Indexing genomic sequences: suffix tree and suffix array

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Example

Let $T = \texttt{CAGAGA\$}$. $T_1 = \texttt{AGAGA\$}$ and $T_{\{3,5\}} = \{\texttt{AGA\$}, \texttt{A\$}\}$.

\$
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AGA\$
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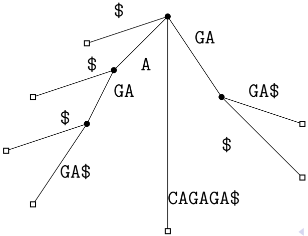

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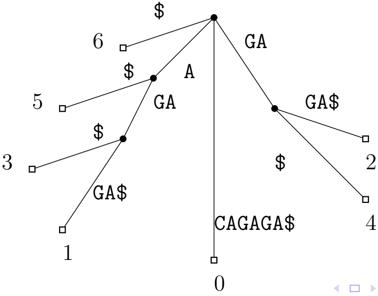

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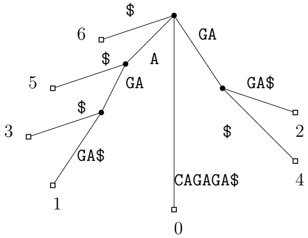

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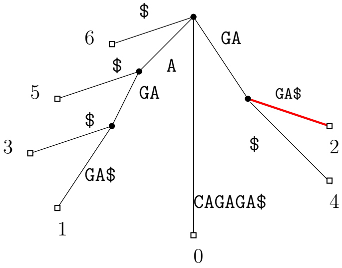

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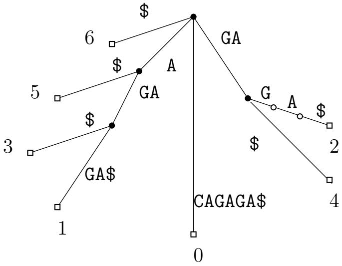

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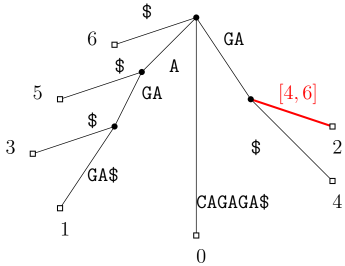

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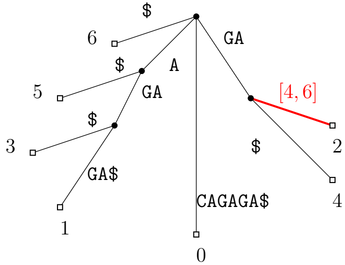

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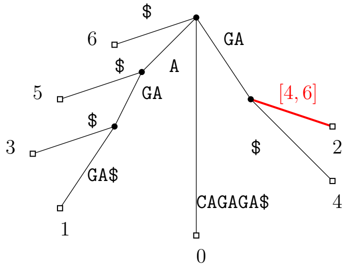

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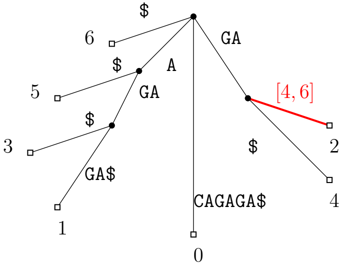

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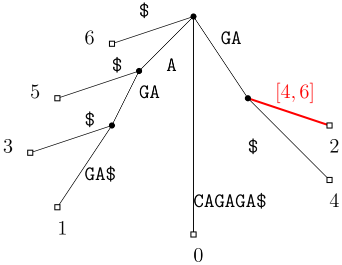

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Space linear in n: $O(n)$.

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Theorem (Farach, FOCS 1997)

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In bioinformatics we usually have that $|\Sigma| = O(1)$.

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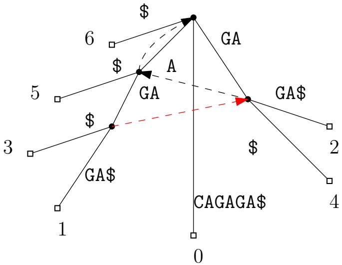
Example

Let $T =$ `CAGAGA$`. str(u) = `AGA` and str(v) = `GA`.



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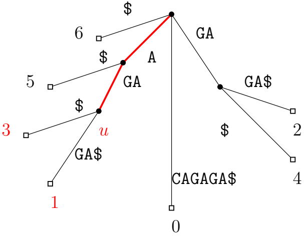

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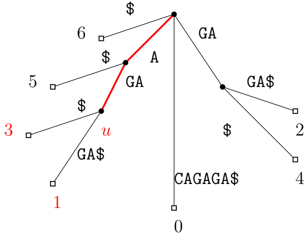

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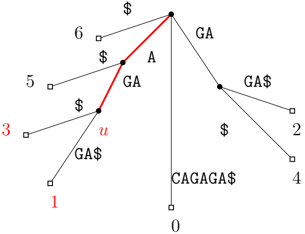

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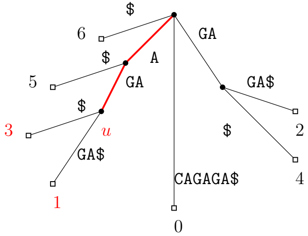

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Theorem

Exact string matching queries can be answered in $O(|P| + occ)$ time after $O(n)$ time preprocessing.

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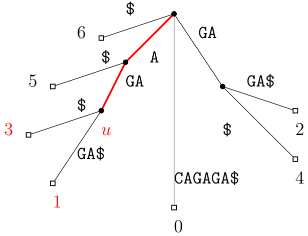

Alternatively: binary search for P in the suffix array of T.

Theorem

Exact string matching queries can be answered in $O(|P| \log n + occ)$ *time using the suffix array.*

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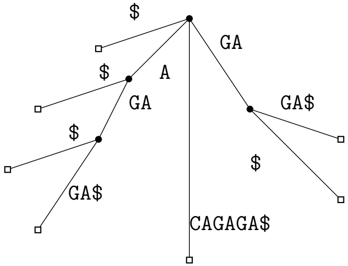

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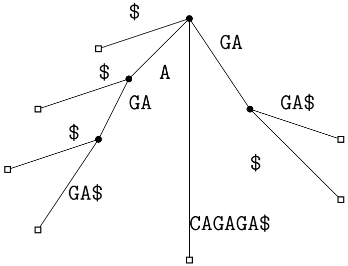

Every *locus* **(node, depth)** in the suffix tree represents a substring of the sequence and every substring is represented by some locus.

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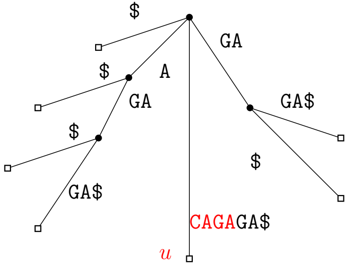
Example

Let $T =$ `CAGAGA$`. Locus $(u, 4)$ represents `CAGA`.



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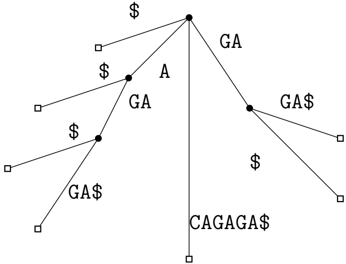

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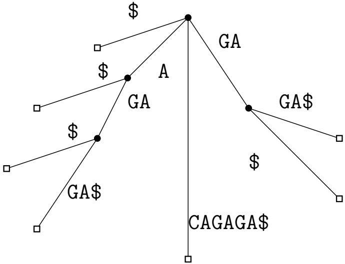

Count the number of distinct loci using a suffix tree traversal.

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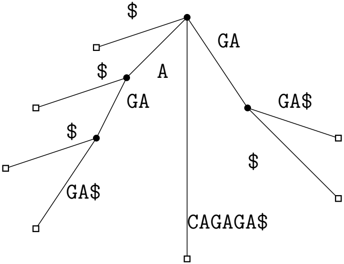

Theorem

The number of distinct substrings can be computed in $O(n)$ time.

Application 3: Longest repeating substring

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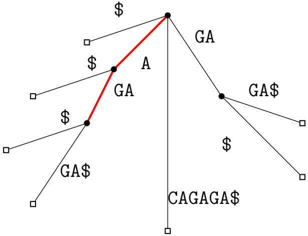
Let $T =$ `CAGAGA$`. The answer is `AGA`.



Find a deepest internal node using a traversal of the suffix tree.

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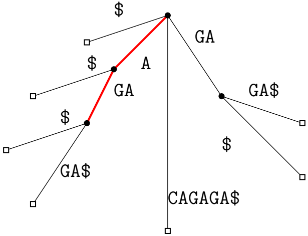

Find a deepest internal node using a traversal of the suffix tree.

Theorem

A longest repeating substring can be found in $O(n)$ time.

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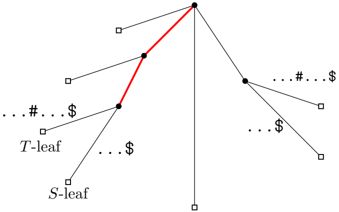
Suffix tree of $T\#S\$$.



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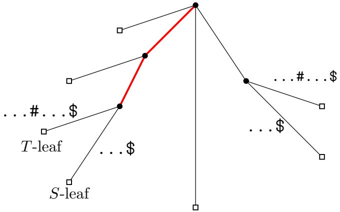

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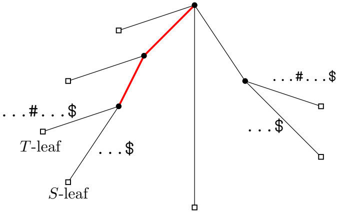

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Theorem

A longest common substring can be found in $O(n + |S|)$ *time.*

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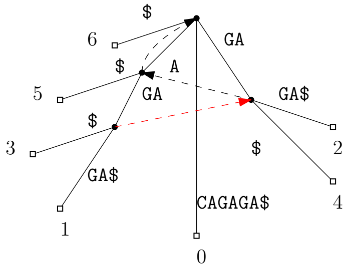

Scan S using the suffix tree of T.

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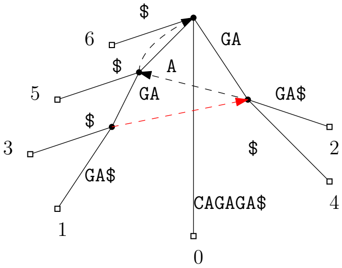


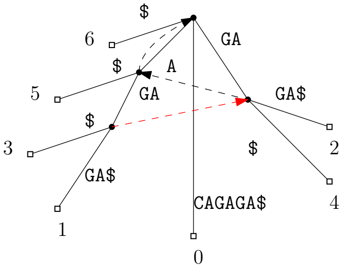
Spell S_i as much as possible;

Application 5: Matching statistics

PREPROCESS: a sequence T
QUERY: a sequence S; return the longest prefix of S_i that is a substring of T, for all $i \in [0, |S| - 1]$

Example

Let $T = \texttt{CAGAGA\$}$.



Spell S_i as much as possible; say $S[i \mathinner{.\,.} j-1]$.

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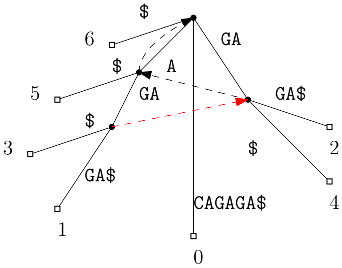

Mismatch at $S[i\,.\,.\,j]$?

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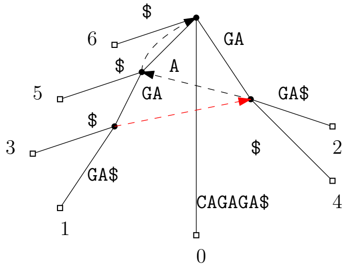

Mismatch at $S[i\,..\,j]$? Use suffix link as the failure transition!

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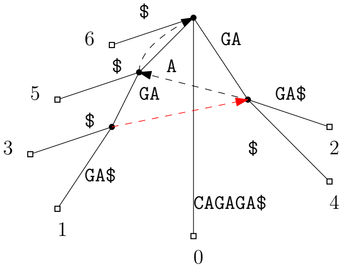

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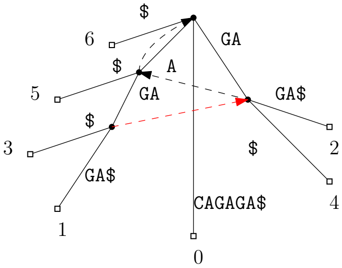

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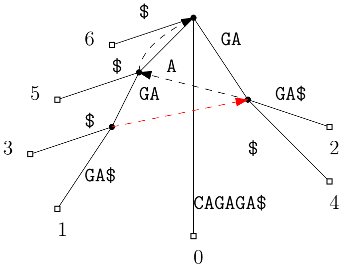

Theorem

Matching statistics of S with respect to T can be computed in $O(|S|)$ time after $O(n)$ time preprocessing.

Application 6: Longest common prefix (LCP)

PREPROCESS: a sequence T
QUERY: a pair (i, j); return the length of the LCP of (T_i, T_j)

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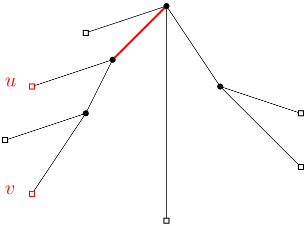

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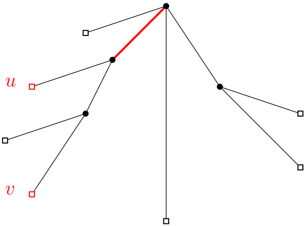

Theorem (Bender and Farach-Colton, LATIN 2000)

Any tree of size $O(N)$ can be preprocessed in $O(N)$ time so that the LCA of any two nodes can be computed in $O(1)$ time.

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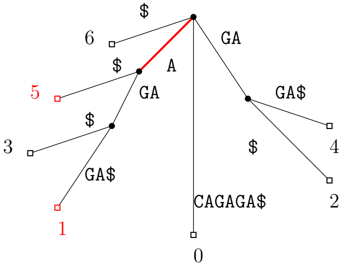
Let $T = \texttt{CAGAGA\$}$. Let $(1, 5)$ be the query. The answer is $1 = |\texttt{A}|$.



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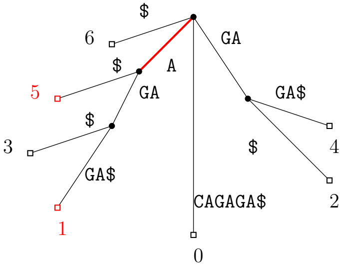

Theorem

Longest common prefix queries can be answered in $O(1)$ time after $O(n)$ time preprocessing.

Application 7: Longest palindromic substring

INPUT: a sequence T
OUTPUT: a longest palindromic substring of T

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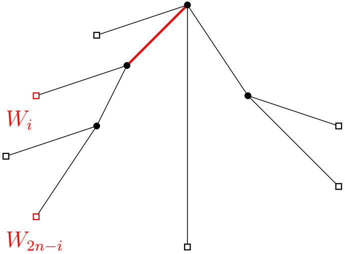

$T = \texttt{C}\underline{\texttt{ATGTA}}\texttt{TT}$

$T^R = \texttt{TTATGTAC}$

$W = T\#T^R\$ = \texttt{CAT}\overset{3}{\underline{\texttt{GTA}}}\underset{\times}{\texttt{T}}\texttt{T\#TTAT}\overset{13}{\underline{\texttt{GTA}}}\underset{\times}{\texttt{C}}\$$

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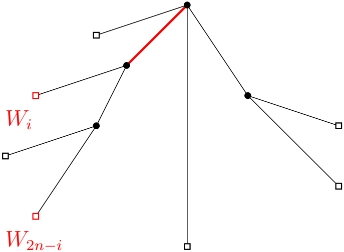

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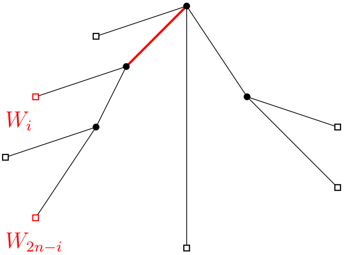
INPUT: a sequence *T*

OUTPUT: a longest palindromic substring of *T*



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- A longest LCP represents a longest odd-length palindrome.

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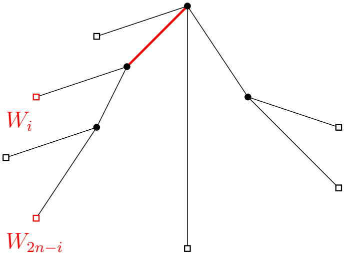

$T = \texttt{C}\underline{\texttt{AT}\textcolor{red}{\texttt{GTA}}}\texttt{TT}$

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- A longest LCP represents a longest odd-length palindrome.
- Even-length palindromes are handled analogously.

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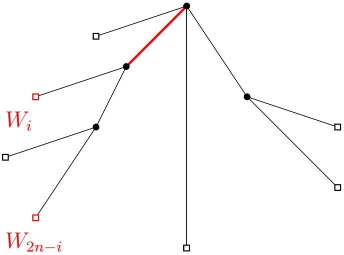

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- Take the longer of the two as the globally longest.

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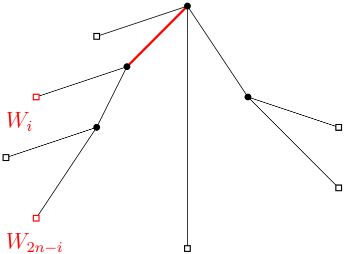

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Theorem

A longest palindromic substring can be computed in $O(n)$ time.

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INPUT: a sequence T, a pattern P, and an integer $k > 0$
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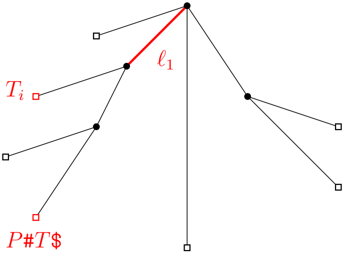

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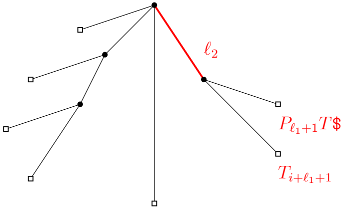

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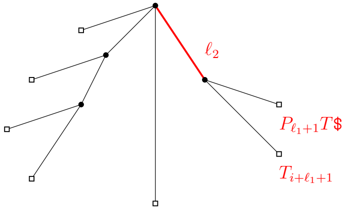

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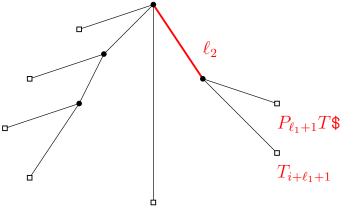

- Answer (at most) $k+1$ LCP queries per i.

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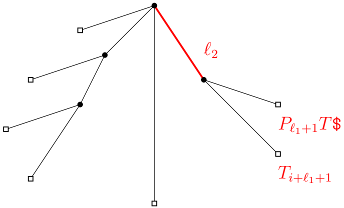

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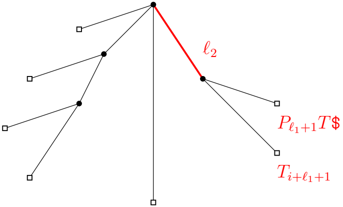

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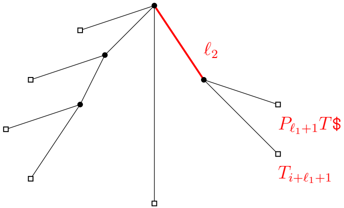

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Theorem (Landau and Vishkin, TCS 1986)

Approximate string matching can be solved in $O(kn)$ time.

Application 9: Shortest unique substring

INPUT: a sequence T
OUTPUT: a shortest unique substring of T

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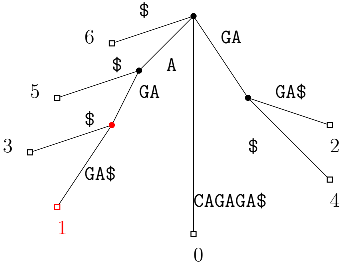
Example

Let $T =$ `CAGAGA$`.



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- str(v) concatenated with the succeeding letter is the shortest unique substring starting at i.

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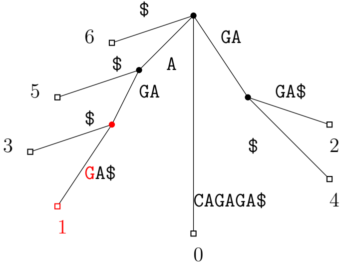
- str(v) concatenated with the succeeding letter is the shortest unique substring starting at i.

Example

Let $T =$ `CAGAGA$`. The shortest unique substring starting at 1 is `AGAG`.



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- Take a shortest substring among all i.

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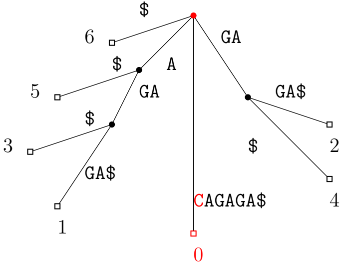
Example

Let $T = \texttt{CAGAGA\$}$. The shortest unique substring is `C`.



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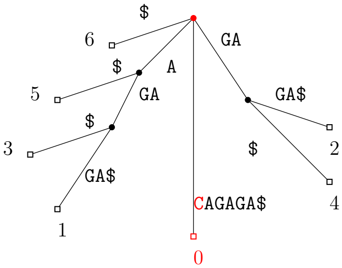
Example

Let $T =$ `CAGAGA$`. The shortest unique substring is `C`.



Theorem

A shortest unique substring can be computed in $O(n)$ time.

Application 10: Lempel-Ziv (LZ) factorization

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OUTPUT: LZ factorization of T

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Example

Let $T = \texttt{abbaabbbaaabab}$.

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Let $T = \texttt{abbaabbbaaabab}$. The LZ factorization of T is

$$\texttt{a} \cdot \texttt{b} \cdot \texttt{b} \cdot \texttt{a} \cdot \texttt{abb} \cdot \texttt{baa} \cdot \texttt{ab} \cdot \texttt{ab}.$$

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Why do we care?

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Why do we care? LZ factorization is a basic and powerful technique for text compression (and string algorithms)!

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- Decorate each internal node with the leftmost starting position the string it represents occurs.

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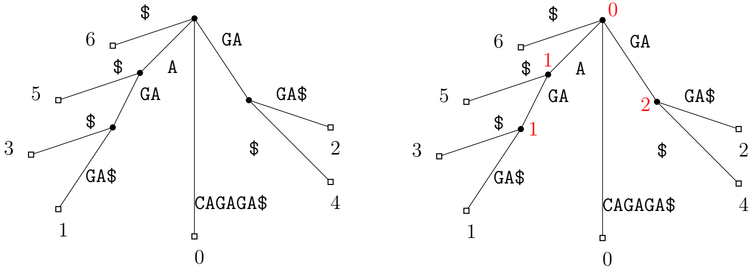

- How?

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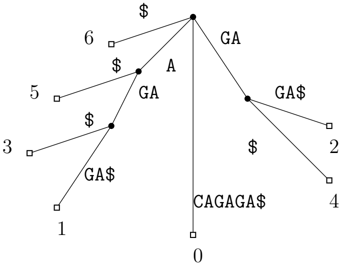

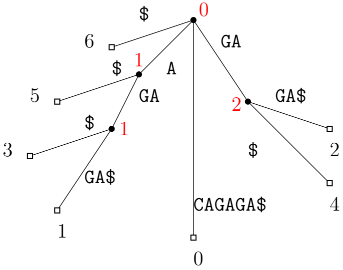

- How? Use a depth-first traversal, propagate the starting positions upwards, and keep the minimum value.

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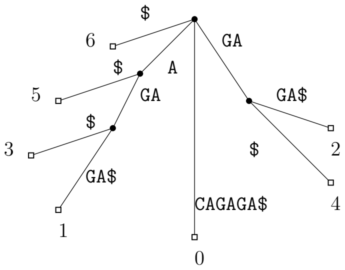

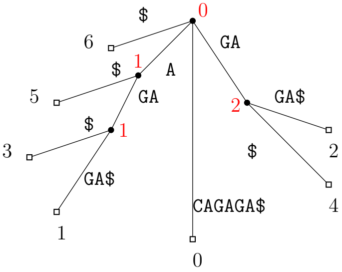

- Spell T, from left to right, in the suffix tree of T.

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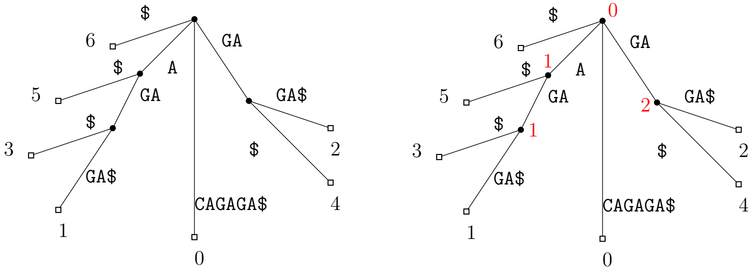

- Spell T, from left to right, in the suffix tree of T.
- For each match $T[i\,..\,j]$ check the leftmost starting position p.

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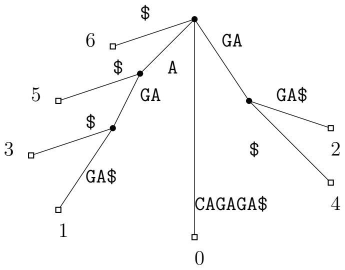

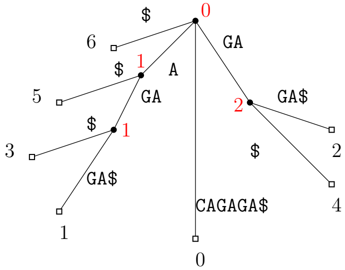

- If $p \geq i$, we have not seen $T[i\,..\,j]$ previously, increment i.

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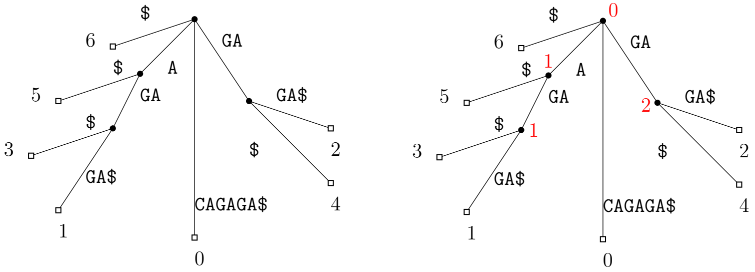

- If $p \geq i$, we have not seen $T[i \mathinner{..} j]$ previously, increment i.
- If $p < i$, we have seen $T[i \mathinner{..} j]$ previously, increment j.

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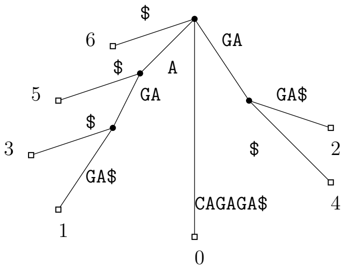

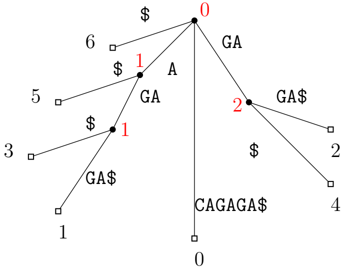

Theorem

The Lempel-Ziv factorization can be computed in $O(n)$ time.