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Introduction to

Algorithm Design and Analysis

[16] Dynamic Programming 1



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In the last class...

- **Single-source shortest paths**
 - From BFS to the Dijkstra algorithm
- **All-pairs shortest paths**
 - BF1, BF2, BF3
 - Floyd-Warshall algorithm

Dynamic Programming

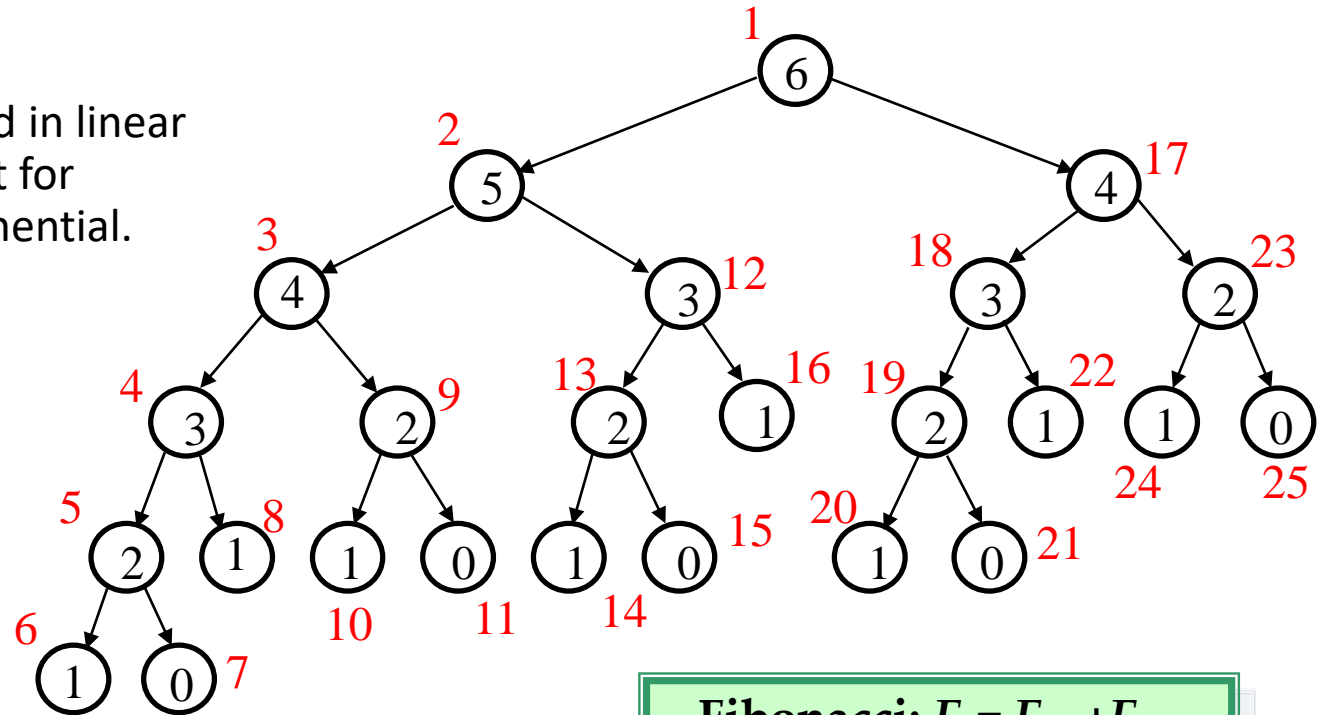
- **Basic Idea of Dynamic Programming (DP)**
 - Smart scheduling of subproblems
- **Minimum Cost Matrix Multiplication**
 - BF1, BF2
 - A DP solution
- **Weighted Binary Search Tree**
 - The “same” DP with matrix multiplication



Brute Force Recursion

The F_n can be computed in linear time easily, but the cost for recursion may be exponential.

The number of activation frames are $2F_{n+1}-1$



Fibonacci: $F_n = F_{n-1} + F_{n-2}$

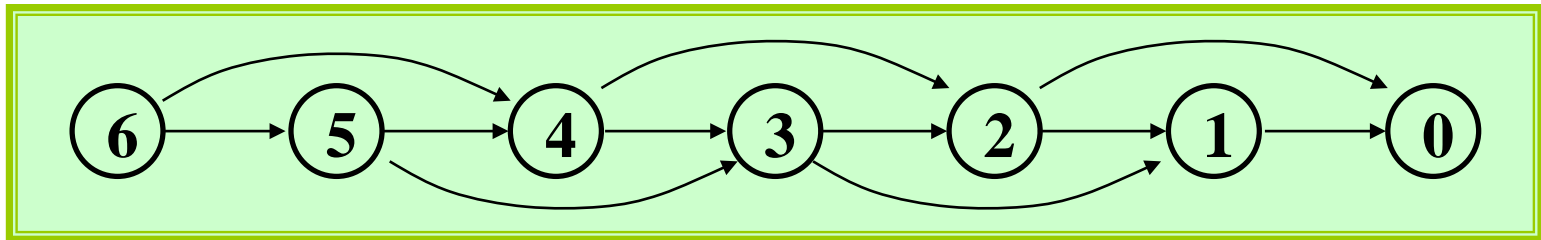
For your reference

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 35, ...

Subproblem Graph

- The **subproblem graph** for a recursive algorithm A of some problem is defined as:
 - vertex: the instance of the problem
 - directed edge: $I \rightarrow J$ if and only if when A invoked on I , it makes a recursive call directly on instance J .
- Portion $A(P)$ of the subproblem graph for Fibonacci function: **here is fib(6)**



Properties of Subproblem Graph

- If A always terminates, the subproblem graph for A is a **DAG**.
 - For each path in the tree of activation frames of a particular call of A , $A(P)$, there is a corresponding path in the subproblem graph of A connecting vertex P and a base-case vertex.
 - The subproblem graph can be viewed as a dependency graph of subtasks to be solved.
- A top-level recursive computation traverse the entire subproblem graph in some **memoryless** style.



Basic Idea of DP

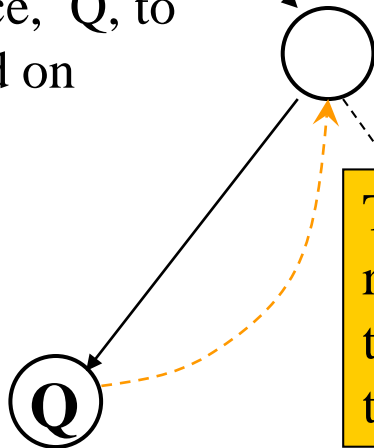
- **Smart recursion**
 - Compute each subproblem **only once**
- **Basic process of a “smart” recursion**
 - Find a reverse **topological order** for the subproblem graph
 - In most cases, the order can be determined **by particular knowledge** of the problem.
 - General method based on DFS is available
 - Scheduling the subproblems according to the reverse topological order
 - **Record** the subproblem solutions in a **dictionary**



Recursion by DP

Case 1: White Q

a instance, Q, to
be called on



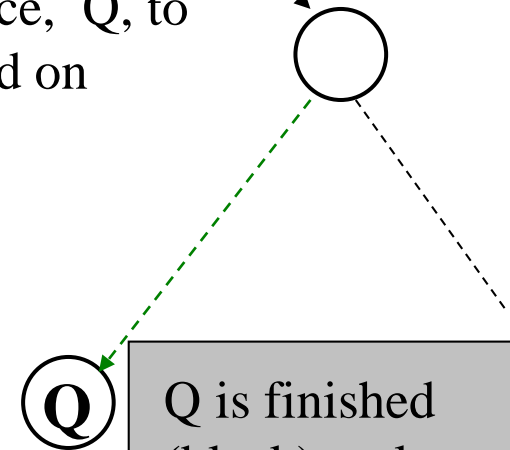
To backtracking,
record the result into
the dictionary (Q,
turned black)

Q is undiscovered
(white), go ahead with
the recursive call

**Note: for DAG, no
gray vertex will be met**

Case 2: Black Q

a instance, Q, to
be called on



Q is finished
(black), only
“checking” the
edge, retrieve
the result from
the dictionary

Fibonacci by DP

```
fibDPwrap(n)
```

```
    Dict soln=create(n);  
    return fibDP(soln,n)
```

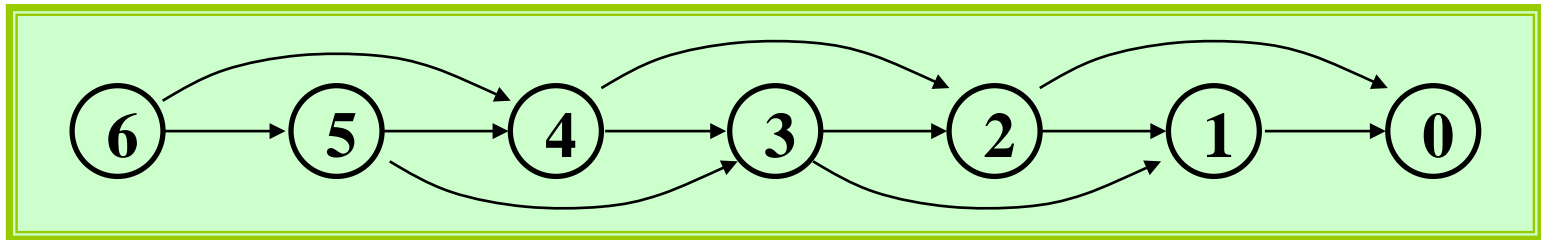
This is the wrapper, which will contain processing existing in original recursive algorithm wrapper.

```
fibDP(soln,k)  
    int fib, f1, f2;  
    if (k<2) fib=k;  
    else  
        if (member(soln, k-1)==false)  
            f1=fibDP(soln, k-1);  
        else  
            f1= retrieve(soln, k-1);  
        if (member(soln, k-2)==false)  
            f2=fibDP(soln, k-2);  
        else  
            f2= retrieve(soln, k-2);  
        fib=f1+f2;  
        store(soln, k, fib);  
    return fib
```

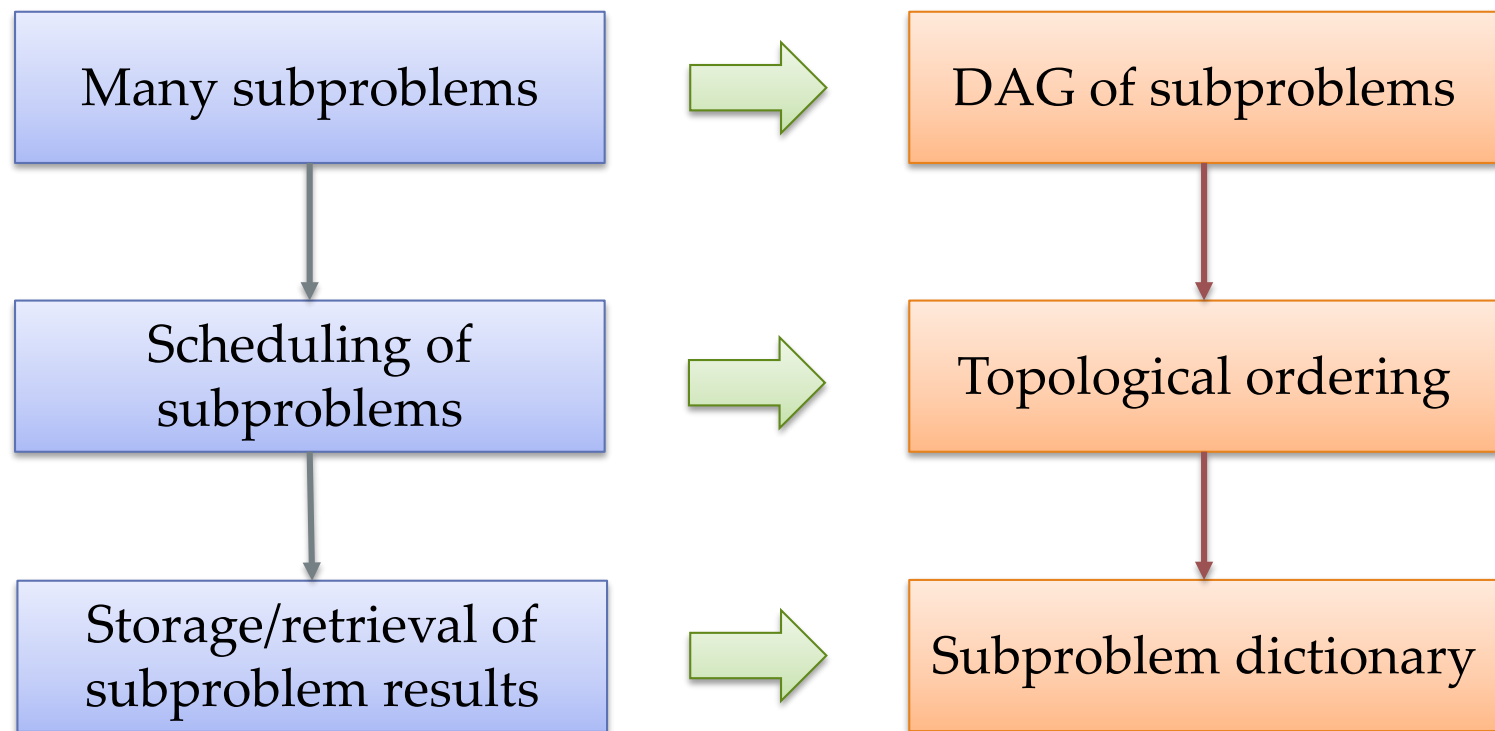


Fibonacci by DP

- **Subproblems**
 - $\text{soln}[0], \text{soln}[1], \dots, \text{soln}[n]$
- **Topology order to the subproblems**
 - $\text{soln}[0] = 0; \text{soln}[1] = 1;$
 - $\text{for}\{i=2; i \leq n; i++\}\{$
 $\text{soln}[i] = \text{soln}[i-1] + \text{soln}[i-2];$
 $\}\ // \text{ end for}$



DP: New Concept Recursion



Matrix Multiplication Order Problem

- The task:

Find the product: $A_1 \times A_2 \times \dots \times A_{n-1} \times A_n$

A_i is 2-dimensional array of different legal sizes

- The issues:

- Matrix multiplication is associative
- Different computing order results in great difference in the number of operations

- The problem:

- Which is the best computing order



Cost for Matrix Multiplication

An example: $A_1 \times A_2 \times A_3 \times A_4$
 $30 \times 1 \quad 1 \times 40 \quad 40 \times 10 \quad 10 \times 25$

Let $C = A_{p \times q} \times A_{q \times r}$

$((A_1 \times A_2) \times A_3) \times A_4$: 20700 multiplications

$A_1 \times (A_2 \times (A_3 \times A_4))$: 11750

$(A_1 \times A_2) \times (A_3 \times A_4)$: 41200

$A_1 \times ((A_2 \times A_3) \times A_4)$: 1400

$$c_{i,j} = \sum_{k=1}^q a_{ik} b_{kj}$$

There are q multiplication

C has $p \times r$ elements as $c_{i,j}$

So, pqr multiplications altogether



Looking for a Greedy Solution

- **Strategy 1: “cheapest multiplication first”**
 - Success: $A_{30 \times 1} \times ((A_{1 \times 40} \times A_{40 \times 10}) \times A_{10 \times 25})$
 - Fail: $(A_{4 \times 1} \times A_{1 \times 100}) \times A_{100 \times 5}$
- **Strategy 2: “largest dimension first”**
 - Correct for the second example above
 - Fail: $A_{1 \times 10} \times A_{10 \times 10} \times A_{10 \times 2}$



Intuitive Solution

- **Matrices:** A_1, A_2, \dots, A_n
- **Dimension:** $\text{dim: } d_0, d_1, d_2, \dots, d_{n-1}, d_n$, for A_i is $d_{i-1} \times d_i$
- **Sub-problem:** $\text{seq: } s_0, s_1, s_2, \dots, s_{\text{len}-1}, s_{\text{len}}$, which means the multiplication of k matrices, with the dimensions: $d_{s_0} \times d_{s_1}, d_{s_1} \times d_{s_2}, \dots$
 - Note: the original problem is: $\text{seq}=(0,1,2,\dots,n)$

Intuitive Solution

```
mmTry1(dim, len, seq)
  if (len<3) bestCost=0
  else
```

```
    bestCost=∞;
```

```
    for (i=1; i≤len-1; i++)
```

```
      c=cost of multiplication at position seq[i];
```

```
      newSeq=seq with ith element deleted;
```

```
      b=mmTry1(Dim, len-1, newSeq);
```

```
      bestCost=min(bestCost, b+c);
```

```
  return bestCost
```

Recursion on index sequence:

(seq): 0, 1, 2, ..., n (len= n)

with the k th matrix is A_k ($k \neq 0$) of the size $d_{k-1} \times d_k$,

and the k th ($k < n$) multiplication is $A_k \times A_{k+1}$.

$$T(n) = (n-1)T(n-1) + n,$$

$$\text{in } \Theta((n-1)!)$$



Subproblem Graph

- **key issue**
 - How can a subproblem be denoted using a **concise identifier**?
 - For mmTry1, the difficulty originates from the **varied intervals** in each newSeq.
- If we look at the **last** (contrast to the first) multiplication, the **two** (not one) resulted subproblems are both contiguous subsequences, which can be uniquely determined by the pair:
<head-index, tail-index>

Improved Recursion

```
mmTry2(dim, low, high)
  if (high-low==1) bestCost=0
  else
    bestCost=∞;
    for (k=low+1; k≤high-1; k++)
      a=mmTry2(dim, low, k);
      b=mmTry2(dim, k, high);
      c=cost of multiplication at position k;
      bestCost=min(bestCost, a+b+c);
  return bestCost
```

Only one matrix

with dimensions:
dim[low], dim[k], and
dim[high]

Still in $\Omega(2^n)$!

Smart Recursion by DP

- DFS can traverse the subproblem graph in time $O(n^3)$
 - At most $n^2/2$ vertices, as $\langle i, j \rangle$, $0 \leq i < j \leq n$.
 - At most $2n$ edges leaving a vertex

```
mmTry2DP(dim, low, high, cost)
```

```
.....
```

```
for (k=low+1; k≤high-1; k++)
```

```
    if (member(low,k)==false) a=mmTry2(dim, low, k);
```

```
        else a=retrieve(cost, low, k);
```

```
    if (member(k,high)==false) b=mmTry2(dim, k, high);
```

```
        else b=retrieve(cost, k, high);
```

```
.....
```

```
store(cost, low, high, bestCost);
```

```
return bestCost
```

Corresponding to the
recursive procedure of DFS

Order of Computation

- Dependency between subproblems

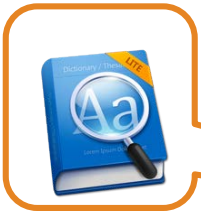
matrixOrder(n , cost, last)

- for ($low=n-1$; $low \geq 1$; $low--$)
- for ($high=low+1$; $high \leq n$; $high++$)

Compute solution of subproblem (low , $high$) and store it in $cost[low][high]$ and $last[low][high]$

- **return $cost[0][n]$**

DP dict



Multiplication Order

- Input: array **dim** = (d_0, d_1, \dots, d_n) , the dimension of the matrices.
- Output: array **multOrder**, of which the i th entry is the index of the i th multiplication in an optimum sequence.

Using the stored results

```
float matrixOrder(int[] dim, int n, int[]
    multOrder)
    <initialization of last, cost, bestcost, bestlast...>
    for (low=n-1; low>=1; low--)
        for (high=low+1; high<=n; high++)
            if (high-low==1) <base case>
                else bestcost=∞;
                for (k=low+1; k<=high-1; k++)
                    a=cost[low][k];
                    b=cost[k][high]
                    c=multCost(dim[low], dim[k],
                        dim[high]);
                    if (a+b+c<bestCost)
                        bestCost=a+b+c; bestLast=k;
                    cost[low][high]=bestCost;
                    last[low][high]=bestLast;
    extrctOrderWrap(n, last, multOrder)
    return cost[0][n]
```

An Example

- Input: $d_0=30$, $d_1=1$, $d_2=40$, $d_3=10$, $d_4=25$

cost as finished

| | | | | |
|---|---|------|-----|-------|
| — | 0 | 1200 | 700 | 1400 |
| — | — | 0 | 400 | 650 |
| — | — | — | 0 | 10000 |
| — | — | — | — | 0 |
| — | — | — | — | — |

Note: $cost[i][j]$ is the least cost of $A_{i+1} \times A_{i+2} \times \dots \times A_j$.

For each selected k , retrieving:

- least cost of $A_{i+1} \times \dots \times A_k$.
 - least cost of $A_{k+1} \times \dots \times A_j$.
- and computing:
- cost of the last multiplication

First entry filled

Last entry filled

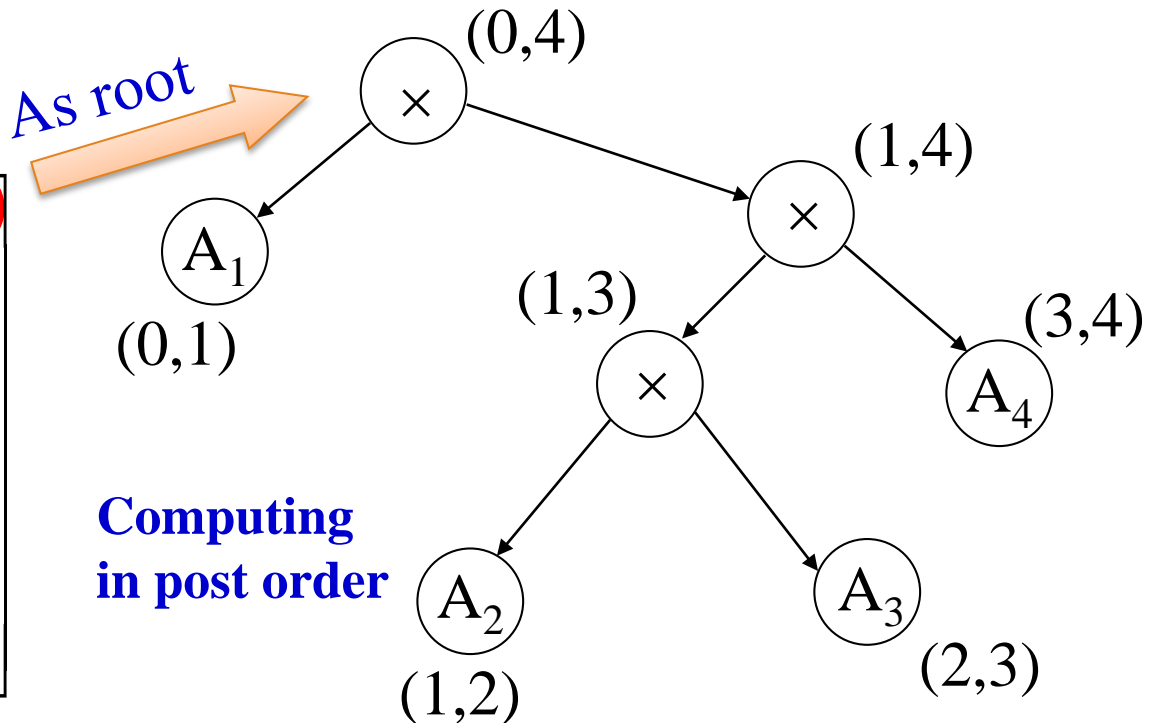


Arithmetic Expression Tree

- Example input: $d_0=30, d_1=1, d_2=40, d_3=10, d_4=25$

last as finished

| | | | | |
|---|----|----|----|----|
| - | -1 | 1 | 1 | 1 |
| - | - | -1 | 2 | 3 |
| - | - | - | -1 | 3 |
| - | - | - | - | -1 |
| - | - | - | - | - |



Getting the Optimal Order

- The core procedure is **extractOrder**, which fills the multiOrder array for subproblem (low,high), using informations in *last* array.

```
extractOrder(low, high, last, multiOrder)
```

```
int k;
```

```
if (high-low>1)
```

```
    k=last[low][high];
```

Just a post-order traversal

```
    extractOrder(low, k, last, multiOrder);
```

```
    extractOrder(k, high, last, multiOrder);
```

```
    multiOrder[multiOrderNext]=k;
```

```
    multiOrderNext++;
```

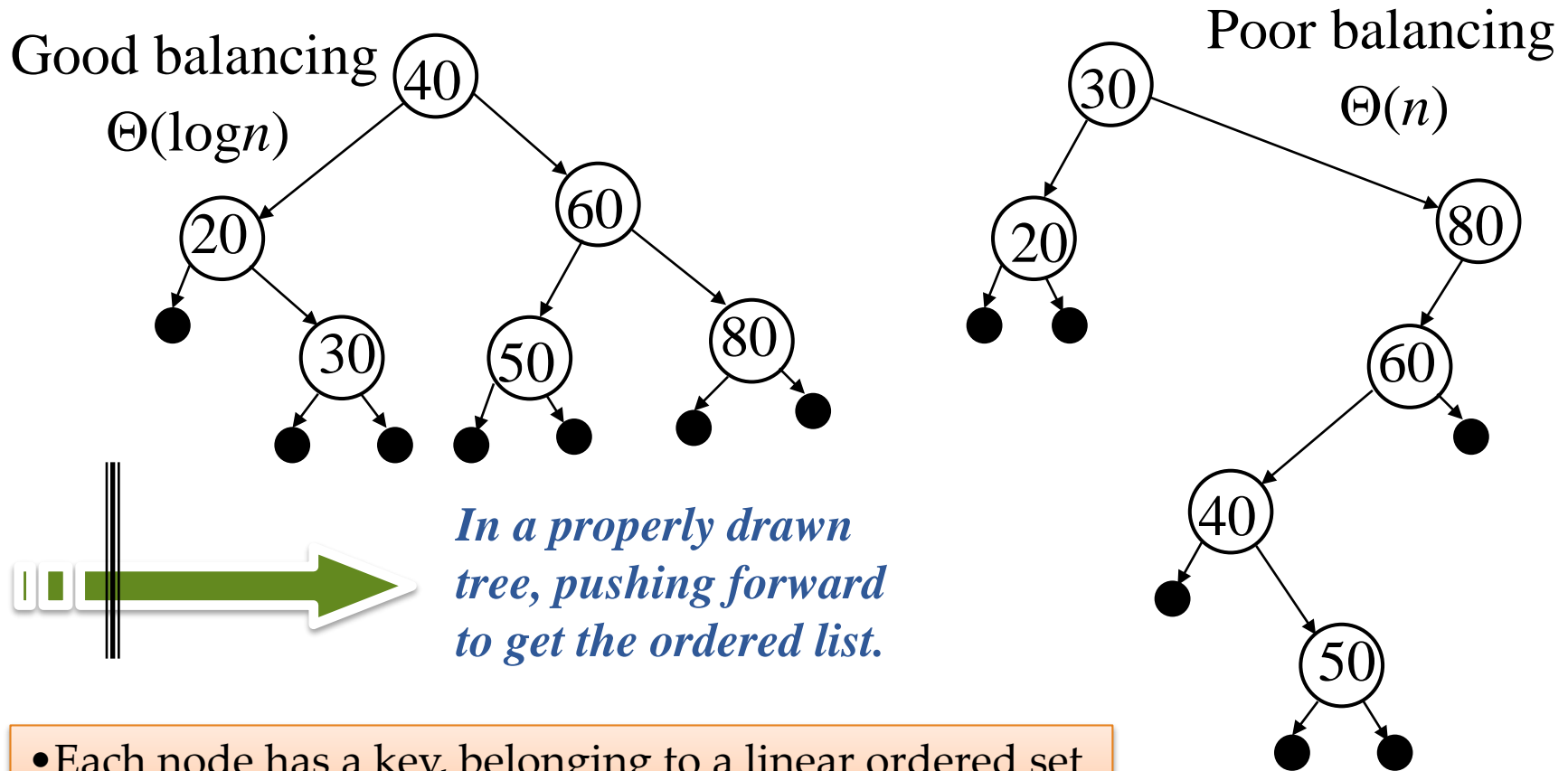
initialized in the wrapper



Analysis of matrixOrder

- **Main body: 3 layer of loops**
 - Time: the innermost processing costs constant, which is executed $\Theta(n^3)$ times.
 - Space: extra space for *cost* and *last*, both in $\Theta(n^2)$
- **Order extracting**
 - There are $2n-1$ nodes in the arithmetic-expression tree. For each node, extractOrder is called once. Since non-recursive cost for extractOrder is constant, so, the complexity of extractOrder is in $\Theta(n)$

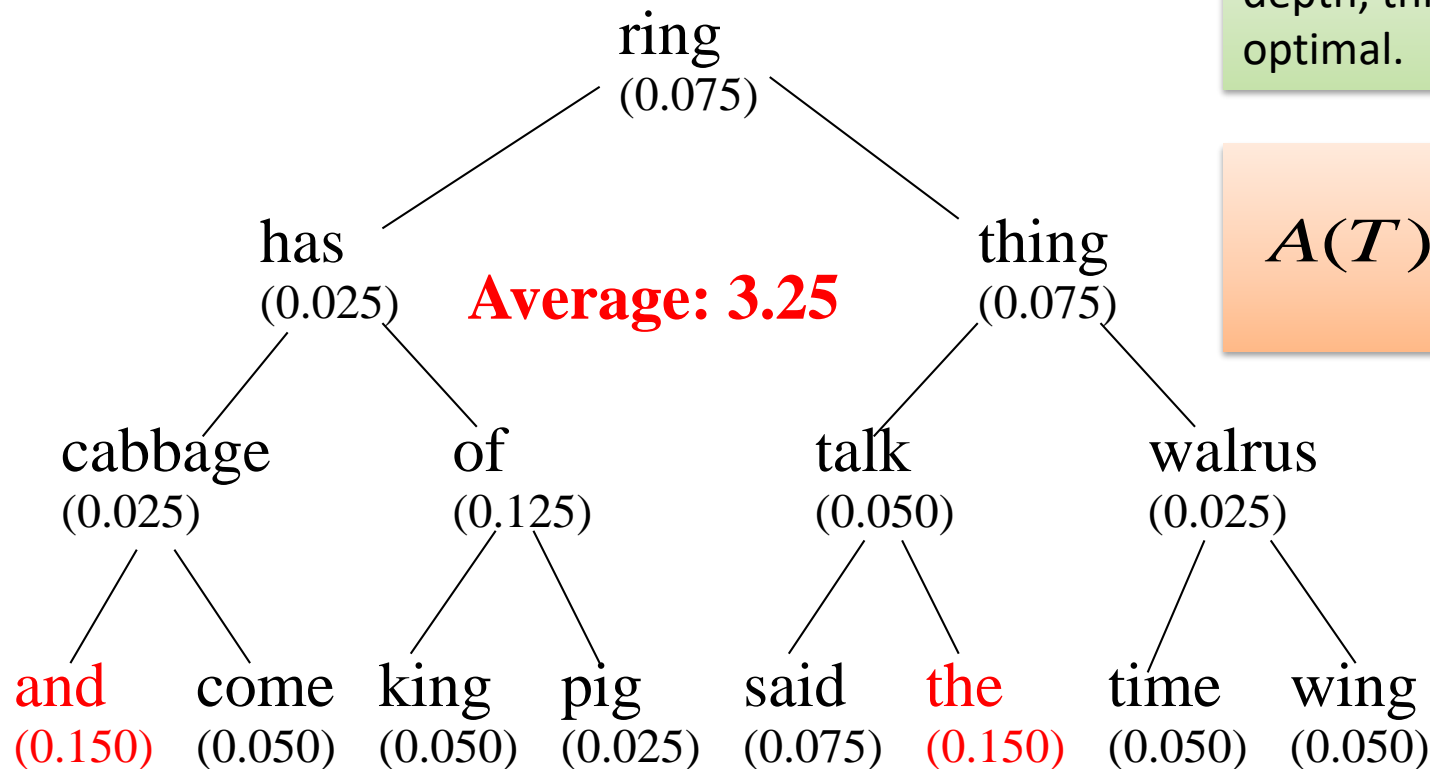
Binary Search Tree



- Each node has a key, belonging to a linear ordered set
- An inorder traversal produces a sorted list of the keys

Keys with Different Frequencies

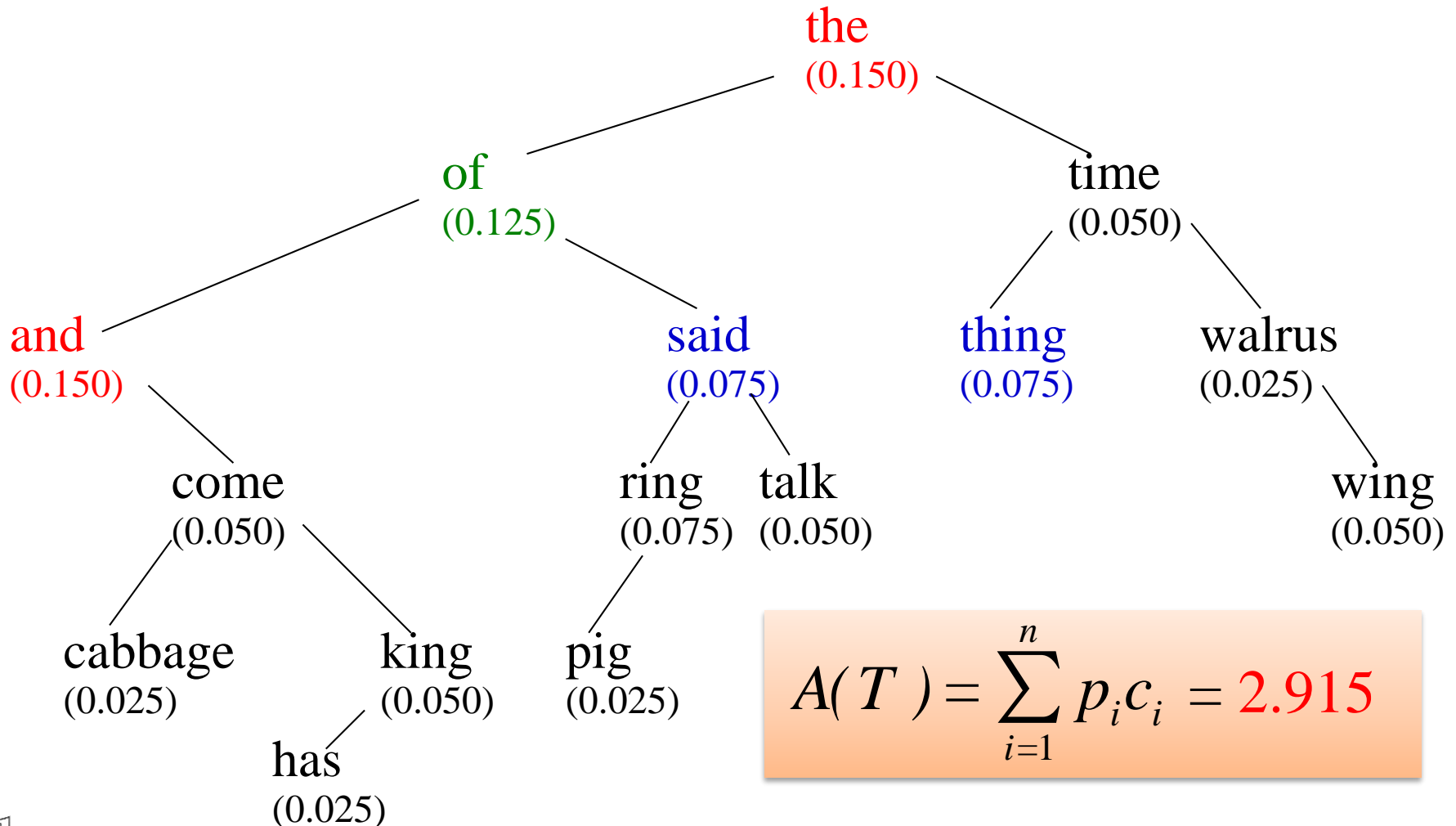
A binary search tree perfectly balanced



Since the keys with larger frequencies have larger depth, this tree is not optimal.

$$A(T) = \sum_{i=1}^n p_i c_i$$

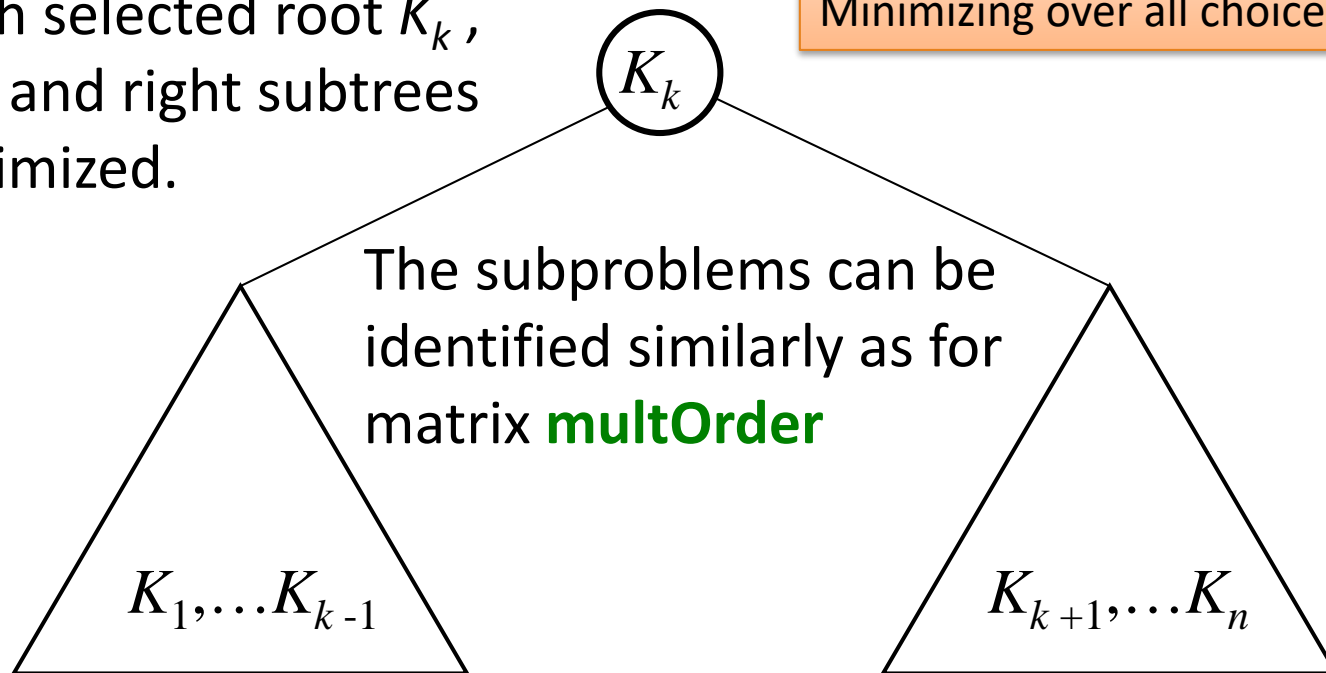
Unbalanced but Improved



Optimal Binary Tree

For each selected root K_k ,
the left and right subtrees
are optimized.

The problem is decomposes by the
choices of the root.
Minimizing over all choices



Subproblems as left and right subtrees

Problem Rephrased

- **Subproblem identification**
 - The keys are in sorted order.
 - Each subproblem can be identified as a pair of index (low, high)
- **Expected solution of the subproblem**
 - For each key K_i , a weight p_i is associated.
Note: p_i is the probability that the key is searched for.
 - The subproblem (low, high) is to find the binary search tree with *minimum weighted retrieval cost*.

Minimum Weighted Retrieval Cost

- $A(\text{low}, \text{high}, r)$ is the minimum weighted retrieval cost for subproblem (low, high) when K_r is chosen as the root of its binary search tree.
- $A(\text{low}, \text{high})$ is the minimum weighted retrieval cost for subproblem (low, high) over all choices of the root key.
- $p(\text{low}, \text{high})$, equal to $p_{\text{low}} + p_{\text{low}+1} + \dots + p_{\text{high}}$, is the weight of the subproblem (low, high).

Note: $p(\text{low}, \text{high})$ is the probability that the key searched for is in this interval.

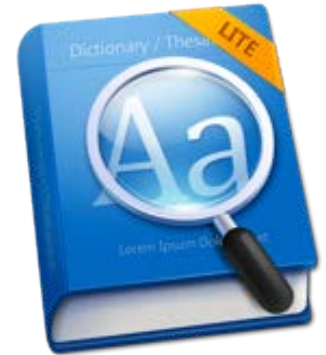
Subproblem Solutions

- **Weighted retrieval cost of a subtree**
 - T contains $K_{\text{low}}, \dots, K_{\text{high}}$, and the weighted retrieval cost of T is W , with T being a whole tree.
 - As a subtree with the root at level 1, the weighted retrieval cost of T will be: **$W+p(\text{low}, \text{high})$**
- **So, the recursive relations are:**
 - $A(\text{low}, \text{high}, r)$
$$= p_r + p(\text{low}, r-1) + A(\text{low}, r-1) + p(r+1, \text{high}) + A(r+1, \text{high})$$
$$= p(\text{low}, \text{high}) + A(\text{low}, r-1) + A(r+1, \text{high})$$
 - $A(\text{low}, \text{high}) = \min\{A(\text{low}, \text{high}, r) \mid \text{low} \leq r \leq \text{high}\}$

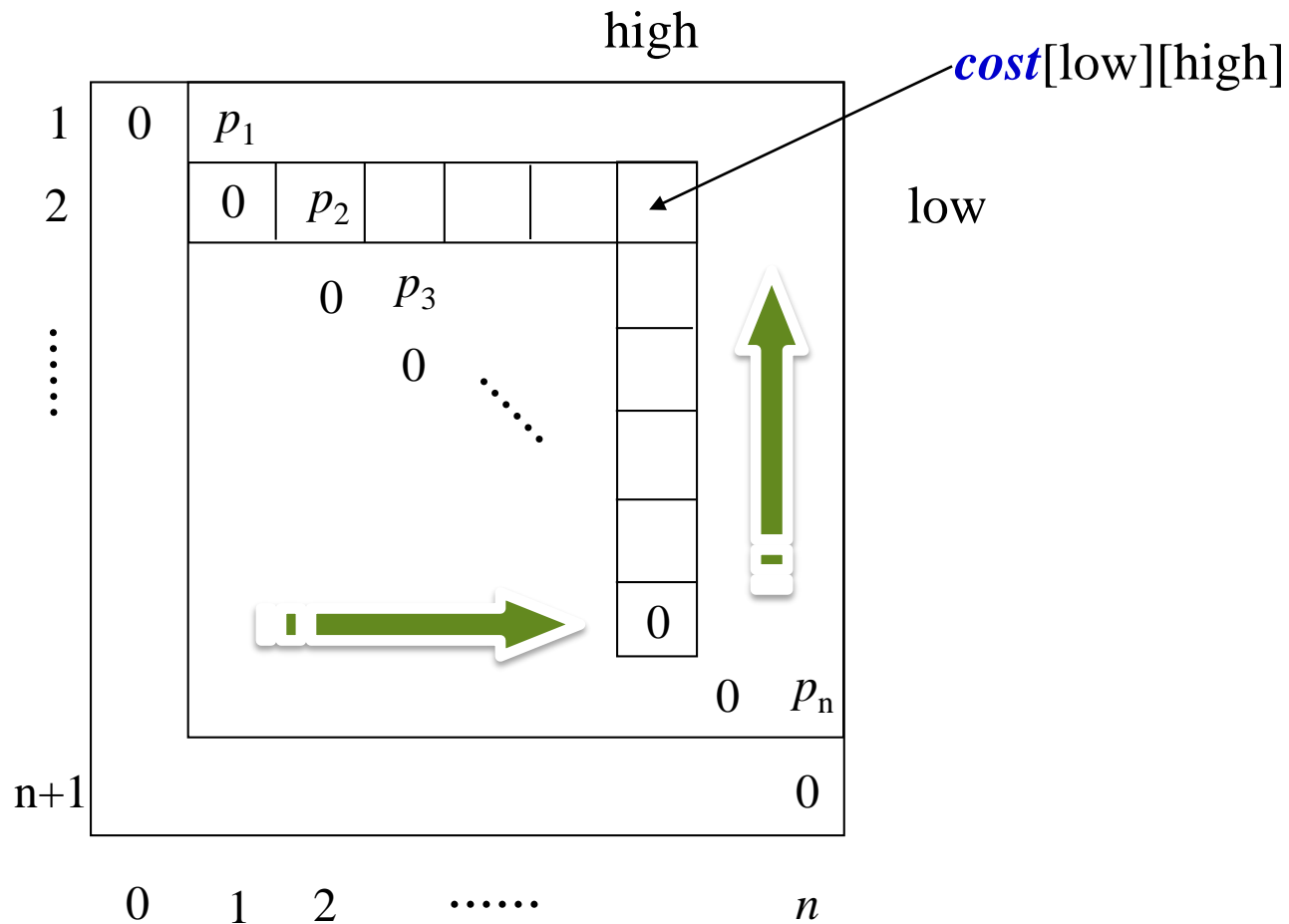
Using DP

- **Array *cost***
 - $Cost[low][high]$ gives the minimum weighted search cost of subproblem $(low, high)$.
 - The $cost[low][high]$ depends upon subproblems with **higher first index** (row number) and **lower second index** (column number)
- **Array *root***
 - $root[low][high]$ gives the best choice of root for subproblem $(low, high)$

DP dictionary



Array cost[]



Optimal BST by DP

```
bestChoice(prob, cost, root, low, high)
```

```
  if (high < low)
```

```
    bestCost = 0;
```

```
    bestRoot = -1;
```

```
  else
```

```
    bestCost = ∞;
```

```
  for (r = low; r ≤ high; r++)
```

```
    rCost = p(low, high) + cost[low][r-1] + cost[r+1][high];
```

```
    if (rCost < bestCost)
```

```
      bestCost = rCost;
```

```
      bestRoot = r;
```

```
    cost[low][high] = bestCost;
```

```
    root[low][high] = bestRoot;
```

```
  return
```

```
  optimalBST(prob, n, cost, root)
```

```
    for (low = n+1; low ≥ 1; low--)
```

```
      for (high = low-1; high ≤ n; high++)
```

```
        bestChoice(prob, cost, root, low, high)
```

```
    return cost
```

in $\Theta(n^3)$

Thank you!

Q & A

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