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Linear Algebra

Assignment # 5

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Q1

$$A = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & -4 \end{bmatrix} \qquad V = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

$$Av = \lambda v$$

$$A \cdot v = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & -4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 12 - 21 + 9 \\ -16 + 15 + 1 \\ 8 - 12 - 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -8 \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} 12 - 21 + 9 \\ -16 + 15 + 1 \\ 8 - 6 + 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}}$$

As $A \cdot v \neq \lambda \cdot v$ for any value of λ so 'v' is not an eigen vector of matrix A.

$$\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

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(Q2) (a) forming new matrix by subtracting 'λ' from diagonal entries $(A - \lambda I)v = 0$.

$$\begin{bmatrix} 1-\lambda & 2 & 0 \\ -1 & -\lambda-1 & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} \Rightarrow (1-\lambda)\left[(-\lambda-1)(1-\lambda)(-1)\right]$$

$$(1-\lambda)(-\lambda+\lambda^2-1+\lambda-1$$

$$(1-\lambda)(\lambda^2) \Rightarrow \lambda^2 - \lambda^3$$

characteristic polynomials $\rightarrow P(\lambda) = \lambda^2 - \lambda^3$

(b) Find eigen values

$$\lambda^2 - \lambda^3 = 0$$

$$\lambda^2(1-\lambda) = 0$$

$\lambda^2 = 0$	$1 - \lambda = 0$
$\lambda = 0$ OR	$\lambda = 1$

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Find eigen space corresponding to $\lambda = 0$

$$E_1 = N(A - 0I)$$

$$A - 0I = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

row reducing $\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A - I\lambda = 0$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 = free variable

$x_2 = 0$

$-x_1 - 2x_2 + x_3 = 0$

$+x_1 = +x_3 \quad \therefore \quad x_1 = x_3$

$$x = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} \cdot x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{\text{thus basis of } E_1 \text{ \& } E_2 \text{ is}}$

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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Q2

(c) for $\lambda = 0$, geometric multiplicity is 1 as eigen space is spanned by 1 non-zero vector.

for $\lambda = 1$, geometric multiplicity is 1 as eigen space is spanned by 1 non-zero vector.

for $\lambda = 0$, algebraic multiplicity $= 2$

for $\lambda = 1$ algebraic multiplicity $= 2$.

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(Q3)

(a)

$\lambda_1 = -1/3$	$\lambda_2 = 1$	$\lambda_3 = 1/3$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

for λ_1 & v_1, $Av_1 = -1/3\, v_1$

for λ_2 & v_2, $Av_2 = 1\, v_2$

for λ_3 & v_3, $Av_3 = 1/3\, v_3$

$$A^K v_1 = \left(-\frac{1}{3}\right)^K v_1, \quad A^K v_2, \ 1^K v_2, \quad A^K v_3 = \left(\frac{1}{3}\right)^K v_3.$$

$\Rightarrow$ To compute $A^K x$ we need to express x as linear combination of v_1, v_2 & v_3.

$$x = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$c_1 = 1$, $c_2 = 2$, $c_3 =$

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$$A^k x = A^k(v_1 + 2v_2 - v_3)$$

$$A^k x = A^k v_1 + 2A^k v_2 - A^k v_3$$

$$A^k x = \left(-\frac{1}{3}\right)^k v_1 + 2v_2 - \left(\frac{1}{3}\right)^k v_3$$

(b)

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$A = PDP^{-1} \rightarrow A^k = PD^kP^{-1}, \quad D^k = \begin{bmatrix} (a_{11})^k & 0 & 0 \\ 0 & (a_{22})^k & 0 \\ 0 & 0 & (a_{33})^k \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} (-1/3)^{50} & 0 & 0 \\ 0 & 1^{50} & 0 \\ 0 & 0 & (1/3)^{50} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

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$$A^{50} \Rightarrow \begin{bmatrix} (1/3)^{50} & 1^{50} & (1/3)^{50} \\ 0 & 1^{50} & (1/3)^{50} \\ 0 & 1^{50} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A^{50} \to \begin{bmatrix} (-1/3)^{50} & (-\frac{1}{3}+\frac{1}{3})^{50} & 1-(1/3)^{50} \\ 0 & (1/3)^{50} & 1-(1/3)^{50} \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A^{50} = \begin{bmatrix} (-\frac{1}{3})^{50} & 0 & 1-(\frac{1}{3})^{50} \\ 0 & (1/3)^{50} & 1-(\frac{1}{3})^{50} \\ 0 & 0 & 1 \end{bmatrix}$$

Q4

$x_1(t)$ = fraction of market held by chanel 1

$x_2(t)$ = fraction " " " " chanel 2

so $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

Assuming $t=0$ as starting point when 2 channels have 50% of market

$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \to \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

we need $x_1(k+1)$ & $x_2(k+1)$ over 1-year period,

channel 1 retains ~~[illegible]~~ 80% of its starting fraction & gains 10% of channel 2 starts fraction.

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Q4

$$x_1(k+1) = 0.8\,x_1(k) + 0.1\,x_2(k)$$

similarly channel 2 retains 90% & gains 20%

$$x_2(k+1) = 0.2\,x_2(k) + 0.9\,x_2(k)$$

$$\therefore \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$x(1) = \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.9 \\ 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.9 \\ 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix}$$

if $t = 1$ then $t = k+1 = 2$

$$x(2) = \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.9 \\ 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix}$$

So

year	channel I	channel 2
1	0.45	0.55
1	0.415	0.585

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Q5

$$A: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} 3x \\ y \\ x-y \end{pmatrix}$$

$$\begin{pmatrix} 3x \\ y \\ x-y \end{pmatrix} = x\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

det of $A - \lambda I = 0$

$$\det\left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & -\lambda \end{bmatrix}\right) = 0 \rightarrow (3-\lambda)\left[(1-\lambda)(-\lambda) - 0\right] = 0$$

$$(3-\lambda)(-\lambda+\lambda^2) = 0$$

$$(\lambda)(3-\lambda)(\lambda-1) = 0$$

As A is 3×3 matrix with 3 distinct eigen values.

$\therefore$ IT IS DIAGNOLIZABLE

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Find eigen values corresponding to each λ :

$\lambda = 3$
$$\begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & -\lambda \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 = free variable

$\boxed{-2x_2 = 0}$

$x_1 - 0 - 3x_3 = 0$

$\boxed{x_1 = 3x_3}$

$$x = \begin{bmatrix} 3x_3 \\ 0 \\ x_3 \end{bmatrix} \Rightarrow x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow \text{eigen vector}$$

$\lambda = 1$
$$\begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & -\lambda \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boxed{\text{Interchanging rows}} \ \& \ \frac{1}{2}R_2 \rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$R_2 - R_1 \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 = free variable

$-x_2 - x_3 = 0$

$\boxed{x_2 = -x_3}$

$x_1 - x_2 - x_3 = 0$

$\boxed{x_1 = 0}$

$$x = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} \rightarrow x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$\rightsquigarrow$ eigen vector

$$\lambda = 0 \quad \begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & \lambda \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_2 - 1/3 R_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 = free variable

$x_2 = 0$

$x_1 = 0$

$$x = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} \rightarrow x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\rightsquigarrow$ eigen vector

$$\text{i.e. } P \text{ matrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$