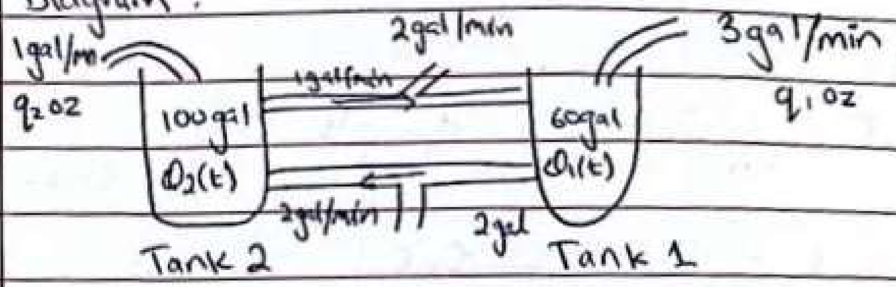
GRAND ASSIGNMENT

1) Diagram :



a. Total Rate = Rate in − Rate out

Tank 1: $\dfrac{dQ_1}{dt} = (3)(q_1) + (1)\dfrac{(Q_2)}{100} - (2)\dfrac{(2\times Q_1)}{60}$

$= 3q_1 + \dfrac{Q_2}{100} - \dfrac{4}{60}Q_1$

$\dfrac{dQ_2}{dt} = (q_2 \times 1) + \dfrac{(2\times Q_1)}{60} - \dfrac{(3\times Q_2)}{100}$

$= \dfrac{2Q_1}{60} - \dfrac{3Q_2}{100} + q_2$

initial conditions : $Q_1(0) = Q_1^{\circ}$ and $Q_2(0) = Q_2^{\circ}$

b. Let $Q_1 = Q_{1E}$ and $Q_2 = Q_{2E}$

~~$Q_1(t)$ 88:~~ $Q_1(t)$ ~~and~~ $Q_2(t) = 0$ ⇐ For equilibrium

$0 = 3q_1 + \dfrac{Q_{2E}}{100} - \dfrac{4Q_{2E}}{60} = q_2 + \dfrac{2Q_{1E}}{60} - \dfrac{3Q_{2E}}{100} = 0$

Solving eq by matrix :

$Ax = B = \begin{bmatrix} -4/60 & 1/100 \\ 2/60 & -3/100 \end{bmatrix} X = \begin{bmatrix} -3q_1 \\ q_2 \end{bmatrix}$

$X = \begin{bmatrix} -4/60 & 1/100 \\ 2/60 & -3/100 \end{bmatrix}^{-1} \begin{bmatrix} -3q_1 \\ q_2 \end{bmatrix}$

$Q_E = \begin{bmatrix} 18 & 6 \\ 20 & 40 \end{bmatrix} \begin{bmatrix} 3q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 54q_1 + 6q_2 \\ 60q_1 + 40q_2 \end{bmatrix}$

c. $Q_1^E = 60$, $Q_2^E = 50$

Prime

Scanned with CamScanner

Day / Date

1) c. not possible becos of -ve sign so :

$$\frac{2}{60} \times 60 - \frac{3}{100} \times 50 + q_2 = \frac{-1}{2}$$

$$q_2 = -\frac{1}{2}$$

d. $\frac{2}{60} Q_1E - \frac{3}{100} Q_2E \le 0$ & $\frac{-4}{3\times 60} Q_1E + \frac{1}{3\times 100} Q_2E \le 0$

$$\boxed{\frac{3}{20} Q_2E \le Q_1E \le \frac{9}{10} Q_2E}$$

2) $\frac{du_1}{dt} = -L_{21}u_1 + L_{12}u_2$

$\frac{du_2}{dt} = L_{21}u_1 - L_{12}u_2$

IVP are $a = 25$, $L_{12} = 1/50$; $L_{21} = 2/25$

$u_1(0) = a$, $u_2(0) = 0$

$u_1' = -L_{21}u_1 + L_{12}u_2$

$u_2' = L_{21}u_1 - L_{12}u_2$

characteristic eq :

$m_1 = -L_{21} + L_{12}$

$m_2 = L_{21} - L_{12}$

$u(t) = c_1 e^{-(L_{21}+L_{12})t} u_1 + c_2 e^{-(L_{21}+L_{12})t} u_2$

using initial conditions :

$c_1 L_{12} + c_2 = a$, $c_1 L_{21} - c_2 = 0$

$c_2 = a - c_1 L_{12}$, $c_2 = c_1 L_{21}$

$c_1 L_{21} = a - c_1 L_{12}$

$c_1 L_{21} + c_1 L_{12} = a$

$c_1 (L_{21} + L_{12}) = a$

$$c_1 = \frac{a}{L_{21} + L_{12}}$$

$$c_2 = c_1 L_{21} = \frac{L_{21}a}{L_{21}+L_{12}}$$

plotting values:

$a = 25,\quad L_{21} = 2/25,\quad L_{12} = 1/50$

$$c_1 = \frac{25}{0.08+0.02} = 250$$

$$c_2 = \frac{2}{0.08+0.02} = 20$$

put in ①: $\boxed{x(t) = 250\, e^{-(L_{21}+L_{12})t} x_1 + 20\, e^{-(L_{21}+L_{12})t} x_2}$

3) $C = 10^{-5}\,F,\ R = 3\times10^2,\ L = 0.2H,\ Q(0) = 10^{-6}\,C$

$I(t) = Q'(0) = Q'_0 = 0A$

There's no voltage $\therefore\ LQ'' + RQ' + \frac{1}{C}C = 0$

$Q(0) = Q_0,\quad Q'(0) = Q'_0$

$0.2Q'' + 3\times10^2 Q' + 10^5 Q = 0$

$Q(0) = 10^{-6}\,H$

$Q'(0) = 0C$

Characteristic eq: $\frac{2}{10}m^2 + 3\times10^2 m + 10^5 = 0$

$m_1 = -500,\quad m_2 = -1000$

$y(t) = c_1 e^{-500t} + c_2 e^{-1000t}$

$Q(0) = 10^{-6}$:

$10^{-6} = c_1 + c_2$

$c_1 = 10^{-6} - c_2$

$Q'(0) = 0$:

$0 = -500c_1 e^0 - 1000c_2 e^0$

$= -500(10^{-6} - c_2) - 1000c_2$

$$c_2 = \frac{-500\times10^{-6}}{500}$$

$\therefore c_2 = -10^{-6}\,C$

$c_1 = 2\cdot10^{-6}\,C$

Prime

$\boxed{Q(t) = 2\cdot10^{-6} e^{-500t} - 10^{-6} e^{-1000t}}$

Day / Date

4) a. Spring mass system: $m\frac{d^2u}{dt^2} = -ku - \beta\frac{du}{dt}$ or $mu'' + \beta u' + ku = f(t)$

$f(t)$ = External force, m = mass, $u(t)$ = displacement at t, β = damping constant

no external force $\therefore f(t) = 0$

$\therefore mu'' + \beta u' + ku = 0$

$m = 2$

$F = ku$

$k = \frac{u}{F} = \frac{3}{0.1} = 30$

$\beta = \frac{3}{5} = 0.6$

$\therefore$ eq $\Rightarrow 2u'' + 0.6u' + 30u = 0$

characteristic eq: $2m^2 + 0.6m + 30 = 0$

$m^2 + 0.3m + 15 = 0$

$m = \frac{-0.3 \pm 7.74i}{2}$ or $m = -0.15 \pm 3.87i$

$\therefore u(t) = e^{-0.15t}\left[c_1\cos(3.87)t + c_2\sin(3.87)t\right]$ — ①

At $t = 0$, $u' = 10$ cm/s or 0.1 m/s

$c_2 = 0.0274$

$u(t) = e^{-0.15t}\left[0.04\cos(3.87)t + 0.0274\sin(3.87)t\right]$

b. Qusai Frequency:

$u(t) = A\sin e^{-\gamma t}\left(\sqrt{w^2-\gamma^2}\,t + \varphi\right)$

where $\gamma = \frac{\beta t}{2\pi}$ and $\varphi = \frac{\sqrt{w^2-\gamma^2}}{2\pi}$

So $A = \sqrt{c_1^2 + c_2^2} = \sqrt{(0.04)^2 + (0.027)^2} = 0.057$

$\varphi = \tan^{-1}\left(\frac{c_2}{c_1}\right) = \frac{0.027}{0.04} = 3.87$

$u(t) = 0.057\sin e^{-0.15t}(3.87t - 0.50)$

quasi frequency $u = 3.87$ rad/s

Day / Date

4) b. Natural Frequency

$$\frac{4}{\omega_0} = \frac{3.87}{\sqrt{\frac{K}{m}}} = \frac{3.87}{\sqrt{\frac{30}{2}}}$$

$$= \boxed{0.099}$$

Phase Portrait : (done at end)

5) $y'' + 2y = \sin 4t$

$y' + 2y = \sin 4t \quad \{y(0) = 1$

$\mathcal{L}\{y'\} + \mathcal{L}\{2y\} = \mathcal{L}\{\sin 4t\}$

$\mathcal{L}\{y'\} = sY(s) - y(0)$

$2\mathcal{L}\{y\} = Y(s)$

$(\{\sin 4t\}) = \frac{4}{s^2+16}$

$sY(s) - y(0) + 2Y(s) = \frac{4}{s^2+16}$

$sY(s) - 1 + 2Y(s) = \frac{4}{s^2+16}$

$Y(s)(s+2) = \frac{4}{s^2+16} + 1$

$Y(s) = \frac{4}{(s+2)(s^2+16)} + \frac{1}{s+2}$

$Y(s) = \frac{4+s^2+16}{(s+2)(s^2+16)} = \frac{4+s^2+16}{(s+2)(s^2+16)}$

$Y(s) = \frac{s^2+20}{(s+2)(s^2+16)}$

$\frac{s^2+20}{(s+2)(s^2+16)} = \frac{Ax+B}{s^2+16} + \frac{C}{s+2}$

$s^2 + 20 = (Ax+B)(s+2) + C(s^2+16)$

$\therefore$ When $s = -2$: $C = 6/5$

When $s = 0$: $B = 2/5$

$\therefore A = -1/5$

$$\frac{s^2+20}{(s+2)(s^2+16)} = \frac{-s+2}{5(s^2+16)} + \frac{6}{5(s+2)}$$

$$Y(s) = \frac{-s+2}{5(s^2+16)} + \frac{6}{5(s+2)}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-s+2}{5(s^2+16)}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{5(s+2)}\right\}$$

$$\boxed{y = \frac{-1}{5}\cos 4t + \frac{\sin 4t}{10} + \frac{6}{5}e^{-2t}}$$

6) $y'' + e^x y' - y = 0$

$$y = \sum_{n=0}^{\infty} c_n (x)^n$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \dots\right)(c_1 + 2c_2 x + 3c_3 x^2 + \dots) - \sum_{n=0}^{\infty} c_n (x)^n$$

expand: $(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots) + (c_1 + [2c_2 + c_1]x + [3c_3 + 2c_2 + \frac{1c_1}{2}] \dots) - (c_0 + c_1 x + c_2 x^2 + \dots)$

$$= [2c_2 + c_1 - c_0] + (6c_3 + 2c_2)x + [12c_4 + 3c_3 + c_2 + \tfrac{1}{2}c_1] + \dots = 0$$

$\therefore$ $2c_2 + c_1 = 0$

$6c_3 + 2c_2 = 0$

$12c_4 + 3c_3 + c_2 + \frac{1}{2}c_1 = 0$

$c_2 = \frac{c_0 - c_1}{2}$, $c_3 = -\frac{1}{3}c_2$, $c_4 = -\frac{1}{4}c_3 + \frac{1}{12}c_1 - \frac{1c_1}{24}$

Put $c_0 = 1$ & $c_1 = 0$

$c_2 = -\frac{1}{2}$, $c_3 = \frac{1}{6}$, $c_4 = \frac{-1}{24}$

$\therefore$ $y = c_1 y_1 + c_2 y_2$

$\boxed{y_1 = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots}$ & $\boxed{y_2 = x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots}$