Towards Curry-Howard Isomorphism

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1 Intro

Scenario Imagine if aliens invaded (or peacefully communicated with) Earth. It's quite intuitive and natural for us to expect our natural languages (eg: English vs Alienese) to be incompatible and for them not to understand English. This is because we know that English is an arbitrary language and is something man-made.

I suppose we then move to other languages, like math. We believe that math is universal and thus, although our notations for π might not be the same nor our representation of numbers, we will still be able to know that we "speak the same language." Again, this falls back to the recurring idea of "universal" vs "arbitrary."

Howabout programming languages? Is C arbitrary? Answer: yes. Is Java? Yes. Is Pyth–? Yes. So programming languages are manmade and arbitrary? No.

Claim There are universal programming languages that we can discover by following indisputable universal laws or uncontroversial axioms. In particular, studying a particular logic gives rise to a natural programming language. In this paper we will sketch out the most famous and earliest discovery of such an equivalence – the Curry Howard isomorphism between natural deduction and the famous lambda calculus. The following are equivalent; pick and choose the most appealing phrase:

Propositions are types
Proofs are Programs
Logical Systems are Programming Languages

One of the key insights that impacted proof theory from a computational perspective is the observation that implication, $A \supset B$ (read: A implies B), is not just $\neg A \vee B$, which is the classical interpretation, which treats implication as a mere result of negation and disjunction.

Instead, say, for entertainment sake, we will interpret implication as an independent logical connective, where $A \supset B$ is to be interpreted more naturally: "if I know A, then I know B." This sort of mindset leads us to intuitionistic logic, which a proof theorist interested in its computational interpretation ought to at least entertain (whether or not you believe in the philosophical aspect of intuitionism is another story).

2 Natural Deduction

In proof theory, we study proofs. Whereas in other disciplines, proofs are mere tools to say something enrichening, in proof theory, we treat proofs as mathematical objects which allow us to rigorously "prove" various properties of proofs. Natural deduction is one such representation (or a "model," not in the technical sense) due to Prawitz.

2.1 Conjunction

We first start with the conjunction (logical and).

$$\frac{A \quad B}{A \wedge B}\ (\wedge I)$$

This is read from top to bottom as a premise to conclusion: the rule says that from A "and" (in a meta sense) B, we can conclude $A \wedge B$. The $(\wedge I)$ is a label and is of no theoretical importance; the I is a shorthand for "introduction". This is an example of a rule of inference, and more specifically, an introduction rule since the logical connective $\wedge$ appears in the conclusion.

Next we have the elimination rules for conjunction. Since an introduction rule "introduces" an $\wedge$ connective to the conclusion, an elimination rule "eliminates" an $\wedge$ connective from the conclusion. At first, we might be tempted to follow a symmetric pattern:

$$\frac{A \wedge B}{A \quad B}\ (\wedge E_{wrong})$$

However, this is not quite what we want – natural deduction is intentionally formulated to "shrink" down, and so instead, we construct two elimination rules:

$$\frac{A \wedge B}{A}\ (\wedge E_1) \quad \frac{A \wedge B}{B}\ (\wedge E_2)$$

There might be a concern that we are losing information, but implicitly, since we know that if we can prove $A \wedge B$ once, we can prove it again in the boring fashion (by copy pasting), we can always recover both A and B by applying $(\wedge E_1)$ and $(\wedge E_2)$ respectively to two copies of $A \wedge B$, so no information is lost.[1]

So far, we have defined the meaning of a conjunction through the introduction and elimination rules. As an example of a typical proof in natural deduction, we will prove the meta theorem that conjunction is associative. That is, we want to show that from the hypothesis $(A \wedge B) \wedge C$, we can conclude $A \wedge (B \wedge C)$ and vice versa. Put concisely, we want to show:

$$(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$$

$$\dfrac{\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge E_1)}{A}\,(\wedge E_1) \qquad \dfrac{\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge E_1)}{B}\,(\wedge E_2) \qquad \dfrac{(A \wedge B) \wedge C)}{C}\,(\wedge E_2)}{B \wedge C}\,(\wedge I)}{A \wedge (B \wedge C)}\,(\wedge I)$$

The opposite direction follows a similar structure (can be an exercise).

2.2 Implication

Adding implication requires an additional construct in our formulation.

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \supset B}\,(\supset I)$$

The premise $\begin{matrix}[A] \\ \vdots \\ B\end{matrix}$ is called a hypothetical judgement, hypothetical derivation, or a hypothetical proof. The $[A]$ is a local hypothesis and hence bound to the hypothetical derivation only – in particular, this enforces the intuitive idea that we cannot use A outside of the hypothetical proof.

Interestingly, modus ponens appears as a natural elimination rule to the implication we defined:

$$\dfrac{A \supset B \quad A}{B}\,(\supset E)$$

At first, we may question the utility of this connective in natural deduction. Afterall, what's wrong with ignoring $A \supset B$ and instead working directly with the hypothetical proof? It is indeed true that from a pure "proof" perspective, a hypothetical judgement is identical to an implication, and structurally, using the hypothetical judgement makes more sense, though may not be economical space-wise. However, do not forget that proof theory is the study of proofs; this seemingly redundant encoding allows us to look beyond mere provability since we can now reference proofs at a meta-level (using hypothetical judgements) and inside the language (using implication).

[1] This notion of "not losing information" is called completeness, and indeed there are rigorous ways to verify that the seemingly arbitrary introduction and elimination rules are correct in that sense.

2.2.1 Exercise

Prove the fragment of logic we have introduced so far is "cartesian closed." [2] That is, prove:

$$(A \wedge B) \supset C \Leftrightarrow A \supset (B \supset C)$$

(implicit parenthesis added for clarity sake)

2.3 Disjunction

Initially, disjunction seems like a dual to conjunction:

$$\frac{A}{A \vee B}\ (\vee I_1) \quad \frac{B}{A \vee B}\ (\vee I_2)$$

However, its elimination rule is a bit complicated:

$$\frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\ (\vee E)$$

This says that to eliminate an instance of an $A \vee B$, we require a motive C such that from A we can prove C and from B we can prove C. This is essentially a case analysis, which of course is how we intuitively handle statements like "A or B."

2.3.1 Exercise

Prove that disjunction and conjunction distributes. That is, prove:

$$(A \vee B) \wedge C \Leftrightarrow (A \wedge C) \vee (B \wedge C)$$

and

$$(A \wedge B) \vee C \Leftrightarrow (A \vee C) \wedge (B \vee C)$$

2.4 Sequent

For economic reasons, we will formulate the two dimensional hypothetical judgements as one dimensional sequents. A sequent $A, B, C, \cdots \vdash D$ consists of a context, a list of antecedents or hypothesis $(A, B, C, \cdots)$, often abbreviated Γ, followed by a single conclusion (D).[3]

For example, recall the introduction rule for implication:

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \supset B}\ (\supset I)$$

[2] jargon anticipating the connection between logic and category theory

[3] An intuitionistic interpretation... classical sequents allow a list of conclusions (interpretted as disjunctions).

and rewrite it as follows:

$$\frac{A \vdash B}{\cdot \vdash A \supset B}\ (\supset I)$$

where $\cdot$ is meant to represent an "empty hypothesis." The notion that the assumption A cannot be used outside the single premise is captured by removing A from the antecedent in its conclusion.

However, this is not quite general enough; in particular, we've ignored the fact that there can be other hypotheses floating about! That is, since the conclusion requires an empty hypothesis, we are no longer able to apply this rule of implication introduction under other hypotheses. Thus, the correct form of implication introduction should allow arbitrary hypothesis (contexts):

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ (\supset I)$$

Again, it's important to note here that A still "disappears" in the conclusion as expected.

This one dimensional presentation of hypothetical judgements might further provoke some dissonance and objections to the role of implication – really, what is the difference between $A \vdash B$ and $A \supset B$? From a syntactical point of view, the difference is clear; $A \vdash B$ is not a proposition and $A \supset B$ is a proposition. However, this might not be too satisfying, since afterall, this distinction is something we artificially created. The reason for this seemingly duplicate work is again because proof theory is a "proof-relevant" theory. We don't only care about something being true or not or being provable or not – there is value in looking at an "implication" (in an informal sense) from the meta perspective (sequents/hypothetical judgements) and from the language perspective ($\supset$).

2.4.1 Exercise

Rewrite all (or an arbitrary selection) of the rules of inference keeping in mind the "arbitrary context" rule. For the most part, it's as simple as adding $\Gamma \vdash$ in front of everything. The only tricky part is the elimination rule for disjunction, where its hypothetical judgements should be rewritten in a similar manner to the one in implication introduction.

Assuming you did above for at least conjunction introduction (if not, please do it), try to intuitively convince yourself that what you wrote is general enough to capture this version of conjunction introduction:

$$\frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$$

(hint, let your $\Gamma := \Gamma_1, \Gamma_2$ and fill in the gaps). The filling in of these "gaps" is called weakening.

2.5 Proof Terms

Are the excercises frustrating enough? So far we have only introduced metavariables for arbitrary propositions (A, B, C) but have not introduced metavariables (or any structural way of reasoning) about proof terms.

$$M : A \qquad M \text{ is a proof of proposition } A$$

Similarly, the context now consists of not just propositions but the proof terms associated with each proposition. Therefore, a typical sequent looks like:

$$x : A, y : B, \cdots \vdash M : A$$

where Γ is still used as a metavariable for the hypotheses.

This change of course means that we have to reformulate all of our inference rules.

2.5.1 Conjunction

Given a proof of A and a proof of B, what constitutes a proof of $A \wedge B$? Well, we can form a pair of proofs.

$$\frac{\Gamma \vdash M : A \quad N : B}{\Gamma \vdash \langle M, N \rangle : A \wedge B}\ (\wedge I)$$

On the other hand, if we have an arbitrary proof of $A \wedge B$, how can we extract a proof of A and a proof of B? Since the proof of $A \wedge B$ is arbitrary, we cannot assume its form. Instead, we require two projection operators π_1 and π_2:

$$\frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash \pi_1 M : A}\ (\wedge E_1) \quad \frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash \pi_2 M : B}\ (\wedge E_2)$$

η expansion Note that for any proof term $M : A \wedge B$, we have

$$M =_\eta \langle \pi_1 M, \pi_2 M \rangle$$

The right hand side is called the η expansion of M. This essentially says that if we know that M is a proof of $A \wedge B$, we can create an equivalent proof by constructing a pair of its first and second projections respectively. This is also known as "extensional" equality, as opposed to structural or intensional equality, where the idea is that we are asserting that they ought to be equal because they are observationally equivalent. This is a formal version of the whole duck principle: if it quacks like a duck, looks like a duck, ..., then it's a duck.

β reduction On the other hand, for a proof term $\langle M, N \rangle : A \wedge B$, we can talk about what happens to the proof term under π_1 and π_2.

$$\pi_1 \langle M, N \rangle =_\beta M$$
$$\pi_2 \langle M, N \rangle =_\beta N$$

This foreshadows the computational interpretation of natural deduction (of intuitionistic logic); a β reduction of a proof corresponds to a single "computational step."

Associativity Proof Recall the lengthy and pedantic proof that $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$. By introducing the syntactical notion of proof terms, we can now construct a witness that proves the above sequent:

$$x : (A \wedge B) \wedge C \vdash \langle \pi_1 \pi_1 x, \langle \pi_2 \pi_1 x, \pi_2 x \rangle \rangle : A \wedge (B \wedge C)$$

2.5.2 Implication

Implication is the most important logical step that gives us a satisfying computational interpretation. Given a hypothetical judgement $M : B$ under hypothesis $x : A$, we can create a lambda term (or an anonymous function) $\lambda x : A.M : A \supset B$. It turns out that specifying the type of x is unnecessary (nevertheless still useful as an annotation), so leaving out the type annotation, here is the introduction rule:

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \supset B} \ (\supset I)$$

The proof theoretic interpretation is still the same: give me a proof of A, which we will call x, and using that, I will prove B. However, the computational interpretation is that of a function: give me some $x : A$ and I'll give you some well-formed $M : B$.

The elimination rule simply puts two proof terms side by side.

$$\frac{\Gamma \vdash M : A \supset B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : B} \ (\supset E)$$

It may be more suggestive to write $M(N)$ to emphasize that N is applied to M. The computational interpretation is a function application, where N is to be interpreted as an argument to M.

For some $M : A \supset B$, we can conclude

$$M =_\eta \lambda x.Mx$$

On the other hand for some M and N, we have

$$(\lambda x.M)N =_\beta [N/x]M$$

where the right hand side is read: "replace all instances of x in M with N."

2.5.3 Disjunction

Whereas conjunction had a notion of projection operators that extracted either the first or the second term in a pair of proofs, disjunction requires a notion of injection operators that injects a proof into a disjunction of two proofs:

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \iota_1 M : A \vee B} \ (\vee I_1) \quad \frac{\Gamma \vdash M : B}{\Gamma \vdash \iota_2 M : A \vee B} \ (\vee I_2)$$

The elimination rule as we had previously discussed is a case analysis:

$$\frac{\Gamma \vdash M : A \vee B \quad \Gamma, x : A \vdash P : C \quad \Gamma, y : B \vdash Q : C}{\Gamma \vdash \mathrm{case}(M; x.P; y.Q) : C} \ (\vee E)$$

The $x.$ and $y.$ at front emphasizes that P and Q depend on x and y respectively.

For some $M : A \supset B$, we can conclude

$$M =_\eta \lambda x.Mx$$