right)^{T} + 6\right)$$

$$E[N] = \frac{1}{5\left(\frac{3}{2}\right)^{T} - 2}\left(5T\left(\tfrac{3}{2}\right)^{T} + 6\right)$$

#

(c)

$T=0$

$$E[N]=\frac{1}{5\left(\frac{3}{2}\right)^{T}-2}\left(\cancel{5T\left(\frac{3}{2}\right)^{T}}+6\right)$$

$\Downarrow$

$5-2=3$

$\underline{E[N]=2}$ #

10.3 Time to empty

(a)

1. Increase by 1 packet with a $p = 0.4$
2. Decrease by 1 packet with a $p = 0.6$

In Markov chain, the system has memoryless property, meaning the expected time from state 2 to state 1 is same as the expected time from state 1 to state 0.

$$T_{2,1} = E[T_{1,0}]$$

$$E[T_{1,0}] = 0.6 \cdot \overset{\text{Time}}{1} + 0.4 \cdot (\overset{\text{Time}}{1} + E[T_{1,0} + T_{2,1}])$$

$$\Rightarrow E[T_{1,0}] = 0.6 \cdot 1 + 0.4 \cdot (1 + E[T_{1,0}])$$

$$\Rightarrow E[T_{1,0}] = 0.6 + 0.4 + 0.4 \times E[T_{1,0}]$$

$$\Rightarrow E[T_{1,0}] - 0.4 \times E[T_{1,0}] = 1$$

$$\Rightarrow 0.6 E[T_{1,0}] = 1$$

$$\Rightarrow E[T_{1,0}] = \frac{1}{0.6} = \frac{5}{3} \quad \#$$

the expected time from state 1 to state 0 is $\frac{5}{3}$.

11.3 Double Exponential

$$X \sim Exp(\mu)$$

$$Y = 2X$$

$$\text{p.d.f. } f_X(x) = \begin{cases} \mu e^{-\mu x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\text{c.d.f. } F_X(x) = \begin{cases} 1 - e^{-\mu x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$F_Y(y) = P(Y \leq y) = P(2X \leq y) = P(X \leq \tfrac{y}{2}) = F_X(\tfrac{y}{2})$$

$$F_Y(y) = 1 - e^{-\mu(\frac{y}{2})} = 1 - e^{-y(\frac{\mu}{2})}$$

$$\underline{Y \sim Exp(\tfrac{\mu}{2})} \quad \#$$

Doubling all the sizes changes the expontial distribution from rate μ to rate μ divded by 2 ($\frac{\mu}{2}$), the new job sizes still follow an expontial distribution but with half the original rate.

11.5 Practice with Definition of Possion Process

(a) (i)

$$E[\text{Green}] = \lambda \times p = 50 \times 0.05 = 2.5 \ \text{packets}/\text{sec}$$

$$E[\text{Yellow}] = \lambda \times q = 50 \times 0.95 = 47.5 \ \text{packets}/\text{sec}$$

In a possion process, the arrivals of different types of events are independent. Therefore, even if an unusually high number of green packets its observed, the expected number of yellow packets remain at 47.5 #.

(a) (ii)

$Y \sim \text{Possion}(47.5)$

$$P\{N(t+s) - N(s) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

$$P\{N(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

$$= \frac{e^{-47.5 \times 1}(47.5 \times 1)^{200}}{200!}$$

The probability is extremely low, almost zero. #

(b) red : $\lambda_1 = 30$ packets/sec

black: $\lambda_2 = 10$ packets/sec

Assume total of $N = 60$ packets arrive in one second.

$$\lambda = \lambda_1 + \lambda_2 = 40 \text{ packets/sec}$$

In a poission process, given N events occuring within time t, these event is uniformly distributed in time.

red packet success Prob.: green packet success Prob.:

$$p = \frac{\lambda_1}{\lambda} = \frac{30}{40} = 0.75, \quad q = 1 - p = 0.25$$

$$X \sim \text{Binomial}(60, 0.75)$$

$$P\{S_n = k\} = \binom{n}{k} p^k q^{n-k}$$

$$= \binom{60}{40} (0.75)^{40} (0.25)^{20}$$

$$= \underline{0.03834} \#$$

(c)

100 packets have arrive within 30 sec.

Given that $N=10$ events occur within time T, there events are uniformly distibuted in time.

$$p = \frac{t}{T} = \frac{10}{30} = \frac{1}{3} \quad , \quad q = 1-\frac{1}{3} = \frac{2}{3}$$

$$N(10) \sim \text{Binomial}\left(100, \frac{1}{3}\right)$$

$$P\{S_n = k\} = \binom{n}{k} p^k q^{n-k}$$

$$= \binom{100}{20}\left(\frac{1}{3}\right)^{20}\left(\frac{2}{3}\right)^{80}$$

$$= \underline{0.0013} \ \#$$

11.7 Malware and Honeypots

(a) $X \sim \text{Exp}(\mu)$, $0 < s < t$

c.d.f : $F_X(x) = P(X \le x) = 1 - e^{-\mu x}$

$$= P(X \le t-s) = \underline{1 - e^{-\mu(t-s)}} \quad \#$$

(b) $P = \frac{1}{t}\int_0^t \left(1 - e^{-\mu(t-s)}\right) ds \qquad u = t-s,\ du = -ds$

$$= \frac{1}{t}\int_t^0 (1 - e^{-\mu u})(-du) = \frac{1}{t}\int_0^t (1 - e^{-\mu u})\,du$$

$$\int_0^t (1 - e^{-\mu u})\,du = \int_0^t 1\,du - \int_0^t e^{-\mu u}\,du$$

$$= t - \left[-\frac{1}{\mu}e^{-\mu u}\right]_0^t = t - \left(-\frac{1}{\mu}e^{-\mu t} + \frac{1}{\mu}e^0\right)$$

$$= t + \frac{1}{\mu}e^{-\mu t} - \frac{1}{\mu}$$

$$P = \frac{1}{t}\left(t + \frac{1}{\mu}e^{-\mu t} - \frac{1}{\mu}\right)$$

$$= 1 - \frac{1}{\mu t} + \frac{1}{\mu t}e^{-\mu t}$$

$$= 1 - \frac{1}{\mu t}(1 - e^{-\mu t})$$

$$= \underline{1 - \frac{1 - e^{-\mu t}}{\mu t}} \quad \#$$

(c)

$$N(t) \sim \text{Possion}(\lambda t)$$

$$E[N_1(t)] \times p = \lambda t p$$

$$\lambda = \frac{E[N_1(t)]}{tp} \approx \frac{N_1(t)}{tp}$$

#

(d) $$N(t) = N_1(t) + N_2(t)$$

$$E[N_2(t)] = E[N(t)] - E[N_1(t)]$$

$$= \lambda t - \lambda t p$$

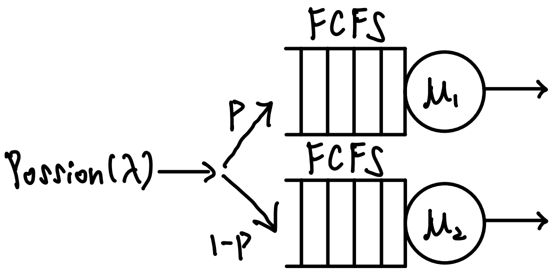
$$= \lambda t(1-p)$$

$$E[N_2(t)] = \left(\frac{N_1(t)}{tp}\right) t(1-p)$$

$$= N_1(t) \frac{1-p}{p}$$

#

13.2 Server Farm



A fraction p of the jobs go to Server 1, which has a service rate of μ_1. And the other fraction $1-p$ go to Server 2, which has a service rate of μ_2.

mean response time formula for an M/M/1 queue:

$$E[T] = \frac{1}{\mu - \lambda}$$

Server 1 $\Rightarrow \lambda_1 = p\lambda$

Server 2 $\Rightarrow \lambda_2 = (1-p)\lambda$

$$E[T_1] = \frac{1}{\mu_1 - py}, \quad E[T_2] = \frac{1}{\mu_2 - (1-p)y}$$

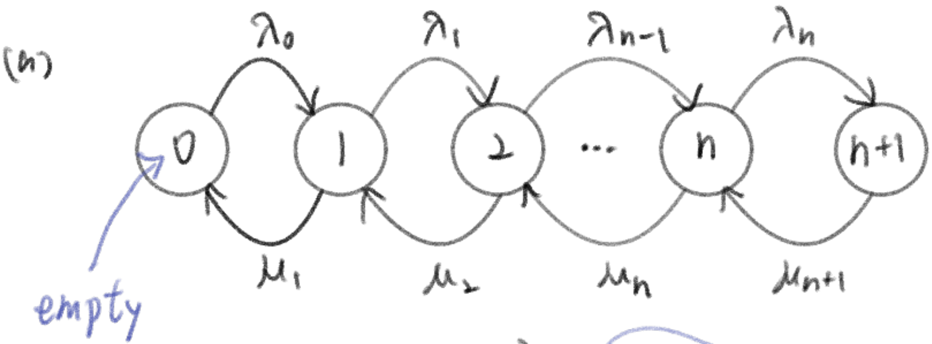
$$E[T] = p \cdot E[T_1] + (1-p) \cdot E[T_2]$$

$$\underline{E[T] = \frac{p}{\mu_1 - p\lambda} + \frac{1-p}{\mu_2 - (1-p)\lambda}}$$

#

13.10 Busy Period in M/M/1

(a)



(b) $E[B] = \underbrace{\frac{1}{\lambda+\mu}} + \underbrace{\left(\frac{\lambda}{\lambda+\mu}\right) E[B_2]}$ → starting from state 2

If it is arrived, the remaining busy period is expected.

The expected time from state 1 to next event.

(c) $E[B_2] = \frac{1}{\mu} + E[B]$

$E[S] = \frac{1}{\mu}$

$E[B] = E[S] + E[N] \times E[B]$

$E[N] = \lambda E[S] = \frac{\lambda}{\mu}$

$\Rightarrow E[B] = \frac{1}{\mu} + \left(\frac{\lambda}{\mu}\right) E[B]$

$\Rightarrow E[B] - \left(\frac{\lambda}{\mu}\right) E[B] = \frac{1}{\mu}$

$\Rightarrow E[B]\left(1 - \frac{\lambda}{\mu}\right) = \frac{1}{\mu}$

$\Rightarrow E[B]\left(\frac{\mu-\lambda}{\mu}\right) = \frac{1}{\mu}$

$\Rightarrow E[B] = \frac{1}{\mu} \times \frac{\mu}{\mu-\lambda} = \frac{1}{\mu-\lambda}$ #

This result show that when $\lambda < \mu$, the expected value of the busy period is similar to the expontial distribution of the parameter $\mu - \lambda$.