A2Q1

1. $E(\tilde{\theta}_1) = E(2\overline{X}) = 2E(\overline{X}) = 2E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$

$= \frac{2}{n}E\left(\sum_{i=1}^{n} X_i\right) = \frac{2}{n}\sum_{i=1}^{n}(E(X_i)) = \frac{2}{n}n\left(\frac{0+\theta}{2}\right)$

$= 2\left(\frac{\theta}{2}\right) = \boxed{\theta}$

$\tilde{\theta}_1$ is an unbiased estimator of θ

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2. $\tilde{\theta}_2 = \max(X_1, \cdots, X_n)$

CDF of $\tilde{\theta}_2$:

$F_{\tilde{\theta}_2}(y) = P(\max(X_1, \cdots, X_n) \leq y)$

$= P(X_1 \leq y \cup X_2 \leq y \cdots \cup X_n \leq y)$

$= \prod_{i=1}^{n} P(X_i \leq y)$ because X_i are independent

$= (y/\theta)^n = y^n/\theta^n$

$f_{\tilde{\theta}_2}(y) = \frac{d}{dy}F_{\tilde{\theta}_2}(y) = n(y^{n-1}/\theta^n)$, $0<y<\theta$

$E(\tilde{\theta}_2) = \int_0^\theta y f_{\tilde{\theta}_2}(y)dy = \int_0^\theta n\left(\frac{y^{n-1}}{\theta^n}\right)y\,dy$

$= \int_0^\theta \frac{n}{\theta^n} y^n dy = \frac{n}{\theta^n}\left[\frac{y^{n+1}}{n+1}\right]_{y=0}^{y=\theta}$

$= \frac{n}{\theta^n}\frac{\theta^{n+1}}{n+1} = \boxed{\frac{n\theta}{n+1}} \neq \theta$

$\tilde{\theta}_2$ is a biased estimator of θ

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3. $Var(\tilde{\theta}_1) = Var(2\overline{X})$

$= 2^2 Var(\overline{X})$

$= 4 Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$

$= \frac{4}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)$

$= \frac{4}{n^2}\sum_{i=1}^{n}(Var(X_i))$ because X_i are independent

$$= \frac{4}{n^2} n\left(\frac{(\theta-0)^2}{12}\right)$$

$$= \frac{4}{n}\frac{\theta^2}{12} = \boxed{\frac{\theta^2}{3n}}$$

4. See simulation.R and simulation.txt

Comparison: The sample mean & sample variance of the 10000 samples of $\tilde{\theta}_1$ are closer to the theoretical results I've obtained for $E(\tilde{\theta}_1)$ and $Var(\tilde{\theta}_1)$

5. See simulation.R and simulation.txt

6. $\tilde{\theta}_1$ over $\tilde{\theta}_2$: The sample mean & sample variance of the 10000 samples of $\tilde{\theta}_1$ are closer to the theoretical results I've obtained for $E(\tilde{\theta}_1)$ and $Var(\tilde{\theta}_1)$. Also, $\tilde{\theta}_1$ is an unbiased estimator of θ whereas $\tilde{\theta}_2$ is biased.

$\tilde{\theta}_2$ over $\tilde{\theta}_1$: From simulation, $\widehat{MSE}(\tilde{\theta}_2) < \widehat{MSE}(\tilde{\theta}_1)$