CSC0056 Data Communication

Markov Chains

Instructor: Chao Wang

Networked Cyber-Physical Systems Laboratory
Department of Computer Science and Information Engineering
National Taiwan Normal University

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Outline

1. References
2. Discrete-time Markov chain (DTMC) [1]
3. From DTMC to the analysis of the M/M/1 systems [2]

References

- Required reading (DTMC, and from DTMC to CTMC)
  1. Harchol-Balter, Mor. Performance modeling and design of computer systems: queueing theory in action. Cambridge University Press, 2013. ISBN 9781107027503. (Chapter 8)
  2. Bertsekas, Dimitri and Gallager, Robert. Data networks (2nd edition). Prentice Hall, 1992. ISBN 0132009161. (Section 3.3–3.3.1 and Appendix A)
  3. A visual explanation for Markov chains: https://setosa.io/blog/2014/07/26/markov-chains/
- Further reading (for CTMC itself)
  1. Harchol-Balter, Mor. Performance modeling and design of computer systems: queueing theory in action. Cambridge University Press, 2013. ISBN 9781107027503. (Section 9.6 and Chapter 12)

DTMC: discrete-time Markov chain [1]

- A discrete-time Markov chain is a stochastic process $\{X_n, n = 0, 1, 2, ...\}$ where X_n denotes the state at discrete time step n and such that, $\forall n \geq 0$, $\forall i, j$, and $\forall i_0, ..., i_{n-1}$,

$$
\begin{aligned}
&P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, ..., X_0 = i_0\} \\
&= P\{X_{n+1} = j | X_n = i\} \\
&= P_{ij} \text{ (by stationarity)},
\end{aligned}
$$

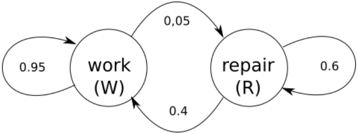
  where P_{ij} is independent of the time step and of past history.
- The Markovian Property: The conditional distribution of any future state X_{n+1}, given past states $X_0, X_1, ..., X_{n-1}$, and given the present state X_n, is independent of past states and depends only on the present state X_n.

Transition probability matrix

- The transition probability matrix associated with any DTMC is a matrix $\mathbf{P}$ whose (i,j)-th entry P_{ij} represents the probability of moving to state j on the next transition, given that the current state is i. Then we have

$$P_{ij}^{n} = \sum_{k=0}^{M-1} P_{ik}^{n-1} P_{kj} \quad \text{and} \quad \mathbf{P}^{n} = \mathbf{P}^{n-1} \cdot \mathbf{P}.$$

- Example:



$$\mathbf{P} = \begin{bmatrix} W \to W & W \to R \\ R \to W & R \to R \end{bmatrix} = \begin{bmatrix} 0.95 & 0.05 \\ 0.4 & 0.6 \end{bmatrix}.$$

Limiting distribution

- In general, for $0 < a, b < 1$, let $\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$, then we will have

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

Now, define the *limiting probability* to be

$$\pi_j = \lim_{n\to\infty} P_{ij}^n = (\lim_{n\to\infty} \mathbf{P}^n)_{ij}.$$

Then for an M-state DTMC, we have

$$\overrightarrow{\pi} = (\pi_0, \pi_1, ..., \pi_{M-1}), \text{ where } \sum_{i=0}^{M-1} \pi_i = 1$$

represents the *limiting distribution* in each state.

Stationary distribution

- A probability distribution $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1})$ is said to be *stationary* for the Markov chain if

$$\vec{\pi} \cdot \mathbf{P} = \vec{\pi} \text{ and } \sum_{i=0}^{M-1} \pi_i = 1.$$

  These equations are called the stationary equations.

- Stationary distribution=Limiting distribution [1]: Given a finite-state DTMC with M states, let $\pi_j = \lim_{n \to \infty} P_{ij}^n > 0$ be the limiting probability of being in state j and let

$$\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1}), \text{ where } \sum_{i=0}^{M-1} \pi_i = 1$$

  be the limiting distribution. Assuming that the limiting distribution exists, then $\vec{\pi}$ is also a stationary distribution and *no other* stationary distribution exists.

Example of the use of the stationary equations

From the example on page 5:

Using Markov chain to analyze the M/M/1 systems [2]

- Let $N(t)$ be the number of messages in the system at time t. Then $\{N(t)|t > 0\}$ is a *Continuous-Time Markov Chain* (CTMC). Each state represents a particular value of $N(t)$.
- To analyze a M/M/1 system, we may instead start with the *Discrete-Time Markov Chain* (DTMC) and then reach CTMC:
  - Consider time points $0, \delta, 2\delta, 3\delta, \ldots$, and let $N_k = N(k\delta)$ denote the number of messages in the system at $k\delta$. Then $\{N_k|k = 0, 1, \ldots\}$ is a DTMC. Then apply $\delta \to 0$ to the analytical results for DTMC to get the results for CTMC.
- In DTMC, denote the transition probability by $P_{ij} = P\{N_{k+1}|N_k = i\}$.
  - Question: how to determine P_{ij}?

The arrival/service statistics for M/M/1

- Messages arrive according to the Poisson process with rate λ.
- The interarrival times are independent and exponentially distributed with parameter λ. For an interval of length τ_n (i.e., the interval between the n-th arrival and the (n+1)-th arrival), we have $P\{\tau_n \leq \delta\} = 1 - e^{-\lambda\delta}$. For $t > 0$, we have (review page 9 of the slides for Poisson process)

$$P\{A(t+\delta) - A(t) = 0\} = e^{-\lambda\delta}\frac{(\lambda\delta)^0}{0!} = 1 - \lambda\delta + \frac{(\lambda\delta)^2}{2} - \cdots$$
$$= 1 - \lambda\delta + o(\delta)$$
$$P\{A(t+\delta) - A(t) = 1\} = e^{-\lambda\delta}\frac{(\lambda\delta)^1}{1!} = \lambda\delta + o(\delta)$$
$$P\{A(t+\delta) - A(t) \geq 2\} = o(\delta)$$

where $o(\delta)$ is a function such that $\lim_{\delta\to 0}\frac{o(\delta)}{\delta} = 0$.

The arrival/service statistics for M/M/1 (cont.)

Accordingly, now we can determine P_{ij} in the following way :)

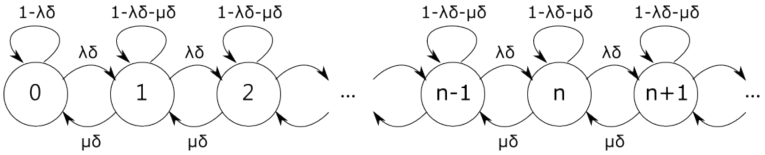
$$\begin{aligned}
P_{00} &= 1 - \lambda\delta + o(\delta) && \\
P_{ii} &= 1 - \lambda\delta - \mu\delta + o(\delta) && i \geq 1 \\
P_{i,i+1} &= \lambda\delta + o(\delta) && i \geq 0 \\
P_{i,i-1} &= \mu\delta + o(\delta) && i \geq 1 \\
P_{ij} &= o(\delta) && i \text{ and } j \neq i, i+1, i-1.
\end{aligned}$$

Some notes on how to obtain the above equations:

- The departure statistics follows the service statistics.
- Beware the difference between the number of arrivals/departures (Poisson distributed) and the interarrival/interdeparture time (Exponential distributed).
- The $o(\delta)$ term includes all higer-order terms asympotically.

The DTMC diagram for M/M/1

Define state n to be the case where there are n messages in the system.



The transistion probabilities shown here are correct up to an $o(\delta)$ term. Let p_n denote the limiting probability of n messages in the system. Then

$$p_n = \lim_{k\to\infty} P\{N_k = n\} = \lim_{t\to\infty} P\{N(t) = n\}.$$

Now, by considering the transition probability and a long traversal between the states, we will have $p_n\lambda\delta + o(\delta) = p_{n+1}\mu\delta + o(\delta)$. Then, as $\delta \to 0$ we will have the following so-called *global balance equations*:

$$p_n\lambda = p_{n+1}\mu.$$

Deriving the average response time of a M/M/1 system

From Little's Law, we have T the average response time equals N/λ, where $N = \sum_{n=0}^{\infty} n p_n$ and p_n can be obtained via global balance equations:

Let $\rho = \lambda/\mu$. Then $p_{n+1} = \rho^{n+1} p_0$. If $\rho < 1$, then $1 = \sum_{n=0}^{\infty} p_n = \frac{p_0}{1-\rho}$.

Therefore, $p_n = \rho^n (1-\rho)$.

$$
\begin{aligned}
N &= \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{\infty} n \rho^n (1-\rho) = \rho(1-\rho) \sum_{n=0}^{\infty} n \rho^{n-1} \\
&= \rho(1-\rho) \frac{\partial}{\partial \rho} \left(\sum_{n=0}^{\infty} \rho^n \right) = \rho(1-\rho) \frac{\partial}{\partial \rho} \left(\frac{1}{1-\rho} \right) = \rho(1-\rho) \frac{1}{(1-\rho)^2} \\
&= \frac{\rho}{1-\rho}. \quad \text{Hence, } T = \frac{N}{\lambda} = \frac{1}{\mu - \lambda}.
\end{aligned}
$$