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STAT 305

Assignment 1

Question 2

2.1 $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\underbrace{\int_{-\infty}^{\infty} c e^{-\frac{|x-3|}{2}}\,dx}_{M} = 1$$

$$M = c\left[\int_{-\infty}^{3} e^{-\frac{|x-3|}{2}}\,dx + \int_{3}^{\infty} e^{-\frac{|x-3|}{2}}\,dx\right]$$

$$= c\left[\int_{-\infty}^{3} e^{-\frac{(3-x)}{2}}\,dx + \int_{3}^{\infty} e^{-\frac{(x-3)}{2}}\,dx\right]$$

$$= c\left[\int_{-\infty}^{3} e^{\frac{1}{2}x-\frac{3}{2}}\,dx + \int_{3}^{\infty} e^{-\frac{1}{2}x+\frac{3}{2}}\,dx\right]$$

$$= c\left[\left(2e^{\frac{1}{2}x-\frac{3}{2}}\right)\Big|_{x=-\infty}^{x=3} + \left(-2e^{-\frac{1}{2}x+\frac{3}{2}}\right)\Big|_{x=3}^{x=\infty}\right]$$

$$= c\left[2e^{0} - (-2e^{0})\right] = 4c$$

$$M = 1 \Rightarrow 4c = 1 \Rightarrow c = \boxed{\frac{1}{4}}$$

2.2 $M_X(t) = E_X(e^{tx})$

$$= \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

$$= C\int_{-\infty}^{\infty} e^{tx} e^{-\frac{|x-3|}{2}}\,dx$$

$$= C\left[\int_{-\infty}^{3} e^{tx} e^{-\frac{(3-x)}{2}}\,dx + \int_{3}^{\infty} e^{tx} e^{-\frac{(x-3)}{2}}\,dx\right]$$

$$= C\left[\int_{-\infty}^{3} e^{(t+\frac{1}{2})x-\frac{3}{2}}\,dx + \int_{3}^{\infty} e^{(t-\frac{1}{2})x+\frac{3}{2}}\,dx\right]$$

$$= C\left(\left[\frac{1}{t+\frac{1}{2}} e^{(t+\frac{1}{2})x-\frac{3}{2}}\right]\Bigg|_{x=-\infty}^{x=3} + \left[\frac{1}{t-\frac{1}{2}} e^{(t-\frac{1}{2})x+\frac{3}{2}}\right]\Bigg|_{x=3}^{x=\infty}\right)$$

$$= C\left(\frac{1}{t+\frac{1}{2}} e^{(t+\frac{1}{2})3-\frac{3}{2}} + \underline{0} - \frac{1}{t-\frac{1}{2}} e^{(t-\frac{1}{2})3+\frac{3}{2}}\right)$$

if $t-\frac{1}{2}<0$

$$= C\left(\frac{1}{t+\frac{1}{2}} - \frac{1}{t-\frac{1}{2}}\right)e^{3t} = \frac{e^{3t}}{4}\left(\frac{1}{t+\frac{1}{2}} - \frac{1}{t-\frac{1}{2}}\right)$$

$$= e^{3t}\left(\frac{1}{4t+2} - \frac{1}{4t-2}\right) = \frac{e^{3t}(4t-2-(4t+2))}{(4t+2)(4t-2)}$$

$$= \frac{e^{3t}(-4)}{16t^2-4} = \boxed{\frac{e^{3t}}{1-4t^2} \text{ where } |t|<\frac{1}{2}}$$

e^{tx} is positive for all x $\Rightarrow M_X(t) = E_X(e^{tX})$ must be >0

$\Rightarrow 1-4t^2>0 \Rightarrow |t|<\frac{1}{2}$

2.3 In 2.2, we found $|t| < \frac{1}{2}$

Also the definition of MGF requires that $M_X(t)$ be defined in a neighbourhood of zero

$\Rightarrow$ interval for t: $-\frac{1}{2} < t < \frac{1}{2}$

2.4 $$\frac{d}{dt} M_X(t) = \frac{d}{dt} \frac{e^{3t}}{(1-4t^2)} = \frac{(1-4t^2)3e^{3t} - e^{3t}(-8t)}{(1-4t^2)^2}$$

$$= \boxed{\frac{e^{3t}(-12t^2+8t+3)}{(1-4t^2)^2}}$$

2.5 $$\frac{d}{dt^2} M_X(t) = \frac{d}{dt} \frac{e^{3t}(-12t^2+8t+3)}{(1-4t^2)^2}$$

$$= \boxed{\frac{(1-4t^2)^2\left[3e^{3t}(-12t^2+8t+3) + e^{3t}(-24t+8)\right] + e^{3t}(-12t^2+8t+3)\cdot 2(1-4t^2)(-8t)}{(1-4t^2)^4}}$$

2.6 $$E(X) = \frac{d}{dt} M_X(0) = \frac{e^0(3)}{1} = \boxed{3}$$

2.7 $$E(X^2) = \frac{d}{dt^2} M_X(0) = \frac{3(3)+8}{1} = 17$$

$$Var(X) = E(X^2) - (E(X))^2 = 17 - 3^2 = \boxed{8}$$