Trampoline Jump Mechanism

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Abstract

This paper attempts to develop a reliable model predicting the launch heights of single or multiple people off a trampoline. Based on given data involving masses of children and respective maximum heights reached the objective was to estimate the maximum height each child can reach when all three children jump together. After multiple failed attempts and analysis of the given data, we developed a model that captures the essence of intrinsic energy each child exerts when jumping. A sequential energy cycle model was developed, which looped through itself representing landing and launching from the trampoline upon repetitive jumps. The solution process was divided into two main stages: finding a realistic amount of energy each child was capable of imparting to the system during each jump, then determining the most profitable distribution of energy among the children, finally resulting in the maximum height each child could attain. Assuming a constant energy loss as a percentage of net system energy during the transfer between the kinetic-elastic-kinetic process, this allowed the height reached as a function of jumps to flatten out, resulting in a steady state height (or maximum height). We determined that when all children are jumping together, they can transfer all the energy (accounting for energy loss) into the kinetic energy of one child by simply pulling their feet upwards off the membrane. This allowed us to calculate maximum heights of 4.13m, 2.58m, and 2.06 m for each child with respective weights 25 kg, 40 kg, and 50 kg.

Contents

1 Introduction

1.1 Problem Background

To most, the trampoline is a familiar tool, used both recreationally and professionally. The basic function of a trampoline can be considered to boost the height reached by a person after jumping on it. Several variables affecting trampoline jump mechanics include the trampoline's factors such as its material, construction, and membrane shape; in addition, the mechanics are also affected by a person's weight, the number of people jumping, and their point of contact (on the membrane).

1.1.1 Problem Restatement

In the given problem, a trampoline of diameter (5m) and three children with fixed masses (25, 40, and 50kg) are given. It is also stated that when jumping alone, the 25kg child can reach a maximum height of 0.5m, the 40kg child can reach a maximum height of 0.8m, and the 50kg child can reach a maximum height of 1.2m above the surface of the trampoline. Given the above fixed parameters, the problem required us to find the maximum height each child could reach when all three children entered the trampoline and jumped on it together. This gavethe us freedom to develop a model suitable for replicating physically expected results of someone jumping on a trampoline, given that the developed model matched the given values for specified input-output conditions.

1.1.2 Given Data

The data given from the question conveys a quadratic relationship between the mass of each child and their maximum heights as seen in figure 1.

Through our research, this plot has acted as our manual to check the validity of our models. Based on these results we assessed and tweaked our models.

2 Tried Models

In this section, we discuss the various models we have tried to work to solve this problem and fit the given data points. We point out these models, as they helped us reach our final solution written up in Sec (4)

2.1 Forced Damped Simple Harmonic Oscillator (SHO)

The first model we implemented was a forced (constant) damped simple harmonic oscillator Morin. We assumed that the trampoline obeyed Hooke's Law:

$$F(y) = -ky$$
$$U_T(y) = \frac{1}{2}k\Delta y^2$$

Through Newton's Second Law, we can generate a differential equation as this:

$$\ddot{y} + \frac{\gamma}{m}\dot{y} + \frac{k}{m}y = -g \tag{1}$$

The solution of this second-order differential equation is then

$$y(t) = e^{-\gamma t/2m}(A\cos(\omega t) + B\sin(\omega t)) - g/\omega^2 \tag{2}$$

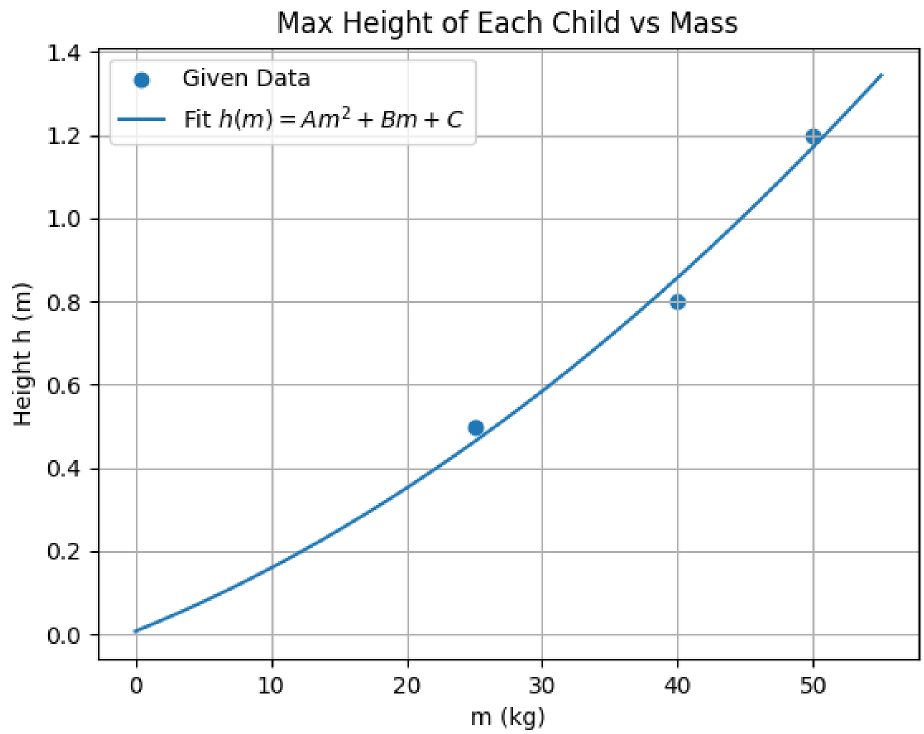

Figure 1: The relationship between the max height of each child and their mass

If we plug in Eq. 5, into the potential energy of the hook's law, we quickly see that the relationship is not quadratic as we see in Fig (1)

2.2 Forced SHO with Drag

In this section, we have added drag, as we thought it might provide us with the quadratic relationship between the maximum height and mass. We modeled drag like in Vargas et al. [2013]. This allowed us to construct the following piece-wise differential equation. Note that $y = 0$ is the plane at which the trampoline's membrane is located when there is no external force acting on it.

$$F_{net} = \left\{ \begin{array}{ll} \ddot{y} + \frac{\gamma}{m}\dot{y} + \frac{k}{m}y = -g & y \leq 0 \\ \ddot{y} + c\dot{y}^2 = -g & y > 0 \end{array} \right\} \tag{3}$$

While this model might provide us with a quadratic relationship, the drag term will not be that impactful as we are moving at relatively low speeds for the drag to make a difference. Thus the dominant force is still the tension and weight which can only provide us with a relationship $h_{max} \propto m^{\leq 1}$

2.3 SHO with external Force $F(m)$

All of the last models prompted us to add an additional term $F(m)$ that depends on the mass of each child. This force is essentially the intrinsic force that each applies on the trampoline. A priori we do not assume its shape, but we just say that it depends on the mass of the child. This is a mathematical way to represent that the heavier can exert a larger force that propels him upwards. We let go of our drag system and the damping and we just wrote the differential equation as such

$$\ddot{y} + \frac{k}{m} y = -g - \frac{1}{m} F(m) \tag{4}$$

which yields an equation of motion:

$$y(t) = A\cos(\omega t) - g/\omega^2 - \frac{1}{mk} F(m) \tag{5}$$

From this, we can measure the max height by substituting into the potential energy equation of a simple harmonic motion, as seen in Ref (Tramp).

While this model might provide us with the most fitting results out of all the previous models, it is still inherently a single-jump model. This means that each child can only get one jump to attain the maximum height. This model also does not account for any loss which is the case in reality. Because of these drawbacks, we have found a new model (Energy cycles) that assumes that each kid jumps a certain amount of time before reaching a max height and that there is a loss function in the system. Please read Sec (4)

3 Assumptions

A key assumption used in the Energy Cycles model, when taking the system as the trampoline and the Earth, included a single main source of energy loss during contact between the children and the membrane. In other words, the trampoline's membrane does not transfer all of its elastic energy back into the children when launching them up. The energy not imparted to the children was assumed to remain in the membrane, causing it to briefly oscillate. The trampoline was assumed to be sufficiently damped such that vibrations leftover in the membrane post-jump died out quickly, resulting in the children always landing onto a membrane at equilibrium. The energy loss term was modeled as a percentage loss of the net energy of the system. The jump procedure was assumed as follows: the child (or children) initially stood on the edge of the trampoline (not causing any disturbance of deformation of the membrane). The child then jumps onto the trampoline, landing perfectly in the center. After the initial landing, the child jumps up at the point when the trampoline membrane sags to its minimum point, thereby doing work on the system during each jump. The work done during each jump was assumed to remain constant among all jumps per child, with heavier children doing more work than lighter ones. Due to the low velocities reached in a trampoline jump, profile drag was assumed to be negligible and ignored.

4 Energy Cycles

4.1 Theory

In this model, we try to accommodate for two factors: the loss function and the fact that a child is able to jump on the trampoline multiple times to reach their maximum respective height. Based on our previous analysis and discussion, we have found this method, albeit numerical in nature, to be the most realistic. Each kid has a respective maximum height that is attained after n number of jumps, ideal $n < 10$. Since the kid would keep jumping at that maximum height at $n > 10$, then we can assume an inverse relationship $H_{max} \propto -\frac{1}{n}$ between the height of each jump and the number of jumps.

Another key idea here that we have taken from our previous attempted models is that each child will perform an intrinsic work $W(m)$ from their body which depends on their mass. We will make the assumption that the heavier kid is older and thus "stronger" and can apply more energy with his legs.

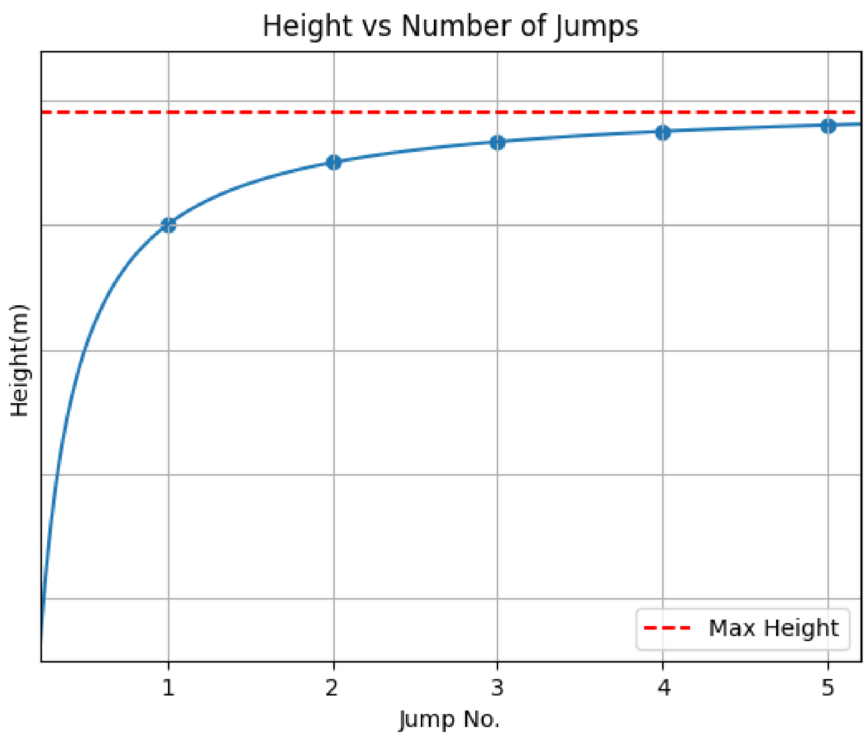

Figure 2: Maximum height

The physical meaning behind the plateau in Fig(2) is the presence of a loss function in the system in general. We estimated a certain loss function ξ, that could depend on the damping forces in the spring as seen in the previous section, slipping or loss of traction when the user touches down on the trampoline, or drag from the air when the user is airborne. To simplify our model we have assumed that xi is a constant amongst all users. While strictly speaking this cannot be true as the velocity in the presence of drag depends on the mass, air resistance is negligible for small speeds $|\vec{V}|$ **??**, which is our case.

$$\xi = \frac{1}{3} \tag{6}$$

We have formulated a diagram Fig.(5) that simulates the energy exchange that happens in the system as the kid jumps each time. We first assume that that kid jumps from outside the trampoline with initial energy W_0 which is translated into the initial kinetic energy KE.

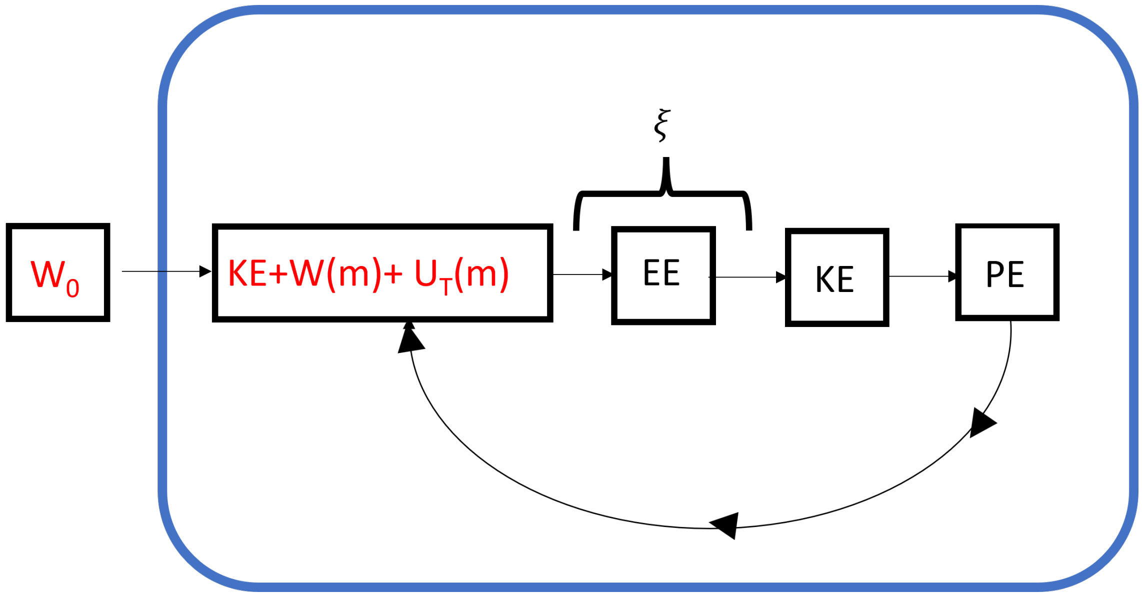

Figure 3: cycle of energy throughout the jumps. $U_T(m)$ is the potential energy due to the tension when the child is at rest on the trampoline. $U_T(m) = \frac{1}{2}Nk(mg/Nk)^2$

Once the kid jumps on the trampoline he then exerts more work $W(m)$, and the trampoline also exerts a work that is quadratic with mass. This is because when the kid is at rest on the trampoline, the displacement z

$$F_{net} = 0 = Nkz - mg$$
$$z = \frac{mg}{Nk}$$

N is the number of springs around the trampoline and k is the spring constant.

It's at this step that we account for the loss function ξ. Hence we obtain a general formula for the available elastic energy (EE) for the kid to harness and jump

$$EE = \left(KE + W(m) + \frac{1}{2}Nk(mg/Nk)^2 = \frac{1}{2}Nk\Delta x^2\right)(1-\xi) \tag{7}$$

Where the last term is the total tension in the system when the kid is at rest, and not exerting any work on the trampoline. After that, this elastic energy is then transformed back to Kinetic energy KE as the kid leaves the trampoline, and is entirely translated into the gravitational potential energy

$$KE = PE = mgh_{max} \tag{8}$$

where m is the mass of the kid, g the gravitational acceleration and h_{max} is the maximum height each kid reaches.

One loop represents one full jump each kid performs. We assume around 10 jumps or energy cycles before each kid reaches their maximum height. We have developed a numerical simulation that performs this task. Please see the appendix below for the code.

4.2 Results

We have performed the loop depticed in Fig(5), with these assumptions

$$N = 25$$
$$k = 1100\ N/m$$
$$g = 9.8\ m/s^2$$

We know how high each kid can jump, and thus we can calculate the final total energy $E_f = mgh_{max}$ each kid needs to have. Given this final total energy, we then do a "self-consistent" calculation altering the added work each kid performs $W(m)$ so that the energy reaches plateaus at the desired value E_f.

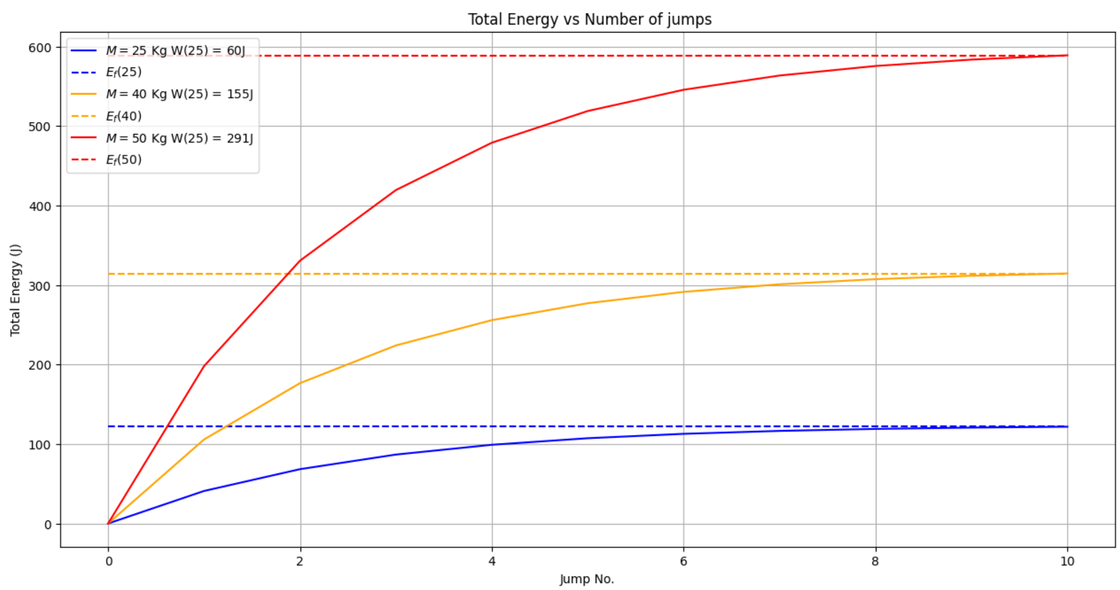

Figure 4: A depiction of the limit of each kid's maximum energy fitted by the given data

The significance of this model is that it allows us to find a numerical estimation of the work each kid exerts at the point of contact with the trampoline $W(m)$. In Table (1), we represent all the final energies, maximum heights, and the needed energy each child must supply for each jump.

Mass (Kg)	n (# Jumps)	$W(m)$ (J)	E_f (J)	h_{max} (m)
25	6	60 ± 13	122.625	0.5
40	7	155 ± 1	313.92	0.8
50	8	291 ± 1	588.6	1.2

Table 1: the energy $W(m)$ exerted by each child to attain the height given in the question

The values in this table allow us to plot a relationship between the $W(m)$ and the mass m of each child.

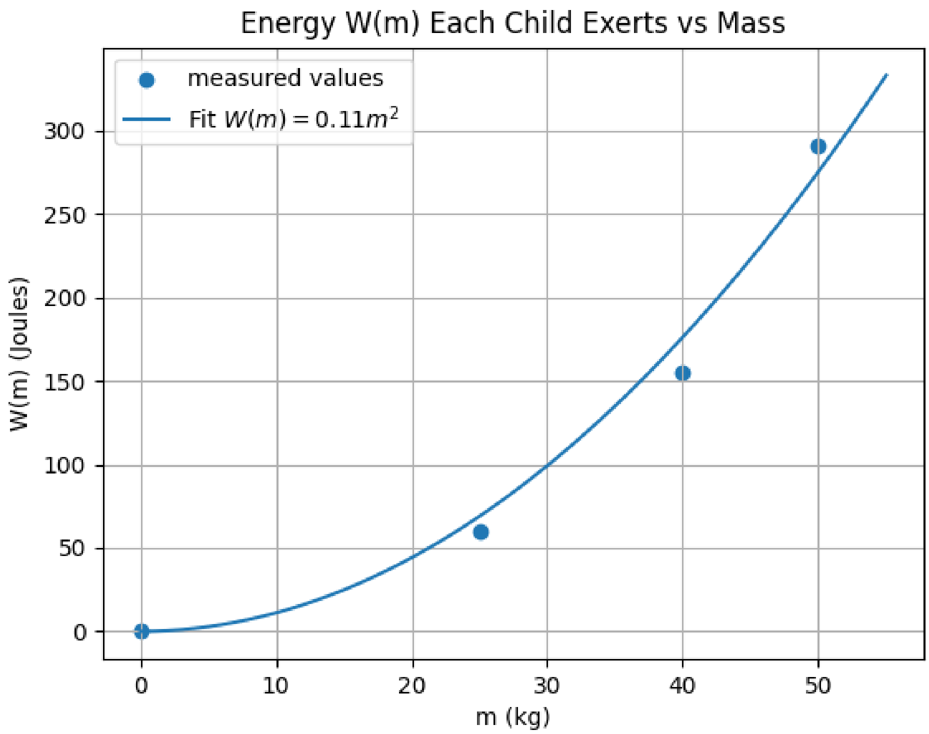

Figure 5: The relationship between the intrinsic work each kid exerts vs mass fits a quadratic relationship with uncertainty $\sigma = 2.8 \times 10^{-5}\ J/Kg^2$

This quadratic relationship will be further discussed in Sec (5). Now that we have $W(m)$ we can find the total energy exerted W_T by all the children on each jump when all three are on the trampoline. The total work exerted on the trampoline is then the sum of all the individual energies

$$W_T = W(25) + W(40) + W(50) = 506 \pm \sqrt{3}\ J \tag{9}$$

Once work inputs each child was capable of producing per jump were determined (figure 5), it was time to tackle the actual problem of all three jumping on the trampoline. The key component making our solution work was that now, for this larger mass, the work done per jump was simply the sum of the work done of each child per jump and not the quadratic mass-work relationship from figure 5 (assuming that all three children are able to jump perfectly in sync, thereby exciting the system in unison). If all three children were to hug each other and jump, their maximum energy achieved was calculated to be 1012J using the iterative method discussed above (bot going forwards). Using the potential energy relation (mgh), the maximum height of the clump of children was computed to be 0.90m above the trampoline surface. It can be readily observed that if all the net energy could be transferred to only a single child, he could attain a much greater height while his friends remained on the membrane. This situation could be achieved by the two children by sharply pulling their legs up after pushing down on the membrane, while the third child located perfectly centered on the membrane keeps his legs straight and absorbs all the (accounting for energy loss) energy transferred from the membrane. Thus, the maximum heights achieved for each child was readily calculated (table 2).

Child (in mass)	H_{max}
25 kg	4.13 m
40 kg	2.58 m
50 kg	2.06 m

Table 2: Maximum height of each child reached

5 Discussion and Conclusion

It was ultimately decided that energy cycle method was the simplest yet most elegant and realistic model. A key assumption of this model, as stated above was that a child jumps repetitively on the trampoline, gaining height after each jump until energy losses balanced out the work input of the child after each jump. Based on figure 2, it was clear that the height (a function of potential energy) was dependent on the elastic energy; the more elastic energy the system had, the greater the final potential energy (remembering that losses were modelled as a fixed percentage regardless of input energy). This elastic energy was increased by the child jumping on the trampoline which involved doing work on the system, resulting in further stretching of the membrane. By referencing analysis on vibrations of thin circular membranes, it was clear that the children should place themselves as close to the center as possible, resulting in the fundamental mode shape of the membrane, corresponding to maximum amplitude. The exponential increase in work done as a function of mass can be deemed unrealistic at first glance, however it was important to remember that the data involved children. A child of mass 25kg is approximately equivalent to a 7 year old while a child of mass 50kg to a 15 year old. Thus it was deemed realistic that a 15 year old can do exponentially more work than a 7 year old. This relationship is not expected to work for adults of varying masses! Maximum heights, as shown in table 2, were observed to increase substantially during group jump compared to a single child jumping. Based on the obtained maximum heights, the best height achieved when the least heavy child was launched (as intuitively expected). As such, the authors of this paper do not recommend the children attempt to launch the smallest child into the air due the alarmingly large height attained of 4.13m and the risk that he may fly off course and land outside the trampoline.

References

David Morin. Oscillations - scholars at harvard. URL https://scholar.harvard.edu/files/david-morin/files/waves_oscillations.pdf.

Mario Vargas, Suthyvann Sor, and Adelaida García Magariño. Drag coefficient of water droplets approaching the leading edge of an airfoil. *5th AIAA Atmospheric and Space Environments Conference*, 2013. doi: 10.2514/6.2013-3054.

Appendix: Loop Iterations

Here's an example of using the lstlisting environment from the listings package:

```
import matplotlib.pyplot as plt
import numpy as np
```

```
import math

def iter_jump(E_0,E_spring, W, loss, list_E):
    tot_e = E_0+W+E_spring
    tot_e = tot_e*(1-loss)
    print(tot_e)
    list_E.append(tot_e)
    if len(list_E) > 10:
        return list_E
    else:
        return (iter_jump(tot_e, E_spring, W, loss, list_E))

N = 25
k = 1100
g = 9.8
#############
M = 25
delta_z = M*g/(N*k)
E_spring = 0.5*N*k*delta_z**2
list_e = (iter_jump(0, E_spring, 60, 0.33, [0]))
plt.plot(list_e, label = r"$M = 25$ Kg W(25) = 60J", color ="b")
plt.hlines(y=122.625, xmin=-0, xmax=10, color='b', r"$E_f(25)$")
#########################
M = 40
delta_z = M*g/(N*k)
E_spring = 0.5*N*k*delta_z**2
list_e = (iter_jump(0, E_spring, 155, 0.33, [0]))
plt.plot(list_e, label = r"$M = 40$ Kg W(25) = 155J", color = "orange")
plt.hlines(y=313.92, xmin=-0, xmax=10, color='orange', r"$E_f(40)$")
#########################
M = 50
delta_z = M*g/(N*k)
E_spring = 0.5*N*k*delta_z**2
list_e = (iter_jump(0, E_spring, 291.1, 0.33, [0]))
plt.plot(list_e, label = r"$M = 50$ Kg W(25) = 291J", color = 'r')
plt.hlines(y=588.6, xmin=-0, xmax=10, color='red',r"$E_f(50)$")

plt.legend(loc = "upper left")
plt.xlabel("Jump No.")
plt.ylabel("Total Energy (J)")
plt.title(r" Total Energy vs Number of jumps" )
plt.grid()

plt.show()
```

Appendix: Fitting $h_{max}(m)$ vs m data

```
import matplotlib.pyplot as plt
from scipy.optimize import curve_fit
import numpy as np

def loss_function(m,A, B, C):
    return A*m**2 + B*m +C


energy = [0,0.5, 0.8,1.2]
m = [0,25,40,50]
plt.scatter([25,40,50],[0.5, 0.8,1.2], label = "Given Data")
plt.xlabel("mass (Kg)")
plt.ylabel("W added (J)")
m_cont =np.linspace(0,55,1000)

pars, cov = curve_fit(f=loss_function, xdata=m, ydata=energy, p0 = [0.5,0.5,0.5])
print(pars, cov)

y_fit = loss_function(np.array(m_cont), pars[0],pars[1],pars[2])


plt.plot(m_cont,y_fit, label = r"Fit $h(m) = Am^2+Bm+C$ ")

plt.ylabel("Height h (m)")
plt.xlabel("m (kg)")
plt.title("Max Height of Each Child vs Mass")
plt.legend()
plt.grid()
plt.show()
```

Appendix: Fitting $W(m)$ vs m data

```
    import matplotlib.pyplot as plt
from scipy.optimize import curve_fit
import numpy as np

def loss_function(m,A):
    return A*m**2


energy = [0,60,155,291]
m = [0,25,40,50]
plt.scatter([0,25,40,50],[0,60,155,291], label = "measured values")
#plt.plot([0,25,40,50],[0,60,155,291])
plt.xlabel("mass (Kg)")
plt.ylabel("W added (J)")
m_cont =np.linspace(0,55,1000)
```

```
pars, cov = curve_fit(f=loss_function, xdata=m, ydata=energy, p0 = [0.5])
print(pars[0], cov)

y_fit = loss_function(np.array(m_cont), pars[0])
y_man = loss_function(np.array(m_cont),0.4)



#plt.scatter([25,40,50],loss)
plt.plot(m_cont,y_fit, label = r"Fit $W(m) = 0.11m^2$ ")
#plt.plot(m_cont,y_man, label = "man")

plt.ylabel("W(m) (Joules)")
#plt.ylim(0,100)
plt.xlabel("m (kg)")
plt.title("Energy W(m) Each Child Exerts vs Mass")
plt.legend()
plt.grid()
plt.show()
```