Neuroprothetics Exercise 5
Multicompartment Model

Chutong Ren

05. Januar 2024

1 Multi-Compartment Model

Multi-Compartment Model can simulate the propagation of action potentials along the axon of a neuron, in which the axoplasmatic resistance (R_a) connect every single compartment. The equivalent circuit is shown in Figure 1. Here, $V_{i,n}$ and $V_{e,n}$ respectively represent the internal and external membrane potential of the n-th compartment (or node).

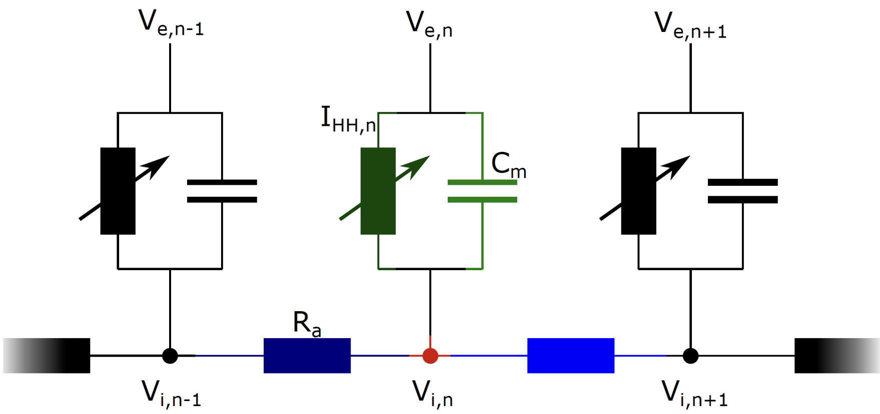


Figure 1: Equivalent Circuit

Kirchhoff's current law states that for any point (node) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of the node. So the current value at the red dot represented by $V_{i,n}$ is zero, which includes the capacitive current $C_m \frac{dV_{m,n}}{dt}$, the ionic current through the Hodgkin-Huxley channels $I_{HH,n}$, and the axial currents between the node $V_{i,n}$ and its neighboring nodes, given by $\frac{V_{i,n}-V_{i,n-1}}{R_a}$ and $\frac{V_{i,n}-V_{i,n+1}}{R_a}$. This results in the derivation of a system of Ordinary Differential Equations(ODEs).

$$0 = C_m \frac{dV_{m,n}}{dt} + I_{HH,n} + \frac{V_{i,n} - V_{i,n-1}}{R_a} + \frac{V_{i,n} - V_{i,n+1}}{R_a} \tag{1}$$

Consequently, $V_{m,n}$ can be determined using Equation 2:

$$\frac{dV_{m,n}}{dt} = -\frac{1}{C_m} I_{HH,n} + \frac{1}{C_m} \frac{V_{i,n-1} - 2V_{i,n} + V_{i,n+1}}{R_a} \tag{2}$$

Based on this foundation, we add a stimulating current $I_{stim,n}$:

$$\frac{dV_{m,n}}{dt} = -\frac{1}{C_m}(-I_{HH,n} + I_{stim,n}) + \frac{1}{C_m} \frac{V_{i,n-1} - 2V_{i,n} + V_{i,n+1}}{R_a} \tag{3}$$

1.1 Resistance

The resistance of the conductor will increase when its length increase or its diameter decrease, and it will decrease if the length decreases or the diameter increases. Therefore, the axonal resistance Ra can be computed using the axonal resistivity $\rho_{\text{axon}} = 1\,\Omega \cdot \text{m}$, the axonal radius $r_{\text{axon}} = 2 \cdot 10^{-6}$ m, and the compartment length $l_{\text{comp}} = 0.1 \cdot 10^{-6}$ m, as detailed in the appendix.

$$R_a = \frac{\rho L}{A} = \frac{\rho L}{\pi r^2} \tag{4}$$

1.2 Vectors

The values of V_m, I_{HH}, I_{stim} and V_i in a 100-compartment model can be represented by column vectors, each of size 100. The system can be described for all indices of n using vectors:

$$\frac{d}{dt}\vec{V}_m = \frac{1}{C_m}(-\vec{I}_{HH} + \vec{I}_{stim} + \frac{1}{C_m R_a} C\vec{V}_i) \tag{5}$$

The matrix C of size $n \times n$, which can represent the influence of neighboring nodes on the current node.

$$C = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$$

1.3 Membrane potential

The membrane potential V_m of neuron is equal to the voltage difference across the cell membrane, which is defined in Equation 6.

$$\vec{V}_m = \vec{V}_i - \vec{V}_e \tag{6}$$

By substituting V_m into Equation 5, we obtain the following expression:

$$\frac{d}{dt}\vec{V}_m = \frac{1}{C_m}(-\vec{I}_{HH} + \vec{I}_{stim} + \frac{1}{C_m R_a} C\vec{V}_m + \frac{1}{C_m R_a} C\vec{V}_e) \tag{7}$$

In this exercise, we assume there is no external potential gradient, which means $\vec{V}_e = 0$. We use Implicit Euler Method to solve this ODE system, which means

$$\vec{V}_m(t+\Delta t) = \vec{V}_m(t) + \frac{\Delta t}{C_m}(-\vec{I}_{HH}(t+\Delta t) + \vec{I}_{stim}(t+\Delta t)) + \frac{\Delta t}{C_m R_a} C\vec{V}_m(t+\Delta t) \tag{8}$$

$$(I - \frac{\Delta t}{C_m R_a} C) \cdot \vec{V}_m(t+\Delta t) = \vec{V}_m(t) + \frac{\Delta t}{C_m}(-\vec{I}_{HH}(t+\Delta t) + \vec{I}_{stim}(t+\Delta t)) \tag{9}$$

This leads to a linear system of equations as follows.

$$A \cdot \vec{x} = \vec{b} \tag{10}$$

So we can use the solve function in Python like x = solve(A, b) or direct x = A \ b in Matlab to solve linear equation 10. $(I - \frac{\Delta t}{C_m R_a} C)$ is A, $\vec{V}_m(t+\Delta t)$ is what we want to get, and the right side of the equation is $\vec{b}$. In this situation, A is a fixed value. If we want to get $\vec{I}_{HH}$ (t + Δt), we must first obtain the values of the gating variables m(t + Δt), n(t + Δt) and h(t + Δt).

2 Experiments

The experiment in this exercise spans 100 ms with $\Delta t = 25\mu$s and temperature = 6.3°C.

1. In Situation 1, we apply a 5 ms rectangular stimulus pulse to the last compartment of the axon starting at t = 5 ms. The resulting action potential propagation is depicted in the top plot of Figure 2. In this exercise we simulate an unbranched, unmyelinated axon of a neuron, so after applying a stimulus current at the terminal end of the axon, the action potential propagates uniformly and continuously towards the first compartment along the axon.

2. In Situation 2, we apply the same pulse as in Situation 1, but at compartment 20 at t = 0 ms and at compartment 80 at t = 15 ms. This propagation is illustrated in the bottom plot of Figure 2. The action potentials are initiated in compartments 20 and 80 and begin to propagate along the axon in opposite directions. Their propagation is interrupted when they meet in the middle of the axon, for the following reason:
   When the two action potentials collide, the region they encounter is already in the absolute refraction period, during which the sodium ion channels are inactivated and cannot immediately respond to new stimulation. This inability to generate

new action potentials results in the cessation of both action potentials' propagation at the point of collision.

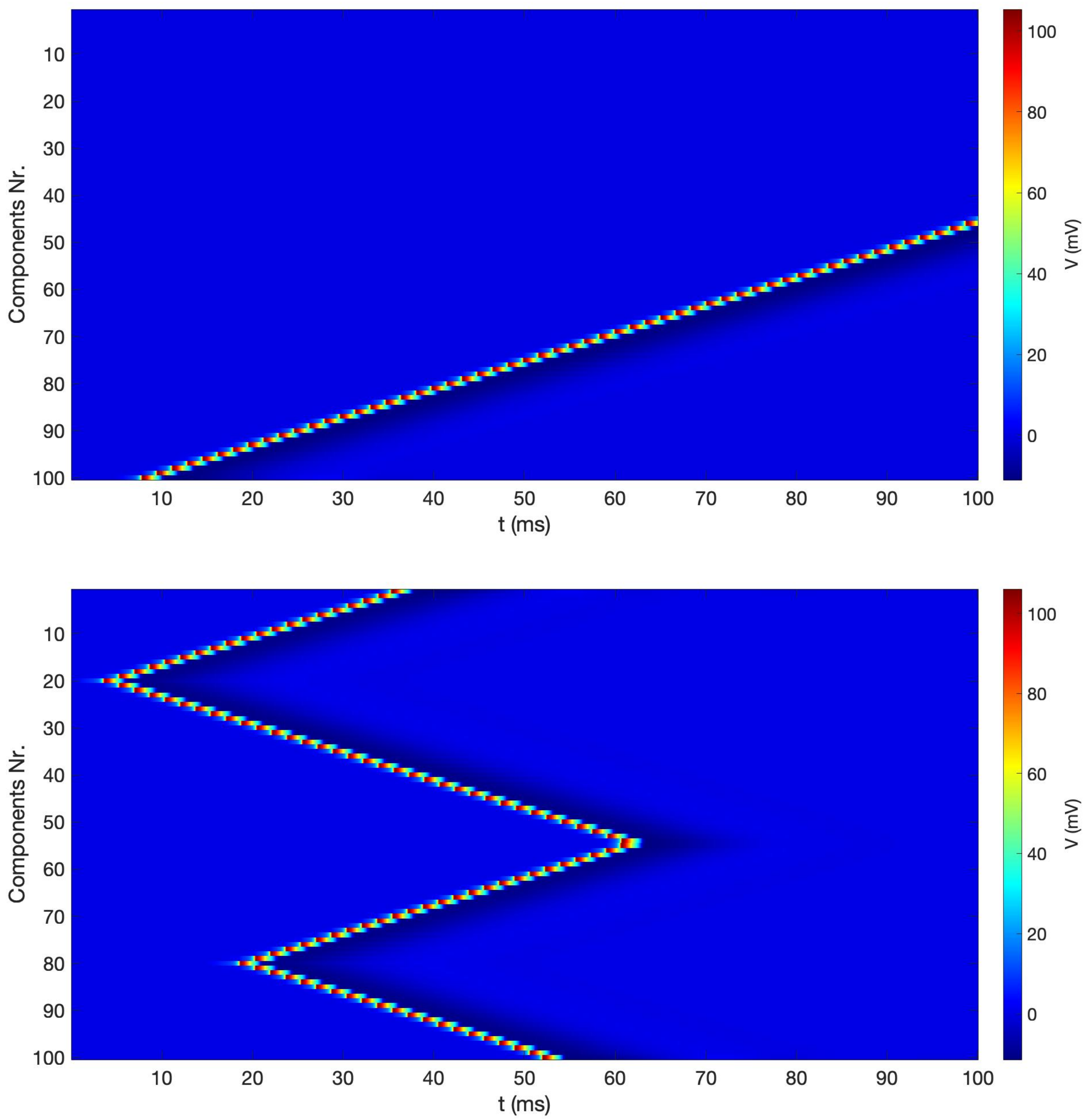
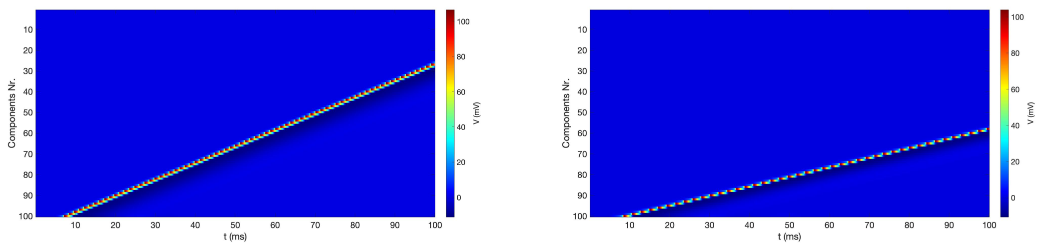


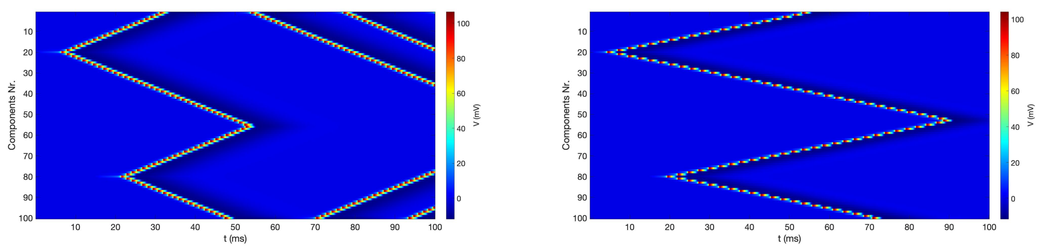
Figure 2: Top: Propagation of the action potential over the length of the axon for stimulation at the last compartment. Bottom: Propagation of the action potential over the length of the axon for stimulation at compartments 20 and 80. The colorbar indicates the membrane potential of the neuron compartments.

3. Higher membrane capacity (C_m) can store more charge, which leads to a slower charging rate and thus slower action potential initiation and propagation. Conversely, lower C_m results in faster charging and quicker action potential propagation. In order to explore how membrane capacity affect action potential propagation, I changed C_m to $0.7C_m$ and $1.3C_m$ in Situation 1($C_m = 1{\cdot}10^{-6}\mu F$), and the results are shown in Figure 3. The left plot demonstrates that when C_m decreases, the slope is steeper, indicating a higher action potential propagation rate. Conversely, the right plot illustrates that when C_m increases, the slope is shallower, demonstrating a slower action potential propagation rate.
   Lower axonal resistance (R_a) reduces resistance to the ionic current, thus speeding up action potential propagation, while higher Ra increases resistance, which can slow down the action potential propagation. Similarly, in order to explore how axonal resistance affect action potential propagation, I changed R_a to $0.5R_a$ and $1.2R_a$ in Situation 2, and the results are shown in Figure 4. The left image demonstrates that, as Ra decreases, the speed of propagation will increase. The right plot clearly shows that when R_a increases, the "colliding" moment is delayed by 20ms compared to the normal R_a, occurring at 90ms.



(a) C_m increase to $0.7C_m$ in Situation 1 (b) C_m increase to $1.3C_m$ in Situation 1

Figure 3: The effect of membrane capacity on action potential propagation



(a) R_a increase to $0.5R_a$ in Situation 2 (b) R_a increase to $1.2R_a$ in Situation 2

Figure 4: The effect of axonal resistance on action potential propagation

4. The myelination of an axon increases its electrical insulation, reducing ion leakage and thereby increasing the effective membrane resistance. Myelination also reduces

the membrane capacity. These changes lead to faster action potential propagation as the action potential jumps between the nodes of Ranvier.
A larger axonal diameter reduces axial resistance because it provides a larger cross-sectional area for the flow of ionic currents, which allows for faster propagation of action potentials due to less resistance. Additionally, a larger diameter also influences the membrane capacity slightly, as it increases the surface area of the cell membrane, but this effect is less significant than the effect on axial resistance.