MATH 6911 Project
Valuing American Options: Least-Squares Monte Carlo Approach

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- An Option is a contract that gives the holder the right, but not the obligation, to buy/sell a specific asset
- European option: can be exercised at only the maturity date
- American option: can be exercised at any time before the maturity date
- Consider a Black-Scholes model

$$S_t = S_0 \exp\left(r - \sigma^2/2\right)t + \sigma W_t$$

- For an American call option, it is never optimal to exercise before maturity, which is similar to a European option
- Goal: Value American put option

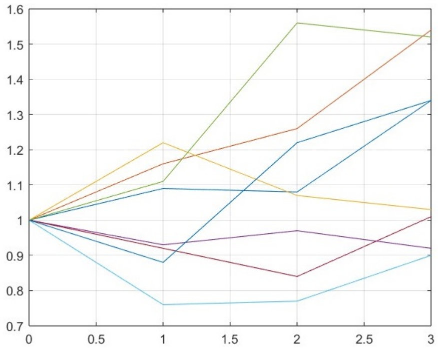
Least Squares Monte Carlo (LSM) Approach

- Recall: In the American option, for each time step we compare the payoff of exercise immediately and the expected payoff of continuation
- Strategy: exercise the option if exercise value $>$ continuation value
- LSM Approach: Estimate conditional expectation of the continuation value using least squares

- Consider an American put option on a share of non-dividend-paying stock.
- Current stock price $S_0 = 1$, Strike price $K = 1.10$, maturity $T = 3$, interest rate $r = 6\%$

Table 1: Stock Price Paths

Path	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34



- Starting from $t = 3$, conditional on not exercising the option before $t = 3$, the cash flows are as follows in Table 2
- Since $t = 3$ is the final expiration date, the cash flows are the same as the European option

Table 2: Cash Flows Matrix at $t = 3$

Path	$t = 1$	$t = 2$	$t = 3$
1	-	-	0
2	-	-	0
3	-	-	0.07
4	-	-	0.18
5	-	-	0
6	-	-	0.20
7	-	-	0.09
8	-	-	0

- At $t = 2$, we need to decide (only if in the money):
  - Exercise immediately
  - Continue the option
- Define X to be the stock price at $t = 2$, and Y to be the discounted cash flows from $t = 3$ if the option is not exercised
- Note: We only include in the money paths here

Table 3: Regression at $t = 2$

Path	Y	X = Stock Price
1	0 x 0.94176	1.08
2	-	-
3	0.07 x 0.94176	1.07
4	0.18 x 0.94176	0.97
5	-	-
6	0.20 x 0.94176	0.77
7	0.09 x 0.94176	0.84
8	-	-

- Exercise value: $K - X = 1.10$ - X
- Continuation value, $E[Y|X]$: Regression of Y
- As an example, regress Y on constant, X, X^2.

$$E[Y|X] = -1.070 + 2.983X - 1.813X^2$$

Table 4: Optimal Early Exercise Decision at $t = 2$

Path	Exercise Value	Continuation Value
1	0.2	0.0369
2	-	-
3	0.3	0.0461
4	0.13	0.1176
5	-	-
6	0.33	0.1520
7	0.26	0.1565
8	-	-

Table 5: Cash Flows Matrix at $t = 2$

Path	$t = 1$	$t = 2$	$t = 3$
1	-	0	0
2	-	0	0
3	-	0	0.07
4	-	0.13	0
5	-	0	0
6	-	0.33	0
7	-	0.26	0
8	-	0	0

- Note: When we exercise the option at $t = 2$, the cash flow in $t = 3$ column will be zero because the option can only be exercised once

- Continue recursively, at $t = 1$

Table 6: Regression at $t = 1$

Path	Y	X = Stock Price
1	0 x 0.94176	1.09
2	-	-
3	-	-
4	0.13 x 0.94176	0.93
5	-	-
6	0.33 x 0.94176	0.76
7	0.26 x 0.94176	0.92
8	0 x 0.94176	0.88

- Same as before, regress Y on constant, X, X^2.

$$E[Y|X] = 2.038 - 3.335X + 1.356X^2$$

Table 7: Optimal Early Exercise Decision at $t = 1$

Path	Exercise Value	Continuation Value
1	0.1	0.0139
2	-	-
3	-	-
4	0.17	0.1092
5	-	-
6	0.34	0.2866
7	0.18	0.1175
8	0.22	0.1533

Table 8: Cash Flows Matrix

Path	$t = 1$	$t = 2$	$t = 3$
1	0	0	0
2	0	0	0
3	0	0	0.07
4	0.17	0	0
5	0	0	0
6	0.34	0	0
7	0.18	0	0
8	0.22	0	0

- Option value: discount each path's cash flow back to $t = 0$, and take the average of all of them
- Value for this American put option: 0.1144
- Value for European put option: 0.0564

LSM Algorithm

- Define $F(\omega; t_k)$ to be the value of continuation at time t_k.
- Formally,

$$F(\omega; t_k) = E_Q[\sum_{j=k+1}^{K} exp(-\int_{t_k}^{t_j} r(\omega, s)ds)C(\omega, t_j; t_k, T)|\mathcal{F}_{tk}]$$

where $C(\omega, s; t, T)$ is the cash flow at time s, conditional on the option not exercised at or before time t, and $r(\omega, t)$ is the interest rate.

- Idea: $F(\omega; t_{K-1})$ can be represented as a linear combination of a countable set of $\mathcal{F}_{t_{K-1}}$ measurable basis functions
- Basis functions: Laguerre, Hermite, Chebyshev, Jacobi, Legendre, Gegenbauer polynomials

- Suppose the value of the asset follows a Markov process, one of possible basis is the weighted Laguerre polynomials:

$$L_0(X) = \exp(-X/2)$$

$$L_1(X) = \exp(-X/2)(1 - X)$$

$$L_2(X) = \exp(-X/2)(1/2)(2 - 4X + X^2)$$

$$L_3(X) = \exp(-X/2)(1/6)(6 - 18X + 9X^2 - X^3)$$

$$L_4(X) = \exp(-X/2)(1/24)(24 - 96X + 72X^2 - 16X^3 + X^4)$$

$$\vdots$$

$$L_n(X) = \exp(-X/2)(1/n!)(n! + n(n!)(-x) + \cdots + n^2(-x)^{n-1} + (-x)^n)$$

- Define as linear combination of $L_j(X)$, where a_j are constant

$$F(\omega; t_{K-1}) = \sum_{j=0}^{\infty} a_j L_j(X)$$

Framework

1. Simulate N_{MC} paths
2. Discretize into K exercisable time as $0 < t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_K = T$
3. At time step $t_k = T$, calculate cash flows, $C(\omega, t_k; t_{k+1}, T)$, for each path
4. Determine which paths are in the money
5. This step and below only consider in the money paths. Calculate exercise value by strike price K - stock price X
6. Calculate continuation value. Approximate $F(\omega; t_{k-1})$ by the first M basis functions of Laguerre polynomials. Regressing the discounted values of $C(\omega, t_k; t_{k+1}, T)$ onto the basis functions for paths that are in the money at time t

Framework

7. Compare exercise & continuation value to decide whether to exercise immediately or not
8. Based on Step 7, calculate the cash flow for time step t_{k-1}
9. Loop through Steps 4-8 for time step $t = T-1, T-2, \ldots, 3, 2$
10. Starting from $t = 0$, continue to move forward until the first non-zero cash flow time and calculate the payoff by discounting the cash flow back to time zero. The same applies for each path.
11. Value of the option is the average of payoff for all paths

Code

Refer Matlab

$S_0 = 105,\ K = 100,\ r = 0.03,\ \sigma = 0.25,\ T = 1,\ Nt = 365,$
$N_{MC} = 1000000$

Table 9: Comparison of LSM and Explicit Scheme* for $S_0 = 105$, $K = 100$

No. of Basis Functions	Option Price (LSM)	Differences
3	6.7616	0.0014
4	6.7621	0.0010
5	6.7576	0.0055

* Option Price from Explicit Scheme: 6.7631

$K = 100,\ r = 0.03,\ \sigma = 0.25,\ Nt = 365,\ N_{MC} = 1000000,\ k = 4$

Table 10: Option Price for $S_0 = 90, 95, 100, 110,\ T = 0.25, 0.5, 1$

S_0	T	Price (LSM)	Price (Explicit Scheme)	Differences
90	0.25	10.9017	10.9080	0.0063
	0.5	12.0458	12.0537	0.0079
	1	13.7916	13.7971	0.0056
95	0.25	7.3623	7.3659	0.0037
	0.5	8.9082	8.9120	0.0039
	1	11.0144	11.0046	0.0099
100	0.25	4.6586	4.6545	0.0041
	0.5	6.3816	6.3952	0.0136
	1	8.6800	8.6744	0.0056
110	0.25	1.5131	1.5198	0.0067
	0.5	3.0155	3.0238	0.0083
	1	5.2164	5.2197	0.0033

Conclusion

- The value estimated from LSM is very close to the Explicit Scheme
- LSM is useful in situations where finite difference can't be used, e.g. value depends on multiple factors
- Monte Carlo simulations are useful as the state variables can follow general stochastic processes, e.g. jump diffusions, non-Markovian processes, general semimartingales
- Practical advantage: Monte Carlo simulations can be implemented with parallel computing

References

Longstaff, F. A., & Schwartz, E. S. (2001).Valuing american options by simulation: A simple least-squares approach. *The Review of Financial Studies, Vol. 14, No. 1., pp. 113-147.*

Wikipedia contributors. (2024). Laguerre polynomials — Wikipedia, the free encyclopedia [[Online; accessed 20-April-2024]].

The method we illustrated assumes that the option is only exercisable in a discrete amount of time. However, it can be shown that LSM applies even for an option that is exercisable continuously.

Proposition 1. *For any finite choice of M, K, and vector $\theta \in R^{M\times(K-1)}$ representing the coefficients for the M basis functions at each of the $K-1$ early exercise dates, let $LSM(\omega; M, K)$ denote the discounted cash flow resulting from following the LSM rule of exercising when the immediate exercise value is positive and greater than or equal to $\widehat{F}_M(\omega_i; t_k)$ as defined by θ. Then the following inequality holds almost surely,*

$$V(X) \geq \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} LSM(\omega_i; M, K).$$

Proof. See the appendix. ■

Appendix: Convergence Results

By selecting a large enough M, which is the number of basis functions (Laguerre polynomials), and let $N \to \infty$, the option value approximated by LSM will be within ϵ of true value. Since ϵ is arbitrary, this means that LSM can be used to converge to any degree of accuracy as desired.

Proposition 2. *Assume that the value of an American option depends on a single state variable X with support on $(0, \infty)$ which follows a Markov process. Assume further that the option can only be exercised at times t_1 and t_2, and that the conditional expectation function $F(\omega; t_1)$ is absolutely continuous and*

$$\int_0^\infty e^{-X} F^2(\omega; t_1) dX < \infty,$$
$$\int_0^\infty e^{-X} F_X^2(\omega; t_1) dX < \infty.$$

Then for any $\epsilon > 0$, there exists an $M < \infty$ such that

$$\lim_{N\to\infty} \Pr\left[\mid V(X) - \frac{1}{N} \sum_{i=1}^{N} LSM(\omega_i; M, K) \mid > \epsilon \right] = 0.$$

Thank you.