CSC0056 Data Communication

Markov Chains

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Outline

1. References
2. Discrete-time Markov chain (DTMC) [1]
3. From DTMC to the analysis of the M/M/1 systems [2]

References

- Required reading (DTMC, and from DTMC to CTMC)

[1] 1. Harchol-Balter, Mor. Performance modeling and design of computer systems: queueing theory in action. Cambridge University Press, 2013. ISBN 9781107027503. (Chapter 8)

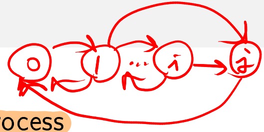
[2] 2. Bertsekas, Dimitri and Gallager, Robert. Data networks (2nd edition). Prentice Hall, 1992. ISBN 0132009161. (Section 3.3–3.3.1 and Appendix A)

3. A visual explanation for Markov chains: https://setosa.io/blog/2014/07/26/markov-chains/

- Further reading (for CTMC itself)

1. Harchol-Balter, Mor. Performance modeling and design of computer systems: queueing theory in action. Cambridge University Press, 2013. ISBN 9781107027503. (Section 9.6 and Chapter 12)

DTMC: discrete-time Markov chain [1]



- A discrete-time Markov chain is a stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$ where X_n denotes the state at discrete time step n and such that, $\forall n \geq 0$, $\forall i, j$, and $\forall i_0, \ldots, i_{n-1}$,

assume now we are at time step n

$$
\begin{aligned}
& P\{X_{n+1} = j | X_n = i, \cancel{X_{n-1} = i_{n-1}, \ldots, X_0 = i_0}\} \\
& = P\{X_{n+1} = j | X_n = i\} \\
& = P_{ij} \text{ (by stationarity)},
\end{aligned}
$$

time-invariant

where P_{ij} is independent of the time step and of past history.

- The Markovian Property: The conditional distribution of any future state X_{n+1}, given past states $X_0, X_1, \ldots, X_{n-1}$, and given the present state X_n, is independent of past states and depends only on the present state X_n.

Transition probability matrix

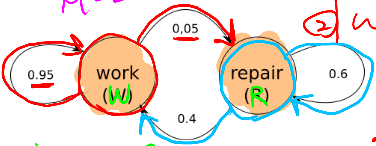
- The transition probability matrix associated with any DTMC is a matrix $\mathbf{P}$ whose (i,j)-th entry P_{ij} represents the probability of moving to state j on the next transition, given that the current state is i. Then we have

$$P_{ij}^n = \sum_{k=0}^{M-1} P_{ik}^{n-1} P_{kj} \quad \text{and} \quad \mathbf{P}^n = \mathbf{P}^{n-1} \cdot \mathbf{P}.$$

$P \cdot P = \begin{bmatrix} 0.95 & 0.05 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.95 & 0.05 \\ 0.4 & 0.6 \end{bmatrix}$

- Example:

$M=2$



① $W \xrightarrow{1} W \xrightarrow{2} W$

② $W \rightarrow R \rightarrow W$

$\Rightarrow P_{ij}^2 = 0.95 \times 0.95 + 0.05 \times 0.4$

$P_{ij}^1 = 0.05$

$P^1 = \begin{bmatrix} 0.95 & 0.05 \\ 0.4 & 0.6 \end{bmatrix}$

$P^2 = \begin{bmatrix} 0.95 & 0.05 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.95 & 0.05 \\ 0.4 & 0.6 \end{bmatrix}$

$= \begin{bmatrix} 0.9225 & 0.0775 \\ 0.62 & 0.38 \end{bmatrix}$

P^x when $x \uparrow$

$\approx \begin{bmatrix} 0.89 & 0.11 \\ 0.89 & 0.11 \end{bmatrix}$

$$\mathbf{P} = \begin{array}{c} \\ W \\ R \end{array} \begin{array}{c} \begin{array}{cc} W & R \end{array} \\ \begin{bmatrix} W \to W & W \to R \\ R \to W & R \to R \end{bmatrix} \end{array} = \begin{array}{c} \\ i \end{array} \begin{array}{c} \begin{array}{cc} & j \end{array} \\ \begin{bmatrix} 0.95 & 0.05 \\ 0.4 & 0.6 \end{bmatrix} \end{array}.$$

Limiting distribution

- In general, for $0 < a, b < 1$, let $\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$, then we will have

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{array}{cc} & \begin{array}{cc} W & R \end{array} \\ \begin{array}{c} W \\ R \end{array} & \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix} \\ & \begin{array}{cc} \pi_W & \pi_R \end{array} \end{array}. \qquad M=2$$

Now, define the *limiting probability* to be

$$\pi_j = \lim_{n\to\infty} P_{ij}^n = (\lim_{n\to\infty} \mathbf{P}^n)_{ij}.$$

Then for an M-state DTMC, we have

$$\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1}), \text{ where } \sum_{i=0}^{M-1} \pi_i = 1$$

represents the *limiting distribution* in each state.

Stationary distribution

stationary probability

- A probability distribution $\vec{\pi} = (\pi_0, \pi_1, ..., \pi_{M-1})$ is said to be *stationary* for the Markov chain if

$$\vec{\pi} \cdot \mathbf{P} = \vec{\pi} \text{ and } \sum_{i=0}^{M-1} \pi_i = 1.$$

These equations are called the stationary equations.

- Stationary distribution=Limiting distribution [1]: Given a finite-state DTMC with M states, let $\pi_j = \lim_{n\to\infty} P_{ij}^n > 0$ be the limiting probability of being in state j and let

$$\vec{\pi} = (\pi_0, \pi_1, ..., \pi_{M-1}), \text{ where } \sum_{i=0}^{M-1} \pi_i = 1$$

be the limiting distribution. Assuming that the limiting distribution exists, then $\vec{\pi}$ is also a stationary distribution and *no other* stationary distribution exists.

Example of the use of the stationary equations

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cc} W & R \end{array} \\ \begin{array}{c} W \\ R \end{array} & \begin{bmatrix} 0.95 & 0.05 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

$$\vec{\pi} = (\pi_W, \pi_R)$$

$$\begin{cases} ① \begin{bmatrix} \pi_W & \pi_R \end{bmatrix} \begin{bmatrix} 0.95 & 0.05 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} \pi_W & \pi_R \end{bmatrix} \\ ② \ \pi_W + \pi_R = 1 \end{cases}$$

From the example on page 5:

$$\Rightarrow \begin{cases} 0.95\pi_W + 0.4\pi_R = \pi_W \\ 0.05\pi_W + 0.6\pi_R = \pi_R \\ \pi_W + \pi_R = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 0.05\pi_W = 0.4\pi_R \\ \pi_W + \pi_R = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \pi_W = 8\pi_R \\ \pi_W + \pi_R = 1 \end{cases} \Rightarrow \begin{cases} \pi_R = \frac{1}{9} \\ \pi_W = \frac{8}{9} \end{cases}$$

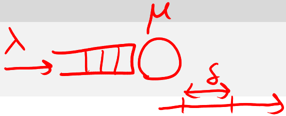
$$\vec{\pi} = (\pi_W, \pi_R) = \left(\frac{8}{9}, \frac{1}{9}\right) = (0.89, 0.11)$$

Using Markov chain to analyze the M/M/1 systems [2]

- Let $N(t)$ be the number of messages in the system at time t. Then $\{N(t)|t > 0\}$ is a *Continuous-Time Markov Chain* (CTMC). Each state represents a particular value of $N(t)$.

- To analyze a M/M/1 system, we may instead start with the *Discrete-Time Markov Chain* (DTMC) and then reach CTMC:
  - Consider time points $0, \delta, 2\delta, 3\delta, \ldots,$ and let $N_k = N(k\delta)$ denote the number of messages in the system at $k\delta$. Then $\{N_k|k = 0, 1, \ldots\}$ is a DTMC. Then apply $\delta \to 0$ to the analytical results for DTMC to get the results for CTMC.
- In DTMC, denote the transition probability by $P_{ij} = P\{N_{k+1}|N_k = i\}$.
  - Question: how to determine P_{ij}?

The arrival/service statistics for M/M/1



- Messages arrive according to the Poisson process with rate λ.
- The interarrival times are independent and exponentially distributed with parameter λ. For an interval of length τ_n (i.e., the interval between the n-th arrival and the (n+1)-th arrival), we have $P\{\tau_n \leq \delta\} = 1 - e^{-\lambda\delta}$. For $t > 0$, we have (review page 9 of the slides for Poisson process)

By Taylor expansion

$$
\begin{aligned}
P\{A(t+\delta) - A(t) = 0\} &= e^{-\lambda\delta} \underbrace{\boxed{\frac{(\lambda\delta)^0}{0!}}}_{=1} = 1 - \lambda\delta + \frac{(\lambda\delta)^2}{2} - \cdots \\
&= 1 - \lambda\delta + o(\delta) \\
P\{A(t+\delta) - A(t) = 1\} &= e^{-\lambda\delta} \frac{(\lambda\delta)^1}{1!} = \lambda\delta + o(\delta) \\
P\{A(t+\delta) - A(t) \geq 2\} &= o(\delta)
\end{aligned}
$$

where $o(\delta)$ is a function such that $\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0$.

The arrival/service statistics for M/M/1 (cont.)

Accordingly, now we can determine P_{ij} in the following way :)

(no arrival) AND (no departure)
$= (e^{-\lambda\delta}) \times (e^{-\mu\delta})$

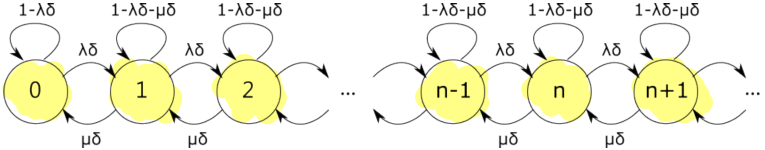
$$
\begin{aligned}
P_{00} &= 1 - \lambda\delta + o(\delta) \\
P_{ii} &= 1 - \lambda\delta - \mu\delta + o(\delta) && i \geq 1 \\
P_{i,i+1} &= \lambda\delta + o(\delta) && i \geq 0 \\
P_{i,i-1} &= \mu\delta + o(\delta) && i \geq 1 \\
P_{ij} &= o(\delta) && i \text{ and } j \neq i, i+1, i-1.
\end{aligned}
$$

Some notes on how to obtain the above equations:

- The departure statistics follows the service statistics.
- Beware the difference between the number of arrivals/departures (Poisson distributed) and the interarrival/interdeparture time (Exponential distributed).
- The $o(\delta)$ term includes all higer-order terms asympotically.

The DTMC diagram for M/M/1

Define state n to be the case where there are n messages in the system.



The transistion probabilities shown here are correct up to an $o(\delta)$ term. Let p_n denote the limiting probability of n messages in the system. Then

$$p_n = \lim_{k\to\infty} P\{N_k = n\} = \lim_{t\to\infty} P\{N(t) = n\}.$$

Now, by considering the transition probability and a long traversal between the states, we will have $p_n\lambda\delta + o(\delta) = p_{n+1}\mu\delta + o(\delta)$. Then, as $\delta \to 0$ we will have the following so-called *global balance equations*:

$$\boxed{p_n\lambda = p_{n+1}\mu.} \rightarrow \quad P_{n+1} = \left(\frac{\lambda}{\mu}\right) \cdot P_n$$

ρ

Deriving the average response time of a M/M/1 system

λ μ

From Little's Law, we have T the average response time equals N/λ, where $N = \sum_{n=0}^{\infty} np_n$ and p_n can be obtained via global balance equations:

Let $\rho = \lambda/\mu$. Then $p_{n+1} = \rho^{n+1} p_0$. If $\rho < 1$, then $1 = \sum_{n=0}^{\infty} p_n = \frac{p_0}{1-\rho}$.

$p_0 = 1 - \rho$

Therefore, $p_n = \rho^n(1-\rho)$.

$$N = \sum_{n=0}^{\infty} np_n = \sum_{n=0}^{\infty} n\rho^n(1-\rho) = \rho(1-\rho)\sum_{n=0}^{\infty} n\rho^{n-1}$$

$$= \rho(1-\rho)\frac{\partial}{\partial\rho}\left(\sum_{n=0}^{\infty}\rho^n\right) = \rho(1-\rho)\frac{\partial}{\partial\rho}\left(\frac{1}{1-\rho}\right) = \rho(1-\rho)\frac{1}{(1-\rho)^2}$$

$= \frac{\rho}{1-\rho}$. Hence, $\boxed{T = \frac{N}{\lambda} = \frac{1}{\mu - \lambda}}$.