Parallel Algorithm Examples

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March 3, 2022

Embarrassingly Parallel

- An embarrassingly parallel computation is a collection of tasks that require *none* or *little* communication among them. In other words, they are *independent*.
- This is "embarrassing" since nothing needs to be done to get good parallel performance.
- People doing parallel processing, e.g. me, are not fond of this kind of computation.

Monte Carlo Method

- The Monte Carlo method repeats a *random process* to compute the answer.
- We generate random numbers as input to the computation to derive the answer from the random process. Note that all computations are *independent*.
- It is important to use different *random seed* with these tasks so that the results from them are *statistically independent*.

Compute π

- Randomly throw darts into a square with an inscribed circle.
- Compute the number of darts that fall into and outside the circle.
- Compute the probability that a dart falls into the circle.
- Multiple the probability by 4 to approximate π.

Notes

- This is not a good example because people will not compute π this way.
- Nevertheless this is an easy-to-understand example to illustrate the independence of tasks.

Parameter Search

- Suppose we want to fins the a set of "best" parameters $x = (x_1, \ldots, x_n)$, which maximize an objective function $y = f(x)$.
- We also assume that the f function is very complex so that we cannot deduce the values of $f(x)$'s for x's that we have not yet computed the function values, from those function values that we have already computed.
- It is easy to dispatch the computation of $f(x)$ to processors to speed up the parameter search, since there is no dependency between these computations.

File Serving

- A web server serves static HTML files to clients.
- The clients' requests are independent, so the web server serves the files in parallel, maybe using multiple threads.
- If there are multiple web servers, they can also serve the files in parallel.

Summary

- Embarrassingly parallel computation has good speedup and efficiency because the tasks are independent and do not require much communication.
- It is trivial to dispatch tasks to processors if they require roughly the same amount of time.
- It is non-trivial to dispatch tasks to processors if they require a different amount of time. In this case, we need *load balancing*.

Discussion

- Give an example of embarrassingly parallel computation.

Divide and Conquer

- Divide-and-conquer is a common parallel algorithm design technique.
- As in a sequential divide-and-conquer algorithm, the problem is first divided into sub-problems.
- Unlike a sequential divide-and-conquer algorithm, a parallel algorithm solves (conquer) the sub-problem *in parallel*.
- Some communication may be necessary since the sub-problems may depend on each other.
- Finally, we combine the answers from individual sub-problems into the final answer.

Sequential Summation

- We want to sum n numbers, and n is huge.
- We add all numbers to the first one, and at the end, it has the sum.
- How many additions do this algorithm use?
- Is it possible to improve this algorithm?

Lower and Upper Bounds

- The lower bound is the minimum amount of operations we need to solve this problem.
- For every algorithm A, there exists an input so that the cost is at least $L(n)$ (as a function of the input size n).
- The upper bound is the minimum number of operations we need to solve for all possible inputs.
- There exists an algorithm $A*$, and for all inputs, the cost is at most $U(n)$ (as a function of the input size n).

Discussion

- How do you find a lower bound?
- How do you find a upper bound?
- Is it possible that $L(n)$ is asumptotically larger than $U(n)$?.

Summation

- We want to sum n numbers, and n is very large.
- It is easy to see that we can apply divide-and-conquer technique to solve the problem in parallel with p processors.

Parallel Summation Algorithm

1. Partition the numbers so that each processor has roughly n/p numbers.
2. Each processor computes the sum of assigned numbers.
3. A processor collects all the partial sums from other processors and computes the final sum.

Analysis

- We assume that first step takes very little time. This is the case for shared memory model, but not necessarily true for distributed memory model.
- The second steps takes $O(\frac{n}{p})$ times.
- The third steps takes $O(p)$ times.

The time complexity is as follows.

$$O(\frac{n}{p} + p) \tag{1}$$

Discussion

- How do we minimize the $O(\frac{n}{p} + p)$ by choosing the right p?

Communication

- Having one processor collects all the answers is not efficient.
- We partition the processor into two groups. Every processor in the first group sends its answer to the corresponding processors in the second group.
- We repeatedly do this until we have only one processor left, who should have the final answer.

Parallel Summation Algorithm

1. Partition the numbers so each processor has n/p numbers.
2. Each processor computes the sum of its numbers.
3. Use the recursive algorithm to compute the final sum.
4. This is similar to the *tree optimization* in synchronization.

Analysis

- We assume that first step takes very little time.
- The second steps takes $O(\frac{n}{p})$ time.
- The third steps takes $O(\log p)$ time because the depth of a complete binary tree of n nodes is about $O(\log n)$.

The final time complexity is as follows.

$$O(\frac{n}{p} + \log p) \tag{2}$$

Discussion

- Describe the difference between the previous two algorithms.

Observation

- The first term ($\frac{n}{p}$) is *computation*. We can *never* reduce this part.
- The second term ($\log p$) is about *communication*. We try our best to reduce this part.
- If we increase p, the computation time decreases and the communication time increases. That means we have more workers to share the workload, but we need to communicate more among more workers.

More Observations

- It is important to balance the load among processors. We want to send $(\frac{n}{p})$ data to each processor for processing.
- A shared memory implementation is significantly easier than a distributed memory implementation.
- The recursive (or tree-like) communication pattern is much more complicated than a naive one, and it requires complicated synchronization.

Analysis

Speedup

$$k = \frac{n}{\frac{n}{p} + \log p} \tag{3}$$

Efficiency

$$e = \frac{n}{n + p \log p} \tag{4}$$

Choice of p

- What is the best p in terms of speedup?
- Set $\frac{n}{p} = \log p$ and solve $p = \frac{n}{\log n}$.
- The minimum parallel execution time $\Theta(\log n)$ is achieved when $p = \Theta(\frac{n}{\log n})$.
- If we set $p = \sqrt{n}$, then the time will be $\Theta(\sqrt{n})$, which is much larger than the optimal $\Theta(\log n)$.

Discussion

- Use calculus to compute the optimal p value.
- Is this *theoretical* optimal p useful in practice?

Discussion

- What will happen if we set p to n?
- What is the commuication time of this algorithm?
- What are the upper and lower bounds of the number of stages this algorithm requires?
- Can you improve the algorithm?

Prefix Sum

Given n numbers $(x_1, \ldots, x_n)$, we want to compute all prefix sums as follows.

$$s_k = \sum_{i=1}^{k} x_i \tag{5}$$

Compact An Array

- The prefix sum has various applications.
- If there are zeros and non-zeros in an array A and we only wish to keep the non-zeros in a new array B, then we can do a prefix sum on another array P with 0 and 1 to determine the positions of non-zeros in B.

Pact an Array

A	before packng	4	0	9	0	0	8	5	7
P	0 and 1	1	0	1	0	0	1	1	1

prefix sum

	new index	1	1	2	2	2	3	4	5
B	after packing	4	9	8	5	7			

Fibonacci's Numbers

- The prefix sum has various applications, and it is not limited to summation.
- We all know Fibonacci's numbers.

$$f_i = \begin{cases} 0 & i = 0 \\ 1 & i = 1 \\ f_{i-1} + f_{i-2} & i \geq 2 \end{cases} \tag{6}$$

Fibonacci's Numbers

$$
\begin{aligned}
f_1 &= f_1 && (7)\\
f_2 &= f_0 + f_1 && (8)\\
\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} && (9)\\
\begin{pmatrix} f_2 \\ f_3 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} && (10)\\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} && (11)\\
\begin{pmatrix} f_i \\ f_{i+1} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^i\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} && (12)
\end{aligned}
$$

Prefix Product

Given n matrices $(x_1, \ldots, x_n)$, we want to compute all prefix products as follows.

$$s_k = \Pi_{i=1}^{k} x_i \tag{13}$$

Discussion

- Give an example of prefix sum application.

Parallel Prefix Sum Algorithm

- Use the k-th processor to compute s_k.

Analysis

- A sequential algorithm can do this easily in $O(n)$ time.
- We assume that we use one processor per data, so $p = n$.
- The k-th processor requires $O(k)$ time.
- The parallel time is the maximum of all processor time, hence $O(n)$.
- The speedup is $O(1)$, and the efficiency is $(\frac{1}{p})$.
- Not very efficient.

Discussion

- What is wrong with the previous algorithm?

Parallel Prefix Sum Algorithm

- To avoid doing duplicated work, we again use the k-th processor to compute s_k, but we get the result from the $k-1$-th processor.

$$s_k = s_{k-1} + x_k \tag{14}$$

Analysis

- Now, we do not duplicate work, but we need to wait.
- The k-th processor cannot compute its sum before receiving s_{k-1}.
- The result will go ripple-like from the first to the last processor like a wave-front.
- The parallel time is the maximum of all processor time, hence $O(n)$.
- The speedup is $O(1)$, and the efficiency is $(\frac{1}{n})$.
- Again not very efficient.

Discussion

- What is wrong with the previous algorithm?

A Better Algorithm

- There are $\log n$ stages.
- In the i-stage every element adds the element 2^i to the left to itself.
  - In the first stage every element adds the element to its left to itself.
  - In the second stage every element adds the element two elements to its left to itself.

Parallel Prefix Sum Algorithm

for $i \leftarrow 0$ TO $\log n - 1$ **do**
  for $k \leftarrow 2^i$ TO n **do**
    x[k] += x[k - 2^i] {The k-th processor receives a partial sum from the $k - i$-th processor.}
  end for
end for

Parallel Prefix

2	1	7	4	2	3	1	5
2	3	8	11	6	5	4	6
2	3	10	14	14	16	10	11
2	3	10	14	16	19	20	25

Analysis

- There are $\log n$ steps, since $p = n$ now.
- A processor does a sum and receives a message, so the time is $O(1)$.
- The total parallel time is $O(logn)$, which is much better than $O(n)$ in previous approaches.
- The speedup is $O(\frac{n}{logn})$, and the efficiency is $(\frac{1}{\log n})$.

Discussion

- Prove that the algorithm is correct.
- What is the possible problem with the previous algorithm?

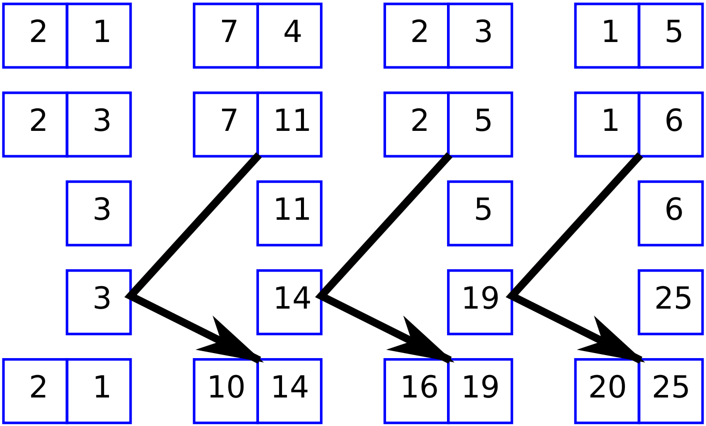
Improvement

- It is not practical to use one processor per data since the number of data is much more than the number of processors in practice.
- We will assume that n is much larger than p; hence we need to partition the data among processors.
- Now, each processor will compute the prefix sum of its data first, then use the previous algorithm to "patch" things up.

Improved Algorithm

1. Partition data among processors.
2. Each processor computes its prefix sum.
3. Use the previous algorithm to compute the prefix sum of the *last* elements from all processors.
4. Use the prefix sum from the last elements to patch up the answers.

Parallel Prefix



Analysis

- A sequential algorithm can do this easily in $O(n)$ time.
- The first step does not take time.
- Both the second and the fourth step take $O(\frac{n}{p})$ time.
- The third step takes $O(\log p)$, as discussed before.

$$T_p = O(\frac{n}{p} + \log p) \tag{15}$$

- Similar optimization can find good p.

Final Notes

- What did we compute?

$$s_4 = (((x_1 + x_2) + x_3) + x_4) \quad (16)$$

$$s_4 = ((x_1 + x_2) + (x_3 + x_4)) \quad (17)$$

Discussion

- What property the operation must have in order for this algorithm to work?
- Does "+" have this property?
- Does "maximum" have this property?
- Does matrix multiplication have this property?

Sorting

- To sort keys between 1 and n in order.
- We try the bucket sort first.

Bucket Sort

- We need b buckets.
- We need to know the range of keys, and we assume that the keys are *evenly* distributed in this range.
- We use an array element to record whether a key *appears* in the input.
- We do not compare!

Bucket Sort

1. Scan the keys and record its appearance in the corresponding buckets.
2. Read the keys from the buckets.

Bucket Sort

Example 1: Bucket sort

```
void bucketsort(int array[], int n, int b)
{
  int i, j = 0;
  int *bucket = calloc(b + 1, sizeof(int));
  for (i = 0; i < n; i++)
    bucket[array[i]]++;
  for (i = 0; i <= b; i++)
    while(bucket[i]--)
      array[j++] = i;
}
```

[1]

[1] http://www.eecs.ucf.edu/courses/cop3502h/spr2007/sorting3.pdf

Bucket Sort

array

2	1	7	4	2	3	1	5

bucket

2	2	1	1	1	0	1	0	0	0
1	2	3	4	5	6	7	8	9	10

array

1	1	2	2	3	4	5	7

Analysis

- The first step takes $O(n)$ time.
- The second step takes $O(n + b)$ time.
- The total complexity is $O(n + b)$.

Discussion

- Why the second step takes $O(n+b)$, not $O(nb)$ time?

Parallel Bucket Sort

1. Every processor has exactly one bucket.
2. Every processor scans all keys to record keys corresponding to its bucket.
3. Read the keys from the buckets.

Bucket Sort

array	2	1	7	4	2	3	1	5

bucket	2	2	1	1	1	0	1	0	0	0
	1	2	3	4	5	6	7	8	9	10

array	1	1	2	2	3	4	5	7

Parallel Bucket Sort

- Every processor takes $O(n)$ time just to scan data.
- Exactly how to "read the keys from the bucket"?

Discussion

- What is wrong with the previous algorithm?

Parallel Bucket Sort

- Every processor reads only $\frac{n}{p}$ keys, and record the appearance into its own set of buckets.
- Now it only takes $O(\frac{n}{p})$ time.

Read Keys

How do we read the keys out in the second step?

1. Every processor remembers the number of keys it places in every bucket.
2. Each processor computes the number of keys that should be in its bucket.
3. Use the parallel prefix sum algorithm to know each bucket's starting position.
4. Read from the bucket.

Bucket Sort

array	2	1	7	4	2	3	1	5		
	1	2	3	4	5	6	7	8	9	10
bucket	1	1	0	0	0	0	0	0	0	0
bucket	0	0	0	1	0	0	1	0	0	0
bucket	0	1	1	0	0	0	0	0	0	0
bucket	1	0	0	0	1	0	0	0	0	0
number	2	2	1	1	1	0	1	0	0	0
total number			5		2			1		0
prefix sum			5		7			8		8
array	1	1	2	2	3	4	5	7		

Analysis

1. The first step takes $O(\frac{n}{p})$ time.
2. The second step takes $O(b)$ because each processor needs to add b numbers.
3. The third step takes $O(\log p)$.
4. the fourth step takes $O(\frac{n}{p})$.[2]

[2] Assuming the keys are evenly distributed among processors.

Final Time Complexity

1. The "read to bucket" takes $O(\frac{n}{p})$ time.
2. The "read from bucket" takes $O(\frac{n}{p} + \log p + b)$ time.
3. The total time complexity is $O(\frac{n}{p} + \log p + b)$.
4. The speedup is $O(\frac{n+b}{\frac{n}{p}+\log p+b})$, which is $O(\frac{n+b}{\log n+b})$ when we set p to $\frac{n}{\log n}$.

Discussion

- What is wrong with the previous algorithm?

Improvement

- The bucket sort uses (and wastes) a lot of memory.
- We will try a "quicker" sort that also uses the divide-and-conquer techniques.
- Again, we assume that the keys have an even distribution, and we know the range.

Sequential "Quicker" Sort

Sort the keys recursively as follows.

- Partition keys into g groups according to $g - 1$ pivots.
- Individually sort the keys in each group recursively.
- Concatenate all the keys from the g groups.

Analysis

- The first step takes $O(n)$ time.
- Let the time to sort n keys be $T(n)$.

$$T(n) = gT(\frac{n}{g}) + n \tag{18}$$

- It is easy to see that $T(n) = O(n \log_g n) = O(n \log n)$.

Discussion

- Explain what will happen when there are only two groups in the "quicker" sort.

Parallel Quicker Sort

Every processor manages a group, which will store a range of keys.

1. Every processor reads only $\frac{n}{p}$ keys and puts them into the corresponding bucket.
2. Each processor sorts the keys in its bucket.
3. Read all keys from processors.

Pact an Array

array	2	1	7	4	2	3	1	5

group	2	1	1	3	2	1	1	2	2	3
group	4	5				4	5			
group	7					7				
group										

array	1	1	2	2	3	4	5	7

Analysis

- The first step takes $O(\frac{n}{p})$ time, without considering the synchronization.
- The second step takes $O(\frac{n}{p} \log \frac{n}{p})$, assuming that the keys are evenly distributed.
- The third step takes $O(\log p)$ from previous analysis on prefix sum.
- The total time is $O(\frac{n}{p} \log \frac{n}{p} + \log p)$.
- The speedup is $O(\frac{n \log n}{\frac{n}{p} \log \frac{n}{p} + \log p})$.

Discussion

- What synchronization mechanism do we need for the first step in the previous algorithm?
- Why we did not need it in the previous parallel bucket sort?

Exchange Sort

- We consider a recursive “exchange” sort that is more suitable for distributed memory multicomputers.
- This algorithm is very suitable for the hypercube.
- We do not require that the keys have a uniform distribution since we can argue that the performance is *statistically* acceptable.

Exchange Sort

- Divide the processors into two groups of equal size.
- Each processor *exchanges* keys with its corresponding processor in the other group according to a pivot – smaller keys go to a group of processors, and bigger keys go to the other group.
- We recursively do the same for both groups.

Exchange Sort

1. Find a pivot.
2. Divide the processors into two groups of equal size.
3. Each processor "exchanges" keys with its corresponding processor in the other group.
4. We recursively do the same for both groups.
5. Finally, each processor sorts its keys.

Pivot

- How to find a pivot?
- This is like the argument sequential quicksort; we randomly pick one, which is good enough.
- Use a binomial trial to argue that the tree depth of a quick sort is bounded by $O(\log p)$ with high probability.

Analysis

- We focus on the number of "movements" as the cost, since basically no computation is involved.
- In each level of the exchange a processor exchanges at most $\frac{n}{p}$ keys.
- From previous argument the depth of the tree is bounded by $O(\log p)$, so the cost of exchange is $O(\frac{n}{p} \log p)$.
- Finally each processor still needs to sort its key with $O(\frac{n}{p} \log \frac{n}{p})$ time.
- The final complexity is $O(\frac{n}{p}(\log p + \log \frac{n}{p})) = O(\frac{n}{p} \log n)$.

Discussion

- What is the theoretical speedup of this algorithm?
- Find out the definition of hypercube and why is this algorithm suitable for hypercube.

Matrix Multiplication

- Multiple two $n \times n$ matrices A, and B and place the result into C.

$$A \times B = C \tag{19}$$

- We assume that the matrix is dense.

Sequential Matrix Multiplication

- For the interest of simplicity we use the standard $O(n^3)$ algorithm, instead of the Stassen[3] algorithm.
- The time complexity is $O(n^3)$.

[3] http://en.wikipedia.org/wiki/Strassen_algorithm

Parallel Matrix Multiplication

- We use p processors to compute the n^3 elements in C.
- Each processor simply computes the answer and no communication among them is necessary for a shared memory implementation.

Analysis

- Each processor computes $\frac{n^2}{p}$ elements.
- Each elements takes $O(n)$ time to computes.
- The parallel time is $O(\frac{n^3}{p})$.
- The speedup is $\frac{n^3}{\frac{n^3}{p}} = p$. It seems to an embarrassingly parallel computation and nothing can be improved.

Discussion

- Why no communication among processors is necessary for a shared memory implementation of the previous algorithm during the computation stage?