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STAT 305

Assignment 1

Question 3

3.1 $Y_n \sim \text{Expon}(\lambda)$ $\quad M_{Y_n}(t) = \frac{\lambda}{\lambda - t}$

$Z_n \sim \text{Bern}(\frac{1}{2})$ $\quad M_{Z_n}(t) = 1 - \frac{1}{2} + \frac{1}{2}e^t = \frac{1}{2}(e^t + 1)$

$$M_{X_1}(t) = E_{X_1}(e^{X_1 t})$$

$$= E_{Y_0, Z_0}(e^{Y_0 Z_0 t})$$

$$= E_{Y_0}(e^{Y_0 \cdot 1 \cdot t})\, Pr(Z_0 = 1) + E_{Y_0}(e^{Y_0 \cdot 0 \cdot t})\, Pr(Z_0 = 0)$$

by applying law of total probability

and because Y_0 and Z_0 are independent

$$= E_{Y_0}(e^{tY_0})(\tfrac{1}{2}) + (1)(\tfrac{1}{2})$$

$$= \tfrac{1}{2} M_{Y_0}(t) + \tfrac{1}{2}$$

$$= \frac{1}{2}\left(\frac{\lambda}{\lambda - t} + 1\right) = \frac{1}{2}\,\frac{\lambda + \lambda - t}{\lambda - t} = \frac{1}{2}\,\frac{2\lambda - t}{\lambda - t} = \boxed{\frac{\lambda - t/2}{\lambda - t}, \; -\infty < t < \lambda}$$

3.2 $X_2 = \frac{1}{2}X_1 + Y_1 Z_1$

$= M + N$ where $M = \frac{X_1}{2}$, $N = Y_1 Z_1 = Y_0 Z_0 = X_1$

$= M_M(t) M_N(t)$ since M and N are independent

$$= E_M(e^{Mt}) E_N(e^{Nt})$$

$$= E_{X_1}(e^{\frac{X_1}{2}t}) E_{X_1}(e^{X_1 t})$$

$$= M_{X_1}(t/2)\, M_{X_1}(t)$$

$$= \frac{1}{2}\left(\frac{\lambda}{\lambda - t/2} + 1\right)\frac{1}{2}\left(\frac{\lambda}{\lambda - t} + 1\right) = \frac{1}{4}\left(\frac{2\lambda}{2\lambda - t} + 1\right)\left(\frac{\lambda + \lambda - t}{\lambda - t}\right)$$

$$= \frac{1}{4}\left(\frac{2\lambda + 2\lambda - t}{2\lambda - t}\right)\frac{2\lambda - t}{\lambda - t} = \frac{1}{4}\,\frac{4\lambda - t}{\lambda - t} = \boxed{\frac{\lambda - t/4}{\lambda - t}, \; -\infty < t < \lambda}$$

3.3 $X_n = \begin{cases} \frac{1}{2} X_{n-1} + Y_{n-1} Z_{n-1} & \text{for } n \geq 2 \\ Y_0 Z_0 & \text{for } n = 1 \end{cases}$

$$M_{X_n}(t) = M_{\frac{1}{2}X_{n-1}}(t) \cdot M_{Y_{n-1}Z_{n-1}}(t)$$

$$= M_{X_{n-1}}(t/2) \cdot M_{Y_{n-1}Z_{n-1}}(t)$$

$$= M_{X_{n-1}}(t/2)\, M_{Y_0 Z_0}(t) = M_{X_{n-1}}(t/2)\, M_{X_1}(t)$$

$$= M_{X_{n-2}}\left(\frac{t/2}{2}\right) M_{X_1}(t/2)\, M_{X_1}(t)$$

$$\vdots$$

$$= M_{X_1}\left(\frac{t}{2^{n-1}}\right) \prod_{k=0}^{n-2} M_{X_1}\left(\frac{t}{2^k}\right)$$

$$= \frac{\lambda - \frac{1}{2}\frac{t}{2^{n-1}}}{\lambda - \frac{t}{2^{n-1}}} \cdot \left(\frac{\cancel{\lambda - \frac{1}{2}\frac{t}{2^0}}}{\lambda - \frac{t}{2^0}} \cdot \frac{\cancel{\lambda - \frac{1}{2}\frac{t}{2^1}}}{\cancel{\lambda - \frac{t}{2^1}}} \cdot \frac{\cancel{\lambda - \frac{1}{2}\frac{t}{2^2}}}{\cancel{\lambda - \frac{t}{2^2}}} \cdots \frac{\lambda - \frac{1}{2}\frac{t}{2^{n-2}}}{\cancel{\lambda - \frac{t}{2^{n-2}}}} \right)$$

$$= \frac{\lambda - \frac{t}{2^n}}{\lambda - \frac{t}{2^{n-1}}} \frac{\left(\lambda - \frac{t}{2^{n-1}}\right)}{(\lambda - t)} = \boxed{\frac{\lambda - \frac{t}{2^n}}{\lambda - t}, \quad -\infty < t < \lambda}$$

3.4 $\lim_{n \to \infty} M_{X_n}(t) = \lim_{n \to \infty} \frac{\lambda - t/2^n}{\lambda - t} = \frac{\lambda}{\lambda - t}$, which is the exponential MGF

Since MGF uniquely identifies a distribution, we can conclude that $X_n \xrightarrow{d} X$ where X is exponentiall distributed