COMP 790-125: Goals for today

- Matching pursuit
- Learning dictionaries
  - batch learning
  - online learning
- Sparse representations, compressive sensing

Dictionary based representations

Last time we talked about sparsely representing a patch

$$\underset{\boldsymbol{\alpha}}{\operatorname{argmin}} \frac{1}{2} \left\| \underbrace{\mathbf{R}_{i,j}\mathbf{y}}_{\text{extracted patch}} - \underbrace{\mathbf{D}\boldsymbol{\alpha}}_{\substack{\text{combination of} \\ \text{dictionary elements}}} \right\|_2^2 + \lambda \underbrace{\|\boldsymbol{\alpha}\|_1}_{\text{sparsity penalty}}$$

We talked about using dictionaries to obtain denoised images

$$\underset{\boldsymbol{\alpha},\mathbf{x}}{\operatorname{argmin}} \frac{\mu}{2} \|\mathbf{x}-\mathbf{y}\|_2^2 + \sum_{i,j} \left\{ \frac{1}{2} \|\mathbf{R}_{i,j}\mathbf{x} - \mathbf{D}\boldsymbol{\alpha}_{i,j}\|_2^2 + \lambda \|\boldsymbol{\alpha}_{i,j}\|_1 \right\}$$

All the while, we assumed that we know $\mathbf{D}$, and we looked at some standard (orthonormal) dictionaries.

Matching pursuit

One of the benefits of having an orthogonal dictionary $\mathbf{D}$ is that

$$\underset{\boldsymbol{\alpha}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{y} - \mathbf{D}\boldsymbol{\alpha}\|_2^2 + \|\boldsymbol{\alpha}\|_1$$

has a closed form solution

$$\alpha_j = \mathbf{d}_j^T \mathbf{y}$$

But, if the dictionary is overcomplete, and consequently cannot possibly be orthonormal, we cannot guarantee that this update is optimal.

Matching pursuit – dictionary coherence

Recall that orthonomal dictionaries are those for which $\mathbf{d}_i^T \mathbf{d}_j = 0, i \neq j$.

Departure from orthonomality called coherence is defined as

$$\mathcal{M}(\mathbf{D}) = \max_{i,j} |\mathbf{d}_i^T \mathbf{d}_j|.$$

If coherence is small then update

$$\alpha_j = \mathbf{d}_j^T \mathbf{y}$$

might still be approximately correct.

Matching pursuit algorithm [2] is based on this idea

Initialize: $\alpha_i = 0, \forall i, \mathbf{r} = \mathbf{y}$
while $\|\mathbf{r}\|_2 > \text{threshold}$ **do**
  $k = \text{argmax}_j |\mathbf{d}_j^T \mathbf{r}|$
  $\alpha_k = \mathbf{d}_k^T \mathbf{r}$
  $\mathbf{r} = \mathbf{r} - \alpha_k \mathbf{d}_k$
end while

Learning dictionaries

In order to make story simpler we will assume that there is a patchset[1] $\mathrm{Data} = \{\mathbf{y}_i : i = 1..n\}$

$$\underset{\boldsymbol{\alpha}, \mathbf{D}}{\text{minimize}} \frac{1}{2} \sum_i \|\mathbf{y}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2 + \lambda \|\boldsymbol{\alpha}_i\|_1$$

We want to simultaneously learn weights for each patch and dictionary.

[1] We do not need to bother with details of patch extraction, since we are not focusing on reconstruction of clean signal. We did that last time.

Learning dictionaries

A problem

$$\mathbf{D}\boldsymbol{\alpha}_i = (\frac{1}{3}\mathbf{D})(3\boldsymbol{\alpha}_i)$$

so we could learn dictionary $\mathbf{D}$ and weights $\boldsymbol{\alpha}_i$ or just scale the dictionary down and weights up.

More broadly, matrix factorization $\mathbf{A} = \mathbf{BC}$ is not unique[2]

For some $\mathbf{T}$ such that $\mathbf{T}\mathbf{T}^T = \mathbf{I}$ we can obtain a different factorization $\mathbf{A} = (\mathbf{BT})(\mathbf{T}^T\mathbf{C})$.

Q: What could we possibly do?

[2] Be on the lookout for such problems when your optimization problem has product terms.

Learning dictionaries

Just constrain the dictionary elements to have norm less or equal to 1

$$\begin{aligned} \underset{\boldsymbol{\alpha}, \mathbf{D}}{\text{minimize}} \quad & \frac{1}{2} \sum_i \|\mathbf{y}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2 + \lambda \|\boldsymbol{\alpha}\|_1 \\ \text{subject to} \quad & \|\mathbf{d}_i\|_2 \leq 1, \forall i \end{aligned}$$

Q: Can you argue that with this constraint optimal dictionary consists of elements with norm exactly 1?

Learning dictionaries

$$\begin{aligned} \underset{\boldsymbol{\alpha}, \mathbf{D}}{\text{minimize}} \quad & \frac{1}{2} \sum_i \|\mathbf{y}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2 + \lambda \|\boldsymbol{\alpha}\|_1 \\ \text{subject to} \quad & \|\mathbf{d}_j\|_2 \leq 1, \forall j \end{aligned}$$

Even though we eliminated spurious weight and dictionary combinations with unit norm constraint a more insidious problem persists.

There exist local minima, solutions that locally look optimal but are not global optima.

The problem is convex in $\boldsymbol{\alpha}$ *given* $\mathbf{D}$ and vice versa.

$$\begin{aligned} \underset{\boldsymbol{\alpha}}{\text{argmin}} \quad & \frac{1}{2} \sum_i \|\mathbf{y}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2 + \lambda \|\boldsymbol{\alpha}\|_1 \\ \underset{\mathbf{D}, \forall j \|\mathbf{d}_j\|_2 \leq 1}{\text{argmin}} \quad & \frac{1}{2} \sum_i \|\mathbf{y}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2 + \lambda \|\boldsymbol{\alpha}\|_1 \end{aligned}$$

Learning dictionaries – an algorithm

for $t = 1, \ldots \mathrm{MAXITERATIONS}$ **do**
  for $i = 1, \ldots T$ **do**
    $$\boldsymbol{\alpha}_i = \underset{\boldsymbol{\alpha}_i}{\operatorname{argmin}} \frac{1}{2} \left\| \mathbf{y}_i - \mathbf{D}\boldsymbol{\alpha}_i \right\|_2^2 + \lambda \left\| \boldsymbol{\alpha}_i \right\|_1$$
  end for
  $$\mathbf{D} = \underset{\mathbf{D}, \forall j \left\| \mathbf{d}_j \right\|_2 \leq 1}{\operatorname{argmin}} \frac{1}{2} \sum_i \left\| \mathbf{y}_i - \mathbf{D}\boldsymbol{\alpha}_i \right\|_2^2$$
end for

Updating dictionaries

A number of different algorithms to solve

$$\boldsymbol{\alpha}_i = \underset{\boldsymbol{\alpha}_i}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{y}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2 + \lambda \|\boldsymbol{\alpha}_i\|_1$$

- Coordinate descent
- Matching Pursuit
- LARS (Least Angle Regression)[3]
- FISTA (Fast Iterative Shrinkage and Thresholding Algorithm)[1].

What about dictionary update

$$\mathbf{D} = \underset{\mathbf{D}, \forall j \left\|\mathbf{d}_j\right\|_2 \leq 1}{\operatorname{argmin}} \frac{1}{2} \sum_i \|\mathbf{y}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2 ?$$

We will use *projected* gradient descent.

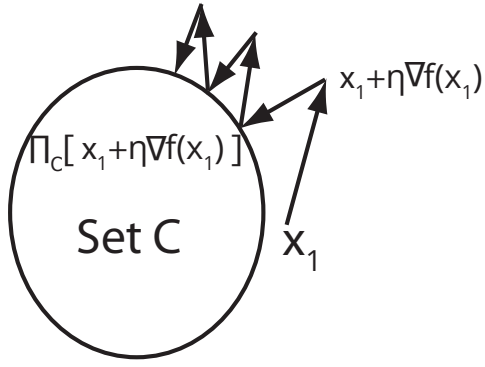
Projected gradient descent

Like gradient descent, but with projection back onto the constrained set.

$$\underset{\mathbf{x} \in C}{\text{minimize}}\, f(\mathbf{x})$$

The algorithm alternates two steps:

for $i = 1, .., \text{MAXIT}$ **do**
  $\mathbf{x}^0 = \mathbf{x} - \eta \nabla f(\mathbf{x})$
  $\mathbf{x} = \Pi_C[\mathbf{x}^0]$
end for



Projection step for unit norm ball

Projection is defined as

$$\Pi_C[\mathbf{x}] = \operatorname*{argmin}_{\mathbf{z} \in C} \|\mathbf{x} - \mathbf{z}\|_2^2$$

a point in set C closest to point $\mathbf{x}$.

In our case for a single column of dictionary $S = \{\mathbf{z} : \|\mathbf{z}\|_2 \leq 1\}$ and

$$\Pi_S[\mathbf{x}] = \frac{\mathbf{x}}{\max(1, \|\mathbf{x}\|_2)}$$

and it amounts to normalizing a vector only if its norm is greater than 1.

For full dictionary $C = \{\mathbf{A} : \forall j \, \|\mathbf{a}_j\|_2 \leq 1\}$ and

$$\Pi_C[\mathbf{D}] = \begin{bmatrix} \Pi_S[\mathbf{d}_1] & \Pi_S[\mathbf{d}_2] & \cdots & \Pi_S[\mathbf{d}_1] \end{bmatrix}$$

So, just normalize columns whose norm is larger than 1.

Dictionary learning update

Hence a single dictionary update, for given $\boldsymbol{\alpha}$ amounts to running this code

for $i = 1, .., \mathrm{MAXIT}$ **do**
$\mathbf{D}^0 = \mathbf{D} + \frac{\eta}{n} \sum_i (\mathbf{y}_i + \sum_i (\mathbf{y}_i - \mathbf{D}\alpha_i)\alpha_i^T$
$\mathbf{D} = \Pi_C[\mathbf{D}^0]$
end for

where n is the number of samples (or patches) and η is a step size.

Full learning algorithm

for $t = 1, ...\text{MAXITERATIONS}$ **do**
  for $i = 1, ...T$ **do**
    $\boldsymbol{\alpha}_i = \underset{\boldsymbol{\alpha}_i}{\text{argmin}} \frac{1}{2} \|\mathbf{y}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2 + \lambda \|\boldsymbol{\alpha}_i\|_1$
  end for
  for $i = 1, .., \text{MAXIT}$ **do**
    $\mathbf{D}^0 = \mathbf{D} + \frac{\eta}{n} \sum_i (\mathbf{y}_i + \sum_i (\mathbf{y}_i - \mathbf{D}\alpha_i)\alpha_i^T$
    $\mathbf{D} = \Pi_C[\mathbf{D}^0]$
  end for
end for

Illustrative applications of dictionary learning – inpainting

Illustrative applications of dictionary learning – completion

Online dictionary learning

Batch algorithms, pass through whole dataset before updating parameters

Online algorithms work on smaller subsets of data at a time.

for $t = 1, ...\mathrm{MAXITERATIONS}$ **do**

randomly choose a sample $\mathbf{z}_t$

$$\boldsymbol{\alpha}_t = \underset{\boldsymbol{\alpha}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{z}_t - \mathbf{D}\boldsymbol{\alpha}\|_2^2 + \lambda \|\boldsymbol{\alpha}\|_1$$

$$\mathbf{D}^t = \underset{\mathbf{D}, \forall j \|\mathbf{d}_j\|_2}{\operatorname{argmin}} \frac{1}{t} \sum_{i=1}^{t} \frac{1}{2} \sum_i \|\mathbf{z}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2$$

end for

A dictionary learning implementation

Use SPArse Modeling Software package.

It is available from:
`http://spams-devel.gforge.inria.fr/downloads.html`

It as interfaces for R, python and Matlab.

It is fairly efficient, and you can try out your ideas in a matter of hours.

Sparse representations

We implicitly assumed that given a patch $\mathbf{y}$ and a dictionary $\mathbf{D}$ we can find a *sparse* $\boldsymbol{\alpha}$ to encode that patch so that

$$\mathbf{y} \approx \mathbf{D}\boldsymbol{\alpha}.$$

We are going to look into when we can guarantee that such sparse representation exists, and in some cases when it is guaranteed to be exact, that is

$$\mathbf{y} = \mathbf{D}\boldsymbol{\alpha}.$$

Johnson-Lindenstrauss Lemma

Lemma

For a set V points in $\mathbf{R}^n$ and some $\epsilon \in (0,1)$ and integer $k = O(1/\epsilon^2 \log |V|)$ there exists a map $f : \mathbf{R}^n \to \mathbf{R}^k$ such that for all $u, v \in V$

$$(1-\epsilon)\|u-v\|^2 \leq \|f(u)-f(v)\|^2 \leq (1+\epsilon)\|u-v\|^2$$

Johnson-Lindenstrauss Lemma

The number of dimensions needed to capture the distances between datapoints with a small distortion is logarithmic in the size of the set.

- No mention of distributions – any set of points
- No mention of the source space dimension n just the number of data points $\log |V|$

Note that the projection f need not preserve the feature identities.

A coordinate in the low dimensional space may combine apples and oranges, e.g. $1/2$*height $+$ $1/3$*weight - $1/4$*age

Practical construction of mapping[3]

We will use a simple linear mapping $f(\mathbf{u}) = \mathbf{Ax}$

First, randomly generate matrix $\mathbf{U} : k \times n$

- rows uniformly sampled basis vectors of length n
- rows sampled from a spherical Gaussian $\mathcal{N}(0, I_n)$
- each entry is randomly sampled from +1/-1

$$p(u_{i,j} = 1) = 1/2 \qquad p(u_{i,j} = -1) = 1/2$$

- each entry is randomly sampled from $\{-1, 0, +1\}$

$$p(u_{i,j} = \frac{1}{\sqrt{q}}) = \frac{1}{2q} \quad p(u_{i,j} = -\frac{1}{\sqrt{q}}) = \frac{1}{2q} \quad p(u_{i,j} = 0) = 1 - \frac{1}{q}$$

From matrix $\mathbf{U}$ construct final projection matrix

$$\mathbf{A} = \frac{1}{\sqrt{k}}\mathbf{U}$$

[3] Johnson and Lindenstrauss 1984, Indyk and Motwani 1998, Achlioptas 2003,Matoušek 2008.

Sparse mapping

The last choice ensured that there are $1 - \frac{1}{q}$ zero entries in the matrix $\mathbf{A}$.

This results in a cheap projection operation with all of the benefits of the JL lemma.

Locality sensitive hashing

In our exploration of JL lemma we came up with a scheme

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

that did dimensionality reduction but preserved the distances between points.

It turns out you can go even further and construct a mapping

$$h_{\mathbf{a}}(\mathbf{x}) = \operatorname{sign}(\mathbf{a}^T\mathbf{x})$$

Each row of matrix $\mathbf{A}$ defines a single hash function.

We will denote the group of hashing functions

$$h_A : \mathbf{R}^N \to \{-1, +1\}^k$$

which produces a k long sign vector. Note that storing this vector takes k bits!

Sparse representations

The framework is very similar to the regression setup.

We are given a set of vectors $(\mathbf{x}_1, \ldots, \mathbf{x}_p)$ that form a basis of some space S or alternatively a frame.[4]

We wish to find a representation of a vector $\mathbf{y}$

$$\mathbf{y} = \sum_i w_i \mathbf{x}_i = \mathbf{X}\mathbf{w}$$

where the number of nonzero ws

$$\|\mathbf{w}\|_0 = \sum_i [w_i \neq 0]$$

is as small as possible.

[4] A frame is a set of possibly linearly dependent vectors that span a space with a constraint that there exist $0 < A \leq B < \infty$ such that $A \|\mathbf{v}\|_2 \leq \sum_i \langle \mathbf{x}_i, \mathbf{v} \rangle \leq B \|\mathbf{v}\|_2$

ℓ_0 is hard but ℓ_1 is not

Solving optimization problem (P_0)

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \|\mathbf{w}\|_0 \\ \text{subject to} \quad & \mathbf{y} = \mathbf{X}\mathbf{w} \end{aligned}$$

is NP-hard, but its convex relaxation (P_1)

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \|\mathbf{w}\|_1 \\ \text{subject to} \quad & \mathbf{y} = X\mathbf{w} \end{aligned}$$

is not. This problem is named Basis Pursuit.

Note that unlike the regression we require exact equality between $\mathbf{y}$ and its representation $\mathbf{X}\mathbf{w}$

Basis Pursuit Denoising is a lasso regression analog.

When does solving (P_1) give solution to (P_0)?

We can define coherence property for our basis/frame

$$\mathcal{M}(\mathbf{X}) = \max_{i \neq j} |\mathbf{x}_i^T \mathbf{x}_j|$$

Theorem

Suppose sparsest solution w^*

$$\|\mathbf{w}^*\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mathcal{M}(\mathbf{X})}\right)$$

then solving (P_1) *gives the same solutions as the original* (P_0) *problem.*

Restricted Isometry Property

Definition

A vector $\mathbf{w} \in \mathbf{R}^n$ is k-sparse if it has at most k non-zero entries.

Definition

If for any k-sparse vector $\mathbf{w}$ it holds that

$$(1 - \delta_k) \|\mathbf{w}\|_2^2 \leq \|\mathbf{X}\mathbf{w}\|_2^2 \leq (1 + \delta_k) \|\mathbf{w}\|_2^2$$

then X has k-Restricted Isometry Property (RIP) with δ_k.

Put differently, using $\mathbf{X}$ to map $\mathbf{w}$ into a different space gives a vector of similar length as $\mathbf{w}$.

Contrast with Johnson Lindenstrauss for $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$

$$(1 - \epsilon) \|\mathbf{u} - \mathbf{v}\|^2 \leq \|\mathbf{A}(\mathbf{u} - \mathbf{v})\|^2 \leq (1 + \epsilon) \|\mathbf{u} - \mathbf{v}\|^2$$

If matrix $\mathbf{A}$ satisfies JL Lemma then RIP can be shown to hold for this matrix as well.

When does solving (P_1) give solution to (P_0)?

$$(1 - \delta_k) \|\mathbf{w}\|_2^2 \leq \|\mathbf{X}\mathbf{w}\|_2^2 \leq (1 + \delta_k) \|\mathbf{w}\|_2^2$$

Theorem

If $\mathbf{X}$ *is k-RIP with* $\delta_{2k} < \sqrt{2} - 1$*, then for all k-sparse vectors* $\mathbf{w}$ *such that* $\mathbf{X}\mathbf{w} = \mathbf{y}$ *the solution for* (P_1) *is equal to solution of* (P_0)*.*

To summarize, if you construct a matrix that is k-RIP then you can represent a vector $\mathbf{y}$ in this basis with some small distortion. Further, recovery of this representation can be done solving a convex optimization problem (P1).

Sensing

If a high-dimensional signal $\mathbf{y}$ is sparsely representable in some basis $\mathbf{X}$

$$\mathbf{y} = \mathbf{X}\mathbf{w}$$

how much of $\mathbf{y}$ do we need to know to reconstruct $\mathbf{w}$ and hence the rest of $\mathbf{y}$.

We will split the issues of representation and sensing:

- representation: $\mathbf{y} = \mathbf{X}\mathbf{w}$ where w is k-sparse vector
- sensing: $\mathbf{z} = \mathbf{R}\mathbf{y} = \mathbf{R}\mathbf{X}\mathbf{w}$ is measured

Sensing

Assuming we only measured some entries of vector $\mathbf{z} = \mathbf{R}\mathbf{y} = \mathbf{R}\mathbf{X}\mathbf{w}$ indexed by set M the recovery problem is

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \|\mathbf{w}\|_1 \\ \text{subject to} \quad & z_i = \langle \mathbf{r}_i, \mathbf{X}\mathbf{w} \rangle, \forall i \in M \end{aligned}$$

How large does set M have to be?

Recovery from compressive sensing measurements

We define coherence between basis

$$\mu(X, R) = \sqrt{n} \max_{i,j} | \langle \mathbf{x}_i, \mathbf{r}_j \rangle |$$

and for any two bases $\mu(\mathbf{A}, \mathbf{B}) \in [1, \sqrt{n}]$

Theorem

Given a representation basis $\mathbf{X}$ *and sensing basis* $\mathbf{R}$ *and a vector* $\mathbf{y} = \mathbf{X}\mathbf{w}$ *with* $\mathbf{w}$ *k-sparse then if*

$$m \geq C \mu^2(\mathbf{X}, \mathbf{R}) k \log \frac{n}{\delta}$$

and we randomly select m values to sense from $\mathbf{z} = \mathbf{R}\mathbf{y}$ *we can recover* $\mathbf{w}$ *with probability* $1 - \delta$.

Practicalities

You don't need to know the k up front, you simply solve the problem

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \|\mathbf{w}\|_1 \\ \text{subject to} \quad & z_i = \langle \mathbf{r}_i, \mathbf{X}\mathbf{w} \rangle, \forall i \in M \end{aligned}$$

The choice of M is random so any subset of vectors in $\mathbf{R}$ is fine.

Note that the problem is not the same as Lasso but you can derive an ADMM algorithm for example.

Compressive sensing further reading

Compressive sensing is a very active field.

Your best bet in exploring applications is to start at `http://dsp.rice.edu/cs`

Interesting ideas for projects relating to compressive sensing would involve learning dictionaries using prior knowledge and possibly giving strong theoretical guarantees about ability to reconstruct signals.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