hw4

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1 Task 1: Missing Tile

- **Theorem:** Any $2^n \times 2^n$ grid with one painted cell can be tiled using L-shaped triominoes such that the entire grid is covered by the triominoes, but no triomino overlaps with another or with the painted cell.
- **Base Case:** Show that $P(1)$ is true.

  For $n = 1$, a $2^1 \times 2^1 = 2 \times 2$ grid has four cells. - If one cell is painted, three cells remain, which can be covered by one L-shaped triomino. - Therefore, $P(1)$ is true.
- **Inductive Step:** Assume that $P(k)$ is true for $k \geq 1$.

  A $2^k \times 2^k$ grid with one painted cell can be covered by triominoes.
- **Want to Show:** $P(k+1)$ is also true.
- **Consider a $2^{k+1} \times 2^{k+1}$ board with one square painted.** Divide the grid into four $2^k \times 2^k$ subgrids by splitting in both the horizontal and vertical directions.
- **By inductive hypothesis(IH)**, each $2^k \times 2^k$ subgrid can be tiled with L-shaped triominoes when one cell is painted. Placing a center triomino covers one cell in each of the three subgrids without the original painted cell, leaving each subgrid with exactly one painted cell. We can now apply the hypothesis to each subgrids tile.
- This completes the tiling of the $2^{k+1} \times 2^{k+1}$ grid without any overlap or covering of the original painted cell.
- Thus, by Mathematical Induction (MI), $P(n)$ is true for all positive integers n. Therefore, any $2^n \times 2^n$ grid with one painted cell can be tiled by L-shaped triominoes.

2 Task 4: Tail Sum of Squares

Goal: Prove that `sumSqr(n)` correctly returns

$$1^2 + 2^2 + 3^2 + \ldots + n^2$$

for $n \geq 1$.

Define a helper function `sumHelper(n, a)` that returns

$$a + 1^2 + 2^2 + \ldots + n^2$$

- **Base Case:** Let $n = 0$.
  Then, `sumHelper(0, a)` returns a, which satisfies the sum because there are no terms to add.
- **Inductive Hypothesis (IH):** Assume that for some $k \geq 0$, `sumHelper(k, a)` correctly computes

  $$a + 1^2 + 2^2 + \ldots + k^2.$$

- **WTS** `sumHelper(k+1, a)` correctly computes

  $$a + 1^2 + 2^2 + \ldots + k^2 + (k+1)^2.$$

- **Proof:** By the Inductive Hypothesis (IH), we know that

  $$\texttt{sumHelper(k, a)} = a + 1^2 + 2^2 + \ldots + k^2.$$

  If we call `sumHelper(k+1, a)`, the function adds $(k+1)^2$ to the sum of the first k squares. Thus,

  $$\texttt{sumHelper(k+1, a)} = a + 1^2 + 2^2 + \ldots + k^2 + (k+1)^2.$$

- **Conclusion:** This matches the desired result, so by Mathematical Induction, `sumHelper(n, a)` correctly computes

  $$a + 1^2 + 2^2 + \ldots + n^2$$

  for all integers $n \geq 0$.

3 Task 5: Mysterious Function

- **Base Case:** Let $n = 1$.
  We have `foo(1)` which returns the pair $(1, 2)$.
  Let $\frac{p}{q} = \frac{1}{2}$.
  Check if $\frac{1}{2} = 1 - \frac{1}{n+1}$ holds when $n = 1$.
  Substitute $n = 1$:

$$1 - \frac{1}{1+1} = 1 - \frac{1}{2} = \frac{1}{2}$$

  Therefore, the base case holds.
- **Inductive Hypothesis (IH):** Assume that for some integer $k \geq 1$,

$$\frac{1}{2} = 1 - \frac{1}{k+1}$$

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- **Inductive Step:** Show that the statement is also true when $n =$ k+1

$$(p, q) = \texttt{foo(k+1)} = (q + p \cdot (k+1)(k+2),\, q \cdot (k+1)(k+2))\,.$$

  Thus, we have:

$$p = q + p \cdot (k+1)(k+2)$$

  and

$$q = q \cdot (k+1)(k+2).$$

  Now, calculate $\frac{p}{q}$:

$$\frac{p}{q} = \frac{q + p \cdot (k+1)(k+2)}{q \cdot (k+1)(k+2)} = \frac{q}{q \cdot (k+1)(k+2)} + \frac{p \cdot (k+1)(k+2)}{q \cdot (k+1)(k+2)}.$$

  Simplify the expression:

$$= \frac{1 + p \cdot (k+1)(k+2)}{q} / (k+1)(k+2).$$

  From the Inductive Hypothesis (IH), we know that:

$$\frac{p}{q} = 1 - \frac{1}{k+1}.$$

  because,

$$\frac{p \cdot (k+1)(k+2)}{q} = \left(1 - \frac{1}{k+1}\right)(k+1)(k+2)$$

$$= \left((k+1) - \frac{k+1}{k+1} \right) (k+2)$$

$$= (k+1) - 1(k+2)$$

$$= k + 1 - 1(k+2)$$

$$= k(k+2)$$

Therefore,

$$\frac{1 + k(k+2)}{(k+1)(k+2)} = \frac{1 + k^2 + 2k}{(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2 - 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2 - 1}{(k+1)(k+2)}$$

because,

$$(k+1)^2 - 1 = k^2 + 2k + 1 - 1 = k^2 + 2k$$

So we have:

$$\frac{k^2 + 2k}{(k+1)(k+2)}$$

Factor out k in the numerator:

$$= \frac{k(k+2)}{(k+1)(k+2)}$$

$$= \frac{k}{k+1}$$

Rewrite as $1 - \frac{1}{k+1}$:

$$= 1 - \frac{1}{k+1}$$

Therefore, by Mathematical Induction (MI),

$$\frac{p}{q} = 1 - \frac{1}{n+1} \quad foralln > 1.$$

4 Task 7: Midway Tower

Subtask 1:

- $N = 0, 1, \ldots, N-1$ -Base Case: $n = 1$

$$P(1) = 2^0 - 1 = 0$$

Therefore, the base case holds.

Inductive Step: Assume that for $n = k$, the process generates exactly $2^k - 1$ lines of instructions.

WTS:

For $n = k + 1$, it will generate exactly $2^{k+1} - 1$ lines of instruction.

Proof for $n = k + 1$:

- Move the top k disks from peg 0 to peg 1. By the Induction Hypothesis (IH), this step generates $2^k - 1$ moves.
- Move the $(k+1)$-th disk (the largest disk) from peg 0 to peg 2. This requires 1 move.
- Move the top k disks from peg 1 to peg 2. Again, by the Induction Hypothesis, this requires $2^k - 1$ moves.

Therefore, the total moves for $k + 1$ disks is:

$$(2^k - 1) + 1 + (2^k - 1) = 2 \cdot 2^k - 1 = 2^{k+1} - 1$$

Thus, by induction, we prove that for any $n \geq 0$, the solution requires $2^n - 1$ moves to transfer n disks in the Tower of Hanoi.