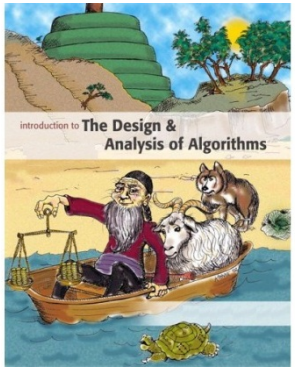
计算机科学与技术系
Department of Computer Science and Technology

南京大学
NANJING UNIVERSITY

Introduction to

Algorithm Design and Analysis

[19] NP Complete Problems



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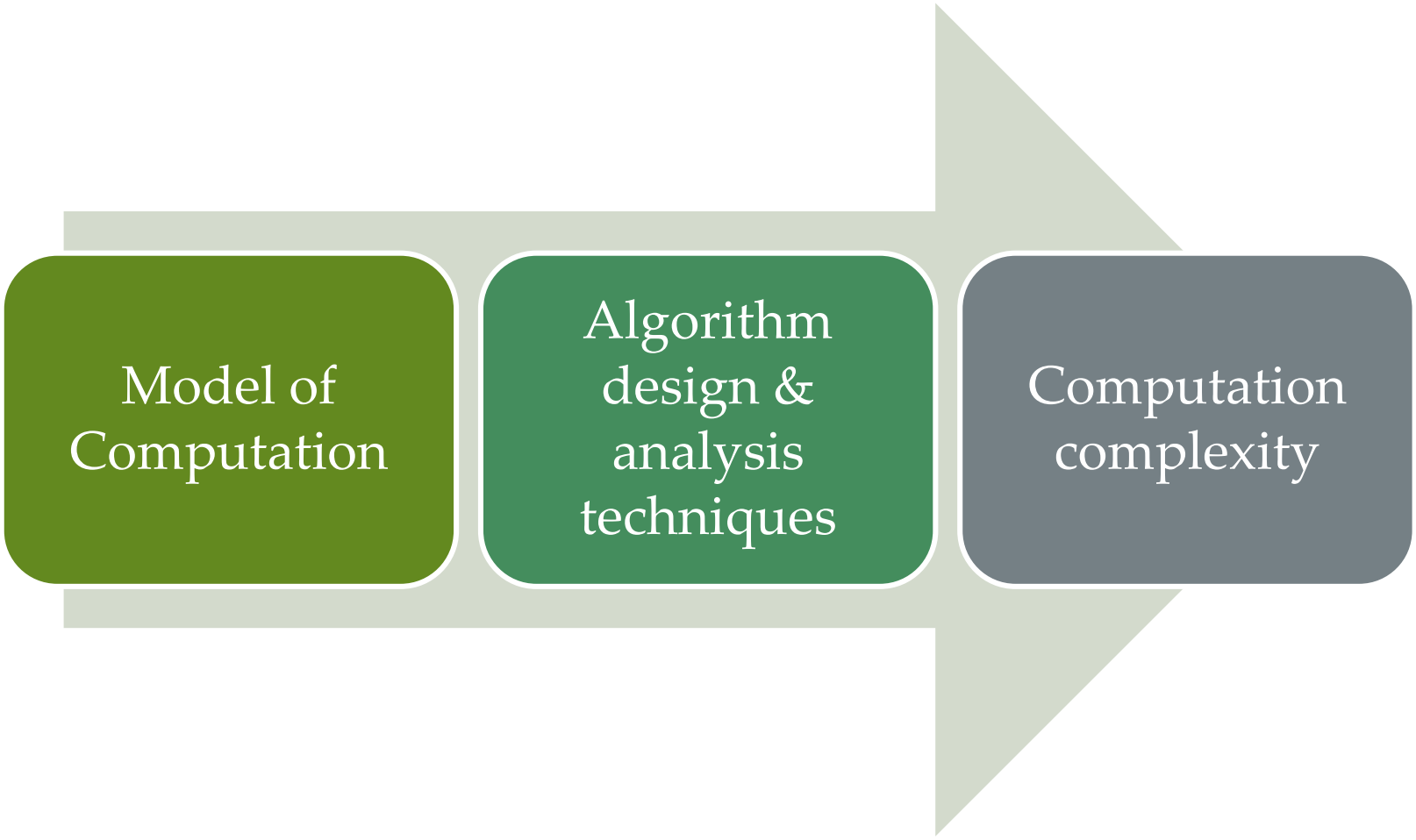
Syllabus


Model of Computation
Algorithm design & analysis techniques
Computation complexity

Hard Games & Puzzles

Complexity of games & puzzles

Complexity of algorithmic problems

NP-Complete Problems

- **P**
  - Decision problem
- **NP**
  - Nondeterministic algorithm
- **NPC**
  - Reduction between problems

How Functions Grow

Algorithm	1	2	3	4	
Time function(ms)	$33n$	$46n \lg n$	$13n^2$	$3.4n^3$	2^n
Input size(n)	Solution time				
10	0.00033 sec.	0.0015 sec.	0.0013 sec.	0.0034 sec.	0.001 sec.
100	0.0033 sec.	0.03 sec.	0.13 sec.	3.4 sec.	4×10^{16} yr.
1,000	0.033 sec.	0.45 sec.	13 sec.	0.94 hr.	
10,000	0.33 sec.	6.1 sec.	22 min.	39 days	
100,000	3.3 sec.	1.3 min.	1.5 days	108 yr.	
Time allowed	Maximum solvable input size (approx.)				
1 second	30,000	2,000	280	67	20
1 minute	1,800,000	82,000	2,200	260	26

Hanoi Tower Revisited

- **It is easy to provide a recursive algorithm to resolve the problem of Hanoi Tower**
  - The solution requires 2^N-1 moves of disc.
- **It is extremely difficult to achieve the result for an input of moderate size**
  - For the input of 64, it takes half a million years even if the Tibetan priest has superhuman strength to move a million discs in a second.

Max Clique - an Example

- **A complete subgraph of a graph G is called a clique, whose size is the number of vertices in it.**
  - **Optimization problem**: Find the maximal clique in a given graph G.
  - **Decision problem**: Has G a clique of size at least *k* for some given *k*?

Decision Problem

- **Statement of a decision problem**
  - Part 1: instance description defining the input
  - Part 2: question stating the actual yes-or-no question
- **A decision problem is a mapping from all possible inputs into the set {yes, no}**

Optimization vs. Decision

- **Usually, an optimization problem can be rephrased as a decision problem.**
- **For some cases, it can be proved that the decision problem can be solved in polynomial time if and only if the corresponding optimization problem can.**
  - We can make the statement that if the decision problem cannot be solved in polynomial time then the corresponding optimization problem cannot either.

Max Clique Revisited

- **The max clique problem can be solved in polynomial time iff. the corresponding decision problem can be solved in polynomial time.**
  - If the size of a max clique can be found in time g(n), the corresponding decision may be settled in that time of course.
  - If deciClique is algorithm for the decision problem with k in the complexity of $f(n)$, then we apply the algorithm at most n time, for k=n, n-1, ..., 2, 1, and we can solve the optimization problem, and with the complexity no worse than $nf(n)$, which is polynomial only if $f(n)$ is polynomial.

Some Typical Decision Problems

- **Graph coloring**
  - Given a undirected graph *G* and a positive integer *k*, is there a coloring of *G* using at most *k* colors?
- **Job scheduling with penalties**
  - Given a group of jobs, each with its execution duration, deadline and penalty for missing the deadline, and a nonnegative integer *k*, is there a schedule with the total penalty bounded by *k*?

Some Typical Decision Problems

- **Bin packing**
  - Given k bins each of capacity one, and n objects with size $s_1, \ldots, s_n$, (where s_i is a rational number in (0,1]). Do the n objects fit in k bins?
- **Knapsack**
  - Given a knapsack of capacity C, n objects with sizes $s_1, \ldots, s_n$ and "profits" $p_1, \ldots, p_n$, and a positive integer k. Is there a subset of the n objects that fits in the knapsack and has total profit at least k?

  (Subset sum as a simplified version)

Some Typical Decision Problems

- ***CNF*-Satisfiability**
  - Given a CNF formula, is there a truth assignment that satisfies it?
- **Hamiltonian cycles or Hamiltonian paths**
  - Given a undirected graph *G*. Does *G* have a Hamiltonian cycle or Hamiltonian path?
- **Traveling salesperson**
  - Given a complete, weighted graph and an integer *k*, is there a Hamiltonian cycle with total weight at most *k*?

Theory of *NP*-Completeness

- **What it cannot do**
  - Provide a method of obtaining polynomial time algorithms for those "hard" problems
  - Negate the existence of algorithms of polynomial complexity for those problems
- **What it can do**
  - Show that many of the problems for which there is no known polynomial time algorithm are computationally related

The Class *P*

- A *polynomially bounded algorithm* is one with its worst-case complexity bounded by a polynomial function of the input size.
- A *polynomially bounded problem* is one for which there is a polynomially bounded algorithm.
- The *class P* is the class of decision problems that are polynomially bounded.

Notes on the Class *P*

- **Class P has a too broad coverage**
  - In the sense that not every problems in P has an acceptable efficient algorithm
- **However**
  - The problem not in P must be extremely expensive and probably impossible to solve in practice.
  - The problems in P have nice "closure" properties for algorithm integration.
  - The property of being in P is independent of the particular formal model of computation used.

Nondeterministic Algorithm

```
void nondetA(String input)
   String s=genCertif();
   Boolean CheckOK=verifyA(input,s);
   if (checkOK)
      Output "yes";
   return;
```

Phase 1 Guessing: generating **arbitrarily** "certificate", i.e. proposed solution

The algorithm may behave differently on the same input in different executions: "yes" or "no output".

Phase 2 Verifying: determining if *s* is a valid description of a object for answer, and satisfying the criteria for solution

Nondeterministic Algorithm

- **For a particular decision problem with input *x*:**
  - The answer computed by a nondeterministic algorithm is defined to be *yes* if and only if there is some execution of the algorithm that gives a *yes* output.
  - The answer is *no*, if for **all** *s*, there is no output.

Nondeterministic Algorithm

In $O(1)$

Note: $\Omega(n)$ for deterministic algorithm

```
void nondetSearch(int k; int[ ] S)
    int i =genCertif();
    if (S[i]=k)
        Output "yes";
    return;
```

In $O(n)$

Note: $\Omega(n\log n)$ for deterministic algorithm

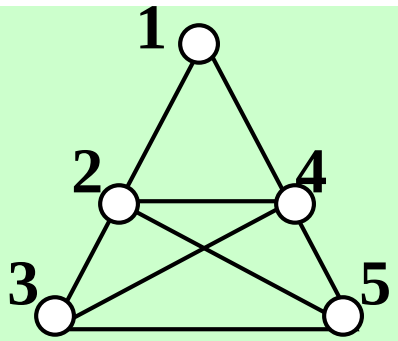
```
void nondetSort(int[ ] S; int n)
    int i, j; int[ ] out=0;
    for i =1 to n do
        j= genCertif();
        if out[j ]≠0 then return;
        out[j]=S[i ];
    for i =1 to n-1 do
        if out[i ]>out[i+1] then return;
    S=out;
    Output(yes);
return
```

Nondeterministic Graph Coloring

Problem instance G

Input string:

4,5,(1,2)(1,4)(2,4)(2,3)(3,5)(2,5)(3,4)(4,5)



s	Output	Reason
RGRBG	*false*	v_2 and v_5 conflict
RGRB	*false*	Not all vertices are colored
RBYGO	*false*	Too many colors used
RGRBY	*true*	**A valid 4-coloring**
R%*,G@	*false*	Bad syntax

verified by phase 2

(G,4)→yes

generated by phase 1

The Class *NP*

- **A polynomial bounded nondeterministic algorithm is one for which there is a (fixed) polynomial function p such that for each input of size n for which the answer is *yes* , there is some execution of the algorithm that produces a *yes* output in at most $p(n)$ steps.**
- **The class *NP* is the class of decision problems for which there is a polynomial bounded nondeterministic algorithm.**

The Class *NP*

- **NP means *Non-deterministic P***
  - From "deterministic" to "non-deterministic"
  - From "solve a problem" to "verify the answer of a problem"
- **What does NP indicate?**
  - Harder problems
  - Not too hard
    - At least, you can quickly understand the answer

Proof of Being in *NP*

- **Graph coloring is in *NP***
  - Description of the input and the certificate
  - Properties to be checked for an answer "*yes*"
    - There are n colors listed: $c_1, c_2, \ldots, c_n$ (not necessarily different)
    - Each c_i is in the range $1,\ldots,k$
    - Scan the list of edges to see if a conflict exists
  - Proving that each of the above statement can be checked in polynomial time.

CLIQUE is in NP

```
void nondeteClique(graph G; int n, k)
    set S=ϕ;
    for int i=1 to k do
        int t=genCertif();
        if t∈S then return;
        S=S∪{t };
    for all pairs (i,j) with i,j in S and i≠j do
        if (i,j ) is not an edge of G
            then return;
        Output("yes");
```

In $O(n)$

In $O(k^2)$

So, we have an algorithm for the maximal clique problem with the complexity of $O(n+k^2)=O(n^2)$

SAT

An example of propositional conjunctive normal form (CNF) is like this:

$$(p \vee q \vee s) \wedge (\bar{q} \vee r) \wedge (\bar{p} \vee r) \wedge (\bar{r} \vee s) \wedge (\bar{p} \vee \bar{s} \vee \bar{q})$$

Satisfiability Problem

Given a CNF formula, is there a truth assignment that satisfies it?

In other words, is there a assignment for the set of propositional variable in the CNF, such that the value of the formula is **true**.

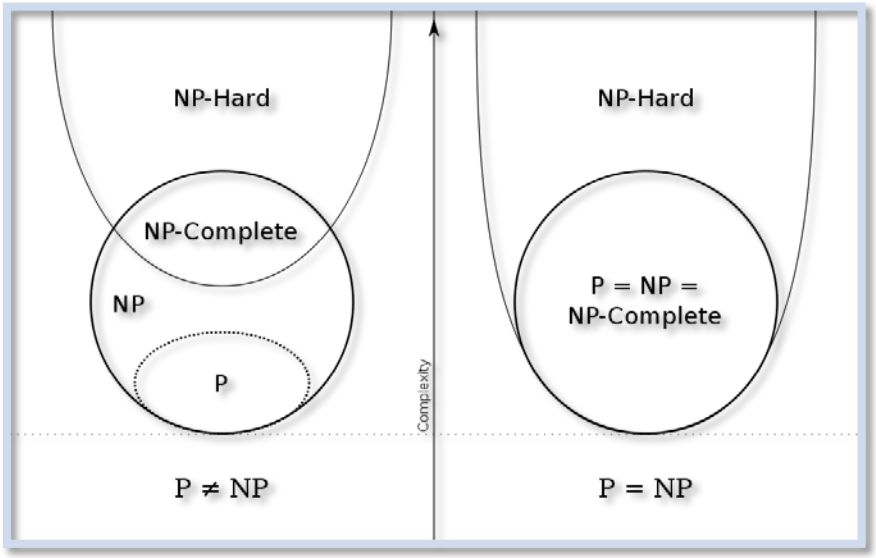
```
void nondetSat(E, n)
    boolean p[ ];
    for int i =1 to n do
        p[i ]= genCertif(true, false);
    if E(p[1], p[2], ..., p[n])=true
        then Output("yes");
```

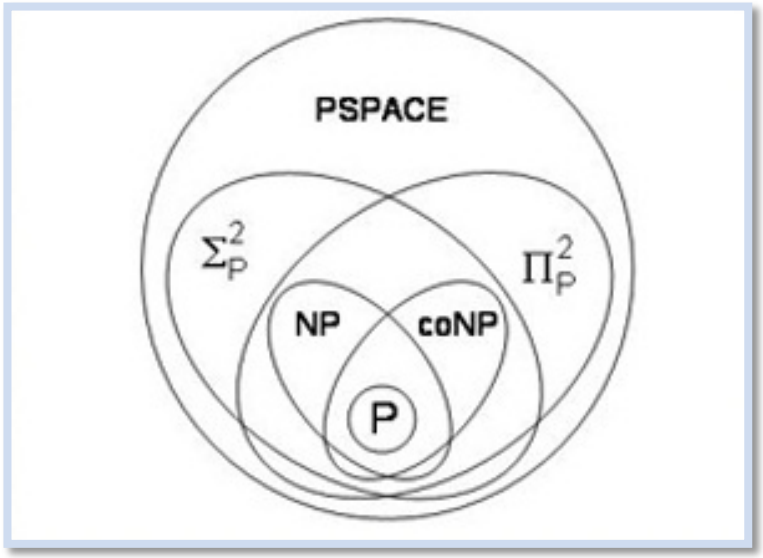
So, the problem is in **NP**

Relation between *P* and *NP*

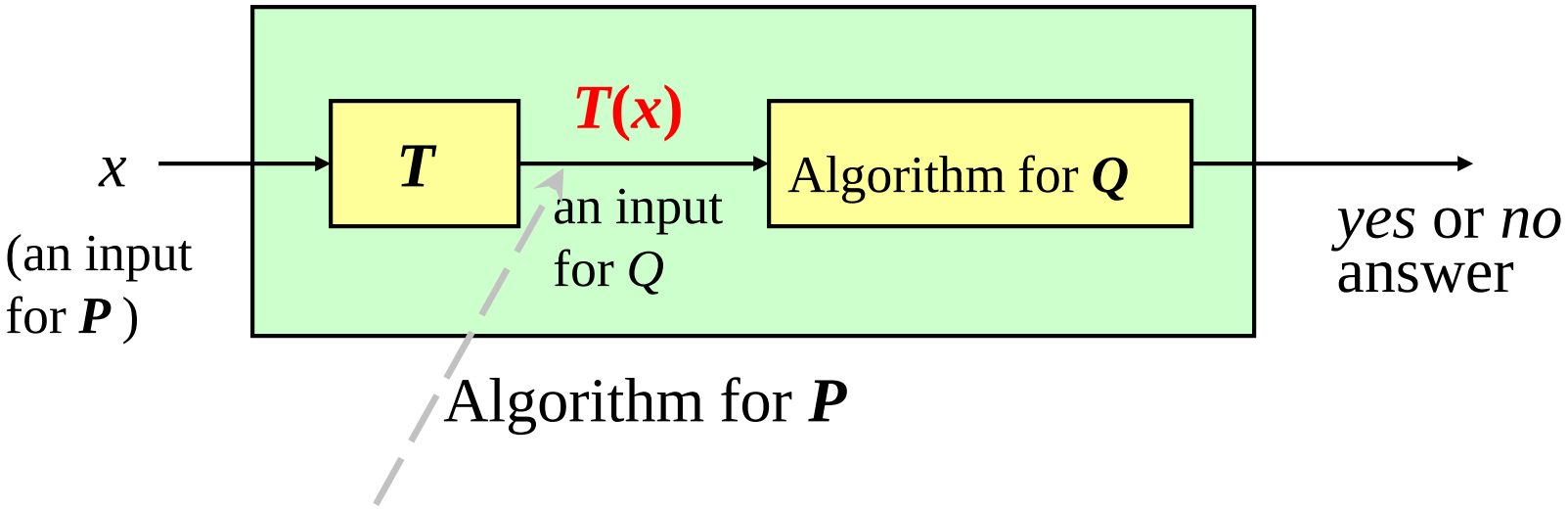
- **An deterministic algorithm for a decision problem is a special case of a nondeterministic algorithm, which means: $P \subseteq NP$**
  - The deterministic algorithm is looked as the phase 2 of a nondeterministic one, which always ignore the *s* the phase 1 has written.
- **Intuition implies that *NP* is a much larger set than *P*.**
  - The number of possible *s* is exponential in *n*.
  - No one problem in *NP* has been proved not in *P*.

P, NP and …


NP-Hard
NP-Complete
NP
P
Complexity
P ≠ NP
NP-Hard
P = NP =
NP-Complete
P = NP


PSPACE
Σ_P^2
Π_P^2
NP
coNP
P

Reduction



The correct answer for ***P*** on x is *yes* **if and only if** the correct answer for ***Q*** on $T(x)$ is *yes*.

Polynomial Reduction

- **Let *T* be a function from the input set for a decision problem *P* into the input set for *Q*. *T* is a polynomial reduction from *P* to *Q* if:**
  - *T* can be computed in polynomial bounded time
  - x is a *yes* input for $P \rightarrow$ $T(x)$ is a *yes* input for *Q*
  - x is a *no* input for $P \rightarrow$ $T(x)$ is a *no* input for *Q*

An example:

***P*: Given a sequence of Boolean values, does at least one of them have the value true?**

***Q*: Given a sequence of integers, is the maximum of them positive?**

$T(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$, where: y_i=1 if x_i=true, and y_i=0 if x_i=false

Relation of Reducibility

- **Problem P is polynomially reducible to Q if there exists a polynomial reduction from P to Q, denoted as: $P \leq_P Q$**
  - If $P \leq_P Q$ and Q is in ***P***, then P is in ***P***
  - If $P \leq_P Q$ and $Q \leq_P R$, then $P \leq_P R$
- **If $P \leq_P Q$, then Q is at least as hard as P**

Relation of Reducibility

- **If $P \leq_P Q$ and Q is in *P*, then P is in *P***
  - The complexity of P is the sum of T, with the input size n, and Q, with the input size $t(n)$, where p is the polynomial bound on T,
  - So, the total cost is: $t(n)+q(t(n))$, where q is the polynomial bound on Q.
- **If $P \leq_P Q$ and $Q \leq_P R$, then $P \leq_P R$**
  - n -> $t_1(n)$ -> $t_1(t_2(n))$

NP-complete Problems

- **A problem Q is *NP-hard* if every problem P in $\mathcal{NP}$ is reducible to Q, that is $P \leq_p Q$.**

  (which means that Q is at least as hard as any problem in $\mathcal{NP}$)

- **A problem Q is *NP-complete* if it is in $\mathcal{NP}$ and is *NP-hard*.**

  (which means that Q is at most as hard as to be solved by a polynomially bounded nondeterministic algorithm)

Example of an *NP*-hard problem

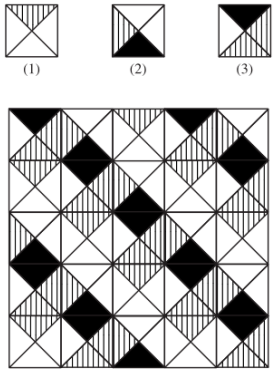
- **Halt problem: Given an arbitrary deterministic algorithm *A* and an input *I*, does *A* with input *I* ever terminate?**
  - A well-known **undecidable** problem, of course not in $\mathcal{NP}$.
  - Satisfiability problem is reducible to it.
    - Construct an algorithm *A* whose input is a propositional formula *X*. If *X* has *n* variables then *A* tries out all 2^n possible truth assignments and verifies if *X* is satisfiable. If it is satisfiable then *A* stops. Otherwise, *A* enters an infinite loop.
    - So, *A* halts on *X* iff. *X* is satisfiable.

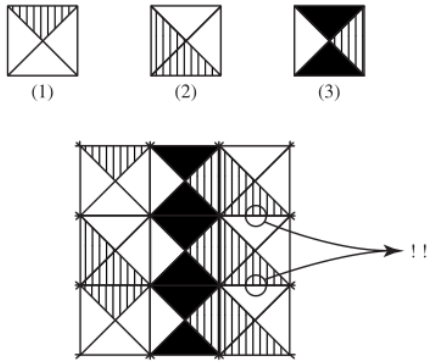
More Undecidable Problems

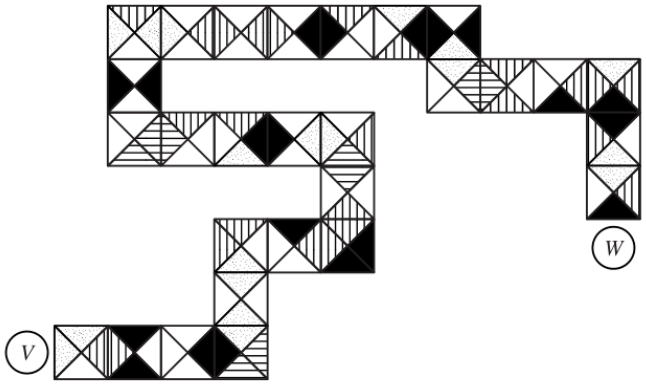
- **Arithmetical SAT**

$$x^3yz + 2y^4z^2 - 7xy^5z = 6$$

- **The *tiling* problem**



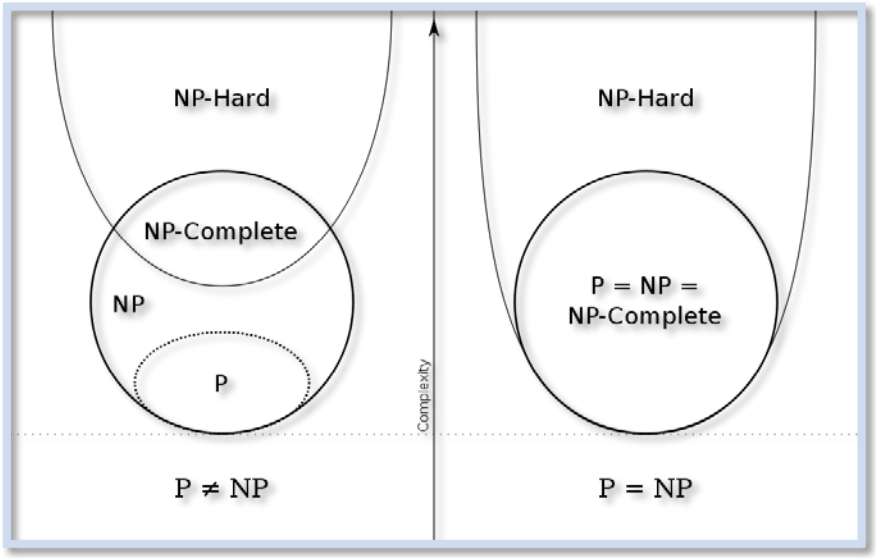


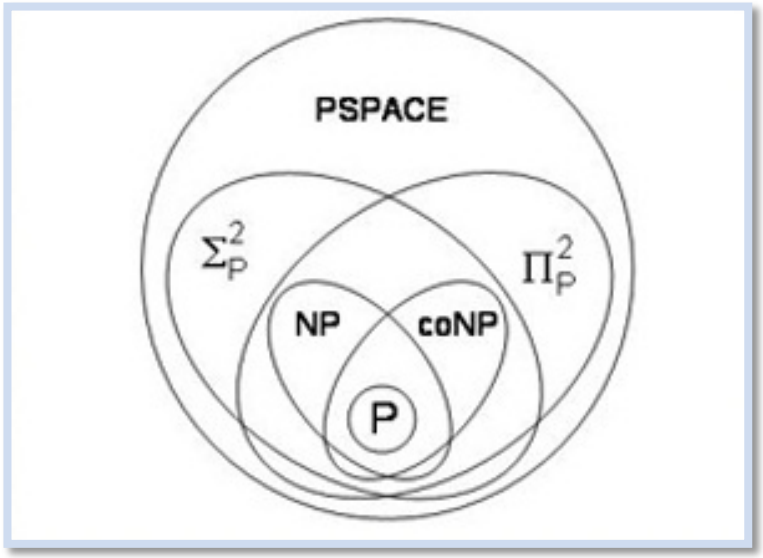


$\mathcal{P}$ and $\mathcal{NP}$ - Revisited

- **Intuition implies that $\mathcal{NP}$ is a much larger set than $\mathcal{P}$.**
  - No one problem in $\mathcal{NP}$ has been proved not in $\mathcal{P}$.
- **If any $\mathcal{NP}$-completed problem is in $\mathcal{P}$, then $\mathcal{NP}$ =$\mathcal{P}$.**
  - Which means that every problems in $\mathcal{NP}$ can be reducible to a problem in $\mathcal{P}$!
  - Much more questionable

$\mathcal{P}$ and $\mathcal{NP}$ - Revisited





Procedure for NP-Completeness

- **Knowledge: P is NPC**
- **Task: to prove that Q is NPC**
- **Approach: to reduce P to Q**
  - For any $R \in \mathbf{NP}$, $R \leq_P P$
  - Show $P \leq_P Q$
  - Then $R \leq_P Q$, by transitivity of reductions
  - Done. Q is NP-complete (given that Q has been proven in NP)

First Known $\mathcal{NPC}$ Problem

- **Cook's theorem:**
  - The SAT problem is NP-complete.
- **Reduction as tool for proving *NP*-completeness**
  - Since *CNF-SAT* is known to be *NP*-hard, then all the problems, to which *CNF-SAT* is reducible, are also *NP*-hard. So, the formidable task of proving *NP*-complete is transformed into relatively easy task of proving of being in $\mathcal{NP}$.

Proof of Cook's Theorem

COOK, S. 1971.

The complexity of theorem-proving procedures.

In

Conference Record of

3rd Annual ACM Symposium on Theory of Computing.

ACM New York, pp. 151–158.

Stephen Arthur Cook: b.1939 in Buffalo, NY. Ph.D of Harvard. Professor of Toronto Univ. 1982 Turing Award winner. The Turing Award lecture: "An Overview of Computational Complexity", CACM, June 1983, pp.400-8

Satisfiability Problem

- **CNF**
  - A literal is a Boolean variable or a negated Boolean variable, as x or $\bar{x}$
  - A clause is several literals connected with ∨ s, as $x_1 \vee \overline{x_2}$
  - A CNF formula is several clause connected with ∧ s
- **CNF-SAT problem**
  - Is a given CNF formula satisfiable, i.e. taking the value TRUE on some assignments for all x_i.
- **A special case: 3-SAT**
  - 3-SAT: each clause can contain at most 3 literals

Proving NPC by Reduction

- **The *CNF-SAT* problem is *NP*-complete.**
- **Prove problem *Q* is *NP*-complete, given a problem *P* known to be *NP*-complete**
  - For all $R \in NP$, $R \leq_P P$;
  - **Show $P \leq_P Q$;**
  - By transitivity of reduction, for all $R \in NP$, $R \leq_P Q$;
  - So, *Q* is *NP*-hard;
  - If *Q* is in *NP* as well, then *Q* is *NP*-complete.

Max Clique Problem is NP

```
void nondeteClique(graph G; int n, k)
    set S=ϕ;
    for int i=1 to k do
        int t=genCertif();
        if t∈S then return;
        S=S∪{t };
    for all pairs (i,j) with i,j in S and i≠j do
        if (i,j ) is not an edge of G
            then return;
        Output("yes");
```

Example 1: Max Clique Problem is NPC

In $O(n)$

In $O(k^2)$

So, we have an algorithm for the maximal clique problem with the complexity of $O(n+k^2)=O(n^2)$

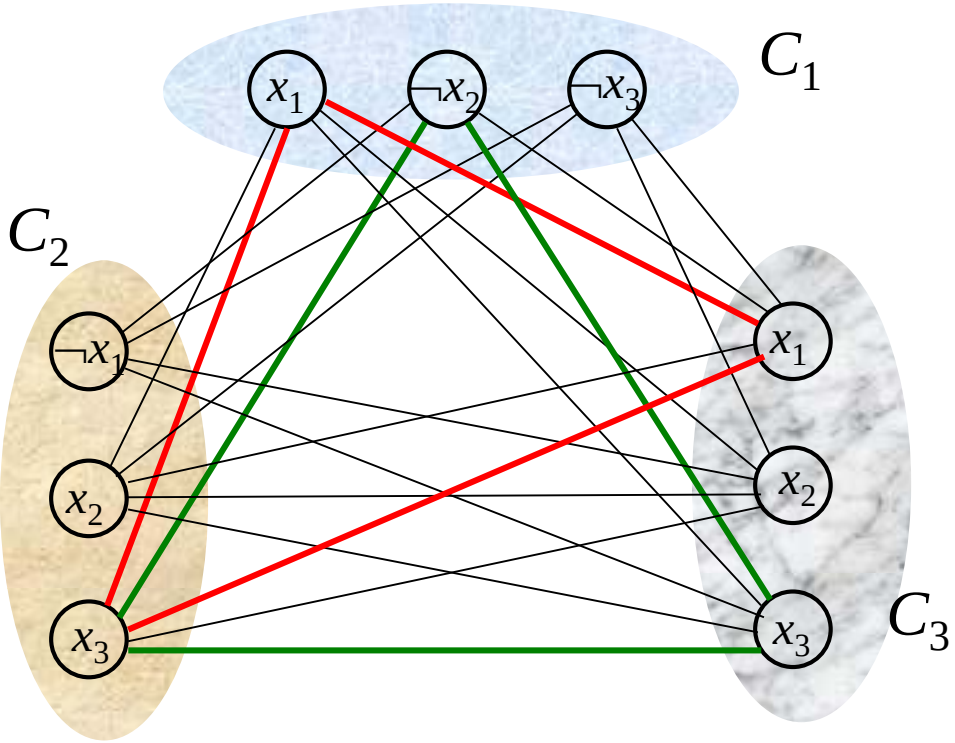
CNF-SAT to Clique

- **Let $\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_k$ be a formula in CNF-3 with k clauses. For $r = 1,2,\ldots,k$, each clause $C_r = (l_1^r \vee l_2^r \vee l_3^r)$, l_i^r is x_i or $\neg x_i$, any of the variables in the formula.**
- **A graph can be constructed as follows. For each C_r, create a triple of vertices v_1^r, v_2^r and v_3^r, and create edges between v_i^r and v_j^s if and only if:**
  - they are in different triples, i.e. $r \neq s$, and
  - they do not correspond to the literals negating each other

(Note: there is no edges within one triple)

3-CNF Graph

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$



Two of satisfying assignments:

x_1=1/0, x_2=0; x_3=1, or

x_1=1, x_2=1/0, x_3=1

For corresponding clique, pick one "true" literal from each triple

Clique Problem is NP-Complete

- **ϕ, with k clauses, is satisfiable iff. The graph G has a clique of size k.**
- **Proof: $\Rightarrow$**
  - Suppose that ϕ has a satisfying assignment.
  - Then **there is at least one "true" literal in each clause**. Picking such a literal from each clause, their corresponding vertices in G can be proved to be a clique, since any two of them are in different triples and cannot be complements to each other(they are both true).

Known NP-Complete Problems

- **Garey & Johnson:** ***Computer and Intractability: A Guide to the Theory of NP-Completeness*****, Freeman, 1979**
  - About 300 problems, grouped in 12 categories:

1. Graph Theory 2. Network Design 3. Set and Partition
4. Storing and Retrieving 5. Sorting and Scheduling
6. Mathematical Planning 7. Algebra and Number Theory
8. Games and Puzzles 9. Logic
10. Automata and Theory of Languages
11. Optimization of Programs 12. Miscellaneous

Advanced Topics

- **Solving hard problems**
  - Approximation algorithms
  - Randomized algorithms
- **Solving more complex problems**
  - Online algorithms
  - External memory models
  - Distributed computation models

Thank you!

Q & A

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