Contents

1 Notations

A large part of robotics is about developing machines that perceives and interact with the environment. For that robots use sensors to collect and process data, and knowing what the data describes is of utmost importance. Imagine obtaining the position of the robot but not knowing what that position is with respect to. Missing data descriptions such as what a position vector is expressing, what is it with respect to and more causes many hours of painful trail and error to extract that information.

In the following section the notation used throughout this document will be described, and it follows closely of that of Paul Furgale's [1]. The aim is to mitigate the ambiguity that arises when describing robot poses, sensor data and more.

A vector expressed in the world frame, $\mathcal{F}_W$, is written as $_W\mathbf{r}$, or more precisely if the vector describes the position of the camera frame, $\mathcal{F}_C$, expressed in $\mathcal{F}_W$, the vector can be written as $_W\mathbf{r}_{WC}$ with W and C as start and end points, or for brevity as $_W\mathbf{r}_C$. Similarly a transformation of a point from $\mathcal{F}_C$ to $\mathcal{F}_W$ can be represented by a homogeneous transform matrix, $\mathbf{T}_{WC}$, where its rotation matrix component is written as $\mathbf{C}_{WC}$ and the translation component written as $_W\mathbf{r}_{WC}$. A rotation matrix that is parametrized by quaternion $\mathbf{q}_{WC}$ is written as $\mathbf{C}\{\mathbf{q}_{WC}\}$.

Position:	$_W\mathbf{r}_{WB}$
Velocity:	$_W\mathbf{v}_{WB}$
Acceleration:	$_W\mathbf{a}_{WB}$
Angular velocity:	$_W\boldsymbol{\omega}_{WB}$
Rotation:	$\mathbf{C}_{WB}$
Transform:	$\mathbf{T}_{WB}$
Point:	$_W\mathbf{p}$

2   Rotations

Z-Y-X rotation sequence:

$$\mathbf{C}_{zyx} = \begin{bmatrix} c(\psi)c(\theta) & c(\psi)s(\theta)s(\phi) - s(\psi)c(\phi) & c(\psi)s(\theta)c(\phi) + s(\psi)s(\phi) \\ s(\psi)c(\theta) & s(\psi)s(\theta)s(\phi) + c(\psi)c(\phi) & s(\psi)s(\theta)c(\phi) - c(\psi)s(\phi) \\ -s(\theta) & c(\theta)s(\phi) & c(\theta)c(\phi) \end{bmatrix} \quad (1)$$

3   Quaternions

A quaternion, $\mathbf{q} \in \mathbb{R}^4$, generally has the following form

$$\mathbf{q} = q_w + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}, \tag{2}$$

where $\{q_w, q_x, q_y, q_z\} \in \mathbb{R}$ and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are the imaginary numbers satisfying

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$
$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \ \ \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \ \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j} \tag{3}$$

corresponding to the Hamiltonian convention. The quaternion can be written as a 4 element vector consisting of a *real (scalar)* part, q_w, and *imaginary (vector)* part $\mathbf{q}_v$ as,

$$\mathbf{q} = \begin{bmatrix} q_w \\ \mathbf{q}_v \end{bmatrix} = \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix} \tag{4}$$

There are other quaternion conventions, for example, the JPL convention. A more detailed discussion between Hamiltonian and JPL quaternion convention is discussed in [2].

3.1   Main Quaternion Properties

3.1.1   Sum

Let $\mathbf{p}$ and $\mathbf{q}$ be two quaternions, the sum of both quaternions is,

$$\mathbf{p} \pm \mathbf{q} = \begin{bmatrix} p_w \\ \mathbf{p}_v \end{bmatrix} \pm \begin{bmatrix} q_w \\ \mathbf{q}_v \end{bmatrix} = \begin{bmatrix} p_w \pm q_w \\ \mathbf{p}_v \pm \mathbf{q}_v \end{bmatrix}. \tag{5}$$

The sum between two quaternions $\mathbf{p}$ and $\mathbf{q}$ is **commutative** and **associative**.

$$\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p} \tag{6}$$

$$\mathbf{p} + (\mathbf{q} + \mathbf{r}) = (\mathbf{p} + \mathbf{q}) + \mathbf{r} \tag{7}$$

3.1.2 Product

The quaternion multiplication (or product) of two quaternions $\mathbf{p}$ and $\mathbf{q}$, denoted by $\otimes$ is defined as

$$\mathbf{p} \otimes \mathbf{q} = (p_w + p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k})(q_w + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}) \tag{8}$$

$$= \begin{array}{ll} (p_wq_w - p_xq_x - p_yq_y - p_zq_z) & \\ (p_wq_x + p_xq_w + p_yq_z - p_zq_y) & \mathbf{i} \\ (p_wq_y - p_yq_w + p_zq_x + p_xq_z) & \mathbf{j} \\ (p_wq_z + p_zq_w - p_xq_y + p_yq_x) & \mathbf{k} \end{array} \tag{9}$$

$$= \begin{bmatrix} p_wq_w - p_xq_x - p_yq_y - p_zq_z \\ p_wq_x + q_xp_w + p_yq_z - p_zq_y \\ p_wq_y - p_yq_w + p_zq_x + p_xq_z \\ p_wq_z + p_zq_w - p_xq_y + p_yq_x \end{bmatrix} \tag{10}$$

$$= \begin{bmatrix} p_wq_w - \mathbf{p}_v^\top \mathbf{q}_v \\ p_w\mathbf{q}_v + q_w\mathbf{p}_v + \mathbf{p}_v \times \mathbf{q}_v \end{bmatrix}. \tag{11}$$

The quaternion product is **not commutative** in the general case[1],

$$\mathbf{p} \otimes \mathbf{q} \neq \mathbf{q} \otimes \mathbf{p} . \tag{12}$$

The quaternion product is however **associative**,

$$\mathbf{p} \otimes (\mathbf{q} \otimes \mathbf{r}) = (\mathbf{p} \otimes \mathbf{q}) \otimes \mathbf{r} \tag{13}$$

and **distributive over the sum**

$$\mathbf{p} \otimes (\mathbf{q} + \mathbf{r}) = \mathbf{p} \otimes \mathbf{q} + \mathbf{p} \otimes \mathbf{r} \quad \text{and} \quad (\mathbf{p} \otimes \mathbf{q}) + \mathbf{r} = \mathbf{p} \otimes \mathbf{r} + \mathbf{q} \otimes \mathbf{r} \tag{14}$$

The quaternion product can alternatively be expressed in matrix form as

$$\mathbf{p} \otimes \mathbf{q} = [\mathbf{p}]_L \mathbf{q} \quad \text{and} \quad \mathbf{p} \otimes \mathbf{q} = [\mathbf{q}]_R \mathbf{p} , \tag{15}$$

where $[\mathbf{p}]_L$ and $[\mathbf{q}]_R$ are the left and right quaternion-product matrices which are derived from (10),

$$[\mathbf{p}]_L = \begin{bmatrix} p_w & -p_x & -p_y & -p_z \\ p_x & p_w & -p_z & p_y \\ p_y & p_z & p_w & -p_x \\ p_z & -p_y & p_x & p_w \end{bmatrix}, \quad \text{and} \quad [\mathbf{q}]_R = \begin{bmatrix} q_w & -q_x & -q_y & -q_z \\ q_x & q_w & q_z & -q_y \\ q_y & -q_z & q_w & q_x \\ q_z & q_y & -q_x & q_w \end{bmatrix}, \tag{16}$$

or inspecting (11) a compact form can be derived as,

$$[\mathbf{p}]_L = \begin{bmatrix} 0 & -\mathbf{p}_v^\top \\ p_w\mathbf{I}_{3\times 3} + \mathbf{p}_v & p_w\mathbf{I}_{3\times 3} - \lfloor \mathbf{p}_v \times \rfloor \end{bmatrix} \tag{17}$$

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[1] There are exceptions to the general non-commutative rule, where either $\mathbf{p}$ or $\mathbf{q}$ is real such that $\mathbf{p}_v \times \mathbf{q}_v = 0$, or when both $\mathbf{p}_v$ and $\mathbf{q}_v$ are parallel, $\mathbf{p}_v \| \mathbf{q}_v$. Only in these cirmcumstances is the quaternion product commutative.

and

$$[\mathbf{q}]_R = \begin{bmatrix} 0 & -\mathbf{q}_v^\top \\ \mathbf{q}_w\mathbf{I}_{3\times3} + \mathbf{q}_v & \mathbf{q}_w\mathbf{I}_{3\times3} - \lfloor\mathbf{q}_v \times\rfloor \end{bmatrix}, \tag{18}$$

where $\lfloor\bullet \times\rfloor$ is the skew operator that produces a matrix cross product matrix, and is defined as

$$\lfloor\mathbf{v} \times\rfloor = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}, \quad \mathbf{v} \in \mathbb{R}^3 \tag{19}$$

3.1.3 Conjugate

The conjugate operator for quaternion, $(\bullet)^*$, is defined as

$$\mathbf{q}^* = \begin{bmatrix} q_w \\ -\mathbf{q}_v \end{bmatrix} = \begin{bmatrix} q_w \\ -q_x \\ -q_y \\ -q_z \end{bmatrix}. \tag{20}$$

This has the properties

$$\mathbf{q} \otimes \mathbf{q}^{-1} = \mathbf{q}^{-1} \otimes \mathbf{q} = q_w^2 + q_x^2 + q_y^2 + q_z^2 = \begin{bmatrix} q_w^2 + q_x^2 + q_y^2 + q_z^2 \\ \mathbf{0} \end{bmatrix}, \tag{21}$$

and

$$(\mathbf{p} \otimes \mathbf{q})^* = \mathbf{q}^* \otimes \mathbf{p}^*. \tag{22}$$

3.1.4 Norm

The norm of a quaternion is defined by

$$\|\mathbf{q}\| = \sqrt{\mathbf{q} \otimes \mathbf{q}^*} \tag{23}$$
$$= \sqrt{\mathbf{q}^* \otimes \mathbf{q}} \tag{24}$$
$$= \sqrt{q_w^2 + q_x^2 + q_y^2 + q_z^2} \in \mathbb{R}, \tag{25}$$

and has the property

$$\|\mathbf{p} \otimes \mathbf{q}\| = \|\mathbf{q} \otimes \mathbf{p}\| = \|\mathbf{p}\|\|\mathbf{q}\| \tag{26}$$

4 Computer Vision

4.1 Pinhole Camera Model

$$K = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \tag{27}$$

4.2 Radial Tangential Distortion

$$
\begin{aligned}
k_{\mathrm{radial}} &= 1 + (k_1 r^2) + (k_2 r^4) \\
x' &= x \cdot k_{\mathrm{radial}} \\
y' &= y \cdot k_{\mathrm{radial}} \\
x'' &= x' + (2p_1 xy + p_2(r^2 + 2x^2)) \\
y'' &= y' + (p_1(r^2 + 2y^2) + 2p_2 xy)
\end{aligned}
\tag{28}
$$

4.3 Equi-distant Distortion

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\theta &= \arctan(r) \\
\theta_d &= \theta(1 + k_1\theta^2 + k_2\theta^4 + k_3\theta^6 + k_4\theta^8) \\
x' &= (\theta_d/r) \cdot x \\
y' &= (\theta_d/r) \cdot y
\end{aligned}
\tag{29}
$$

4.4 Bundle Adjustment

Let $\mathbf{z} \in^2$ be the image measurement and $\mathbf{h}(\cdot) \in^2$ be the projection function that produces an image projection $\tilde{\mathbf{z}}$. The reprojection error e is defined as the euclidean distance between $\mathbf{z}$ and $\tilde{\mathbf{z}}$.

$$\mathbf{e} = \mathbf{z} - \tilde{\mathbf{z}} \tag{30}$$

Our aim given image measurement $\mathbf{z}$ is to find the image projection $\tilde{\mathbf{z}}$ that minimizes the reprojection e. The image projection $\tilde{\mathbf{z}}$ in pixels can be represented in homogeneous coordinates with u, v, w as

$$\tilde{\mathbf{z}} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \end{bmatrix} \tag{31}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathrm{w}\mathbf{P} \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{C}_{\mathrm{WC}}[\mathrm{w}\mathbf{p} - \mathrm{w}\mathbf{r}_{\mathrm{WC}}] \tag{32}$$

where u, v, w is computed by projecting a landmark position $_\mathrm{W}\mathbf{p}$ in the world frame to the camera's image plane with projection matrix $\mathbf{P}$. The projection matrix, $\mathbf{P}$, can be decomposed into the camera intrinsics matrix, $\mathbf{K}$, the camera rotation, $\mathbf{C}_\mathrm{WC}$, and camera position, $_\mathrm{W}\mathbf{r}_\mathrm{WC}$, expressed in the world frame. The orientation of the camera in (32) is represented using a rotation matrix. To reduce the optimization parameters the rotation matrix can be parameterized by a quaternion by using the following formula,

$$\mathbf{C}_\mathrm{WC} = \begin{bmatrix} q_w^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_x q_y - q_w q_z) & 2(q_x q_z + q_w q_y) \\ 2(q_x q_y + q_w q_z) & q_w^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_y q_z - q_w q_x) \\ 2(q_x q_z - q_w q_y) & 2(q_y q_z + q_w q_x) & q_w^2 - q_x^2 - q_y^2 + q_z^2 \end{bmatrix}. \quad (33)$$

By parameterzing the rotation matrix with a quaternion, the optimization parameters for the camera's orientation is reduced from 9 to 4.

Our objective is to optimize for the camera rotation $\mathbf{C}_\mathrm{WC}$, camera position $_\mathrm{W}\mathbf{r}_\mathrm{WC}$ and 3D landmark position $_\mathrm{W}\mathbf{p}$ in order to minimize the cost function,

$$\underset{\mathbf{C}_\mathrm{WC}, _\mathrm{W}\mathbf{r}_\mathrm{WC}, _\mathrm{W}\mathbf{p}}{\mathrm{argmin}} \ \|\mathbf{z} - \mathbf{h}(\mathbf{C}_\mathrm{WC}, _\mathrm{W}\mathbf{r}_\mathrm{WC}, _\mathrm{W}\mathbf{p})\|^2 \quad (34)$$

$$\underset{\mathbf{C}_\mathrm{WC}, _\mathrm{W}\mathbf{r}_\mathrm{WC}, _\mathrm{W}\mathbf{p}}{\mathrm{argmin}} \ \|\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} u/w \\ v/w \end{bmatrix}\|^2. \quad (35)$$

The cost function above assumes only a single measurement, if there are N measurements corresponding to N unique landmarks the cost function can be rewritten as a maximum likelihood estimation problem as,

$$\underset{\mathbf{C}_\mathrm{WC}, _\mathrm{W}\mathbf{r}_\mathrm{WC}, _\mathrm{W}\mathbf{p}}{\mathrm{argmin}} \ \sum_{j=1}^{N} \|\mathbf{z}_j - \mathbf{h}(\mathbf{C}_\mathrm{WC}{}^j, _\mathrm{W}\mathbf{r}_\mathrm{WC}{}^j, _\mathrm{W}\mathbf{p})\|^2 \quad (36)$$

under the assumption that the observed landmark, $_\mathrm{W}\mathbf{p}$, measured in the image plane, z, are corrupted by a **zero-mean Gaussian noise**.

For the general case of M images taken at different camera poses the cost function can be further extended to,

$$\underset{\mathbf{C}_\mathrm{WC}, _\mathrm{W}\mathbf{r}_\mathrm{WC}, _\mathrm{W}\mathbf{p}}{\min} \ \sum_{i=1}^{M} \sum_{j=1}^{N} \|\mathbf{z}_{i,j} - \mathbf{h}(\mathbf{C}_\mathrm{WC}{}^i, _\mathrm{W}\mathbf{r}_\mathrm{WC}{}^i, _\mathrm{W}\mathbf{p})\|^2 \quad (37)$$

The optimization process begins by setting the first image camera pose as world origin, and subsequent $\mathbf{C}_{\mathrm{WC}i}$ and $_\mathrm{W}\mathbf{r}_{\mathrm{WC}i}$ will be relative to the first camera pose.

Jacobians

The Jacobian for the optimization problem for a **single measurement** has the form:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial \mathbf{h}}{\partial \mathbf{C}_\mathrm{WC}} \dfrac{\partial \mathbf{C}_\mathrm{WC}}{\partial \mathbf{q}} & \dfrac{\partial \mathbf{h}}{\partial {}_\mathrm{W}\mathbf{r}_\mathrm{WC}} & \dfrac{\partial \mathbf{h}}{\partial {}_\mathrm{W}\mathbf{p}} \end{bmatrix} \quad (38)$$

If there are two measurements the Jacobian is stacked with the following pattern:

$$\mathbf{J} = \begin{bmatrix} \text{Image 1}_{2\times 7} & \mathbf{0}_{2\times 7} & \text{3D Point}_{2\times 3} \\ \mathbf{0}_{2\times 7} & \text{Image 2}_{2\times 7} & \text{3D Point}_{2\times 3} \end{bmatrix} \tag{39}$$

Derivation for $\dfrac{\partial \mathbf{h}}{\partial \mathbf{C_{WC}}}$

$$\frac{\partial \mathbf{h}}{\partial \mathbf{C}_{\text{WC}}} = \begin{bmatrix} \dfrac{w\dfrac{\partial u}{\partial \mathbf{C}_{\text{WC}}} - u\dfrac{\partial w}{\partial \mathbf{C}_{\text{WC}}}}{w^2} \\[3em] \dfrac{w\dfrac{\partial v}{\partial \mathbf{C}_{\text{WC}}} - v\dfrac{\partial w}{\partial \mathbf{C}_{\text{WC}}}}{w^2} \end{bmatrix}_{2\times 9} \tag{40}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{K}\mathbf{C}_{\text{WC}}[\mathbf{wp} - \mathbf{wr}_{\text{WC}}]$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f_x & 0 & p_x \\ 0 & f_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} [\mathbf{wp} - \mathbf{wr}_{\text{WC}}]$$

$$\frac{\partial u}{\partial \mathbf{C}_{\text{WC}}} = \begin{bmatrix} f_x(\mathbf{wp} - \mathbf{wr}_{\text{WC}}) & \mathbf{0}_{1\times 3} & p_x(\mathbf{wp} - \mathbf{wr}_{\text{WC}}) \end{bmatrix}_{1\times 9} \tag{41}$$

$$\frac{\partial v}{\partial \mathbf{C}_{\text{WC}}} = \begin{bmatrix} \mathbf{0}_{1\times 3} & f_y(\mathbf{wp} - \mathbf{wr}_{\text{WC}}) & p_y(\mathbf{wp} - \mathbf{wr}_{\text{WC}}) \end{bmatrix}_{1\times 9} \tag{42}$$

$$\frac{\partial w}{\partial \mathbf{C}_{\text{WC}}} = \begin{bmatrix} \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} & (\mathbf{wp} - \mathbf{wr}_{\text{WC}}) \end{bmatrix}_{1\times 9} \tag{43}$$

Derivation for $\dfrac{\partial \mathbf{C_{WC}}}{\partial \mathbf{q}}$

$$\frac{\partial \mathbf{C}_{\text{WC}}}{\partial \mathbf{q}} = \begin{bmatrix} \dfrac{\partial \mathbf{C}_{\text{WC}11}}{\partial \mathbf{q}} \\[1.2em] \dfrac{\partial \mathbf{C}_{\text{WC}12}}{\partial \mathbf{q}} \\[1.2em] \vdots \\[0.8em] \dfrac{\partial \mathbf{C}_{\text{WC}33}}{\partial \mathbf{q}} \end{bmatrix}_{9\times 4} \tag{44}$$

$$\mathbf{C}_{\text{WC}} = \begin{bmatrix} q_w^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_xq_y - q_wq_z) & 2(q_xq_z + q_wq_y) \\ 2(q_xq_y + q_wq_z) & q_w^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_yq_z - q_wq_x) \\ 2(q_xq_z - q_wq_y) & 2(q_yq_z + q_wq_x) & q_w^2 - q_x^2 - q_y^2 + q_z^2 \end{bmatrix}$$

$$\frac{\mathbf{C}_{\text{WC}11}}{\partial \mathbf{q}} = \begin{bmatrix} 0 & -4q_y & -4q_z & 0 \end{bmatrix} \tag{45}$$

$$\frac{\mathbf{C}_{\text{WC}12}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_y & 2q_x & -2q_w & -2q_z \end{bmatrix} \tag{46}$$

$$\frac{\mathbf{C}_{\text{WC}13}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_z & 2q_w & 2q_x & 2q_y \end{bmatrix} \tag{47}$$

$$\frac{\mathbf{C}_{\text{WC}21}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_y & 2q_x & 2q_w & 2q_z \end{bmatrix} \tag{48}$$

$$\frac{\mathbf{C}_{\text{WC}22}}{\partial \mathbf{q}} = \begin{bmatrix} 4q_x & 0 & 4q_z & 0 \end{bmatrix} \tag{49}$$

$$\frac{\mathbf{C}_{\text{WC}23}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_w & 2q_z & 2q_y & 2q_x \end{bmatrix} \tag{50}$$

$$\frac{\mathbf{C}_{\text{WC}31}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_z & -2q_w & 2q_x & -2q_y \end{bmatrix} \tag{51}$$

$$\frac{\mathbf{C}_{\text{WC}32}}{\partial \mathbf{q}} = \begin{bmatrix} -4q_x & -4q_y & 0 & 0 \end{bmatrix} \tag{52}$$

$$\frac{\mathbf{C}_{\text{WC}33}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_w & 2q_z & 2q_y & 2q_x \end{bmatrix} \tag{53}$$

Derivation for $\dfrac{\partial \mathbf{h}}{\partial \mathbf{w} \mathbf{p}}$

$$\frac{\partial \mathbf{h}}{\partial \text{w}\mathbf{p}} = \begin{bmatrix} \dfrac{w \dfrac{\partial u}{\partial \text{w}\mathbf{p}} - u \dfrac{\partial w}{\partial \text{w}\mathbf{p}}}{w^2} \\[2em] \dfrac{w \dfrac{\partial v}{\partial \text{w}\mathbf{p}} - v \dfrac{\partial w}{\partial \text{w}\mathbf{p}}}{w^2} \end{bmatrix}_{2 \times 3} \tag{54}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{K} \mathbf{C}_{\text{WC}} [\text{w}\mathbf{p} - \text{w}\mathbf{r}_{\text{WC}}]$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f_x & 0 & p_x \\ 0 & f_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} [\text{w}\mathbf{p} - \text{w}\mathbf{r}_{\text{WC}}]$$

$$\frac{\partial u}{\partial_{\mathrm{W}}\mathbf{p}} = \begin{bmatrix} f_x \mathbf{C}_{\mathrm{WC}11} + c_x \mathbf{C}_{\mathrm{WC}31} & f_x \mathbf{C}_{\mathrm{WC}12} + c_x \mathbf{C}_{\mathrm{WC}32} & f_x \mathbf{C}_{\mathrm{WC}13} + c_x \mathbf{C}_{\mathrm{WC}33} \end{bmatrix} \tag{55}$$

$$\frac{\partial v}{\partial_{\mathrm{W}}\mathbf{p}} = \begin{bmatrix} f_y \mathbf{C}_{\mathrm{WC}21} + c_y \mathbf{C}_{\mathrm{WC}31} & f_y \mathbf{C}_{\mathrm{WC}22} + c_y \mathbf{C}_{\mathrm{WC}32} & f_y \mathbf{C}_{\mathrm{WC}23} + c_y \mathbf{C}_{\mathrm{WC}33} \end{bmatrix} \tag{56}$$

$$\frac{\partial w}{\partial_{\mathrm{W}}\mathbf{p}} = \begin{bmatrix} \mathbf{C}_{\mathrm{WC}31} & \mathbf{C}_{\mathrm{WC}32} & \mathbf{C}_{\mathrm{WC}33} \end{bmatrix} \tag{57}$$

Derivation for $\dfrac{\partial h}{\partial_{\mathbf{W}}\mathbf{r_{WC}}}$

$$\frac{\partial \mathbf{h}}{\partial_{\mathrm{W}}\mathbf{r_{WC}}} = \begin{bmatrix} \dfrac{w\dfrac{\partial u}{\partial_{\mathrm{W}}\mathbf{r_{WC}}} - u\dfrac{\partial w}{\partial_{\mathrm{W}}\mathbf{r_{WC}}}}{w^2} \\[2em] \dfrac{w\dfrac{\partial v}{\partial_{\mathrm{W}}\mathbf{r_{WC}}} - v\dfrac{\partial w}{\partial_{\mathrm{W}}\mathbf{r_{WC}}}}{w^2} \end{bmatrix}_{2\times 3} \tag{58}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{K}\mathbf{C}_{\mathrm{WC}}[_{\mathrm{W}}\mathbf{p} - {_{\mathrm{W}}}\mathbf{r_{WC}}]$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f_x & 0 & p_x \\ 0 & f_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} [_{\mathrm{W}}\mathbf{p} - {_{\mathrm{W}}}\mathbf{r_{WC}}]$$

$$\frac{\partial u}{\partial_{\mathrm{W}}\mathbf{r_{WC}}} = - \begin{bmatrix} f_x \mathbf{C}_{\mathrm{WC}11} + c_x \mathbf{C}_{\mathrm{WC}31} & f_x \mathbf{C}_{\mathrm{WC}12} + c_x \mathbf{C}_{\mathrm{WC}32} & f_x \mathbf{C}_{\mathrm{WC}13} + c_x \mathbf{C}_{\mathrm{WC}33} \end{bmatrix} \tag{59}$$

$$\frac{\partial v}{\partial_{\mathrm{W}}\mathbf{r_{WC}}} = - \begin{bmatrix} f_y \mathbf{C}_{\mathrm{WC}21} + c_y \mathbf{C}_{\mathrm{WC}31} & f_y \mathbf{C}_{\mathrm{WC}22} + c_y \mathbf{C}_{\mathrm{WC}32} & f_y \mathbf{C}_{\mathrm{WC}23} + c_y \mathbf{C}_{\mathrm{WC}33} \end{bmatrix} \tag{60}$$

$$\frac{\partial w}{\partial_{\mathrm{W}}\mathbf{r_{WC}}} = - \begin{bmatrix} \mathbf{C}_{\mathrm{WC}31} & \mathbf{C}_{\mathrm{WC}32} & \mathbf{C}_{\mathrm{WC}33} \end{bmatrix} \tag{61}$$

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5   Calibration

$$\underset{\mathbf{T}_{\text{WS}}, \mathbf{T}_{\text{SC}}, \mathbf{T}_{\text{WF}}}{\operatorname{argmin}} \|\mathbf{z} - \mathbf{h}(\mathbf{T}_{\text{WS}}, \mathbf{T}_{\text{SC}}, \mathbf{T}_{\text{WF}})\|^2 \tag{62}$$

$$\hat{\mathbf{z}} = \mathbf{h}(\mathbf{T}_{\text{WS}}, \mathbf{T}_{\text{SC}}, \mathbf{T}_{\text{WF}}) \tag{63}$$

$$\mathbf{e} = \mathbf{z} - \hat{\mathbf{z}} \tag{64}$$

$$\hat{\mathbf{z}} = \begin{bmatrix} {}_{\text{C}}X/{}_{\text{C}}Z \\ {}_{\text{C}}Y/{}_{\text{C}}Z \end{bmatrix} \tag{65}$$

$$_{\text{C}}\mathbf{p} = \begin{bmatrix} {}_{\text{C}}X \\ {}_{\text{C}}Y \\ {}_{\text{C}}Z \end{bmatrix} \tag{66}$$

$$\frac{\partial \mathbf{h}}{\partial {}_{\text{C}}\mathbf{p}} = \begin{bmatrix} 1 & 0 & {}_{\text{C}}X/{}_{\text{C}}Z^2 \\ 0 & 1 & {}_{\text{C}}Y/{}_{\text{C}}Z^2 \end{bmatrix} \tag{67}$$

$$_{\text{C}}\mathbf{p} = \mathbf{T}_{\text{SC}}{}^{-1}\mathbf{T}_{\text{WS}}{}^{-1}\mathbf{T}_{\text{WF}}\ {}_{\text{F}}\mathbf{p} \tag{68}$$

5.1   Jacobian w.r.t Sensor Pose, $\mathbf{T_{WS}}$

$$_{\text{W}}\mathbf{p} = \mathbf{T}_{\text{WS}}\ {}_{\text{S}}\mathbf{p} \tag{69}$$

$$\frac{\partial \mathbf{h}}{\partial {}_{\text{W}}\mathbf{p}} = \frac{\partial \mathbf{h}}{\partial {}_{\text{C}}\mathbf{p}} \frac{\partial {}_{\text{C}}\mathbf{p}}{\partial {}_{\text{W}}\mathbf{p}} \tag{70}$$

$$\frac{\partial \mathbf{h}}{\partial {}_{\text{C}}\mathbf{p}} = \begin{bmatrix} 1 & 0 & {}_{\text{C}}X/{}_{\text{C}}Z^2 \\ 0 & 1 & {}_{\text{C}}Y/{}_{\text{C}}Z^2 \end{bmatrix} \tag{71}$$

$$\frac{\partial {}_{\text{C}}\mathbf{p}}{\partial {}_{\text{W}}\mathbf{p}} = \mathbf{C}_{\text{CW}} \tag{72}$$

$$\frac{\partial \mathbf{h}}{\partial \mathbf{T}_{\text{WS}}} = \begin{bmatrix} \dfrac{\partial \mathbf{h}}{\partial {}_{\text{W}}\mathbf{r}_{\text{WS}}} & \dfrac{\partial \mathbf{h}}{\partial \mathbf{C}_{\text{WS}}} \end{bmatrix} \tag{73}$$

$$\frac{\partial \mathbf{h}}{\partial {}_{\text{W}}\mathbf{r}_{\text{WS}}} = \frac{\partial \mathbf{h}}{\partial {}_{\text{C}}\mathbf{p}} \frac{\partial {}_{\text{C}}\mathbf{p}}{\partial {}_{\text{S}}\mathbf{p}} \frac{\partial {}_{\text{S}}\mathbf{p}}{\partial {}_{\text{W}}\mathbf{r}_{\text{WS}}} \tag{74}$$

$$\frac{\partial \mathbf{h}}{\partial \mathbf{C}_{\text{WS}}} = \frac{\partial \mathbf{h}}{\partial {}_{\text{C}}\mathbf{p}} \frac{\partial {}_{\text{C}}\mathbf{p}}{\partial {}_{\text{S}}\mathbf{p}} \frac{\partial {}_{\text{S}}\mathbf{p}}{\partial \mathbf{C}_{\text{WS}}} \tag{75}$$

$$\frac{\partial {}_{\text{S}}\mathbf{p}}{\partial {}_{\text{W}}\mathbf{r}_{\text{WS}}} = \mathbf{I} \tag{76}$$

$$\frac{\partial {}_{\text{S}}\mathbf{p}}{\partial \mathbf{C}_{\text{WS}}} = -\lfloor \mathbf{C}_{\text{WS}}\ {}_{\text{S}}\mathbf{p}\ \times \rfloor \tag{77}$$

5.2 Jacobian w.r.t Sensor-Camera Extrinsics, $\mathbf{T_{SC}}$

$$\text{s}\mathbf{p} = \mathbf{T}_{\text{SC}}\ \text{c}\mathbf{p} \tag{78}$$

$$\frac{\partial \mathbf{h}}{\partial \text{s}\mathbf{p}} = \frac{\partial \mathbf{h}}{\partial \text{c}\mathbf{p}} \frac{\partial \text{c}\mathbf{p}}{\partial \text{s}\mathbf{p}} \tag{79}$$

$$\frac{\partial \mathbf{h}}{\partial \text{c}\mathbf{p}} = \begin{bmatrix} 1 & 0 & \text{c}X/\text{c}Z^2 \\ 0 & 1 & \text{c}Y/\text{c}Z^2 \end{bmatrix} \tag{80}$$

$$\frac{\partial \text{c}\mathbf{p}}{\partial \text{s}\mathbf{p}} = \mathbf{C}_{\text{CS}} \tag{81}$$

$$\frac{\partial \mathbf{h}}{\partial \mathbf{T}_{\text{SC}}} = \begin{bmatrix} \dfrac{\partial \mathbf{h}}{\partial \text{s}\mathbf{r}_{\text{SC}}} & \dfrac{\partial \mathbf{h}}{\partial \mathbf{C}_{\text{SC}}} \end{bmatrix} \tag{82}$$

$$\frac{\partial \mathbf{h}}{\partial \text{s}\mathbf{r}_{\text{SC}}} = \frac{\partial \mathbf{h}}{\partial \text{c}\mathbf{p}} \frac{\partial \text{c}\mathbf{p}}{\partial \text{s}\mathbf{p}} \frac{\partial \text{s}\mathbf{p}}{\partial \text{s}\mathbf{r}_{\text{SC}}} \tag{83}$$

$$\frac{\partial \mathbf{h}}{\partial \mathbf{C}_{\text{SC}}} = \frac{\partial \mathbf{h}}{\partial \text{c}\mathbf{p}} \frac{\partial \text{c}\mathbf{p}}{\partial \text{s}\mathbf{p}} \frac{\partial \text{s}\mathbf{p}}{\partial \mathbf{C}_{\text{SC}}} \tag{84}$$

$$\frac{\partial \text{s}\mathbf{p}}{\partial \text{s}\mathbf{r}_{\text{SC}}} = \mathbf{I} \tag{85}$$

$$\frac{\partial \text{s}\mathbf{p}}{\partial \mathbf{C}_{\text{SC}}} = -\lfloor \mathbf{C}_{\text{SC}}\ \text{s}\mathbf{p}\ \times \rfloor \tag{86}$$

5.3 Jacobian w.r.t Fiducial Pose, $\mathbf{T_{WF}}$

$$\begin{aligned} \text{w}\mathbf{p} &= \mathbf{T}_{\text{WF}}\ \text{F}\mathbf{p} \\ &= \mathbf{C}_{\text{WF}}\ \text{F}\mathbf{p} + \text{w}\mathbf{r}_{\text{WF}} \end{aligned} \tag{87}$$

$$\frac{\partial \mathbf{h}}{\partial \text{w}\mathbf{p}} = \frac{\partial \mathbf{h}}{\partial \text{c}\mathbf{p}} \frac{\partial \text{c}\mathbf{p}}{\partial \text{w}\mathbf{p}} \tag{88}$$

$$\frac{\partial \mathbf{h}}{\partial \text{c}\mathbf{p}} = \begin{bmatrix} 1 & 0 & \text{c}X/\text{c}Z^2 \\ 0 & 1 & \text{c}Y/\text{c}Z^2 \end{bmatrix} \tag{89}$$

$$\frac{\partial \text{c}\mathbf{p}}{\partial \text{w}\mathbf{p}} = \mathbf{C}_{\text{CW}} \tag{90}$$

$$\frac{\partial \mathbf{h}}{\partial \delta\theta} = \frac{\partial \mathbf{h}}{\partial \text{w}\mathbf{p}} \frac{\partial \text{w}\mathbf{p}}{\partial \delta\theta} \tag{91}$$

$$\frac{\partial \mathbf{h}}{\partial \text{w}\mathbf{p}} = \begin{bmatrix} 1 & 0 & \text{c}X/\text{c}Z^2 \\ 0 & 1 & \text{c}Y/\text{c}Z^2 \end{bmatrix} \mathbf{C}_{\text{CF}} \tag{92}$$

$$\frac{\partial \text{w}\mathbf{p}}{\partial \delta\theta} = \mathbf{I} \tag{93}$$

$$\frac{\partial \mathbf{h}}{\partial_{\mathrm{W}} \mathbf{r}_{\mathrm{WF}}} = \frac{\partial \mathbf{h}}{\partial_{\mathrm{W}} \mathbf{p}} \frac{\partial_{\mathrm{W}} \mathbf{p}}{\partial_{\mathrm{W}} \mathbf{r}_{\mathrm{WF}}} \tag{94}$$

$$\frac{\partial \mathbf{h}}{\partial_{\mathrm{W}} \mathbf{p}} = \begin{bmatrix} 1 & 0 & {}_{\mathrm{C}}X/{}_{\mathrm{C}}Z^2 \\ 0 & 1 & {}_{\mathrm{C}}Y/{}_{\mathrm{C}}Z^2 \end{bmatrix} \mathbf{C}_{\mathrm{CF}} \tag{95}$$

$$\frac{\partial_{\mathrm{W}} \mathbf{p}}{\partial_{\mathrm{W}} \mathbf{r}_{\mathrm{WF}}} = -\lfloor \mathbf{C}\{\delta\theta\}_{\mathrm{F}} \mathbf{p} \times \rfloor \tag{96}$$

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