



Introduction to

Algorithm Design and Analysis

[17] Dynamic Programming 2



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In the Last Class...

- **Basic idea of DP**
- **Least cost matrix multiplication**
 - BF1, BF2
 - A DP solution
- **Weighted binary search tree**
 - The same DP solution

DP - II

- **From the DP perspective**
 - All-pairs shortest paths; SSSP over DAG
- **More DP problems**
 - Edit distance
 - Highway restaurants; Separating sequence of words
 - Changing coins
- **Elements of DP**



All-pairs Shortest Paths

- **BF2**
 - Path length k
 - k in $[1, n]$
- **Floyd algorithm**
 - Index range k
 - k in $[1, n]$



BF2

$$\text{dist}(u, v, k) = \begin{cases} 0 & \text{if } u = v \\ \infty & \text{if } k = 0 \text{ and } u \neq v \\ \min_x (\text{dist}(u, x, k-1) + w(x \rightarrow v)) & \text{otherwise} \end{cases}$$

Length of the shortest path of at most k edges

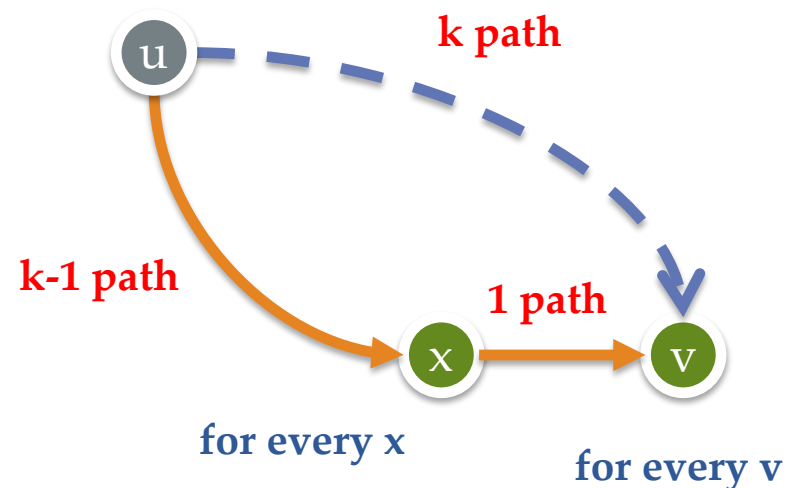
APSP(V, E, w):

```

for all vertices u
  for all vertices v
    if u = v
      dist[u, v, 0] ← 0
    else
      dist[u, v, 0] ← ∞
  for k ← 1 to V - 1
    for all vertices u
      for all vertices v
        dist[u, v, k] ← ∞
      for all vertices x
        if dist[u, v, k] > dist[u, x, k-1] + w(x → v)
          dist[u, v, k] ← dist[u, x, k-1] + w(x → v)
    
```

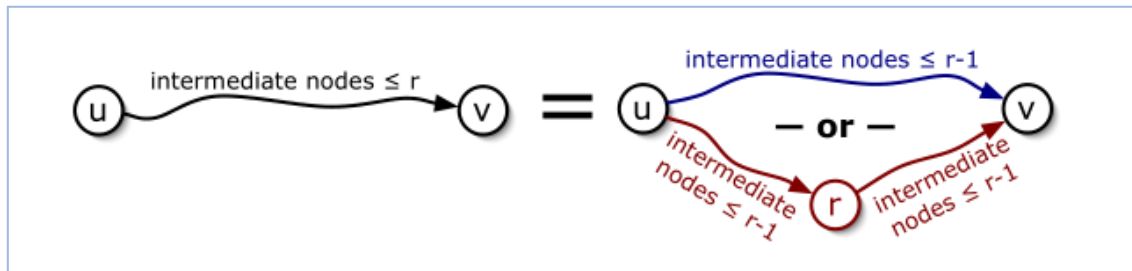
O(n⁴)

for every u



Floyd Algorithm

- Basic idea



- Smart recursion

$$\text{dist}(u, v, r) = \begin{cases} w(u \rightarrow v) & \text{if } r = 0 \\ \min \{ \text{dist}(u, v, r-1), \text{dist}(u, r, r-1) + \text{dist}(r, v, r-1) \} & \text{otherwise} \end{cases}$$

Floyd Algorithm

- Basic DP (3-dimensional)

```
FLOYDWARSHALL( $V, E, w$ ):  
  for all vertices  $u$   
    for all vertices  $v$   
       $dist[u, v, 0] \leftarrow w(u \rightarrow v)$   
  
  for  $r \leftarrow 1$  to  $V$   
    for all vertices  $u$   
      for all vertices  $v$   
        if  $dist[u, v, r - 1] < dist[u, r, r - 1] + dist[r, v, r - 1]$   
           $dist[u, v, r] \leftarrow dist[u, v, r - 1]$   
        else  
           $dist[u, v, r] \leftarrow dist[u, r, r - 1] + dist[r, v, r - 1]$ 
```

$O(n^3)$

- Improved DP (2-dimensional)

```
FLOYDWARSHALL2( $V, E, w$ ):  
  for all vertices  $u$   
    for all vertices  $v$   
       $dist[u, v] \leftarrow w(u \rightarrow v)$   
  
  for all vertices  $r$   
    for all vertices  $u$   
      for all vertices  $v$   
        if  $dist[u, v] > dist[u, r] + dist[r, v]$   
           $dist[u, v] \leftarrow dist[u, r] + dist[r, v]$ 
```

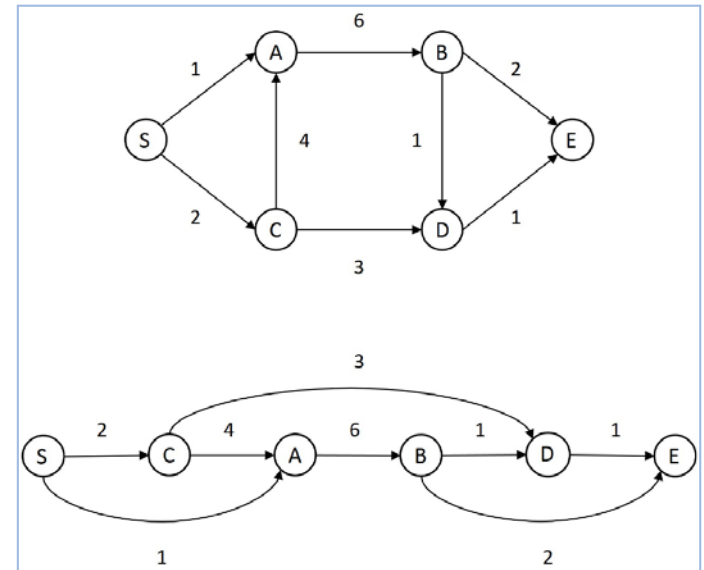
$O(n^3)$



SSSP over a DAG

- **Subproblems**
 - One problem for each node
 - $dis[1..n]$
- **Dynamic programming**
 - Topological ordering of nodes in a DAG
- **More than SSSP**
 - As long as the recursion succeeds

$$D.dis = \min \{ B.dis + 1, C.dis + 3 \}$$



Edit Distance

- You can edit a word by
 - Insert, Delete, Replace
- Edit distance
 - Minimum number of edit operations
- Problem
 - Given two strings, compute the edit distance

F	O	O		D
M	O	N	E	Y

The edit distance is 4

4 op: **R** **R** **I** **R**

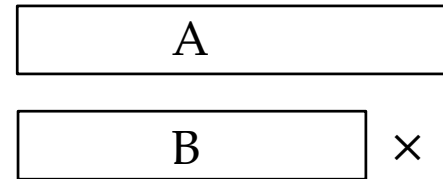
3 op: not possible

“BF” Recursion

- **Case 1**

- 1.1 Insert
- 1.2: dual of case 1.1

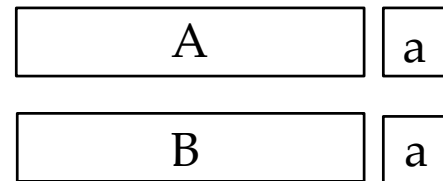
Case 1.1



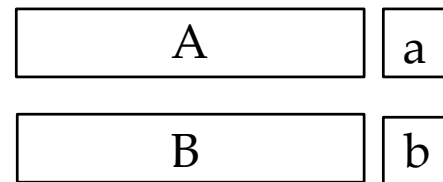
- **Case 2**

- 2.1 $a=a$
- 2.2 $a \neq b$

Case 2.1



Case 2.2



“BF” Recursion

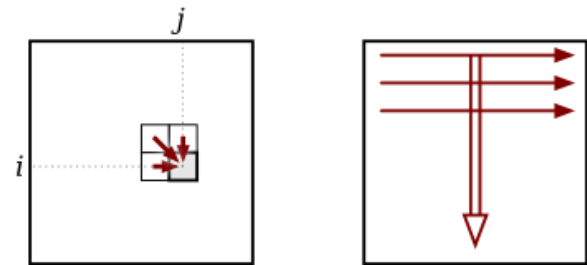
- **EditDis(i,j)**
 - Base case:
 - If $i=0$, $\text{EditDis}(i,j)=j$
 - If $j=0$, $\text{EditDis}(i,j)=i$
 - Recursion:

$$\text{EditDis}(A[1..m], B[1..n]) = \min \begin{cases} \text{EditDis}(A[1..m-1], B[1..n]) + 1 \\ \text{EditDis}(A[1..m], B[1..n-1]) + 1 \\ \text{EditDis}(A[1..m-1], B[1..n-1]) + I\{A[m] \neq B[n]\} \end{cases}$$

Smart Programming

- DP dict
 - EditDis[1..m, 1..n]
- DP algorithm

dependencies



EDITDISTANCE($A[1..m], B[1..n]$):

for $j \leftarrow 1$ to n
 $Edit[0, j] \leftarrow j$

for $i \leftarrow 1$ to m
 $Edit[i, 0] \leftarrow i$

for $j \leftarrow 1$ to n

 if $A[i] = B[j]$

$Edit[i, j] \leftarrow \min \{Edit[i-1, j] + 1, Edit[i, j-1] + 1, Edit[i-1, j-1]\}$

 else

$Edit[i, j] \leftarrow \min \{Edit[i-1, j] + 1, Edit[i, j-1] + 1, Edit[i-1, j-1] + 1\}$

return $Edit[m, n]$

Example

algorithm

vs.

altruistic

		A	L	G	O	R	I	T	H	M
	0	1	2	3	4	5	6	7	8	9
A	1	0	1	2	3	4	5	6	7	8
L	2	1	0	1	2	3	4	5	6	7
T	3	2	1	1	2	3	4	4	5	6
R	4	3	2	2	2	2	3	4	5	6
U	5	4	3	3	3	3	3	4	5	6
I	6	5	4	4	4	4	3	4	5	6
S	7	6	5	5	5	5	4	4	5	6
T	8	7	6	6	6	6	5	4	5	6
I	9	8	7	7	7	7	6	5	5	6
C	10	9	8	8	8	8	7	6	6	6

DP in One Dimension

- **Highway restaurants**
 - n possible locations on a straight line
 - $m_1, m_2, m_3, \dots, m_n$
 - At most one restaurant at one location
 - Expected profit for location i is p_i
 - Any two restaurants should be at least k miles apart
- **How to arrange the restaurants**
 - To obtain the maximum expected profit



Highway Restaurants

- **The recursion**

- $P(j)$: the max profit achievable using only first j locations
 - $P(0)=0$
- $\text{prev}[j]$: largest index before j and k miles away

$$P(j) = \max(p_j + P(\text{prev}[j]), P(j - 1))$$



Highway Restaurants

- One dimension DP algorithm
 - Fill in $P[0], P[1], \dots, P[n]$

(First compute the $\text{prev}[\cdot]$ array)

$i = 0$

for $j = 1$ to n :

 while $m_{i+1} \leq m_j - k$:

$i = i + 1$

$\text{prev}[j] = i$

(Now the dynamic programming begins)

$P[0] = 0$

for $j = 1$ to n :

$P[j] = \max(p_j + P[\text{prev}[j]], P[j - 1])$

return $P[n]$



Words into Lines

- **Words into lines**
 - Word-length w_1, w_2, \dots, w_n and line-width: W
- **Basic constraint**
 - If w_i, w_{i+1}, \dots, w_j are in one line, then $w_i + w_{i+1} + \dots + w_j \leq W$
- **Penalty for one line: some function of X . X is:**
 - 0 for the last line in a paragraph, and
 - $W - (w_i + w_{i+1} + \dots + w_j)$ for other lines
- **The problem**
 - How to make the penalty of the paragraph, which is the sum of the penalties of individual lines, minimized

Greedy Solution

i	word	w
1	Those	6
2	who	4
3	cannot	7
4	remember	9
5	the	4
6	past	5
7	are	4
8	condemned	10
9	to	3
10	repeat	7
11	it.	4

Solution by greedy strategy

words	(1,2,3)	(4,5)	(6,7)	(8,9)	(10,11)
X	0	4	8	4	0
penalty	0	64	512	64	0

Total penalty is **640**

An improved solution

words	(1,2)	(3,4)	(5,6,7)	(8,9)	(10,11)
X	7	1	4	4	0
penalty	343	1	64	64	0

Total penalty is **472**

W is 17, and penalty is X^3



Problem Decomposition

- **Representation of subproblem:** a pair of indexes (i,j) , breaking words i through j into lines with minimum penalty.
- **Two kinds of subproblem**
 - (k, n) : the penalty of the last line is 0
 - all other subproblems
- **For some k , the combination of the optimal solution for $(1,k)$ and $(k+1,n)$ gives a optimal solution for $(1,n)$.**
- **Subproblem graph**
 - About n^2 vertices
 - Each vertex (i,j) has an edge to about $j-i$ other vertices, so, the number of edges is in $\Theta(n^3)$



Simpler Identification of Subproblems

- If a subproblem concludes the paragraph, then (k,n) can be simplified as (k)
 - About k subproblems
- Can we eliminate the use of (i,j) with $j < n$?
 - Put the first k words in the first line (with the basic constraint satisfied), the subproblem to be solved is $(k+1,n)$
 - Optimizing the solution over all k 's. (k is at most $W/2$)



One-dimension Recursion

One-dimension problem space

- $(1,n), (2,n), \dots, (n,n)$

Subproblem (i,n)

Algorithm: $\text{lineBreak}(w, W, i, n, L)$

the current line

if $w_i + w_{i+1} + \dots + w_n \leq W$ **then**

 <Put all words on line L , set penalty to 0> ;

else

for $k = 1; w_i + \dots + w_{i+k-1} \leq W; k++$ **do**

$X = W - (w_i + \dots + w_{i+k-1})$;

$kPenalty = \text{lineCost}(X) + \text{lineBreak}(w, W, i+k, n, L+1)$;

 <Set penalty always to the minimum $kPenalty$ > ;

 <Updating k_{min} , which records the k part that produced the minimum penalty> ;

 <Put words i through $i + k_{min} - 1$ on line L > ;

return $penalty$;

Dynamic Programming

Topological ordering of subproblems

- $\text{Penalty}[n] \rightarrow \text{Penalty}[n-1] \rightarrow \dots \rightarrow \text{Penalty}[1]$

Algorithm: lineBreakDP

```
for  $i = n; i \geq 1; i--$  do
    if all words through  $w_i$  to  $w_n$  can be put in one line then
         $\text{Penalty}[i] = 0$  ;
        <put all words through  $i$  to  $n$  in one line> ;
    else
        for  $k = 1; w_i + \dots + w_{i+k-1} \leq W; k++$  do
            calculate the penalty  $\text{Cost}_{cur}$  of putting  $k$  words in this line ;
             $\text{minCost} = \min\{\text{minCost}, \text{Cost}_{cur} + \text{Penalty}[i+k]\}$  ;
            <Updating  $k_{min}$ , which records the  $k$  part that produced the minimum
            penalty> ;
            <Put words  $i$  through  $i + k_{min} - 1$  on one line> ;
         $\text{Penalty}[i] = \text{minCost}$  ;
```

Analysis of lineBreakDP

- Each subproblem is identified by only one integer k , for (k, n)
 - Number of vertex in the subproblem graph: at most n
 - So, in **DP** version, the recursion is executed at most n times.
- So, the running time is in $\Theta(Wn)$
 - The loop is executed at most $W/2$ times.
 - In fact, W , the line width, is usually a constant. So, **$\Theta(n)$** .
 - The extra space for the dictionary is in **$\Theta(n)$** .

Making Change: Revisited

- **How to pay a given amount of money?**
 - Using the smallest possible number of coins
 - With certain systems of coinage
- **We have known that the greedy strategy fails sometimes**



Subproblems

- **Assumptions**

- Given n different denominations
- A coin of denomination i has d_i units
- The amount to be paid: N .

- **Subproblem $[i, j]$**

- The minimum number of coins required to pay an amount of j units, using only coins of denominations 1 to i .

- **The problem**

- Figure out subproblem $[n, N]$ (as $c[n, N]$)



Dependency of Subproblems

- $c[i,0]$ is 0 for all i
- When we are to pay an amount j using coins of denominations 1 to i , we have two choices:
 - No coins of denomination i is used: $c[i-1, j]$
 - One coins of denomination i is used: $1+c[i, j-d_i]$
- So, $c[i,j] = \min (c[i-1, j], 1+c[i, j-d_i])$

Data Structure

Define a array $\text{coin}[1..n, 0..N]$ for all $c[i, j]$

an example

	0	1	2	3	4	5	6	7	8
$d_1=1$	0	1	2	3	4	5	6	7	8
$d_2=4$	0	1	2	3	1	2	3	4	2
$d_3=6$	0	1	2	3	1	2	1	2	2

direction of computation

The Procedure

```
int coinChange(int N, int n, int[][] coin)
```

```
    int denomination[]=[ $d_1, d_2, \dots, d_n$ ];
```

```
    for ( $i=1; i \leq n; i++$ )
```

```
        coin[ $i,0$ ]=0;
```

```
        for ( $i=1; i \leq n; i++$ )
```

```
            for ( $j=1; j \leq N; j++$ )
```

```
                if ( $i=1 \ \&\& \ j < \text{denomination}[i]$ ) coin[ $i,j$ ]= $+\infty$  ;
```

```
                else if ( $i=1$ ) coin[ $i,j$ ]= $1 + \text{coin}[1, j - \text{denomination}[1]]$ ;
```

```
                else if ( $j < \text{denomination}[i]$ ) coin[ $i,j$ ]=cost[ $i-1, j$ ];
```

```
                else coin[ $i,j$ ]=min(coin[ $i-1, j$ ],  $1 + \text{coin}[i, j - \text{denomination}[i]]$ ;
```

```
    return coin[ $n,N$ ];
```

in $\Theta(n^M)$,
 n is usually a constant

Other DP Problems

- **Text string problems**
 - Longest common subsequence, ...
 - Variations of standard text string problems, ...
- **One dimensional problems**
 - Arrangements along a straight line, ...
- **Graph problems**
 - Vertex cover, ...
- **Hard problems**
 - Knapsack problems and variations, ...

Principle of Optimality

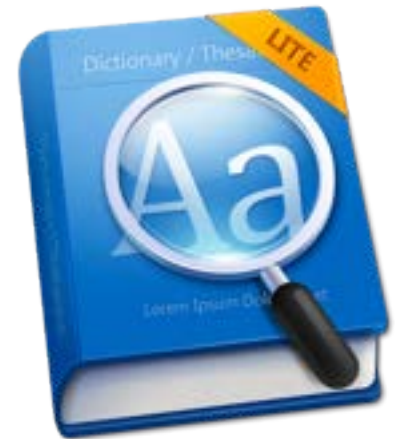
- Given an optimal sequence of decisions, each *subsequence must be optimal by itself*
 - Positive example: shortest path
 - Counterexample: (simple) path
- DP relies on the principle of optimality
 - The optimal solution to any nontrivial instance of a problem is a combination of optimal solutions to some of its sub-instances.
 - It is often not obvious which sub-instances are relevant to the instance under consideration.

Optimal
Substructure

Elements of Dynamic Programming

- Symptoms of DP
 - Overlapping subproblems
 - Optimal substructure
- How to use DP
 - **“Brute force”** recursion
 - Overlapping subproblems
 - **“Smart”** programming
 - Topological ordering of subproblems

DP Dictionary



Thank you!

Q & A

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