PREVIEW PROBLEMS Discrete Time Markov Chains

8.1 [Solving for Limiting Distributions]

Consider the program analysis problem from Section 8.3.3. Determine the limiting distribution, (π_C, π_M, π_U), by solving the stationary equations.

Answer: Recall from Section 8.3.3: A program has three types of instructions: CPU instructions (C), Memory instructions (M), and User interaction instructions (U). In analyzing the program, we note that a C instruction is followed by another C instruction with probability 0.7, but with probability 0.2 is followed by an M instruction and with probability 0.1 is followed by a U instruction. We also note that an M instruction is followed with probability 0.1 by another M instruction, but with probability 0.8 is followed by a C instruction, and with probability 0.1 is followed by a U instruction. Finally, a U instruction, is followed by a C instruction with probability 0.9, and with probability 0.1 is followed by an M instruction.

The program can be represented as a Markov chain with the transition probability matrix, $\boldsymbol{P}$:

$$\boldsymbol{P} = \begin{array}{c} \\ C \\ M \\ U \end{array} \begin{array}{c} \begin{array}{ccc} C & M & U \end{array} \\ \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.8 & 0.1 & 0.1 \\ 0.9 & 0.1 & 0 \end{pmatrix} \end{array}$$

The stationary equations are then of the form

$$\begin{aligned} \pi_C &= \pi_C(0.7) + \pi_M(0.8) + \pi_U(0.9) \\ \pi_M &= \pi_C(0.2) + \pi_M(0.1) + \pi_U(0.1) \\ \pi_U &= \pi_C(0.1) + \pi_M(0.1) + \pi_U(0) \end{aligned}$$

together with the normalization equation $1 = \pi_C + \pi_M + \pi_U$. This readily gives

$$\pi_C = 0.74, \pi_M = 0.17, \pi_U = 0.09$$

Applying Theorem 8.6 tells us that the stationary probabilities are equal to the limiting probabilities, assuming the limition distribution exists.

8.2 [Powers of Transition Matrix]

Given any finite-state transition matrix, $\boldsymbol{P}$, prove that for any integer n, $\boldsymbol{P}^n$ maintains the property that each row sums to 1.

Answer: Take any two square matrices $\boldsymbol{P}$ and $\boldsymbol{P}'$ whose row sum-up to 1. We will show that their product also has rows that sum-up to 1. Let

$$\boldsymbol{P} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} \quad \text{and} \quad \boldsymbol{P}' = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1N} \\ b_{21} & b_{22} & \dots & b_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ b_{N1} & b_{N2} & \dots & b_{NN} \end{pmatrix}$$

where by assumption $\forall i$, $\sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N} b_{ij} = 1$. Considering next the product $\boldsymbol{P}'' = \boldsymbol{P}\boldsymbol{P}'$, we have

$p''_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj}$. Focusing on the i^{th} row of $\boldsymbol{P}''$, we get

$$\begin{aligned}
\sum_{j=1}^{N} p''_{ij} &= \sum_{j=1}^{N}\sum_{k=1}^{N} a_{ik} b_{kj} \\
&= \sum_{k=1}^{N} a_{ik} \sum_{j=1}^{N} b_{kj} \\
&= \sum_{k=1}^{N} a_{ik} \\
&= 1
\end{aligned}$$

where we have used the fact that $\forall i,\ \sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N} b_{ij} = 1$. Hence, since the product of two matrices whose rows sum-up to 1 keeps that property, so will any power of the matrix $\boldsymbol{P}$.

Problem S5.1: Consider the scenario of Problem S3.1 restated as follows. A student's life alternates between sleep, work, and drinking coffee/eating. When at home, the student wakes up at the end of each hour with probability $\frac{1}{5}$ and gets up to go to work, and with probability $\frac{4}{5}$ she goes on to sleep another hour. Conversely, when at work, at the end of each hour the student goes back home to sleep with probability $\frac{1}{4}$, stays at work with probability $\frac{2}{3}$, or goes to get coffee at the coffee house with probability $\frac{1}{12}$. Finally, once at the coffee house, after each hour there the student leaves with probability $\frac{3}{4}$ to get back to work, or stays for another hour with probability $\frac{1}{4}$.

Find the fraction of time the student spends sleeping, at work, and at the coffee house, *i.e.,* the probabilities that the student is at home, work and the coffee house. Find also the average durations of the blocks of time the student spends at home (sleeping), at work, and at the coffee house before heading somewhere else.

Answer: The student's life is governed by the finite, discrete time Markov chain shown below.

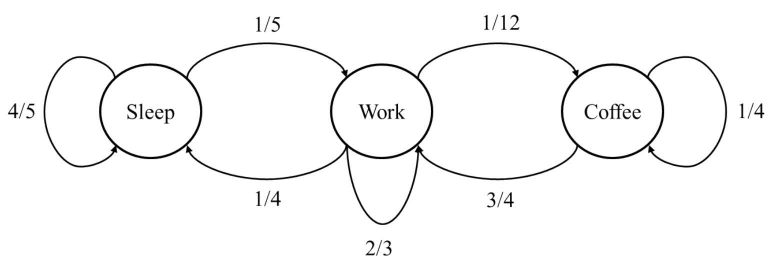

Figure 1: Markov chain representation of student life.

This translates into the following system of equation for the stationary state probabilities.

$$\begin{aligned}
\pi_{\text{Sleep}} &= \pi_{\text{Sleep}}(0.8) + \pi_{\text{Work}}(0.25) + \pi_{\text{Coffee}}(0) \\
\pi_{\text{Work}} &= \pi_{\text{Sleep}}(0.2) + \pi_{\text{Work}}(0.6666) + \pi_{\text{Coffee}}(0.75) \\
\pi_{\text{Coffee}} &= \pi_{\text{Sleep}}(0) + \pi_{\text{Work}}(0.0833) + \pi_{\text{Coffee}}(0.25)
\end{aligned}$$

You can either solve the above equations together with the normalization condition of $\pi_{\text{Sleep}} + \pi_{\text{Work}} + \pi_{\text{Coffee}} = 1$ or raise the transition probability matrix to a sufficiently high power so that all its rows are equal, which gives $\pi_{\text{Sleep}} = 0.529, \pi_{\text{Work}} = 0.424, \pi_{\text{Coffee}} = 0.047$

In order to compute the average duration of the blocks opf time the student spends at home, work and in the coffee shop, consider the fact that the distribution the time the student spends at each location is a geometric distribution with a probability of leaving equal to

$$
\begin{aligned}
p_{\text{leave home}} &= 0.2 \\
p_{\text{leave work}} &= 0.33333 \\
p_{\text{leave coffee house}} &= 0.75
\end{aligned}
$$

Since the mean of a geometrically distributed random variable with parameter p is $\frac{1}{p}$, this implies that we have

$$
\begin{aligned}
E[\text{sleep period}] &= \text{5 hours} \\
E[\text{work period}] &= \text{3 hours} \\
E[\text{coffee period}] &= \text{1 hour 20 minutes}
\end{aligned}
$$