Neuroprothetics Exercise 3
Mathematical Basics 2

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1 Solver Implementation and Solver Functions

By initiating at a selected point and tracing a curve that follows the direction indicated by the slope field, then methodically progressing to the next point and continuing this process, one can effectively construct an approximate outline of the entire integral curve. During this process, solver functions play crucial roles. Solver functions calculate the next point V_{n+1} based on the current slope and the time step size Δt.
We obtain the value at time n by evaluating $V(t_n)$, and similarly, we find the value at step $n+1$ by evaluating $V(t_{n+1})$. So we get the equations:

$$V_{\mathrm{n+1}} = t_n + \Delta t \tag{1}$$

$$V_{\mathrm{n}} = V(t_n) \tag{2}$$

$$V_{\mathrm{n+1}} = V(t_{\mathrm{n+1}}) \tag{3}$$

We will interpret and implement the following three Solver Functions: Forward (Explicit) Euler, Heun (2. Order) and Exponential Euler (1. Order).

1.1 Forward (Explicit) Euler

As shown in equation 4, the Forward (Explicit) Euler method directly directly use the slope f (V_n, tn) times Δt to compute the position of the next point $V_{\mathrm{n+1}}$.

$$V_{\mathrm{n+1}} = V_n + f(V_n, t_n) \cdot \Delta t \tag{4}$$

1.2 Heun (2. Order)

As shown in equation 8, the Heun (2. Order) method use the average slope of f(V_n, t_n) and one more support point to compute the position of the next point $V_{\mathrm{n+1}}$.

$$A = f(V_n, t_n) \tag{5}$$

$$\tilde{V} = V_n + A \cdot \Delta t \tag{6}$$

$$B = f(\tilde{V}, t_{n+1}) \tag{7}$$

$$V_{\text{n+1}} = V_n + \frac{A + B}{2} \cdot \Delta t \tag{8}$$

1.3 Exponential Euler (1. Order)

In the case of the 1st order exponential euler method such an equation can be solved by:

$$V_{\text{n+1}} = V_n e^{A \cdot \Delta t} + \frac{B(V, t_n)}{A}(e^{A \cdot \Delta t} - 1) \tag{9}$$

1.4 Plotting Differential Equations

In this subsection, we solve the differential equation $\frac{dV}{dt} = 1 - V - t$, where $V(t = -4\text{s}) = V_0 = -3V$, using the above-mentioned solvers. The simulations run for 9 s with varying step sizes of 1.5 s, 0.75 s, and 0.1 s. The results are shown in Fig 1.

1.5 Conlusion

(1) the differences between solvers
The Forward Euler method is simple but may not be as stable and accurate as other methods, especially for larger step sizes. Heun(2. Order) method is more accurate and stable than Forward Euler method. The Exponential Euler method offers higher accuracy and generally more stable for stiff problems with exponential changes, but is more computationally complex.
(2) the impact of changing the step size
Smaller step sizes generally lead to higher accuracy in the solution, because a smaller step size can better capture rapid changes and fine details of the equation's solution. However, the smaller the step size, the more iterations are required for the computation, which will increase the consumption of computational time and resources.
(3) the tradeoff when choosing a stepsize in numerical solving and why not go for infinitesimal stepsize
When choosing a step size in numerical solving, the tradeoff involves balancing accuracy and computational efficiency. Smaller step sizes (approaching infinitesimal) increase accuracy but significantly raise computational cost and time due to a higher number of iterations. Extremely small step sizes can also introduce rounding errors due to the finite precision of computer arithmetic, potentially reducing overall accuracy.

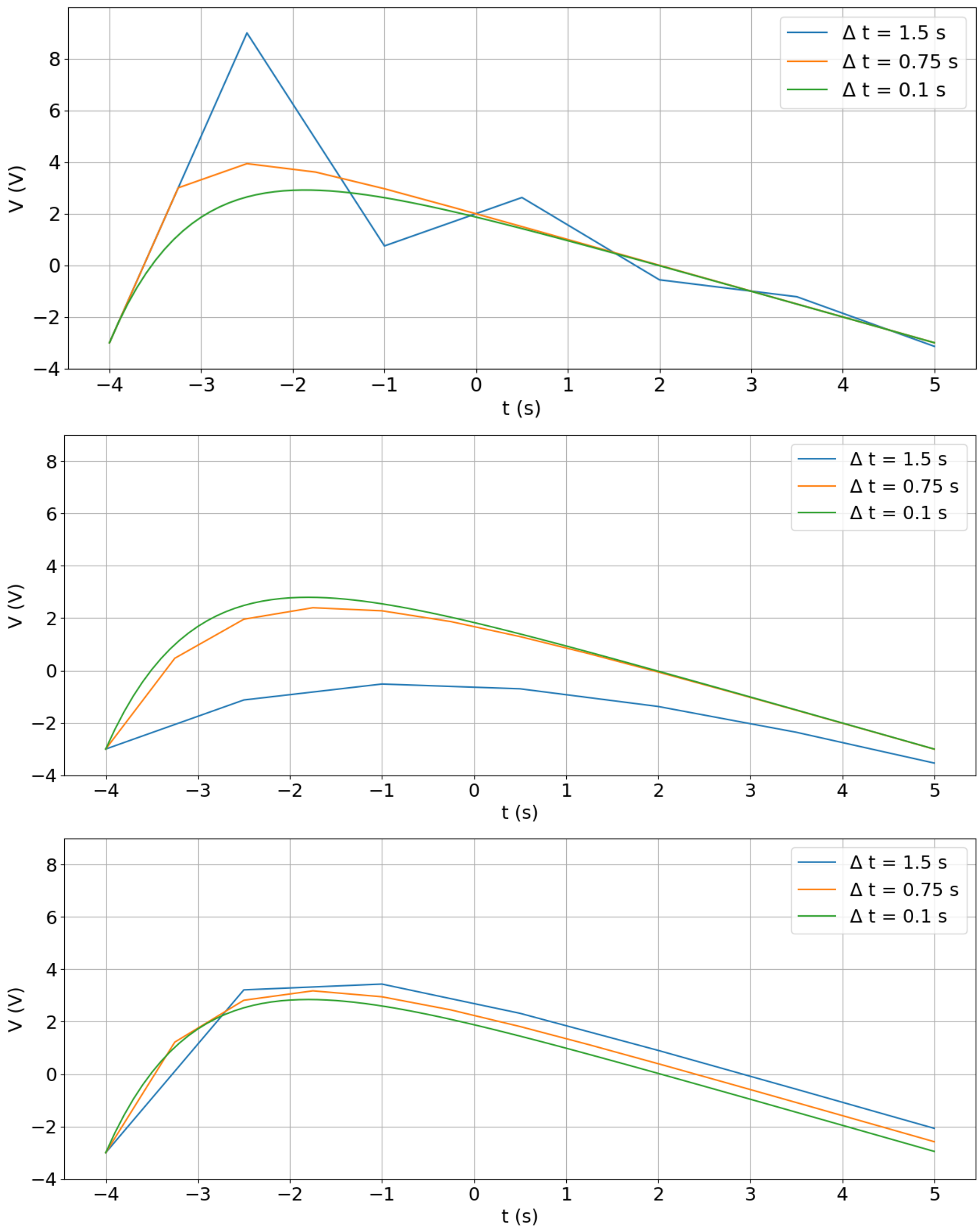


Figure 1: Approximations of the given differential equation in section 2 with different solver methods and step sizes. Top: Forward-Euler-Method. Middle: Heun-Method. Bottom: Exponential-Euler Method.

2 The Leaky Integrate and Fire Neuron

When the cell membrane voltage is below the spiking threshold voltage V_{thr}, the neuron is in its integration mode. During this phase, the membrane potential increases as input currents charge the membrane capacitance. When the membrane potential reaches the threshold, it is set to the spike voltage V_{spike}, representing the firing of an action potential. Afterward, the membrane potential is reset to the resting potential V_{rest}, simulating the neuron's return to a rest state post-firing. Equation 10 illustrates a Leaky Integrate and Fire Neuron Model:

$$V_{n+1} = \begin{cases} V_n + \frac{\Delta t}{C_m}(-g_{leak}(V_n - V_{rest})) + I_{input}(t_n)) & V_n < V_{thr} \\ V_{spike} & V_{thr} \le V_n < V_{spike} \\ V_{rest} & V_{spike} \le V_n \end{cases} \tag{10}$$

2.1 Plotting the current I_{input}

I_{input} in equation 10 is a rectified sinusoidal current. The equation of I_{input} can be expressed as equation 11. We get the plot of the current I_{input} in Fig 2.

$$I_{input}(t) = A \cdot |\sin(2\pi f t)| \tag{11}$$

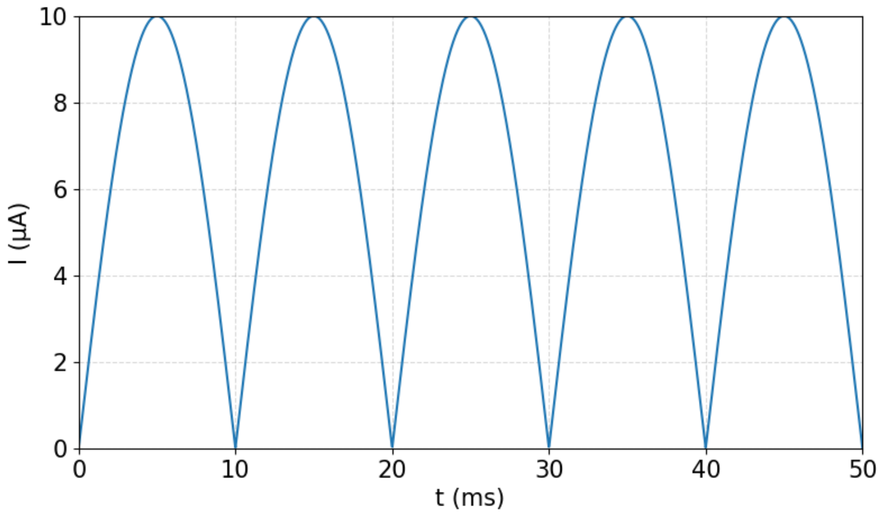
2.2 Plotting the LIF model

In this exercise, the initial values of all variables are set as follows: C_m = 1 μF, g_{leak} = 100 μS, V_{rest} = -60 mV, V_{thr} = -20 mV, V_{spike} = 20 mV, t_{start} = 0 ms, t_{end} = 50 ms, Δt = 25 μs, and two different rectified sine wave inputs with amplitudes of A_1 = 10 μA, A_2 = 30 μA. According to the equation 7 and all the above variables initial values, we get the plot of the Leaky Integrate and Fire Neuron model in Fig 3.

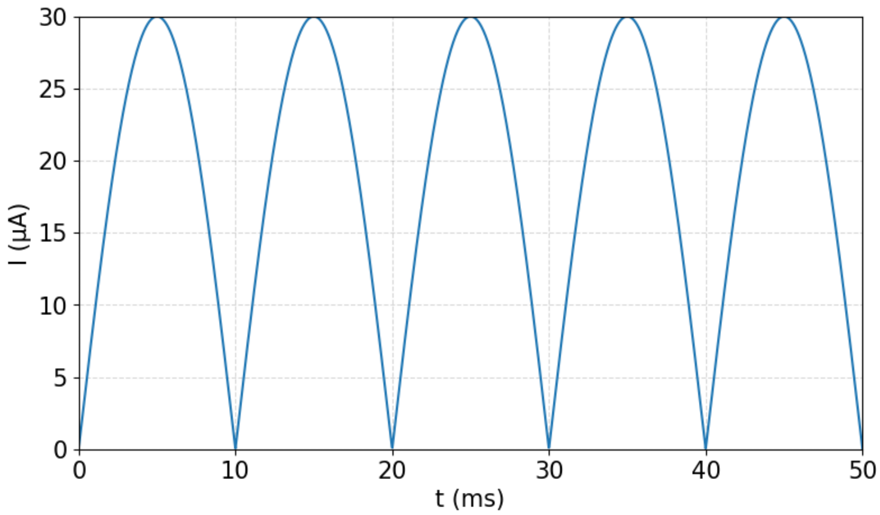
2.3 Conclusions

(1) visualizing the leaky, the integrating, and the firing property of LIF model in the plots

- the leaky property: This property is visible as the gradual decline in the voltage between spikes. Following each spike, the voltage doesn't remain at the peak, instead, it decreases towards the resting potential due to the neuron's inherent leakiness.

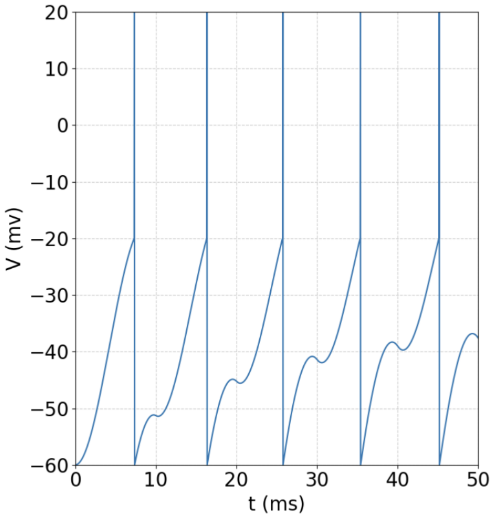


(a) Rectified sine current input for the LIF model with an amplitude of $10\mu A$

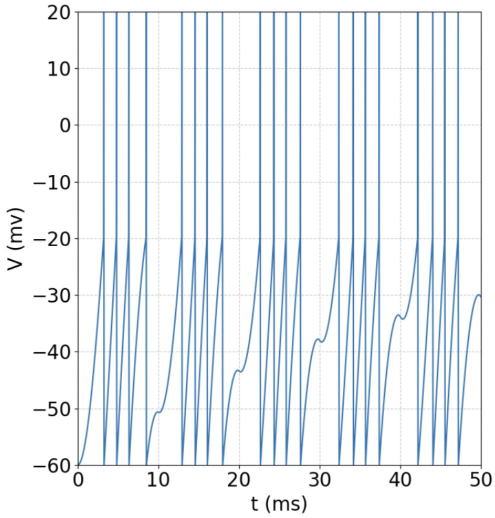


(b) Rectified sine current input for the LIF model with an amplitude of $30\mu A$

Figure 2: Current inputs for the LIF-Model



(a) Rectified sine input current with an amplitude of $10\mu A$



(b) Rectified sine input current with an amplitude of $30\mu A$

Figure 3: Cell membrane voltage of a LIF-Model using different current inputs

- the integrating property: As the input current charges the neuron cells, the cell membrane voltage incrementally increases until it reaches the threshold. This is seen as the rising phases on the plot, where the slope indicates the integration of the input current over time.

- the firing property: The sharp vertical rises in the plot represent the neuron firing action potentials. Whenever the membrane voltage reaches the threshold voltage V_{thr}, a spike is generated, and the voltage shoots up to V_{spike}.

(2) the differences caused by different input currents

As shown in Fig 3, higher input currents lead to a shorter integration time for the membrane potential, resulting in more frequent neuron firing and reduced leakage effects on the membrane potential between spikes. This can be observed in the plots as shorter intervals between spikes and a steeper slope during the integration phase before each spike.