HAMMAD - JAVAID 121-1661 (DS-M)

Differential equations Assignment # 5

Q1 (i) Using curve $y = \cos(c_1 x + c_2)$ — ①

$$\frac{dy}{dx} = -\sin(c_1 x + c_2)\, x\, c_1 \quad — (a)$$

$$\frac{d^2y}{dx^2} = \underbrace{-\cos((c_1 x) + c_3)}_{y} \times c_1^2 \,\cancel{\text{[illegible]}}$$

$\because y = \cos(c_1 x + c_2) \quad \therefore \frac{d^2y}{dx^2} = -y c_1^2$ — (b)

Finding expression for c_1 using (a)

$$c_1 = \frac{1}{-\sin(c_1 x + c_2)} \left(\frac{dy}{dx}\right)$$

$$c_2^2 = \frac{1}{1-y^2} \left(\frac{dy}{dx}\right)^2$$

replacing c_1^2 in (b)

$$\frac{d^2y}{dx^2} = \frac{-(-y)}{-(1-y^2)} \left(\frac{dy}{dx}\right)^2$$

$$\frac{d^2y}{dx^2} = \frac{y}{y^2-1} \left(\frac{dy}{dx}\right)^2$$

(ii) Using curve $y = ae^x + bxe^x$

$$\frac{dy}{dx} = ae^x + bxe^x + be^x$$

$$\frac{dy}{dx} = (a+b)e^x + bxe^x \text{ ——— (a)}$$

$$\frac{dy}{dx} = ae^x + bxe^x + be^x$$

$$\frac{d^2y}{dx^2} = ae^x + bxe^x + be^x + be^x$$

$$= \underbrace{(a+b)e^x + bxe^x}_{\text{(b)}} + be^x$$

$$\therefore \frac{d^2y}{dx^2} = \frac{dy}{dx} + be^x$$

From $\frac{dy}{dx} \rightarrow \quad \frac{dy}{dx} = y + be^x$

$$\text{(c) ——— } \boxed{be^x = \frac{dy}{dx} - y}$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + be^x$$

$\Rightarrow$ Replace be^x with (c)

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{dy}{dx} - y$$

$$\therefore \frac{d^2y}{dx} = 2\frac{dy}{dx} - y$$

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Q2

(i) $y_1(x) = 1$, $y_1'(x) = 0$, $y_1''(x) = 0$

Substituting $y_1(x)$, $y_1'(x)$ & $y_1''(x)$ in given equation

$$\Rightarrow yy'' + (y')^2 = 0$$

$$1(0) + (0)^2 = 0 \quad \boxed{\text{hence proven}}$$

(ii) $y_2(x) = x^{1/2}$ | $y_2'(x) = \frac{1}{2x^{1/2}}$ | $y_2''(x) = -\frac{1}{4x^{3/2}}$

Substituting $y_2(x)$, $y_2'(x)$ & $y_2''(x)$ in given equation

$$\Rightarrow yy'' + (y')^2 = 0$$

$$x^{1/2}\left(-\frac{1}{4x^{3/2}}\right) + \left(\frac{1}{2x^{1/2}}\right)^2 = 0$$

$$-\frac{1}{4x} + \frac{1}{4x} = 0 \quad \boxed{\text{hence proven}}$$

(iii) $y = c_1 + c_2x^{1/2}$ | $y' = \frac{c_2}{2x^{1/2}}$ | $y'' = \frac{-c_2}{4x^{3/2}}$

Substituting into given equation

$$\Rightarrow yy'' + (y')^2 = 0$$

$$(c_1 + c_2x^{1/2})\left(\frac{-c_2}{4x^{3/2}}\right) + \left(\frac{c_2}{2x^{1/2}}\right)^2 \neq 0$$

as $\frac{-c_1c_2}{4x^{3/2}} - \cancel{\frac{c_2^2}{4x}} + \cancel{\frac{c_2^2}{4x}} \Rightarrow -\frac{c_1c_2}{4x^{3/2}} \neq 0$

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hence $y = c_1 + c_2\sqrt{x}$ is not a solution of $yy'' + (y')^2 = 0$

→ This is non-linear DE so superposition principle does not hold true

Q3 (a) $(x+3)y'' + xy' + (\ln|x|)y = 0$

$y(1) = 0,$
$y'(1) = 1$

dividing equation by $x+3$

$$\frac{\cancel{(x+3)}y''}{\cancel{(x+3)}} + \frac{x}{x+3}y' + \frac{\ln|x|}{x+3}y = 0$$

$$y'' + \frac{x}{x+3} + \frac{\ln|x|}{x+3}y = 0$$

$$P(x) = \frac{x}{x+3}, \quad Q(x) = \frac{\ln|x|}{x+3}$$

→ $P(x)$ is discontinuous at $x = -3$

→ $Q(x)$ is discontinuous at $x = -3$ & $x = 0$

⇒ Interval for unique solution is $(0, \infty)$ as it doesn't contain any discontinuity.

(b) $(1-x^2)y'' - 2xy' + \left[\alpha(\alpha+1) + \frac{\mu^2}{1-x^2}\right]y = 0$

$y(0) = y_0, \quad y'(0) = y_1$

÷ by $(1-x^2)$

$$\frac{\cancel{(1-x^2)}y''}{\cancel{(1-x^2)}} - \frac{2x}{1-x^2}y' + \left[\frac{\alpha(\alpha+1)}{(x+1)} + \frac{\mu^2}{(1-x^2)^2}\right]y = 0$$

$$y'' - \frac{2x}{1-x^2}y' + \left[\frac{\alpha(\alpha+1)}{1-x^2} + \frac{\mu^2}{(1-x^2)^2}\right]y = 0$$

$$P(x) = \frac{-2x}{1-x^2} \quad Q(x) = \frac{\alpha(\alpha+1)}{1-x^2} + \frac{\mu^2}{(1-x^2)^2}$$

$P(x)$ & $Q(x)$ are both continuous at $x = \pm 1$.

Interval for unique solution is $(-1, +1)$.

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Q4)

$$W(f,g) = fg' - f'g$$
$$= t\cos t - \sin t$$

$u' = f' + 3g'$
$v' = f' - g'$

$$w(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$
$$= uv' - vu'$$

$$= (f+3g)(f'-g') - (f'+3g')(f-g)$$

$$\cancel{ff'} - fg' + 3gf' - \cancel{3gg'} + \cancel{ff'} + f'g - 3fg' + \cancel{3gg'}$$

$$= -4fg' + 4f'g$$

$$= -4\underbrace{(fg' - f'g)}_{w(f,g)}$$

$$\Rightarrow -4(t\cos t - \sin t)$$

$$= 4\sin t - 4t\cos t$$

Q5 (a) (i) $y_1'' - y_1' - 2y_1$

$y_1(t) = e^{-t}$
$y_1'(t) = -e^{-t}$
$y_1''(t) = e^{t}$

$$e^{-t} - (-e^{-t}) - 2(e^{-t})$$
$$= 2e^{-t} - 2e^{-t} = 0 \quad \boxed{\text{hence proven}}$$

(ii) $y_2'' - y_2' - 2y_2$

$y_2(t) = e^{2t}$
$y_2'(t) = 2e^{2t}$
$y_2''(t) = 4e^{2t}$

$$= 4e^{2t} - 2e^{2t} - 2e^{2t}$$
$$= 4e^{2t} - 4e^{2t} = 0$$

$$\boxed{\text{hence proven}}$$

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(Q5) (a)(i) For fundamental solution, use wronskian

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \rightarrow \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix}$$

$$= e^{-t} \cdot 2e^{2t} + e^{2t} \cdot e^{-t}$$

$$= 2e^{t} + e^{t} \rightarrow 3e^{t} \neq 0$$

$$\forall t \in (-\infty, \infty)$$

$\Rightarrow$ As $y_1(t)$ & $y_2(t)$ are solutions to DE & $w(y_1, y_2) \neq 0$)

so $y_1(t)$ & $y_2(t)$ form fundamental solutions.

(b)(i) $y_3(t) = -2e^{2t}$, $y_3'(t) = -4e^{2t}$, $y_3''(t) = -8e^{2t}$

$$y_3'' - y_3' - 2y_3 = -8e^{2t} + 4e^{2t} + 2(2e^{2t})$$

$$= -8e^{2t} + 8e^{2t} \Rightarrow 0$$

$\boxed{\text{hence proven}}$

(ii) $y_4(t) = y_1(t) + 2y_2(t)$, $y_4'(t) = y_1'(t) + 2y_2'(t)$, $y_4''(t) = y_1''(t) + 2y_2''(t)$

$$y_4''(t) - y_4' - 2y_4 \Rightarrow (y_1'' + 2y_2'') - (y_1' + 2y_2') - 2(y_1 + 2y_2)$$

$$= (y_1'' - y_1' + 2y_1) + 2(y_2'' - y_2' - 2y_2)$$

$\Rightarrow$ from previous solution, $y'' - y_1' - 2y_1 = 0$

$$\& \; y_2'' - y_2' - 2y_2 = 0$$

$\Rightarrow 0 + 2(0) = 0$ $\boxed{\text{hence proven}}$

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(C)(ii) $[y_2(t), \ y_3(t)]$ $\quad W(y_2, y_3) = \begin{bmatrix} y_2 & y_3 \\ y_2' & y_3' \end{bmatrix}$

$$= \begin{bmatrix} e^{2t} & -2e^{2t} \\ 2e^{2t} & -4e^{2t} \end{bmatrix}$$

$$= e^{2t}(-4e^{2t}) - (-2e^{2t})2e^{2t}$$

$$= -4e^{4t} + 4e^{4t} \neq 0 \quad \forall t \in (-\infty, \infty)$$

y_2 & y_3 are solutions to DE but $W(y_2, y_3) = 0$
so it doesn't form a fundamental set of solutions.

(iii) $W(y_1, y_4) \Rightarrow \begin{bmatrix} e^{-t} & e^{-t} + 2e^{2t} \\ -e^{-t} & -e^{-t} + 4e^{2t} \end{bmatrix}$

$$= e^{-t}(-e^{-t} + 4e^{2t}) - (-e^{-t})(e^{-t} + 2e^{2t})$$

$$= \cancel{-e^{-2t}} + 4e^{t} + \cancel{e^{-2t}} + 2e^{t} = 6e^{t} \neq 0$$

$$\forall t \in (-\infty, \infty)$$

y_1 & y_4 are solutions of DE & so $W(y_1, y_2)$
& they also form fundamental set of solutions.

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Q5 (b) (iii)

$y_5(t) = 2y_1(t) - 2y_3(t)$, $y_5'(t) = 2y_1'(t) - 2y_3'(t)$

$y_5''(t) = 2y_1''(t) - 2y_3''(t)$

$\rightarrow y_5'' - y_5' - 2y_5$

$= \left[2y_1''(t) - 2y_3''(t)\right] - \left[2y_1'(t) - 2y_3'(t)\right]$

$- 2\left(2y_1(t) - 2y_3(t)\right)$

$4e^{2t}(2e^{-t})$

$= \left[2y_1''(t) - 2y_1'(t) - 4y_1(t)\right] - \left[2y_3''(t) - 2y_3'(t) - 4y_3(t)\right]$

$= 2\left[y_1''(t) - y_1'(t) - 2y_1(t)\right] - 2\left(y_3''(t) - y_3'(t) - 2y_3(t)\right]$

$= 2(0) - 0 \Rightarrow = 0$ $\boxed{\text{hence proven}}$

(c) (i) $W(y_1, y_2) = \begin{bmatrix} y_1 & y_3 \\ y_1' & y_3' \end{bmatrix} \Rightarrow \begin{bmatrix} e^{-t} & -2e^{2t} \\ -e^{-t} & -4e^{2t} \end{bmatrix}$

$= e^{-t}(-4e^{2t}) - (-e^{-t})(-2e^{2t})$

$= -4e^{2t}\cdot e^{-t} - 2e^{2t}\cdot e^{-t} \Rightarrow -4e^{t} - 2e^{t}$

$= -6e^{t} \neq 0$

$\forall t \in (-\infty, +\infty)$

y_1 & y_2 are solutions to DE &

as $w(y_1, y_2) \neq 0$ so y_1 & y_3 are fundamental solutions

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(Q5) (c)(iv) $w(y_4, y_5) = \begin{bmatrix} y_4 & y_5 \\ y_4' & y_5' \end{bmatrix} \Rightarrow \begin{bmatrix} e^{-t}+2e^{2t} & 2e^{-t}+4e^{2t} \\ -e^{-t}+4e^{2t} & -2e^{-t}+8e^{2t} \end{bmatrix}$

$(e^{-t}+2e^{2t})(-2e^{-t}+8e^{2t}) - (-e^{-t}+4e^{2t})(2e^{-t}+4e^{2t})$

$= \left(-2e^{-t}.e^{-t} + 8e^{2t}.e^{-t} + 2e^{2t}.-2e^{-t} + 2e^{2t}.8e^{2t}\right) - \left(-e^{-t}.2e^{-t} + 4e^{2t}.-e^{-t} + 4e^{2t}(2e^{-t}) + 4e^{2t}.4e^{2t}\right)$

$= \left(-2e^{-2t} + 8e^{t} - 4e^{t} + 16e^{4t}\right) - \left(-2e^{-2t} - 4e^{t} + 8e^{t} + 16e^{4t}\right) \Rightarrow 0$

$\forall t \in (-\infty, \infty)$

y_4 & y_5 are solutions to DE but as

$w(y_4, y_5) = 0$ $\therefore$ they arenot fundamental set of solutions.

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Q6

$$t^2y'' + \alpha t y' + \beta y = 0 \qquad u = \ln t$$

$$\frac{du}{dt} = \frac{1}{t} \quad \text{—①} \qquad , \qquad e^u = t \quad \text{—②}$$

$$\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} \Rightarrow \frac{dy}{dt} = \frac{dy}{du}\left(\frac{1}{t}\right) \Rightarrow \frac{dy}{du}\left(\frac{1}{e^u}\right)$$

$$\frac{dy}{dt} = e^{-u}\frac{dy}{du}$$

$$\frac{d^2y}{du^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) \Rightarrow \frac{du}{dt} \times \frac{d}{du}\left(e^{-u}\frac{dy}{du}\right)$$

substituting ①

$$\frac{1}{t}\left(-e^{-u}\frac{dy}{du} + e^{-u}\frac{d^2y}{du^2}\right)$$

$$= \frac{1}{e^u}\left(-e^{-u}\frac{dy}{du} + e^{-u}\frac{d^2y}{du^2}\right)$$

$$\Rightarrow t^2y'' + \alpha y' + \beta y = 0$$

$$e^{2u}y'' + \alpha e^u y' + \beta y = 0$$

$$e^{2u} \cdot \frac{1}{e^u}\left(-e^{-u}\frac{dy}{du} + e^{-u}\frac{d^2y}{du^2}\right) + \alpha e^u \cdot e^{-u}\frac{dy}{du} + \beta y = 0$$

$$-\frac{dy}{du} + \frac{d^2y}{du^2} + \alpha\frac{dy}{du} + \beta y = 0$$

$$\frac{d^2y}{du^2} + (\alpha - 1)\frac{dy}{du} + \beta y = 0$$

$\boxed{Q6}$ $t^2y'' + 3ty' + 1.25y = 0$

$a = 2, b = 1.25$

$$\frac{d^2y}{dt^2} + \frac{2\,dy}{dt} + 1.25y = 0$$

$\boxed{\text{let } y = e^{mt}, \; y' = me^{mt} \qquad y'' = m^2e^{mt}}$

$$m^2e^{mt} + 2me^{mt} + 1.25e^{mt} = 0$$

$$e^{mt}(m^2 + 2m + 1.25) = 0$$

$e^{mt} \neq 0$ so $m^2 + 2m + 1.25 = 0$ using $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$m = \frac{-2 \pm \sqrt{-1}}{2}$$

$\boxed{m_1 = -1 + \frac{i}{2}, \; m_2 = -1 - \frac{i}{2}}$

$$y = t^{\alpha}\left[c_1 \cos(\beta \ln t) + c_2 \sin(\beta \ln t)\right]$$

$$= t^{-1}\left[c_1 \cos\left(\tfrac{1}{2}\ln t\right) + c_2 \sin\left(\tfrac{1}{2}\ln t\right)\right]$$

as $x = \ln t$

$$\frac{1}{e^x}\left[c_1 \cos\left(\tfrac{1}{2}x\right) + c_2 \sin\left(\tfrac{x}{2}\right)\right]$$

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(17) $y_1(x) = e^{-\delta x^2/2}$, $y_1'(x) = -e^{-\delta x^2/2}\,\delta x$ & $y''(x) =$
$= -\delta x e^{-\delta x^2/2}$

$-\delta e^{-\delta x^2/2} + \delta^2 x^2 e^{-\delta x^2/2}$

$$y'' + \delta(xy' + y) = 0$$

$$= -\delta e^{-\delta x^2/2} + \delta^2 x^2 e^{-\delta x^2/2} + \delta\left(x\left(e^{-\delta x^2/2}, -\delta x\right) + e^{-\delta x^2/2}\right)$$

$$= \cancel{-\delta e^{-\delta x^2/2}} + \cancel{\delta^2 x^2 e^{-\delta x^2/2}} - \cancel{\delta^2 x^2 e^{-\delta x^2/2}} + \cancel{\delta e^{-\delta x^2/2}}$$

$\Rightarrow 0$ $\boxed{\text{hence proven}}$

$\Rightarrow y'' + \delta(xy' + y) = 0$ $\quad p(x) = \delta x$ & $q(x) = \delta$

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2}\,dx \Rightarrow e^{-\delta x^2/2} \int \frac{e^{-\int \delta x\, dx}}{\left(e^{-\delta x^2/2}\right)^2}\,dx$$

$$= e^{-\delta x^2/2} \int \frac{\cancel{e^{-\delta x^2/2}}}{\left(e^{-\delta x^2/2}\right)^{\cancel{2}}}\,dx$$

$$= e^{-\delta x^2/2} \int e^{\delta x^2/2}\,dx$$

$\Rightarrow$ General solution in form of integral , $y = y_1 + y_2$

$$y = e^{-\delta x^2/2} + e^{-\delta x^2/2} \times \int e^{\delta x^2/2}\,dx$$

$$= e^{-\delta x^2/2}\left(1 + \int e^{\delta x^2/2}\,dx\right)$$

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(Q8) $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$

$$\frac{2}{2}Q'' + \frac{24}{2}Q' + \frac{200}{2}Q = \frac{12}{2}\sin 10t$$

$$Q'' + 12Q' + 100Q = 6\sin 10t$$

let $Q = e^{mt}$

$Q' = me^{mt}$,

$Q'' = m^2e^{mt}$

Auxiliary equation = $m^2e^{mt} + 12me^{mt} + 100e^{mt} = 0$

$$e^{mt}(m^2 + 12m + 100) = 0$$

$$m^2 + 12m + 100 = 0 \quad \text{as } e^{mt} \neq 0$$

using $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \quad m = \frac{-12 \pm \sqrt{-256}}{2}$

$$= -6 \pm 8i$$

$g(x) = 6\sin 10t \qquad (D^2 + 10^2) \rightarrow$ annihilator

$$(D^2 + 100)(D^2Q + 12DQ + 100Q) = (D^2 + 100)(6\sin 10t)$$

$$(D^2 + 100)(D^2Q + 12DQ + 100Q) = 0$$

$$(D^2 + 100)(D^2 + 12D + 100)Q = 0$$

$$D^2 + 100 = 0 \quad \& \quad D^2 + 12D + 100 = 0$$

$$D = \pm 10i, \quad D = -6 \pm 8i$$

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$$Q_P = C_3 \sin 10t + C_4 \cos 10t + \underbrace{e^{-6t}(C_5 \sin 8t + C_6 \cos 8t)}_{\to y_c}$$

After eliminating commonality

$$Q_p = A \cos 10t + B \sin 10t \quad , \quad Q_p' = -10A \sin 10t + 10B \cos 10t$$

$$Q_p'' = -100 A \cos 10t - 100 B \sin 10t$$

$$Q'' + 12Q' + 100 Q = 6 \sin t$$

$$-100 \cancel{A \cos 10t} - \cancel{100 B \sin 10t} + 120 B \cos 10t - 120 A \sin 10t + \cancel{100 A \cos 10t} + \cancel{100 B \sin 10t} = 6 \sin t$$

$$120 B \cos 10t - 120 A \sin t = 6 \sin t$$

$$20 B \cos 10t - 20 A \sin 10t = \sin t$$

$$A = -1/20$$

$$B = 0$$

Let $y = Q$

$$y = y_c + y_p \longrightarrow y = e^{-6t}(C_1 \sin 8t + C_2 \cos 8t) - \frac{1}{20} \cos 10t$$

as $y = Q$

$$Q = e^{-6t}(C_1 \sin 8t + C_2 \cos 8t) - \frac{1}{20} \cos 10t$$

$$Q = C_1 e^{-6t} \sin 8t + C_2 e^{-6t} \cos 8t - \frac{1}{20} \cos 10t$$

$$Q' = C_1 \left[-6e^{-6t} \sin 8t + 8e^{-6t} \cos 8t\right) + C_2\left[-6e^{-6t} \cos 8t - 8e^{-6t} \sin 8t\right] + \frac{1}{20} \sin 10t$$

at $Q(0) = 0.001$

$$0.001 = C_1 e^0 \sin 0 + C_2 e^0 \cos 0 - \frac{1}{20} \cos 0$$

$$0.001 = 0 + C_2(1) - \frac{1}{20} \Rightarrow \boxed{C_2 = 0.051}$$

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at $\phi'(0) = 0$

$$0 = c_1\left(-6e^0\sin 0 + 8e^0\cos 0\right) + c_2\left(-6e^0\cos 0 + 8e^0\sin 0\right) + \frac{1}{2}\sin 0$$

$$0 = 8c_1 - 6c_2$$

$$c_1 = \frac{6(0.051)}{8}$$

$$\boxed{c_1 = 0.03825}$$

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