Linear Algebra

…and other useful Math

Handout 1

Introduction to Robotics

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1

Contents

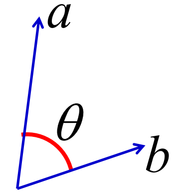
2

Dot product

- The dot (scalar) product of 2 vectors ***a*** and ***b*** is a single number
- Definition

$$a = (a_x, a_y, a_z), b = (b_x, b_y, b_z)$$

$$a \cdot b = a_x b_x + a_y b_y + a_z b_z$$



- Geometrical definition

$$a \cdot b = \|a\| \|b\| \cos \theta$$

$$\theta = \arccos \frac{a \cdot b}{\|a\| \|b\|}$$

3

A square matrix

- Same number of rows and columns
- An n-by-n matrix is a square matrix of order n
- Any two square matrices of the same order can be added and multiplied

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

4

A square matrix: Main diagonal

- Main diagonal: the entries a_{ii} (i = 1, ..., n=4)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4\times 4}$$

- Anti-diagonal or counter-diagonal

Diagonal Matrix

- The entries a_{ji}=0 for i,j = 1, ..., n with i≠j

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Triangular matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

If all entries above (below) the main diagonal are zero, A is called …

…a lower (upper) triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

7

The Identity Matrix

- The identity matrix I_n of size n is the n-by-n matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to 0, e.g:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Multiplication with I_n leaves a matrix unchanged

8

Symmetric or skew-symmetric matrix

- A symmetric matrix:
  a square matrix A that is equal to its transpose

$$A = A^t$$

- A skew-symmetric matrix:
  a square matrix A is equal to the negative of its transpose

$$A = -A^t$$

9

Invertible matrix and its inverse

- A is an invertible (or non-singular) matrix
  if there exists a matrix B such that

$$AB = BA = I_N$$

- If such B exists, it is unique and is called the inverse matrix of A, denoted

$$B = A^{-1}$$

i.e. $$AA^{-1} = A^{-1}A = I_N$$

10

Orthogonal matrix (1 of 2)

- A square matrix with real entries whose columns and rows are orthogonal unit vectors (i.e., orthonormal vectors)

$$Q^t Q = Q Q^t = I_N$$

- A matrix Q is orthogonal if its transpose is equal to its inverse

$$Q^t = Q^{-1}$$

11

Orthogonal matrix (2 of 2)

- The determinant of any orthogonal matrix is either +1 or −1
- Orthogonal group O(n) - the set of n × n orthogonal matrices
- The subgroup SO(n) consisting of orthogonal matrices with determinant +1 is called the special orthogonal group
  - each of its elements is a special orthogonal matrix
- As a linear transformation, every special orthogonal matrix acts as a rotation

12

The rotation groups

- **SO(2)** – the group of all rotations around a point in 2D
- **SO(3)** – the group of all rotations in 3D
  - around a line in 3D
  - or about the origin of 3D Euclidean space
- A rotation about the origin is a transformation that preserves the origin, Euclidean distance, and orientation
- Composing two rotations results in another rotation
- Every rotation has a unique inverse rotation

Determinant

- A value associated with a square matrix
- Can be computed from the entries of the matrix by a specific arithmetic expression
- The determinant of a matrix with integer coefficients will be an integer

Properties of Determinant

1. $\det(I_n) = 1$

2. $\det(A^t) = \det(A)$

3. $\det(A^{-1}) = \dfrac{1}{\det(A)} = \det(A)^{-1}$

4. For square matrices A and B of equal size

$$\det(AB) = \det(A)\det(B)$$

15

Singular Matrix

- A square matrix that does not have a matrix inverse
- A matrix is singular iff its determinant is 0

$$\det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}$$

16

Matrix Multiplication (1 of 2)

- The number of columns in the first matrix must be equal to the number of rows in the second matrix, i.e. the inner dimensions must be the same

  $A_{m\times n} \times B_{n\times p} = C_{m\times p}$

- Properties:
  - Associativity of Multiplication A (BC) = (AB) C
  - Multiplicative Identity IA = AI = A
  - Matrix Multiplication is not Commutative AB ≠ BA

17

Matrix Multiplication (2 of 2)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}$$

$$AB = \begin{bmatrix} (ab)_{11} & (ab)_{12} & \cdots & (ab)_{1p} \\ (ab)_{21} & (ab)_{22} & \cdots & (ab)_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ (ab)_{n1} & (ab)_{n2} & \cdots & (ab)_{np} \end{bmatrix}$$

$$(ab)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

the "dot product" of rows and columns

18

Matrix Multiplication Example (1 of 2)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

"Dot Product"

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & \\ & \end{bmatrix}$$

The "Dot Product" is where we multiply matching members, then sum up:

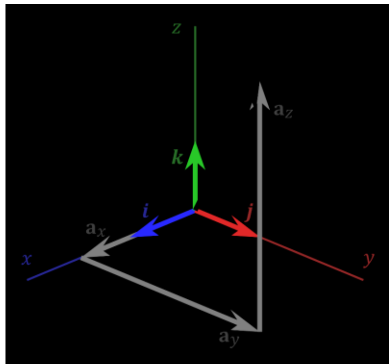
(1, 2, 3) • (7, 9, 11) = 1×7 + 2×9 + 3×11 = 58

Matrix Multiplication Example (2 of 2)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix} \checkmark$$

The standard basis {i, j, k}



- Satisfies the following equalities:

$$i = j \times k, \quad j = k \times i, \quad k = i \times j$$

$$-i = k \times j, \quad -j = i \times k, \quad -k = j \times i$$

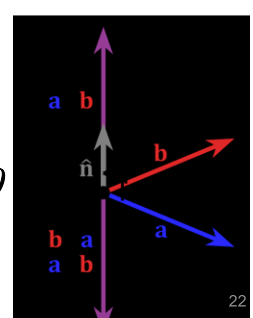
$$i \times i = j \times j = k \times k = 0$$

21

Cross product (1 of 2)

- The cross (vector) product of 2 vectors ***a*** and ***b*** is a binary operation on two vectors in 3D space and is denoted by ***a × b***
- Defined **only** in three-dimensional space
- A vector that is perpendicular to both ***a*** and ***b***

$$|a \times b| = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \|a\|\|b\| \sin\theta$$



Cross product (2 of 2)

$$a \times b = a_1 b_1 0 + a_1 b_2 k - a_1 b_3 j -$$
$$a_2 b_1 k - a_2 b_2 0 + a_2 b_3 i +$$
$$a_3 b_1 j - a_3 b_2 i - a_3 b_3 0 +$$
$$= (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k$$

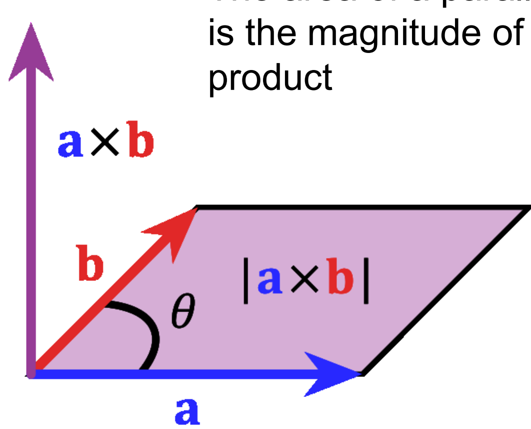
The cross product can also be expressed as the formal determinant:

$$a \times b = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} =$$

$$(a_y b_z i + a_z b_x j + a_x b_y k) - (a_z b_y i + a_x b_z j + a_y b_x k)$$

Geometric meaning

The area of a parallelogram is the magnitude of a cross product



Measure of perpendicularity

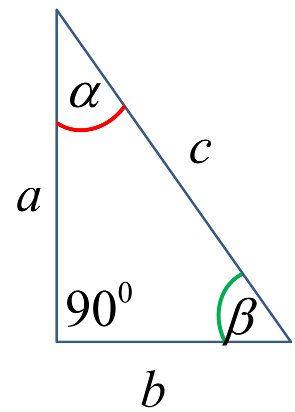
Basic Trigonometry

$$\alpha + \beta = 90^0$$

$$a^2 + b^2 = c^2$$

$$\cos\alpha = \frac{a}{c} \quad \sin\alpha = \frac{b}{c}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$



25

Useful Trigonometry Formulas

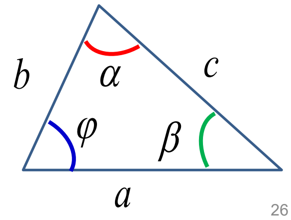
$$\sin^2\varphi + \cos^2\varphi = 1$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\varphi$$



26