Contents

1 Notations

Frames:

$$\text{body frame: B} \tag{1}$$
$$\text{robot frame: R} \tag{2}$$
$$\text{world frame: W} \tag{3}$$
$$\text{global frame: G} \tag{4}$$

Translation:

$$\mathbf{t} \tag{5}$$
$${}_{\text{G}}\mathbf{t}_{\text{GB}} \tag{6}$$

Position:

$$\mathbf{p} \tag{7}$$
$${}_{\text{G}}\mathbf{p}_{\text{GB}} \tag{8}$$

Velocity:

$$\mathbf{v} \tag{9}$$
$${}_{\text{G}}\mathbf{v}_{\text{GB}} \tag{10}$$

Angular Velocity:
Acceleration:

$$\mathbf{a} \tag{11}$$
$${}_{\text{G}}\mathbf{a}_{\text{GB}} \tag{12}$$

Rotation:

$$\mathbf{R} \tag{13}$$
$$\mathbf{R}_{\text{BG}} \tag{14}$$

Transforms:

$$\mathbf{T} \tag{15}$$
$$\mathbf{T}_{\text{BG}} \tag{16}$$

2 Rotation Matrix

Z-Y-X rotation sequence:

$$\mathbf{R}_{zyx} = \begin{bmatrix} c(\psi)c(\theta) & c(\psi)s(\theta)s(\phi) - s(\psi)c(\phi) & c(\psi)s(\theta)c(\phi) + s(\psi)s(\phi) \\ s(\psi)c(\theta) & s(\psi)s(\theta)s(\phi) + c(\psi)c(\phi) & s(\psi)s(\theta)c(\phi) - c(\psi)s(\phi) \\ -s(\theta) & c(\theta)s(\phi) & c(\theta)c(\phi) \end{bmatrix} \tag{17}$$

3 Quaternions

A quaternion, $\mathbf{q} \in \mathbb{R}^4$, generally has the following form

$$\mathbf{q} = q_w + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}, \tag{18}$$

where $\{q_w, q_x, q_y, q_z\} \in \mathbb{R}$ and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are the imaginary numbers satisfying

$$\begin{gathered} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \;\; \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \;\; \mathbf{ki} = -\mathbf{ik} = \mathbf{j} \end{gathered} \tag{19}$$

corresponding to the Hamiltonian convention. The quaternion can be written as a 4 element vector consisting of a *real* (*scalar*) part, q_w, and *imaginary* (*vector*) part $\mathbf{q}_v$ as,

$$\mathbf{q} = \begin{bmatrix} q_w \\ \mathbf{q}_v \end{bmatrix} = \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix} \tag{20}$$

There are other quaternion conventions, for example, the JPL convention. A more detailed discussion between Hamiltonian and JPL quaternion convention is discussed in [1].

3.1 Main Quaternion Properties

3.1.1 Sum

Let $\mathbf{p}$ and $\mathbf{q}$ be two quaternions, the sum of both quaternions is,

$$\mathbf{p} \pm \mathbf{q} = \begin{bmatrix} p_w \\ \mathbf{p}_v \end{bmatrix} \pm \begin{bmatrix} q_w \\ \mathbf{q}_v \end{bmatrix} = \begin{bmatrix} p_w \pm q_w \\ \mathbf{p}_v \pm \mathbf{q}_v \end{bmatrix}. \tag{21}$$

The sum between two quaternions $\mathbf{p}$ and $\mathbf{q}$ is **commutative** and **associative**.

$$\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p} \tag{22}$$

$$\mathbf{p} + (\mathbf{q} + \mathbf{r}) = (\mathbf{p} + \mathbf{q}) + \mathbf{r} \tag{23}$$

3.1.2 Product

The quaternion multiplication (or product) of two quaternions $\mathbf{p}$ and $\mathbf{q}$, denoted by $\otimes$ is defined as

$$\mathbf{p} \otimes \mathbf{q} = (p_w + p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k})(q_w + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}) \tag{24}$$

$$= \begin{matrix} (p_w q_w - p_x q_x - p_y q_y - p_z q_z) & \\ (p_w q_x + p_x q_w + p_y q_z - p_z q_y) & \mathbf{i} \\ (p_w q_y - p_y q_w + p_z q_x + p_x q_z) & \mathbf{j} \\ (p_w q_z + p_z q_w - p_x q_y + p_y q_x) & \mathbf{k} \end{matrix} \tag{25}$$

$$= \begin{bmatrix} p_w q_w - p_x q_x - p_y q_y - p_z q_z \\ p_w q_x + q_x p_w + p_y q_z - p_z q_y \\ p_w q_y - p_y q_w + p_z q_x + p_x q_z \\ p_w q_z + p_z q_w - p_x q_y + p_y q_x \end{bmatrix} \tag{26}$$

$$= \begin{bmatrix} p_w q_w - \mathbf{p}_v^\top \mathbf{q}_v \\ p_w \mathbf{q}_v + q_w \mathbf{p}_v + \mathbf{p}_v \times \mathbf{q}_v \end{bmatrix}. \tag{27}$$

The quaternion product is **not commutative** in the general case[1],

$$\mathbf{p} \otimes \mathbf{q} \neq \mathbf{q} \otimes \mathbf{p} \ . \tag{28}$$

The quaternion product is however **associative**,

$$\mathbf{p} \otimes (\mathbf{q} \otimes \mathbf{r}) = (\mathbf{p} \otimes \mathbf{q}) \otimes \mathbf{r} \tag{29}$$

and **distributive over the sum**

$$\mathbf{p} \otimes (\mathbf{q} + \mathbf{r}) = \mathbf{p} \otimes \mathbf{q} + \mathbf{p} \otimes \mathbf{r} \quad \text{and} \quad (\mathbf{p} \otimes \mathbf{q}) + \mathbf{r} = \mathbf{p} \otimes \mathbf{r} + \mathbf{q} \otimes \mathbf{r} \tag{30}$$

The quaternion product can alternatively be expressed in matrix form as

$$\mathbf{p} \otimes \mathbf{q} = [\mathbf{p}]_L \mathbf{q} \quad \text{and} \quad \mathbf{p} \otimes \mathbf{q} = [\mathbf{q}]_R \mathbf{p} \ , \tag{31}$$

where $[\mathbf{p}]_L$ and $[\mathbf{q}]_R$ are the left and right quaternion-product matrices which are derived from (26),

$$[\mathbf{p}]_L = \begin{bmatrix} p_w & -p_x & -p_y & -p_z \\ p_x & p_w & -p_z & p_y \\ p_y & p_z & p_w & -p_x \\ p_z & -p_y & p_x & p_w \end{bmatrix}, \quad \text{and} \quad [\mathbf{q}]_L = \begin{bmatrix} q_w & -q_x & -q_y & -q_z \\ q_x & q_w & q_z & -q_y \\ q_y & -q_z & q_w & q_x \\ q_z & q_y & -q_x & q_w \end{bmatrix}, \tag{32}$$

or inspecting (27) a compact form can be derived as,

$$[\mathbf{p}]_L = \begin{bmatrix} 0 & -\mathbf{p}_v^\top \\ \mathbf{p}_w \mathbf{I}_{3\times 3} + \mathbf{p}_v & \mathbf{p}_w \mathbf{I}_{3\times 3} - \lfloor \mathbf{p}_v \ \times \rfloor \end{bmatrix} \tag{33}$$

[1] There are exceptions to the general non-commutative rule, where either $\mathbf{p}$ or $\mathbf{q}$ is real such that $\mathbf{p}_v \times \mathbf{q}_v = 0$, or when both $\mathbf{p}_v$ and $\mathbf{q}_v$ are parallel, $\mathbf{p}_v || \mathbf{q}_v$. Only in these cirmcumstances is the quaternion product commutative.

and

$$[\mathbf{q}]_R = \begin{bmatrix} 0 & -\mathbf{q}_v^\top \\ \mathbf{q}_w \mathbf{I}_{3\times 3} + \mathbf{q}_v & \mathbf{q}_w \mathbf{I}_{3\times 3} - \lfloor \mathbf{q}_v \ \times \rfloor \end{bmatrix}, \tag{34}$$

where $\lfloor \bullet \ \times \rfloor$ is the skew operator that produces a matrix cross product matrix, and is defined as

$$\lfloor \mathbf{v} \ \times \rfloor = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}, \quad \mathbf{v} \in \mathbb{R}^3 \tag{35}$$

4 Computer Vision

4.1 Pinhole Camera Model

$$K = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \tag{36}$$

4.2 Radial Tangential Distortion

$$\begin{aligned} k_{\text{radial}} &= 1 + (k_1 r^2) + (k_2 r^4) \\ x' &= x \cdot k_{\text{radial}} \\ y' &= y \cdot k_{\text{radial}} \\ x'' &= x' + (2 p_1 x y + p_2 * (r^2 + 2x^2)) \\ y'' &= y' + (p_1 (r^2 + 2y^2) + 2 p_2 x y) \end{aligned} \tag{37}$$

4.3 Equi-distant Distortion

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan(r) \\ \theta_d &= \theta(1 + k_1 \theta^2 + k_2 \theta^4 + k_3 \theta^6 + k_4 \theta^8) \\ x' &= (\theta_d / r) \cdot x \\ y' &= (\theta_d / r) \cdot y \end{aligned} \tag{38}$$

4.4 Bundle Adjustment

Let $\mathbf{z} \in \mathbb{R}^2$ be the image measurement and $\mathbf{h}(\cdot) \in \mathbb{R}^2$ be the projection function that produces an image projection $\tilde{\mathbf{z}}$. The reprojection error e is defined as the euclidean distance between $\mathbf{z}$ and $\tilde{\mathbf{z}}$.

$$\mathbf{e} = \mathbf{z} - \tilde{\mathbf{z}} \tag{39}$$

Our aim given image measurement $\mathbf{z}$ is to find the image projection $\tilde{\mathbf{z}}$ that minimizes the reprojection e. The image projection $\tilde{\mathbf{z}}$ in pixels can be represented in homogeneous coordinates with u, v, w as

$$\tilde{\mathbf{z}} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \end{bmatrix} \tag{40}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{L} \\ 1 \end{bmatrix} = \mathbf{KR}[\mathbf{L} - \mathbf{C}] \tag{41}$$

where u, v, w is computed by projecting a landmark position $\mathbf{L}$ in the world frame to the camera's image plane with projection matrix $\mathbf{P}$. The projection matrix, $\mathbf{P}$, can be decomposed into the camera intrinsics matrix, $\mathbf{K}$, the camera rotation, $\mathbf{R}$, and camera position, $\mathbf{C}$, expressed in the world frame. The orientation of the camera in (**??**) is represented using a rotation matrix. To reduce the optimization parameters the rotation matrix can be parameterized by a quaternion by using the following formula,

$$\mathbf{R} = \begin{bmatrix} q_w^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_x q_y - q_w q_z) & 2(q_x q_z + q_w q_y) \\ 2(q_x q_y + q_w q_z) & q_w^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_y q_z - q_w q_x) \\ 2(q_x q_z - q_w q_y) & 2(q_y q_z + q_w q_x) & q_w^2 - q_x^2 - q_y^2 + q_z^2 \end{bmatrix}. \tag{42}$$

By parameterzing the rotation matrix with a quaternion, the optimization parameters for the camera's orientation is reduced from 9 to 4.

Our objective is to optimize for the camera rotation $\mathbf{R}$, camera position $\mathbf{C}$ and 3D landmark position $\mathbf{L}$ in order to minimize the cost function,

$$\underset{\mathbf{R},\mathbf{C},\mathbf{L}}{\operatorname{argmin}} \; |\mathbf{z} - \mathbf{h}(\mathbf{R}, \mathbf{C}, \mathbf{L})|^2 \tag{43}$$

$$\underset{\mathbf{R},\mathbf{C},\mathbf{L}}{\operatorname{argmin}} \; \left| \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} u/w \\ v/w \end{bmatrix} \right|^2. \tag{44}$$

The cost function above assumes only a single measurement, if there are N measurements corresponding to N unique landmarks the cost function can be rewritten as a maximum likelihood estimation problem as,

$$\underset{\mathbf{R},\mathbf{C},\mathbf{L}}{\operatorname{argmin}} \sum_{j=1}^{N} |\mathbf{z}_j - \mathbf{h}(\mathbf{R}_j, \mathbf{C}_j, \mathbf{L}_j)|^2, \tag{45}$$

under the assumption that the observed landmark, $\mathbf{L}$, measured in the image plane, z, are corrupted by a **zero-mean Gaussian noise**.

For the general case of M images taken at different camera poses the cost function can be further extended to,

$$\min_{R,C,X} \sum_{i=1}^{M} \sum_{j=1}^{N} |\mathbf{z}_{i,j} - \mathbf{h}(\mathbf{R}_i, \mathbf{C}_i, \mathbf{L}_j)|^2 \tag{46}$$

The optimization process begins by setting the first image camera pose as world origin, and subsequent $\mathbf{R}_i$ and $\mathbf{C}_i$ will be relative to the first camera pose.

Jacobians

The Jacobian for the optimization problem for a **single measurement** has the form:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{h}}{\partial \mathbf{R}} \frac{\partial \mathbf{R}}{\partial \mathbf{q}} & \frac{\partial \mathbf{h}}{\partial \mathbf{C}} & \frac{\partial \mathbf{h}}{\partial \mathbf{L}} \end{bmatrix} \tag{47}$$

If there are two measurements the Jacobian is stacked with the following pattern:

$$\mathbf{J} = \begin{bmatrix} \text{Image } 1_{2\times7} & \mathbf{0}_{2\times7} & \text{3D Point}_{2\times3} \\ \mathbf{0}_{2\times7} & \text{Image } 2_{2\times7} & \text{3D Point}_{2\times3} \end{bmatrix} \tag{48}$$

Derivation for $\frac{\partial \mathbf{h}}{\partial \mathbf{R}}$

$$\frac{\partial \mathbf{h}}{\partial \mathbf{R}} = \begin{bmatrix} \dfrac{w\dfrac{\partial u}{\partial \mathbf{R}} - u\dfrac{\partial w}{\partial \mathbf{R}}}{w^2} \\ \\ \dfrac{w\dfrac{\partial v}{\partial \mathbf{R}} - v\dfrac{\partial w}{\partial \mathbf{R}}}{w^2} \end{bmatrix}_{2\times 9} \tag{49}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}[\mathbf{L} - \mathbf{C}]$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f_x & 0 & p_x \\ 0 & f_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} [\mathbf{L} - \mathbf{C}]$$

$$\frac{\partial u}{\partial \mathbf{R}} = \begin{bmatrix} f_x(\mathbf{L} - \mathbf{C}) & \mathbf{0}_{1\times 3} & p_x(\mathbf{L} - \mathbf{C}) \end{bmatrix}_{1\times 9} \tag{50}$$

$$\frac{\partial v}{\partial \mathbf{R}} = \begin{bmatrix} \mathbf{0}_{1\times 3} & f_y(\mathbf{L} - \mathbf{C}) & p_y(\mathbf{L} - \mathbf{C}) \end{bmatrix}_{1\times 9} \tag{51}$$

$$\frac{\partial w}{\partial \mathbf{R}} = \begin{bmatrix} \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} & (\mathbf{L} - \mathbf{C}) \end{bmatrix}_{1\times 9} \tag{52}$$

Derivation for $\frac{\partial \mathbf{R}}{\partial \mathbf{q}}$

$$\frac{\partial \mathbf{R}}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial \mathbf{R}_{11}}{\partial \mathbf{q}} \\ \frac{\partial \mathbf{R}_{12}}{\partial \mathbf{q}} \\ \vdots \\ \frac{\partial \mathbf{R}_{33}}{\partial \mathbf{q}} \end{bmatrix}_{9\times 4} \tag{53}$$

$$\mathbf{R} = \begin{bmatrix} q_w^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_x q_y - q_w q_z) & 2(q_x q_z + q_w q_y) \\ 2(q_x q_y + q_w q_z) & q_w^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_y q_z - q_w q_x) \\ 2(q_x q_z - q_w q_y) & 2(q_y q_z + q_w q_x) & q_w^2 - q_x^2 - q_y^2 + q_z^2 \end{bmatrix}$$

$$\frac{\mathbf{R}_{11}}{\partial \mathbf{q}} = \begin{bmatrix} 0 & -4q_y & -4q_z & 0 \end{bmatrix} \tag{54}$$

$$\frac{\mathbf{R}_{12}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_y & 2q_x & -2q_w & -2q_z \end{bmatrix} \tag{55}$$

$$\frac{\mathbf{R}_{13}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_z & 2q_w & 2q_x & 2q_y \end{bmatrix} \tag{56}$$

$$\frac{\mathbf{R}_{21}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_y & 2q_x & 2q_w & 2q_z \end{bmatrix} \tag{57}$$

$$\frac{\mathbf{R}_{22}}{\partial \mathbf{q}} = \begin{bmatrix} 4q_x & 0 & 4q_z & 0 \end{bmatrix} \tag{58}$$

$$\frac{\mathbf{R}_{23}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_w & 2q_z & 2q_y & 2q_x \end{bmatrix} \tag{59}$$

$$\frac{\mathbf{R}_{31}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_z & -2q_w & 2q_x & -2q_y \end{bmatrix} \tag{60}$$

$$\frac{\mathbf{R}_{32}}{\partial \mathbf{q}} = \begin{bmatrix} -4q_x & -4q_y & 0 & 0 \end{bmatrix} \tag{61}$$

$$\frac{\mathbf{R}_{33}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_w & 2q_z & 2q_y & 2q_x \end{bmatrix} \tag{62}$$

Derivation for $\frac{\partial \mathbf{h}}{\partial \mathbf{L}}$

$$\frac{\partial \mathbf{h}}{\partial \mathbf{L}} = \begin{bmatrix} \dfrac{w\dfrac{\partial u}{\partial \mathbf{L}} - u\dfrac{\partial w}{\partial \mathbf{L}}}{w^2} \\ \\ \dfrac{w\dfrac{\partial v}{\partial \mathbf{L}} - v\dfrac{\partial w}{\partial \mathbf{L}}}{w^2} \end{bmatrix}_{2\times 3} \tag{63}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}[\mathbf{L} - \mathbf{C}]$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f_x & 0 & p_x \\ 0 & f_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} [\mathbf{L} - \mathbf{C}]$$

$$\frac{\partial u}{\partial \mathbf{L}} = \begin{bmatrix} f_x\mathbf{R}_{11} + c_x\mathbf{R}_{31} & f_x\mathbf{R}_{12} + c_x\mathbf{R}_{32} & f_x\mathbf{R}_{13} + c_x\mathbf{R}_{33} \end{bmatrix} \tag{64}$$

$$\frac{\partial v}{\partial \mathbf{L}} = \begin{bmatrix} f_y\mathbf{R}_{21} + c_y\mathbf{R}_{31} & f_y\mathbf{R}_{22} + c_y\mathbf{R}_{32} & f_y\mathbf{R}_{23} + c_y\mathbf{R}_{33} \end{bmatrix} \tag{65}$$

$$\frac{\partial w}{\partial \mathbf{L}} = \begin{bmatrix} \mathbf{R}_{31} & \mathbf{R}_{32} & \mathbf{R}_{33} \end{bmatrix} \tag{66}$$

Derivation for $\frac{\partial h}{\partial \mathbf{C}}$

$$\frac{\partial \mathbf{h}}{\partial \mathbf{C}} = \begin{bmatrix} \dfrac{w\dfrac{\partial u}{\partial \mathbf{C}} - u\dfrac{\partial w}{\partial \mathbf{C}}}{w^2} \\ \dfrac{w\dfrac{\partial v}{\partial \mathbf{C}} - v\dfrac{\partial w}{\partial \mathbf{C}}}{w^2} \end{bmatrix}_{2\times 3} \tag{67}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}[\mathbf{L} - \mathbf{C}]$$
$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f_x & 0 & p_x \\ 0 & f_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} [\mathbf{L} - \mathbf{C}]$$

$$\frac{\partial u}{\partial \mathbf{C}} = -\begin{bmatrix} f_x\mathbf{R}_{11} + c_x\mathbf{R}_{31} & f_x\mathbf{R}_{12} + c_x\mathbf{R}_{32} & f_x\mathbf{R}_{13} + c_x\mathbf{R}_{33} \end{bmatrix} \tag{68}$$
$$\frac{\partial v}{\partial \mathbf{C}} = -\begin{bmatrix} f_y\mathbf{R}_{21} + c_y\mathbf{R}_{31} & f_y\mathbf{R}_{22} + c_y\mathbf{R}_{32} & f_y\mathbf{R}_{23} + c_y\mathbf{R}_{33} \end{bmatrix} \tag{69}$$
$$\frac{\partial w}{\partial \mathbf{C}} = -\begin{bmatrix} \mathbf{R}_{31} & \mathbf{R}_{32} & \mathbf{R}_{33} \end{bmatrix} \tag{70}$$

5 Calibration

References

[1] J. Solà. Quaternion kinematics for the error-state Kalman filter, November 2017.