CSC0056 Data Communication

Week 6, Part I - Poisson Process

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Outline

1. References

2. Background

3. Poisson process

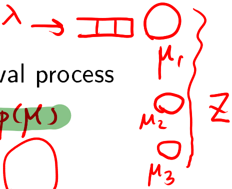
References

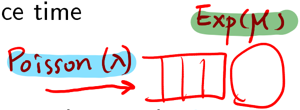
1. Harchol-Balter, Mor. Performance modeling and design of computer systems: queueing theory in action. Cambridge University Press, 2013. ISBN 9781107027503. (Chapters 3 and 11) Poisson process
2. Bertsekas, Dimitri and Gallager, Robert. Data networks (2nd edition). Prentice Hall, 1992. ISBN 0132009161. (Section 3.3 to 3.3.1, before Markov chain formulation)
3. William Feller. An Introduction to Probability Theory and Its Applications, Volume I (3rd edition). Wiley, 1968. ISBN 0471257087.
4. 曹亮吉, Poisson 分佈. http://episte.math.ntu.edu.tw/articles/sm/sm_16_07_1/

Motivation for the following math: the M/M/1 system

M/G/1

- The Kendall notation: a X/Y/Z queueing system
  - X: the distribution of interarrival times for the arrival process
  - Y: the distribution of the service time
  - Z: the number of servers





- The M/M/1 queueing system:
  - Arrival statistics: data arrives according to the Poisson process with rate λ: $\Rightarrow$ The interarrival times follow an exponential distribution (the first M in M/M/1; M for *memoryless*)
  - Service statistics: the data service times follow an exponential distribution with rate μ (the second M in M/M/1; M for *memoryless*)
  - With the above two statistical assumptions, the number of data items in the next moment will only depend on the current number of data items in the system. Thus, Markov chain theory can be applied here to analyze the probability for a system to have N data items, and subsequently we may get a prediction on the response time by applying Little's Law :)

Markovian property

Background: Bernoulli trials and binomial distribution [3]

- Definition: Bernoulli trials are repeated independent trials that have only two possible outcomes for each trial and their probabilities remain the same throughout trials.
  - Example: tosses of a coin

p

$q = 1 - p$

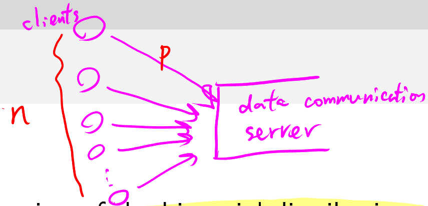
- Definition: Let $b(k; n, p)$ be the probability that n Bernoulli trials (with probability p for success and $q = 1 - p$ for failure) result in k successes and $n - k$ failures. Then

$$P\{S_n = k\} = b(k; n, p) = \binom{n}{k} p^k q^{n-k}$$

$$\frac{n!}{k!(n-k)!}$$

is called the binominal distribution of S_n.

Poisson distribution [3]



- Poisson distribution is an approximation of the binomial distribution. For large n and small p, we have

$\binom{n}{k} p^k q^{n-k}$

$$b(k; n, p) \approx \frac{\lambda'^k}{k!} e^{-\lambda'}$$

where we define parameter $\lambda' = np$. Now, let

$$\boxed{p(k; \lambda') = \frac{\lambda'^k}{k!} e^{-\lambda'}}$$

and we call $p(k; \lambda')$ the Poisson distribution.

Proof of $b(k; n, p) \approx \frac{\lambda'^k}{k!} e^{-\lambda'}$

let $\lambda' = n \cdot p$

From the definition of $b(k; n, p) = \binom{n}{k} p^k q^{n-k}$, for small p and large n we have

$$\frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-(k-1)}} = \frac{b(k; n, p)}{b(k-1; n, p)} = \frac{\lambda' - (k-1)p}{kq} = \frac{\lambda' - (k-1)\overset{\to 0}{p}}{k(1-\underset{\to 0}{p})} \approx \frac{\lambda'}{k}$$

$$= \frac{p(n-(k-1))}{qk}$$

$$\frac{n!}{0!\,n!}$$

which implies

$$b(k; n, p) \approx \frac{\lambda'^k}{k!} b(0; n, p).$$

$b(0; n, p) = \binom{n}{0} p^0 q^n$

$\Rightarrow b(0; n, p) = (1-p)^n$

$\Rightarrow b(0; n, p) = (1 - \frac{\lambda'}{n})^n$

Now, for large n we have $b(0; n, p) \approx e^{-\lambda'}$ using Taylor expansion (i.e., $\ln(1 + t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \ldots$). And therefore, we write $\Rightarrow \ln b(0; n, p) = n \cdot \ln(1 - \frac{\lambda'}{n})$

and let $-\frac{\lambda'}{n} = t$

$\frac{\lambda'}{n} = -t$

$\Rightarrow \ln b(0; n, p) = n \cdot \ln(1+t)$

$\approx n \cdot t$

$$b(k; n, \underbrace{p}_{\lambda'}) \approx \frac{\lambda'^k}{k!} e^{-\lambda'}.$$

Exponential distribution

X

t_0 t_1 timeline

- A random variable X is said to be distributed exponentially with rate λ if X has the following probability density function:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

$P\{X > s\} = 1 - P\{X \leq s\}$

$= e^{-\lambda s}$

- The cumulative distribution function is given by

$\int_0^s \lambda e^{-\lambda x} dx$

$$F_x(s) = P\{X \leq s\} = \int_0^s f(x)\,\mathrm{d}x = 1 - e^{-\lambda s}.$$

② t t_0 timeline

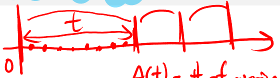
- The mean is given by $E[X] = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x = \frac{1}{\lambda}$.
- Exponential distribution is *memoryless* in the sense that, for $s, t > 0$,

① s t t_0 timeline

$$P\{X > s + t | X > s\} = \frac{P\{X > s + t\}}{P\{X > s\}} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P\{X > t\}.$$

① ②

Poisson process, definition [2]



$A(t) = \#$ of arrivals in $[0, t]$

- A *Poisson process with rate* λ is a stochastic process $\{A(t) | t \geq 0\}$ where
  1. $A(t)$ is a counting process that represents the total # of arrivals in $[0, t]$, and $A(0) = 0$;
  2. Number of arrivals in disjoint time intervals are independent;
  3. Number of arrivals in any intervals of length τ is Poisson distributed with parameter $\lambda\tau$. That is, for all $t, \tau > 0$,

independent increment

stationary increment

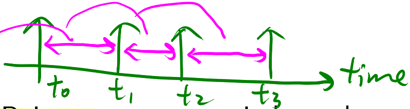
$$P\{A(t+\tau) - A(t) = k\} = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}$$

where k is the number of arrivals in the interval.

(Reminder: a stochastic process is a sequence of random variables)

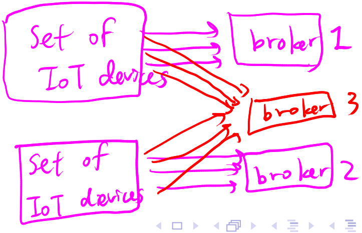
Poisson process, properties [1][2]

See Sections 11.5 and 11.3 in [2]

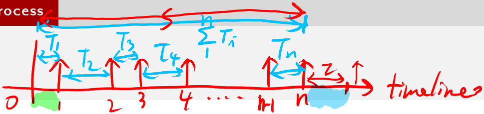


- Property 1: the interarrival times of a Poisson process are independent and exponentially distributed with rate λ.
- Property 2: if two or more independent Poisson processes $A_1, \ldots, A_n$ are merged into a single process $A = A_1 + A_2 + \ldots + A_n$, then A is a Poisson process with a rate equal to the sum of the rates of A_i for $i = 1 \ldots n$.

Section 11.6 in [2]



Proof of Property 1 [1]



Let $T_1, T_2, \ldots, T_n$ be the interarrival times of a sequence of events. By the definition of Poisson process, we have

$$P\{T_1 > \tau\} = P\{A(\tau) = 0\} = \frac{e^{-\lambda\tau}(\lambda\tau)^0}{0!} = e^{-\lambda\tau}.$$

Now, consider

$$P\{T_{n+1} > \tau | \sum_{i=1}^{n} T_i = s\} = P\{\text{zero arrival in } (s, s+\tau] | \sum_{i=1}^{n} T_i = s\}$$
$$= P\{\text{zero arrival in } (s, s+\tau]\}$$
$$(\text{by } \textit{independent increments})$$
$$= e^{-\lambda\tau} \ (\text{by } \textit{stationary increments}),$$

and this shows that

$$P\{T_{n+1} > \tau | \sum_{i=1}^{n} T_i = s\} = P\{T_1 > \tau\}.$$