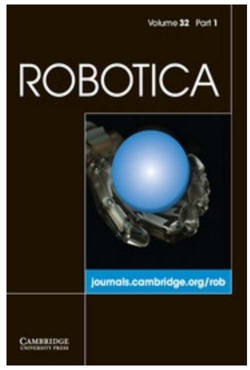

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Kinematic calibration using the product of exponentials formula

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SUMMARY

We present a method for kinematic calibration of open chain mechanisms based on the product of exponentials (POE) formula. The POE formula represents the forward kinematics of an open chain as a product of matrix exponentials, and is based on a modern geometric interpretation of classical screw theory. Unlike the kinematic representations based on the Denavit-Hartenberg (D-H) parameters, the kinematic parameters in the POE formula vary smoothly with changes in the joint axes, *ad hoc* methods designed to address the inherent singularities in the D-H parameters are therefore unnecessary. Another important advantage is that simple closed-form expressions can be obtained for the derivatives of the forward kinematic equations with respect to the kinematic parameters. After introducing the POE formula, we derive a least-squares kinematic calibration algorithm for general open chain mechanisms. Simulation results with a 6-axis open chain are presented.

KEYWORDS: Exponentials formula; Kinematic calibration; Open-chain mechanisms.

1. INTRODUCTION

In practice the actual kinematic parameters of a computer-controlled mechanism rarely agree with those specified by the manufacturer; planar mechanisms, for example, are never quite-planar, and the axes of a spherical mechanism never quite intersect at a single point. These errors can usually be attributed to imprecisions in the manufacturing process or to the natural wear and tear of daily usage. Some form of regular and automatic calibration is therefore usually necessary to maintain reliable and accurate performance. For the PUMA 560 robot, endpoint positioning accuracies before kinematic calibration are of the order of 1 cm, whereas after kinematic calibration the accuracy is improved to 0.3 mm.[1] This represents a significant improvement of almost two orders of magnitude.

Kinematic calibration typically begins with a linearization of the forward kinematic equations. If the nominal forward kinematics is given by the equation $x = f(\theta, p)$, where θ and p are the joint position and kinematic parameter vectors, respectively, and x represents local coordinates for the end-effector position and orientation, the linearized equations are of the form

$$dx = \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial p} dp$$

Calibration involves making several measurements of the endpoint position and orientation error dx, and determining the optimal $d\theta$ and dp that minimize the error in some least-squares sense, *e.g.*, minimizing

$$\left\| dx - \frac{\partial f}{\partial \theta} d\theta - \frac{\partial f}{\partial p} dp \right\|^2.$$

The Denavit-Hartenberg (D-H) parameters are easily the most popular set of parameters used for describing the kinematics of mechanisms. Although they are attractive because of their minimal set property, it is well-known that the D-H parameters are singular when neighboring joint axes are nearly parallel: because the common normal varies wildly with small changes in the axis orientation, these parameters are ill-conditioned. Various modifications have been proposed to handle the parallel axis case. Hayati[2] introduces a modified set of four parameters as well as a five parameter representation that also handles prismatic joints. Stone[3] adds two parameters to the D-H model to allow for arbitrary placement of link frames. Zhuang, Roth and Hamano[4] propose a different six-parameter kinematic representation, in which two additional parameters allow for arbitrary placement of the link frame. The authors emphasize that their model, unlike the D-H or certain other kinematic models, is continuous and parametrically complete. A closely related approach to that proposed in this article is the zero-reference model of Mooring and Tang.[5] Once a zero position for the mechanism has been defined, the location and direction of each joint axis in the zero position is described relative to the fixed reference frame. This model is both complete and parametrically continuous, but Zhuang *et al* point out that constructing the error model (including linearizing the equations) is not straightforward.

While many of the above calibration techniques overcome the problem of singularities in the D-H parameters, most of these methods rely on *ad hoc* kinematic representations. One result is that the calibration procedure is now complicated by arbitrarily defined parameters, and computational routines that depend on whether the joint is revolute or prismatic. Finding $\frac{\partial f}{\partial p}$, the derivative of the kinematic map with respect to the kinematic parameters, can also be a very

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painstaking task: even a symbolic mathematics program will usually produce pages and pages of complicated expressions involving sines and cosines of the D-H parameters. A final observation is that the majority of these kinematic models are developed expressly for calibration, so that their usefulness for other applications is not always clear.

In this article we propose a kinematic calibration method based on representing the forward kinematics as a product of matrix exponentials of the form

$$f(x_1, \ldots, x_n) = e^{A_1x_1}e^{A_2x_2}\cdots e^{A_nx_n}M$$

Here $x_1, \ldots, x_n$ represent the joint variables, M is a 4×4 homogeneous transformation representing the tool frame in its home position, and the A_i are 4×4 constant matrices of a special structure that completely specify the kinematics of the mechanism. Unlike the D-H representations, the A_i vary *smoothly* with changes in the joint axes, and form a complete set of parameters for modeling mechanisms with both revolute and prismatic joints. This representation, called the *product-of-exponentials* (POE) formula, is based on the concept of a one-parameter subgroup of a Lie group, and provides a modern geometric interpretation of classical screw theory.[6] Like the model of Mooring and Tang, the POE formula is a zero reference model, in which local reference frames attached to each link are unnecessary. By drawing upon well-established principles of classical screw theory and matrix groups, the POE formula eliminates many of the *ad hoc* features found in existing calibration algorithms. An especially important advantage of the POE formula for kinematic calibration is that simple closed-form expressions can be obtained for the derivative of the kinematic map with respect to the kinematic parameters. It also offers a general kinematic representation that is useful for applications other than kinematic calibration.[7]

The article is organized as follows: In Section 2 we review the necessary background on SE(3), the Lie group of Euclidean motions, and introduce the POE formula for modeling the kinematics of open chains. In Section 3 we linearize the POE formula with respect to its kinematic parameters, and derive a least-squares algorithm for kinematic calibration. In Section 4 simulation results for a 6-axis revolute open chain using the POE formula are presented. Although we do not assume any prior knowledge of Lie groups and Lie algebras on the part of the reader, we suggest consulting Curtis[8] for additional background.

2. GEOMETRIC BACKGROUND

2.1. The lie group SE(3)

For our purposes it is sufficient to think of SE(3), the Euclidean group of rigid-body motions (or the *Special Euclidean Group*, also known in the robotics literature as the *homogeneous transformations*), as consisting of matrices of the form

$$\begin{bmatrix} \Theta & b \\ 0 & 1 \end{bmatrix} \tag{1}$$

where $\Theta \in SO(3)$ and $b \in \mathcal{R}^3$. Here SO(3) denotes the group of 3×3 rotation matrices. SE(3) has the structure of both a differentiable manifold and an algebraic group, and is an example of a *Lie group*.

Let p be a point on a matrix Lie group $\mathbf{G}$, and $X(t)$ a differentiable curve on $\mathbf{G}$ such that $X(0)=p$. The derivative $\dot{X}(0)$ is said to be a *tangent vector* to $\mathbf{G}$ at p; the set of all tangent vectors at p, denoted $T_p\mathbf{G}$, forms a vector space, called the *tangent space to* $\mathbf{G}$ *at* p. The tangent space at the identity $p=I$ is given a special name, the *Lie algebra* of $\mathbf{G}$, and denoted by the lower-case $\mathbf{g}$. On SO(3) it is easily seen that its Lie algebra so(3) consists of the 3×3 skew-symmetric matrices: if $\Theta(t)$ is a curve on SO(3) such that $\Theta(0)=I$, then differentiating both sides of $\Theta^T(t)\Theta(t)=I$, it follows that $\dot{\Theta}^T(0)+\dot{\Theta}(0)=0$, so that elements of so(3) are matrices of the form

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \triangleq [\omega] \tag{2}$$

where $\omega = \mathcal{R}^3$. Note that an element $[\omega] \in so(3)$ can also be represented as a vector $\omega \in \mathcal{R}^3$. A simple calculation shows that se(3), the Lie algebra of SE(3), consists of 4×4 matrices of the form

$$\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \tag{3}$$

where $[\omega] \in so(3)$ and $v \in \mathcal{R}^3$.

The generalized velocity of a moving rigid-body can be described as an element of se(3) as follows. Let $X(t) = \begin{bmatrix} \Theta(t) & b(t) \\ 0 & 1 \end{bmatrix}$ be a curve in SE(3) describing the motion of a rigid body relative to an inertial reference frame. There exist two natural ways in which the tangent vector $\dot{X}(t)$ can be identified with an element of se(3). It can be verified that both

$$\dot{X}X^{-1} = \begin{bmatrix} \dot{\Theta}\Theta^{-1} & \dot{b} - \dot{\Theta}\Theta^{-1}b \\ 0 & 0 \end{bmatrix} \tag{4}$$

and

$$X^{-1}\dot{X} = \begin{bmatrix} \Theta^{-1}\dot{\Theta} & \Theta^{-1}\dot{b} \\ 0 & 0 \end{bmatrix} \tag{5}$$

are elements of se(3). (Observe that both $\dot{\Theta}\Theta^{-1}$ and $\Theta^{-1}\dot{\Theta}$ are skew-symmetric, and therefore elements of so(3)). The latter is referred to as the *body-fixed velocity* representation of $\dot{X}$, since $\Theta^{-1}\dot{\Theta}$ is the angular velocity of the rigid body relative to its body-fixed frame. By a similar argument we call $\dot{X}X^{-1}$ the *inertial velocity* representation of $\dot{X}$.

An important connection between a Lie group and its Lie algebra is the *exponential mapping*—defined on each Lie algebra is the exponential mapping into the corresponding Lie group. On matrix groups the exponential mapping corresponds to the usual matrix exponential, *i.e.* if A is an element of the Lie algebra, then $\exp A = I + A + \frac{A^2}{2!} + \cdots$ is an element of the Lie group. On so (3) and se(3) the exponential mapping is *onto*; *i.e.*, for every $X \in SO(3)$ (resp., SE(3)) there exists an $x \in so(3)$ (resp., se(3)) such that $\exp x = X$. The following explicit formula for the exponential mapping on se(3) is derived in Park.[9]

Lemma 1. *Let* $[\omega] \in so(3)$ *and* $v \in \mathfrak{R}^3$, *and* $\|\omega\|^2 = w_1^2 + w_2^2 + w_3^2$. *Then*

$$\exp\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \exp[\omega] & Av \\ 0 & 1 \end{bmatrix} \tag{6}$$

is an element of SE(3), *where*

$$\exp[\omega] = I + \frac{\sin\|\omega\|}{\|\omega\|}\cdot[\omega] + \frac{1-\cos\|\omega\|}{\|\omega\|^2}\cdot[\omega]^2 \tag{7}$$

$$A = I + \frac{1-\cos\|\omega\|}{\|\omega\|^2}\cdot[\omega] + \frac{\|\omega\| - \sin\|\omega\|}{\|\omega\|^3}\cdot[\omega]^2 \tag{8}$$

Note that if A is an element of some Lie algebra, then e^{At}, $t \in \mathfrak{R}$, itself forms a group, in this case a subgroup of the Lie group. Such groups are called *one-parameter subgroups* of a Lie group, and play a special role in the kinematic description of mechanisms.

Explicit formulas for the inverse map $\log: SE(3) \to se(3)$ can also be derived. Although SE(3) is not a vector space, se(3) is: the log formula provides a set of *canonical coordinates* for representing neighborhoods of SE(3) as open sets in a vector space.

Lemma 2. *Let* $\Theta \in SO(3)$ *such that* $\mathrm{Tr}(\Theta) \neq -1$, *and let* ϕ *satisfy* $1 + 2\cos\phi = \mathrm{Tr}(\Theta)$, $|\phi| < \pi$. *Finally, suppose* $b \in \mathfrak{R}^3$. *Then*

$$\log\begin{bmatrix} \Theta & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} [\omega] & A^{-1}b \\ 0 & 0 \end{bmatrix} \tag{9}$$

where

$$[\omega] = \frac{\phi}{2\sin\phi}(\Theta - \Theta^T) \tag{10}$$

$$A^{-1} = I - \frac{1}{2}\cdot[\omega] + \frac{2\sin\|\omega\| - \|\omega\|(1+\cos\|\omega\|)}{2\|\omega\|^2\sin\|\omega\|}\cdot[\omega]^2 \tag{11}$$

Also, $\|\log\Theta\|^2 = \phi^2$.

Remark 1. The matrix exponential and logarithm show explicitly the connection between SE(3) and the classical screw theory of rigid-body motions: the screw $(\omega, v) \in \mathfrak{R}^6$ defines a rigid-body motion given by the matrix exponential of Lemma 1.

An element of a Lie group can also be identified with a linear mapping between its Lie algebra as follows: Let **G** be a matrix Lie group with Lie algebra **g**. For every $X \in \mathbf{G}$ the mapping $\mathrm{Ad}_X: \mathbf{g} \to \mathbf{g}$ defined by $\mathrm{Ad}_X(x) = XxX^{-1}$ is linear. Ad_X is known as the *adjoint map* (or *adjoint representation*) of X. If $X = \begin{bmatrix} \Theta & b \\ 0 & 1 \end{bmatrix}$ is an element of SE(3), then its adjoint map acting on an element $x = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$ of se(3) is given by

$$\begin{aligned} \mathrm{Ad}_X(x) &= \begin{bmatrix} \Theta & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Theta^T & -\Theta^T b \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \Theta[\omega]\Theta^T & [b]\Theta\omega + \Theta v \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{12}$$

It is not difficult to verify that $\Theta[\omega]\Theta^T = [\Theta\omega]$. Alternatively, if x is regarded as a six-dimensional column vector the mapping Ad_X admits the 6×6 matrix representation

$$\mathrm{Ad}_X(x) = \begin{bmatrix} \Theta & 0 \\ [b]\Theta & \Theta \end{bmatrix}\begin{bmatrix} \omega \\ v \end{bmatrix} \tag{13}$$

It is also easily verified that $\mathrm{Ad}_X^{-1} = \mathrm{Ad}_{X^{-1}}$ and $\mathrm{Ad}_X \mathrm{Ad}_Y = \mathrm{Ad}_{XY}$ for any $X, Y \in S'E(3)$. These identities will be useful in linearizing the POE formula.

2.2. The product-of-exponentials formula

The product-of-exponentials (POE) formula is a useful theoretical and computational tool for modeling the kinematics of open chains[6,7,10] in which the connections between kinematics and Lie theory are shown explicitly. If a right-handed reference frame is fixed at the tip of each link of the chain, then the Euclidean transformation which describes the position and orientation of the i^{th} frame in terms of the $(i-1)^{\text{th}}$ frame is $f_{i-1,i} = e^{P_i x_i} M_i$, where $M_i \in SE(3)$, $P_i \in se(3)$, and $x_i \in \mathfrak{R}$ is the joint variable, $i = 1, 2, \ldots, n$. In terms of the Denavit-Hartenberg (D-H) parameters the matrices M_i and P_i for a revolute joint are, following the convection of Paul,[11]

$$M_i = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

where $c\alpha_i = \cos\alpha_i$ and $s\alpha_i = \sin\alpha_i$. For a prismatic joint

$$M_i = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & 0 & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

Thus, the fram fixed at the tip is related to that at the base by the product

$$f(x_1, x_2, \ldots, x_n) = e^{P_1 x_1} M_1 e^{P_2 x_2} M_2 \cdots e^{P_n x_n} M_n \tag{16}$$

This equation can be simplified using the fact that $M^{-1}e^P M = e^{M^{-1}PM}$, so that by repeatedly applying the identity $Me^P = e^{MPM^{-1}}M$, f can be written

$$f(x_1, x_2, \ldots, x_n) = e^{A_1 x_1} e^{A_2 x_2} \cdots e^{A_n x_n} M \tag{17}$$

where $A_1 = P_1$, $A_2 = M_1 P_2 M_1^{-1}$, $A_3 = (M_1M_2)P_3(M_1M_2)^{-1}$, *etc.* Similarly, one can apply the identity $e^{A}M = Me^{M^{-1}AM}$ to rewrite the above as

$$f(x_1, x_2, \ldots, x_n) = Me^{B_1x_1}e^{B_2x_2}\cdots e^{B_nx_n} \tag{18}$$

where $B_i = M^{-1}A_iM$, $i = 1, 2, \ldots, n$. The matrix exponentials in this formula can be easily computed from the earlier closed-form expressions.

The POE formula can alternatively be derived independently of the D-H parameters, by appealing to the interpretation of the A_i (B_i) as the screw parameters for joint i's motion.[10] Specifically, denote the rotational and translational components of A_i (B_i) by ω_i and v_i, respectively. Then ω_i is a unit vector in the direction of joint axis i, expressed in inertial (tip) frame coordinates; v_i is a vector, also described in inertial (tip) frame coordinates, such that the pitch of the screw motion generated by joint i is $\omega_i^T v_i$. Note that for revolute joints the pitch is zero, and $\omega_i \times v_i$ is required to be a point lying on the joint axis i; for prismatic joints $\omega_i = 0$ and v_i is the direction of movement.

Example 1. Let the forward kinematics for the 3R open chain of Figure 1 be of the form $f(x_1, x_2, x_3) = e^{A_1x_1}e^{A_2x_2}e^{A_3x_3}M$. Setting the joint variables to zero, the tip frame $M = (\Theta, b)$ relative to the base frame is given by $\Theta = I$, $b = (L_1 + L_2, 0, 0)$. Joint 1 has a screw axis in the direction $\omega_1 = (0, 0, 1)$; since $\omega_1 \times v_1$ must be a point lying on the joint axis, such that v_1 is normal to ω_1, it follows that $v_1 = (0, 0, 0)$. Similarly, for joints 2 and 3 we have $\omega_2 = (0, -1, 0)$, $v_2 = (0, 0, 0)$, and $\omega_3 = (0, -1, 0)$, $v_3 = (0, 0, -L_1)$.

As the above example illustrates, the POE formula for an open chain can be obtained directly without resorting to the D-H parameters. The POE formula also has a number of modeling advantages over the D-H representations which make it attractive for calibration. First, it is a type of *zero reference position description* of the kinematics: once a tip and inertial frame have been chosen, and a zero position selected for each of the joints, there exists a unique set of constant matrices $A_1, A_2, \ldots, A_n \in \mathrm{se}(3)$ and $M \in \mathrm{SE}(3)$ that describes the forward kinematics of the mechanism. Unlike the D-H representations it is not necessary to attach reference frames to each link. Second, the POE formula treats both revolute and prismatic joints in a uniform way; recall that using the D-H parameters the joint variable can be either θ_i or d_i depending on whether the joint is revolute or prismatic. Third, as discussed earlier, the D-H parameters are extremely sensitive to small kinematic variations when neighbouring joint axes are nearly parallel. This sensitivity is what makes kinematic calibration routines based on D-H parameters unnecessarily complicated. On the other hand the POE formula is a continuous parametric model, in the sense that the A_i's vary smoothly with variations in the joint axes.

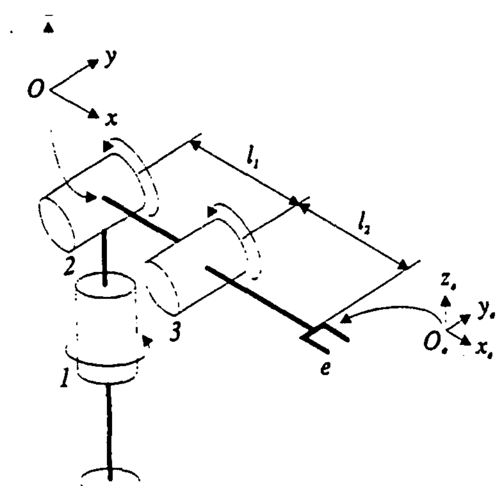

Fig. 1. A 3R elbow manipulator.

Another attractive feature of the POE formula is the compact expression for the Jacobian. If f describes the position and orientation of the tip frame relative to the inertial frame, then recall that $\dot{f}f^{-1}$ is an element of se(3) corresponding to the generalized velocity of the tip frame relative to the inertial frame. Using the fact that $(e^{Ax})^{-1} = e^{-Ax}$, a direct calculation shows that

$$\dot{f}f^{-1} = A_1\dot{x}_1 + e^{A_1x_1}A_2e^{-A_1x_1}\dot{x}_2 + \cdots + e^{A_1x_1}\cdots \times e^{A_{n-1}x_{n-1}}A_ne^{-A_{n-1}x_{n-1}}\cdots e^{-A_1x_1}\dot{x}_n \tag{19}$$

If $\dot{f}f^{-1}$ is rearranged as a six-dimensional vector, then the right-hand side can be expressed as the product of a $6 \times n$ Jacobian matrix $J(x)$ with an n-dimensional joint velocity vector $(\dot{x}_1, \ldots, \dot{x}_n)$, so that each term in the right-hand side can be identified with a column of the Jacobian matrix.

3. KINEMATIC CALIBRATION USING THE POE FORMULA

3.1. Linearizing the POE formula

The following result is the key to our POE-based calibration algorithm:

Lemma 3. *Let $X(t) = e^{A(t)}$ be a curve in* SE(3). *Then*

$$\dot{X}X^{-1} = \int_0^1 e^{A(t)s}\dot{A}(t)e^{-A(t)s}\,ds \tag{20}$$

A proof of this result can be found in Park.[9]

Remark 2. This integral can be evaluated to the following explicit formula. Denoting $A(t)$ by $\begin{bmatrix} [\omega(t)] & v(t) \\ 0 & 0 \end{bmatrix}$ and $e^{A(t)}$ by $\begin{bmatrix} \Theta(t) & b(t) \\ 0 & 1 \end{bmatrix}$, a direct calculation shows that $\dot{\Theta}\Theta^{-1} = [R\dot{\omega}]$, where

$$R = I + \frac{1 - \cos\|\omega\|}{\|\omega\|^2}\cdot[\omega] + \frac{\|\omega\| - \sin\|\omega\|}{\|\omega\|^3}\cdot[\omega]^2$$

Also, $b = Rv$, from which a closed-form expression for $\dot{b} - \dot{\Theta}\Theta^{-1}b$ can be obtained. This formula leads directly to the closed-form expression of the derivatives of the forward kinematic map with respect to the kinematic parameters.

We apply this result to linearize the POE formula with respect to its kinematic parameters. As before let the forward kinematics be given by

$$f(x_1, x_2, \ldots, x_n) = e^{A_1x_1}e^{A_2x_2}\cdots e^{A_nx_n}M,$$
$$A_i = \begin{bmatrix} [\omega_i] & v_i \\ 0 & 0 \end{bmatrix} \tag{21}$$

Note that M can also be expressed using the exponential formula as $M = e^{\Gamma}$ for some constant $\Gamma \in \mathrm{se}(3)$. Then

$$\begin{aligned} df = {} & d(e^{A_1x_1})e^{A_2x_2}\cdots e^{A_nx_n} \\ & + e^{A_1x_1}d(e^{A_2x_2})e^{A_3x_3}\cdots e^{A_nx_n}M \\ & + \cdots + e^{A_1x_1}\cdots e^{A_{n-1}x_{n-1}}d(e^{A_nx_n})M \\ & + e^{A_1x_1}\cdots e^{A_nx_n}(dM) \end{aligned} \tag{22}$$

and

$$\begin{aligned} df \cdot f^{-1} = {} & d(e^{A_1x_1})e^{-A_1x_1} \\ & + e^{A_1x_1}d(e^{A_2x_2})e^{-A_2x_2}e^{-A_1x_1} \\ & + \cdots + e^{A_1x_1}\cdots e^{A_{n-1}x_{n-1}}d(e^{A_nx_n}) \\ & \times e^{-A_nx_n}\cdots e^{-A_1x_1} \\ & + e^{A_1x_1}\cdots e^{A_nx_n}(dM)M^{-1} \\ & \times e^{-A_nx_n}\cdots e^{-A_1x_1} \end{aligned} \tag{23}$$

From Lemma 3, each $d(e^{A_ix_i})e^{-A_ix_i}$ in (23) is expanded as

$$\begin{aligned} d(e^{A_ix_i})e^{-A_ix_i} &= \int_0^1 e^{A_ix_is}d(A_ix_i)e^{-A_ix_is}\,ds \\ &= \int_0^1 e^{A_ix_is}(dA_ix_i + A_idx_i)e^{-A_ix_is}\,ds \\ &= \int_0^1 e^{A_ix_is}\,dA_ix_ie^{-A_ix_is}\,ds \\ &\quad + \int_0^1 e^{A_ix_is}A_idx_ie^{-A_ix_is}\,ds \\ &= x_i\int_0^1 e^{A_ix_is}\,dA_ie^{-A_ix_is}\,ds + A_idx_i \end{aligned} \tag{24}$$

Similarly, $(dM)M^{-1}$ in (23) becomes

$$(dM)M^{-1} = d(e^{\Gamma})e^{-\Gamma} = \int_0^1 e^{\Gamma s}\,d\Gamma e^{-\Gamma s}\,ds \tag{25}$$

In terms of the adjoint map (23) can now be expressed as

$$\begin{aligned} df \cdot f^{-1} = {} & A_1\,dx_1 + \mathrm{Ad}_{e^{A_1x_1}}(A_2)\,dx_2 \\ & + \cdots + \mathrm{Ad}_{e^{A_1x_1}\cdots e^{A_{n-1}x_{n-1}}}(A_n)\,dx_n \\ & + x_1\int_0^1 \mathrm{Ad}_{e^{A_1x_1s}}(dA_1)\,ds \\ & + x_2\mathrm{Ad}_{e^{A_1x_1}}\left(\int_0^1 \mathrm{Ad}_{e^{A_2x_2s}}(dA_2)\,ds\right) \\ & + \cdots + x_n\mathrm{Ad}_{e^{A_1x_1}\cdots e^{A_{n-1}x_{n-1}}} \\ & \times \left(\int_0^1 \mathrm{Ad}_{e^{A_nx_ns}}(dA_n)\,ds\right) \\ & + \mathrm{Ad}_{e^{A_1x_1}\cdots e^{A_nx_n}}\left(\int_0^1 \mathrm{Ad}_{e^{\Gamma s}}(d\Gamma)\,ds\right) \end{aligned} \tag{26}$$

The linearized equation above can also be expressed in the usual matrix form $\mathbf{Ap} = \mathbf{y}$, where $\mathbf{p} \in \mathcal{R}^{7n-6}$ is the vector of kinematic parameters

$$\mathbf{p} = [dx_1\,dx_2\cdots dx_n\,d\omega_1^T\,dv_1^T\cdots d\omega_n^T\,dv_n^T\,d\omega_M^T\,dv_M^T] \tag{27}$$

and $\mathbf{y}$ is the six-dimensional representation of $df \cdot f^{-1}$. Let

$$e^{A_ix_i} = \begin{bmatrix} \Theta_i & b_i \\ 0 & 1 \end{bmatrix},\quad e^{A_ix_is} = \begin{bmatrix} R_i(s) & d_i(s) \\ 0 & 1 \end{bmatrix} \tag{28}$$

$$M = e^{\Gamma} = \begin{bmatrix} \Theta_M & b_M \\ 0 & 1 \end{bmatrix},\quad e^{\Gamma s} = \begin{bmatrix} R_M(s) & d_M(s) \\ 0 & 1 \end{bmatrix} \tag{29}$$

with $e^{A_0x_0}$ defined to be the identity matrix, and

$$r_k = \left(\prod_{i=0}^{k-1}\begin{bmatrix} \Theta_i & 0 \\ [b_i]\Theta_i & \Theta_i \end{bmatrix}\right)\begin{bmatrix} \omega_k \\ v_k \end{bmatrix} \tag{30}$$

$$\begin{aligned} Q_k = {} & \left(\prod_{i=0}^{k-1}\begin{bmatrix} \Theta_i & 0 \\ [b_i]\Theta_i & \Theta_i \end{bmatrix}\right)x_k \\ & \times \int_0^1 \begin{bmatrix} R_k(s) & 0 \\ [d_k(s)]R_k(s) & R_k(s) \end{bmatrix} ds \end{aligned} \tag{31}$$

$$\begin{aligned} Q_M = {} & \left(\prod_{i=0}^{n}\begin{bmatrix} \Theta_i & 0 \\ [b_i]\Theta_i & \Theta_i \end{bmatrix}\right) \\ & \times \int_0^1 \begin{bmatrix} R_M(s) & 0 \\ [d_M(s)]R_M(s) & R_M(s) \end{bmatrix} ds \end{aligned} \tag{32}$$

Then (26) can be expressed as

$$\begin{aligned} \mathbf{y} &= [r_1\,|r_2|\cdots|r_n|\,Q_1\,|Q_2|\cdots|Q_n|\,Q_M]\mathbf{p} \\ &\triangleq \mathbf{Ap} \end{aligned} \tag{33}$$

Next we consider the left-hand side $df \cdot f^{-1}$. Let T_n and T_a be the nominal and actual end-effector frames, respectively, where T_a is obtained from measurement data and T_n is computed using the nominal parameter values. Then $df \cdot f^{-1} = (T_a - T_n)T_n^{-1} = TaT_n^{-1} - I$. If $T_aT_n^{-1} = I + \Omega + \Omega^2/2! + \cdots$, where $\Omega = \log(T_aT_n^{-1})$, and T_a and T_n are sufficiently close, then to first order

$$df \cdot f^{-1} = \Omega = \log(T_aT_n^{-1})$$

3.2. A Least-squares algorithm for kinematic calibration

Calibration proceeds by positioning the mechanism in many points of the workspace and combining the error vectors $\mathbf{y}_i$ and the Jacobians $\mathbf{A}_i$ into a single equation:

$$\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_m \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_m \end{bmatrix}\mathbf{p} \tag{34}$$

or more compactly,

$$\mathcal{Y} = \mathcal{A}\mathbf{p} \tag{35}$$

The least-squares solution for $\mathbf{p}$ is

$$\mathbf{p} = (\mathcal{A}^T\mathcal{A})^{-1}\mathcal{A}^T\mathcal{Y} \tag{36}$$

If **P** denotes the vector of original kinematic parameters, the updated kinematic parameter values **P'** are obtained by

$$\mathbf{P}' = \mathbf{P} + \mathbf{p} \tag{37}$$

Since this is a nonlinear estimation problem, this procedure is iterated until the variations **p** approach zero and the parameters **P** have converged to some stable values. At each iteration the $\mathbf{A}_i$'s are evaluated with the current parameters.

Note that each $A_i = \begin{bmatrix} [\omega_i] & v_i \\ 0 & 0 \end{bmatrix}$ in the POE formula (21) models a general helical joint, in which the joint axis is directed along ω_i, and the pitch is $\omega_i^T v_i$. Hence, purely revolute joints must satisfy the condition $\omega_i^T v_i = 0$, while for prismatic joints $\omega_i = 0$. If revolute joints are to be modeled as being perfectly revolute (*i.e.* without any kinematic imperfections that may result in a slightly helical motion) then this additional constraint can be imposed. Hence, to calibrate an n-revolute joint manipulator we minimize

$$J(\mathbf{p}) = \|\mathscr{A}\mathbf{p} - \mathscr{Y}\|^2$$

subject to the equality constraints $\omega_i^T v_i = 0, i = 1, \ldots, n$. For prismatic joints only v_i need be identified.

4. A CALIBRATION EXAMPLE

In this section simulation experiments are presented for our calibration algorithm with a 6R Puma 560-type manipulator. We assume zero joint offsets, so that the dx_i in the **p** vector of (27) need not be identified (and, consequently, the first n columns of **A** in (33) are eliminated.) The structure of the 6R manipulator is shown in Figure 2. The link lengths are: $l_1 = 150$, $l_2 = 100$, $l_3 = 50$, and $l_4 = 20$ mm. The nominal and actual kinematic parameters for the manipulator are given in Figure 3.

The simulation proceeds by taking several "measurements" of the end-effector frame, by calculating the forward kinematics in various poses using the actual kinematic parameters. To check the sensitivity of our identification algorithm to measurement noise, uniformly distributed random noise is added to each measurement. Specifically, if $\Theta_a(x)$ and $p_a(x)$ denote the measured end-effector orientation and position while in joint position x, uniformly distributed noise in the range $(-0{\cdot}1, 0{\cdot}1)$ is added to each component of $p_a(x)$ for the actual position measurement. also, uniformly distributed noise in the range $(-0{\cdot}001, 0{\cdot}001)$ is added to each of the three independent components of $\log \Theta_a(x)$.

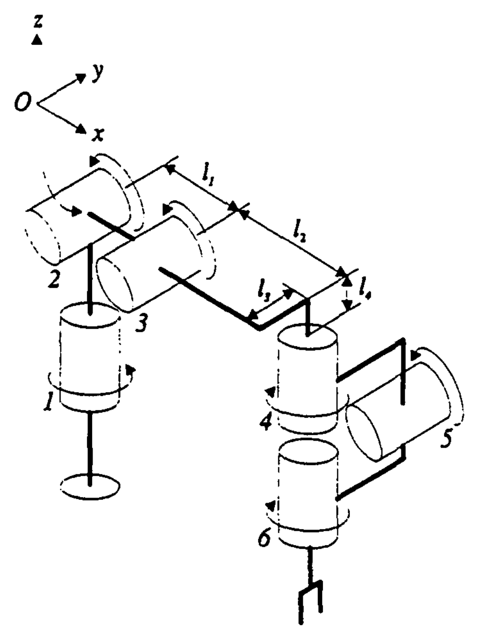

Fig. 2. A 6R Puma 560-type manipulator.

	nominal values	actual values
ω_1	(0,0,1)	(0.04,-0.02,0.999)
v_1	(0,0,0)	(0.02,0.04,0)
ω_2	(0.-1,0)	(0,-1.00002,0)
v_2	(0,0,0)	(-0.02,0,0.05)
ω_3	(0,-1,0)	(0.178,-0.984,-0.001)
v_3	(0,0,-100)	(-0.07,0.009,-101)
ω_4	(0,0,-1)	(0.062,0.013,-0.998)
v_4	(-50,250,0)	(-51,249,0.0752)
ω_5	(0,-1,0)	(0.001,-1.00004,0)
v_5	(-20,0,-250)	(-20.6,-0.0206,-249)
ω_6	(0,0,-1)	(0.095,0.031,-0.995)
v_4	(-50,250,0)	(-51,249,0)
ω_T	(0,0,0)	(0.02,-0.01,0.01)
v_T	(250,50,-20)	(249,51,-20.6)

Fig. 3. POE kinematic parameters for the Puma 560-type manipulator.

These measurements are then used in the calibration algorithm to update the nominal parameters. Each updated set of parameters is referred to as the calibrated parameters. The end-effector frame is evaluated using the calibrated parameters; this frame is denoted T_c. Error is then measured by computing the distances of orientation and position between T_a and T_c in the following way, respectively: if

$$T_a = \begin{bmatrix} \Theta_a & b_a \\ 0 & 1 \end{bmatrix}, \quad T_c = \begin{bmatrix} \Theta_c & b_c \\ 0 & 1 \end{bmatrix}$$

then $d(\Theta_a, \Theta_c) \triangleq \|\log(\Theta_a^{-1}\Theta_c)\|$ and $d(b_a, b_c) \triangleq \|b_c - b_a\|$. $d(b_a, b_c)$ is simply a vector norm and it can be shown that $d(\Theta_a, \Theta_c)$ is a well-defined metric in SO(3) that is invariant with respect to choice of inertial frame.

If the parameters are identified properly then T_a will be very close to T_c, and hence $\Theta_a\Theta_c^{-1}$ will very nearly be the identity; in that case the error measure above will be nearly zero.

For the simulation we assume that the joints are exactly revolute, so that for each $A_i = \begin{bmatrix} [\omega_i] & v_i \\ 0 & 0 \end{bmatrix}$ that we attempt to identify in the POE formula, the constraint $\omega_i^T v_i = 0$ is imposed. We also require $\|\omega_i\| = 1$. The simulation is performed with MATLAB.

4.1. Discussion of simulation results

The orientations and positions are randomly chosen and measured for calibration at each iteration. Also, 50 orientations and positions are randomly taken for

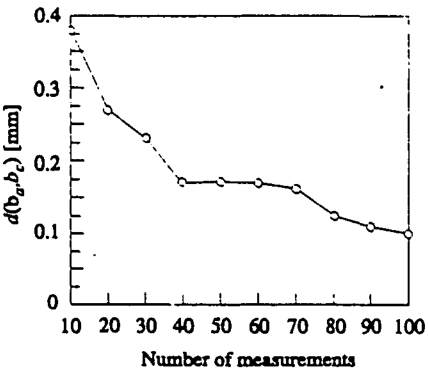

Fig. 4. The distance between b_a and b_c.

verification and averages of $d(b_a, b_c)$ and $d(\Theta_a, \Theta_c)$ of them are computed.

First, the simulation is carried out without measurement noise. 50 orientations and positions are randomly taken for calibration. Then the result is that both $d(b_a, b_c)$ and $d(\Theta_a, \Theta_c)$ become almost zero after the fourth or fifth iteration. Next, simulation is performed about the number of measurements used in calibration. The averages of $d(b_a, b_c)$ and $d(\Theta_a, \Theta_c)$ after the eighth, ninth, and tenth iterations are taken to cancel out the fluctuation resulting from the added noise and plotted in Figure 4 and Figure 5. As shown in Figure 4 and Figure 5, the distances become smaller as the number of measurements used for calibration increases.

The 6R manipulator used in simulation has joint axes that are nearly parallel to each other as shown in Figure 2 and its POE kinematic parameters in Figure 3. This is the very case that calibration with the D-H parameters would fail. The POE formula, by comparison, has no difficulty with parallel axes since the POE kinematic parameters A_i vary smoothly with variations in the joint axes.

5. CONCLUSIONS

In this article we have formulated a kinematic calibration algorithm based on the product of exponentials formula for open kinematic chains. The POE formula enjoys a number of modeling advantages over other commonly used representations, and calibration can be performed in a straightforward and uniform way, without the *ad hoc* methods of other calibration algorithms. In particular, no special procedures are required for adjacent joints whose axes are nearly parallel. Closed-form analytic expressions for the derivative of the forward kinematic map with respect to its kinematic parameters are also obtained. The POE formula can be viewed as a modern geometric formulation of classical screw theory, and as such is a general tool that is useful for a range of applications besides calibration. For example, using the POE formula the dynamic equations can be formulated elegantly and in a form that admits easy factorization and differentiation.[12]

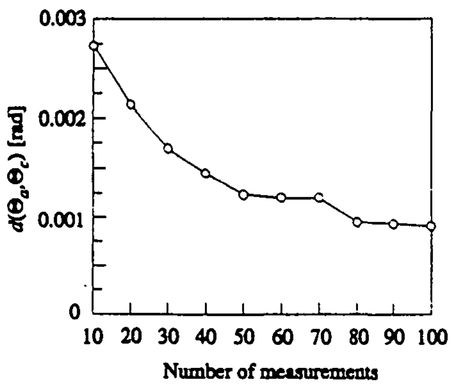

Fig. 5. The distance between Θ_a and Θ_c.

The calibration algorithm discussed in this article determines the least-squares solution, but in principle other estimation algorithms can be used, *e.g.*, Levenberg-Marquardt methods, total least-squares, *etc.* Also, only geometric errors have been considered; for more accurate modeling non-geometric effects such as backlash, gear transmission error, and joint compliance can be included. Many of these issues are discussed in Hollerbach[13] and the references cited, and one interesting possibility for future work is to extend the POE calibration method to include these effects.

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