NANJING UNIVERSITY

Introduction to

Algorithm Design and Analysis

[5] HeapSort

Yu Huang
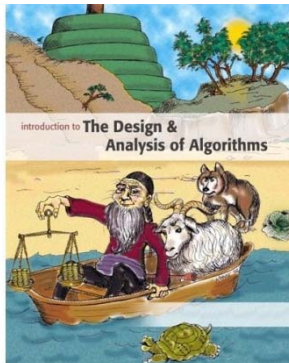
http://cs.nju.edu.cn/yuhuang

Institute of Computer Software

Nanjing University

In the last class …

- **The *sorting* problem**
  - Assumptions
- **InsertionSort**
  - Design
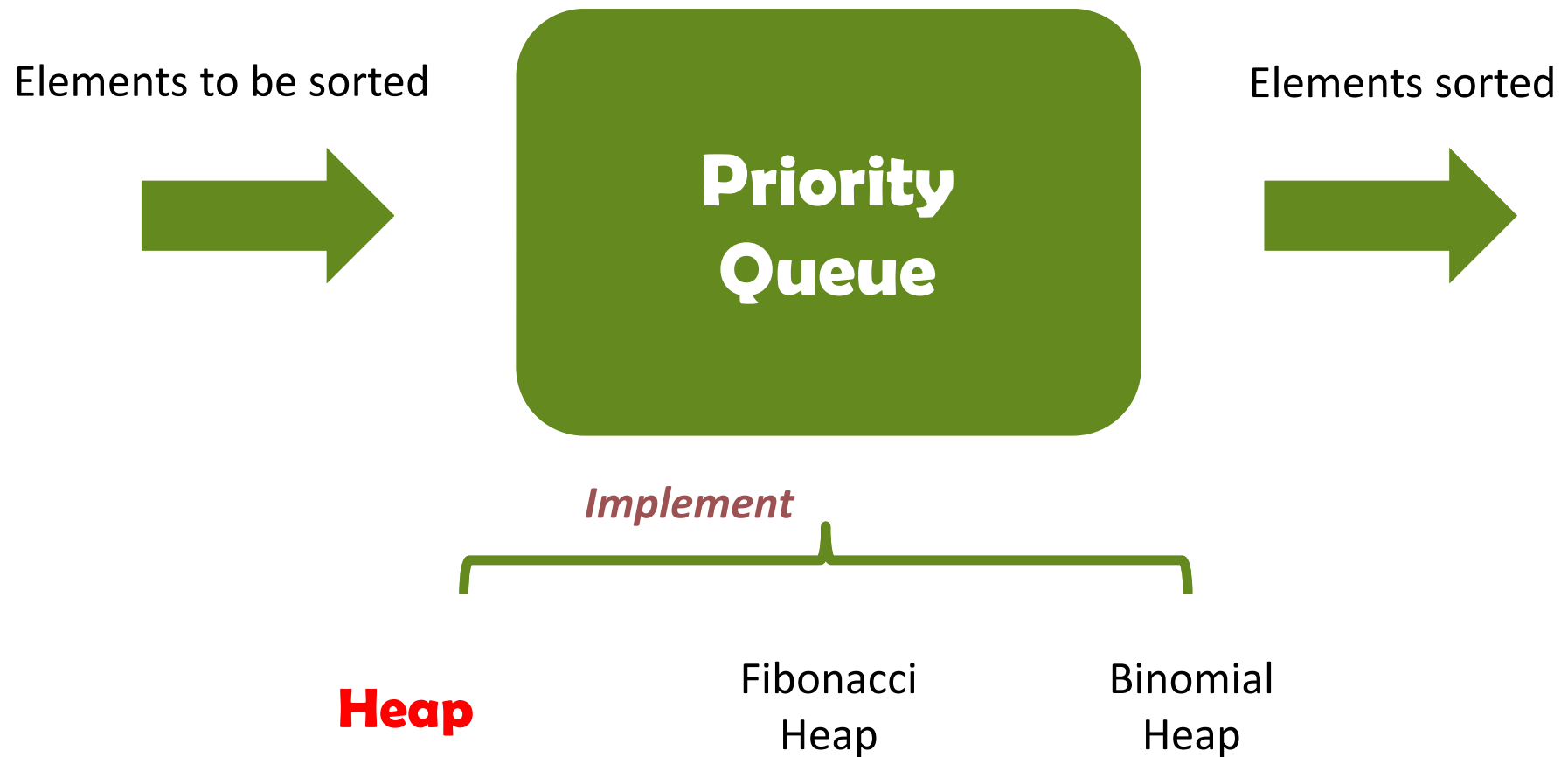 - Analysis: inverse
- **QuickSort**
  - Design
  - Analysis

Heapsort

- **Heap**
- **HeapSort**

- **FixHeap**
- **ConstructHeap**

- **Complexity of Heapsort**
- **Accelerated Heapsort**

How HeapSort Works

Elements to be sorted

Priority Queue

Elements sorted

Implement

Heap

Fibonacci Heap

Binomial Heap

Elementary Priority Queue ADT

- **"FIFO"** in some special sense. The "first" means some kind of "priority", such as value(largest or smallest)
  - **PriorityQ** create()
    - Precondition: none
    - Postconditions: If pq=create(), then, pq refers to a newly created object and isEmpty(pq)=**true**
  - **boolean** isempty(PriorityQ pq)
    - precondition: none
  - **int** getMax(PriorityQ pq)
    - precondition: isEmpty(pq)=**false**
    - postconditions: ******
  - **void** insert(PriorityQ pq, **int** id, **float** w)
    - precondition: none
    - postconditions: isEmpty(pq)=**false; ****
  - **void** delete(PriorityQ pq)
    - precondition: isEmpty(pq)=**false**
    - postconditions: value of isEmpty(pq) updated; ******
  - **void** increaseKey(PriorityQ pq, **int** id, **float** newKey)

> ****** pq can always be thought as a sequence of pairs (id_i, w_i), in non-decreasing order of w_i

Heap: an Implementation of Priority Queue

- **A binary tree T is a *heap structure* if:**
  - T is complete at least through depth h-1
  - All leaves are at depth h or h-1
  - All path to a leaf of depth h are to the left of all path to a leaf of depth h-1

- **Partial order tree *property***
  - A tree T is a (maximizing) partial order tree if and only if the key at any node is greater than or equal to the keys at each of its children (if it has any).

Heap: Examples

The maximal key is always with the root

Heap: an Implementation of Priority Queue

Heap structure

+

Partial order property

➜
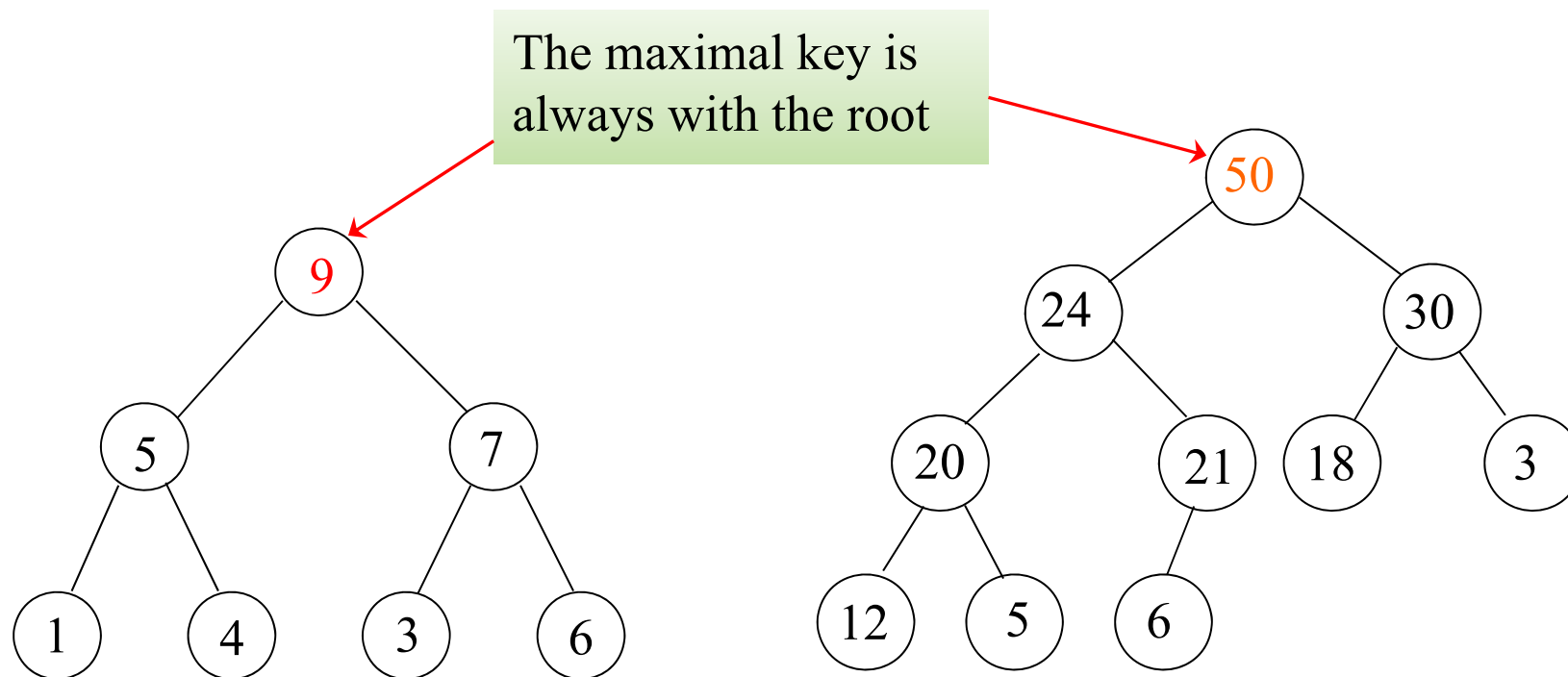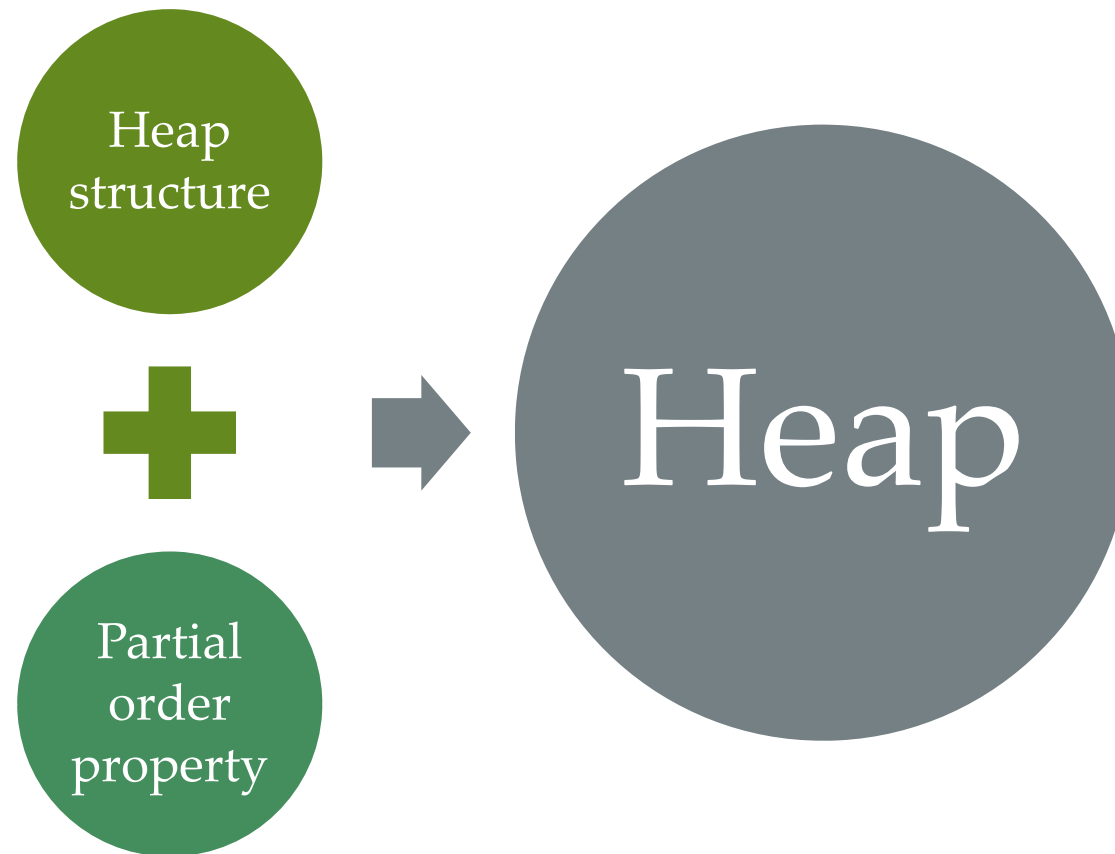
Heap

HeapSort: the Strategy

heapSort(E,n)
   Construct H from E, the set of n elements to be sorted;
   for (i=n;i≥1;i--)
      curMax = getMax(H);
      deleteMax(H);
      E[i] = curMax

deleteMax(H)
   Copy the rightmost element on the lowest level of H into K;
   Delete the rightmost element on the lowest level of H;
   fixHeap(H,K)

FixHeap

- **Input: A nonempty binary tree H with a "vacant" root and its two subtrees in partial order. An element K to be inserted.**
- **Output: H with K inserted and satisfying the partial order tree property.**

- **Procedure:**

  fixHeap(H,K)

     if (H is a leaf) insert K in root(H);

     else

       Set *largerSubHeap*;

       if (K.key≥root(largerSubHeap).key) insert K in root(H)

       else

         insert root(largerSubHeap) in root(H);

        fixHeap(largerSubHeap, K);

     return

One comparison:
largerSubHeap is left- or right-Subtree(H), the one with larger key at its root.
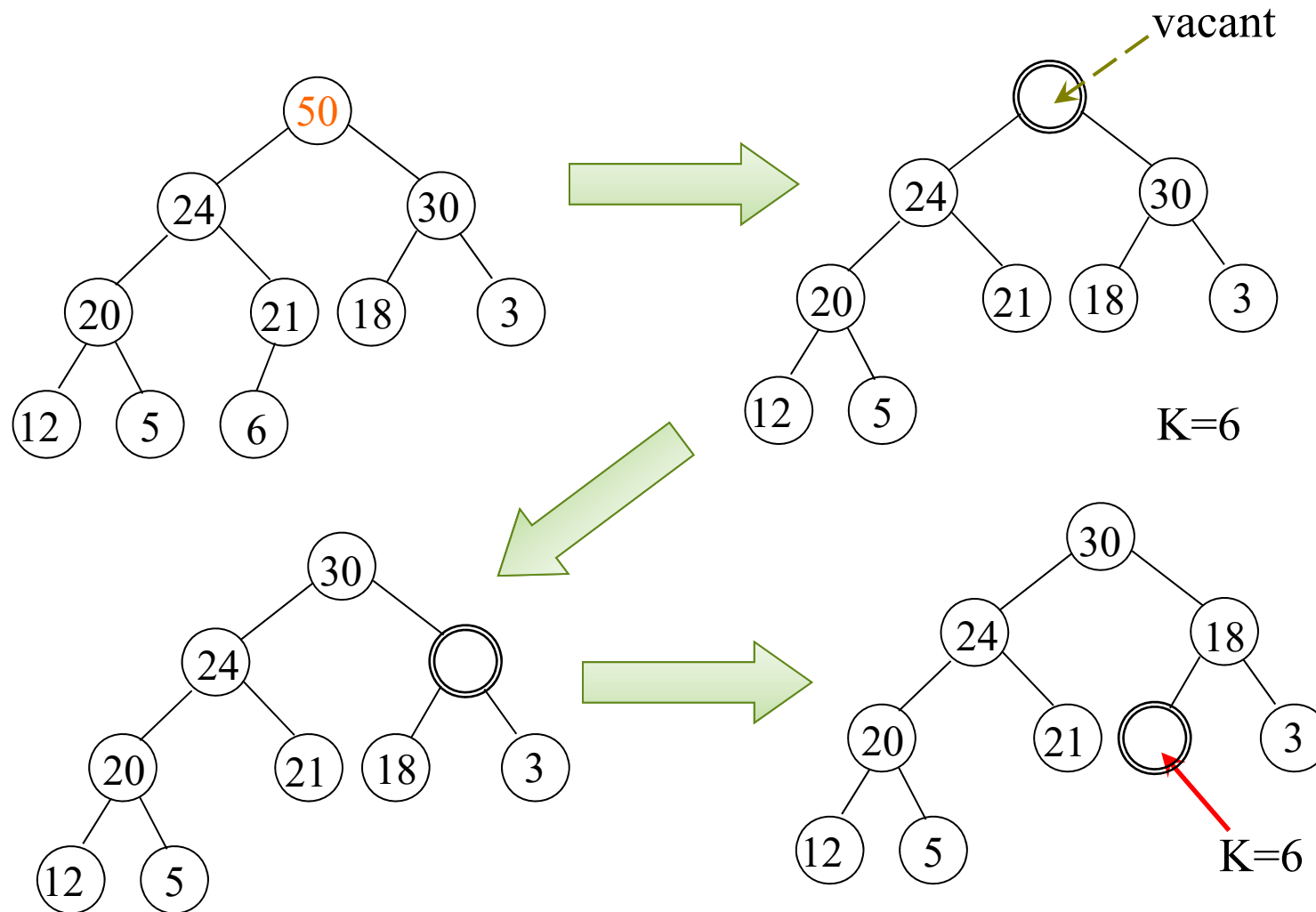Special case: rightSubtree is empty

Recursion

"Vacant" moving down

FixHeap: an Example

Worst Case Analysis for fixHeap

- *2 comparisons* at most in one activation of the procedure
- The tree *height decreases by one* in the recursive call
- So, _2h comparisons are needed in the worst case_, where h is the height of the tree

- **Procesure:**
  fixHeap(H,K)
      if (H is a leaf) insert K in root(H)
      else
          Set *largerSubHeap*;
          if (K.key≥root(largerSubHeap).key) insert K in root(H)
          else
              insert root(largerSubHeap) in root(H);
              fixHeap(largerSubHeap, K);
      return

One comparison:
largerSubHeap is left- or right-Subtree(H), the one with larger key at its root.
Special case: rightSubtree is empty

Recursion

"Vacant" moving down

Heap Construction

- **Note:** *if* left subtree and right subtree **both satisfy** the partial order tree property, then **fixHeap(H,root(H))** *gets the thing done.*

- **We begin from a Heap Structure** *H*:

```
void constructHeap(H)
    if (H is not a leaf)
        constructHeap(left subtree of H);
        constructHeap(right subtree of H);
        Element K=root(H);
        fixHeap(H,K)
    return
```

root

Post-order Traversal

left   right
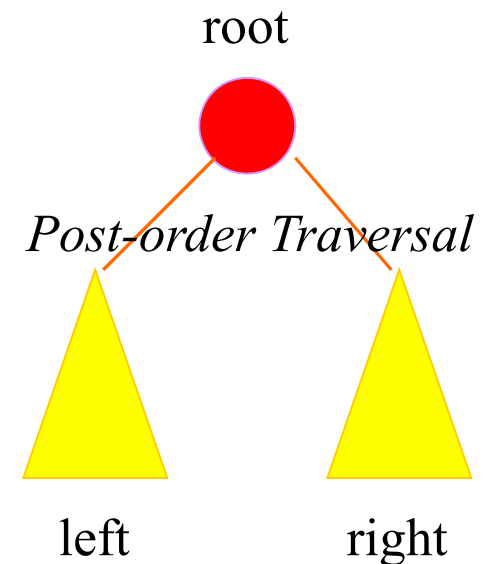
Correctness of *constructHeap*

- **Specification**
  - Input: A heap structure *H,* not necessarily having the partial order tree property.
  - Output: *H* with the same nodes rearranged to satisfy the partial order tree property.

```
void constructHeap(H)
    if (H is not a leaf)
        constructHeap(left subtree of H);
        constructHeap(right subtree of H);
        Element K=root(H);
        fixHeap(H,K)
    return
```

H is a leaf: base case, satisfied trivially.

Preconditions hold respectively?

Postcondition of *constructHeap* satisfied?

Linear Time Heap Construction!

- **The recursion equation:**

  W(n)=W(n-r-1)+W(r)+2log(n)

  Number of nodes in right subheap

  Cost of fixHeap

- **A special case:** *H* is a complete binary tree:
  - The size $N=2^d-1$,

    (then, for arbitrary *n*, $N/2 < n \leq N \leq 2n$, so $W(n) \leq W(N) \leq W(2n)$)
  - Note: $W(N)=2W((N-1)/2)+2log(N)$
  - The **Master Theorem** applies, with b=c=2, and the critical exponent E=1, $f(N)=2log(N)$
  - Note: $$\lim_{N \to \infty} \frac{2log(N)}{N^{1-\varepsilon}} = \lim_{N \to \infty} \frac{2\log N}{N^{1-\varepsilon} \log 2} = \lim_{N \to \infty} \frac{2N^{\varepsilon}}{((1-\varepsilon)\log 2)N}$$
  - When $0<\varepsilon<1$, this limit is equal to zero
  - So, $2log(N) \in O(N^{E-\varepsilon})$, case 1 satisfied, we have $W(n) \in \Theta(n)$

Direct Analysis of Heap Construction

- **Heap construction**
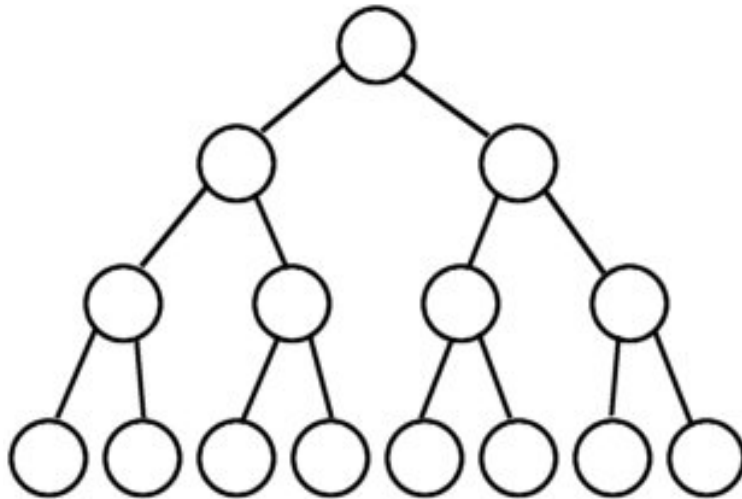 - From *recursion* to *iteration*
  - Sum of rowsums

$$\text{Cost} = \sum_{h=0}^{\lfloor logn \rfloor} n \frac{O(h)}{2^{h+1}} = O(n)$$

c = logn fix; h = logn; # = 1

c = 2 fix; h = 2; # = n/8
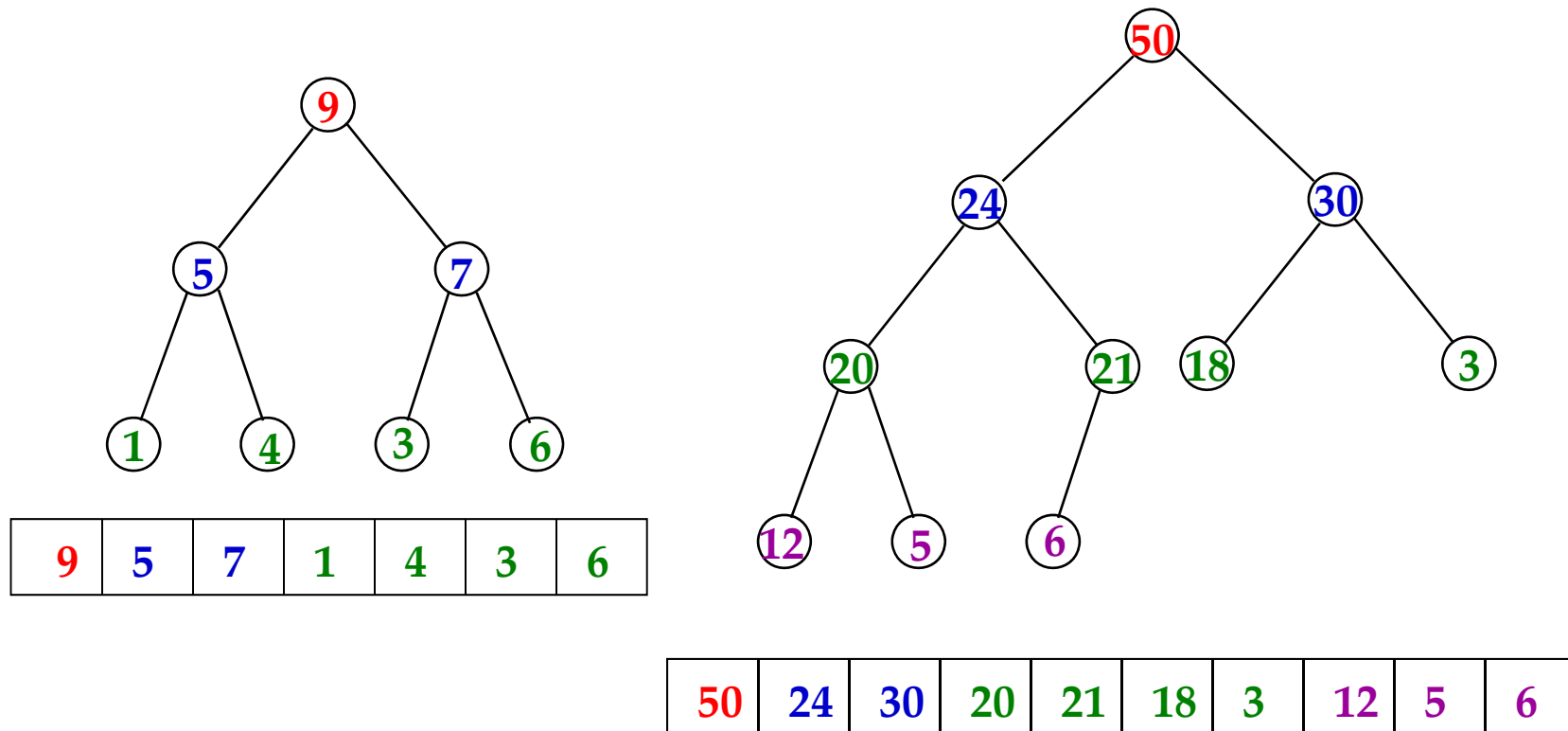
c = 1 fix; h = 1; # = n/4

c = 0 fix; h = 0; # = n/2

1 fix = 2 comparisons

Understanding the Heap

- **Where is the kth element in the heap?**
  - 1st? 2nd ? 3rd?
  - kth? at what cost?
- **Sum of heights**
  - At most n-1
  - When the sum reaches n-1?

Implementing Heap Using Array

Looking for the Children Quickly

Starting from 1, not _zero_, then the _j_ th level has 2^{j-1} elements. and there are $2^{j-1}-1$ elements in the proceeding _j_-1 levels altogether.

So, If E[i] is the kth element at level _j_, then i=$(2^{j-1}-1)$+k, and the index of its left child (if existing) is

$$i+(2^{j-1}-k)+2(k-1)+1=2i$$

The number of node on the right of E[i] on level j

The number of children of the nodes on level j on the left of E[i]

50	24	30	20	21	18	3	12	5	6

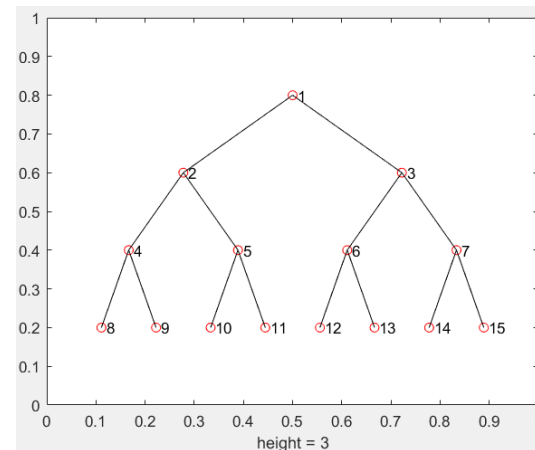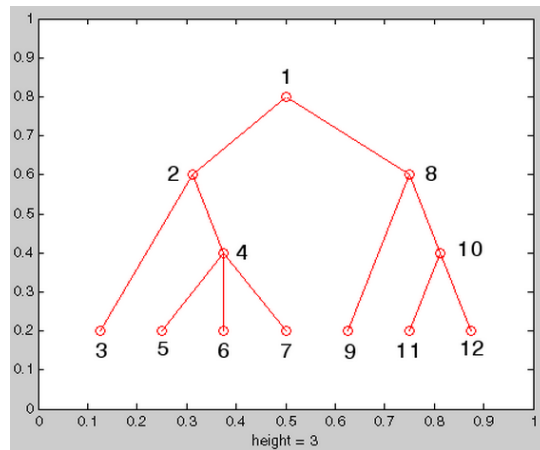<5>   <10>
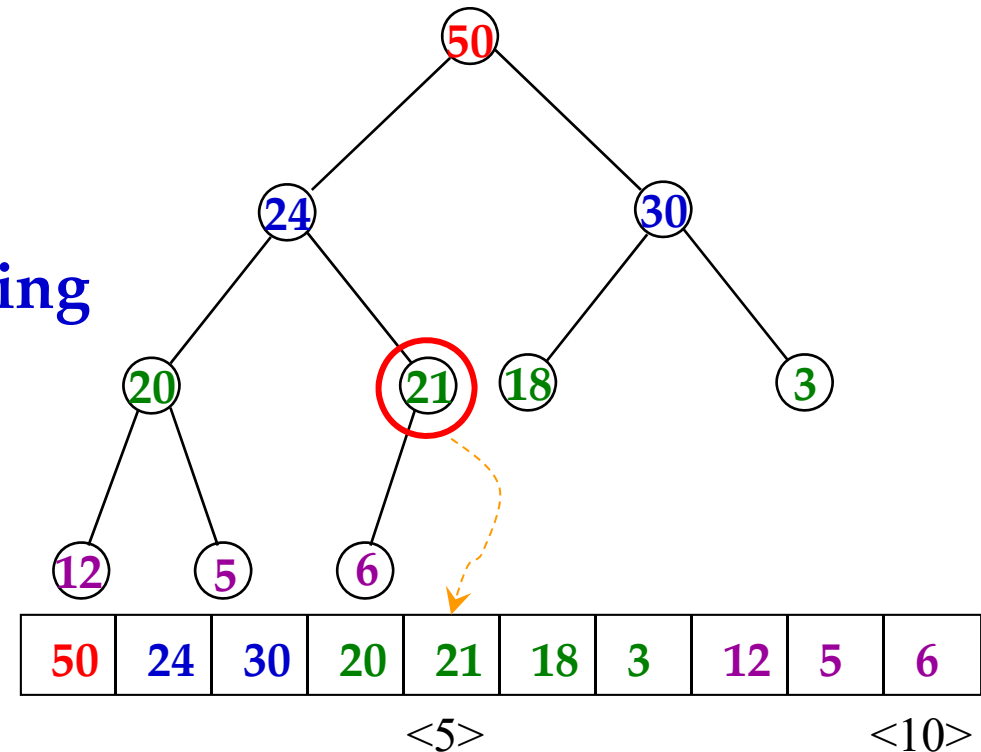
For E[i]:
Left subheap: E[2i]
right subheap: E[2i+1]

An Example from Matlab

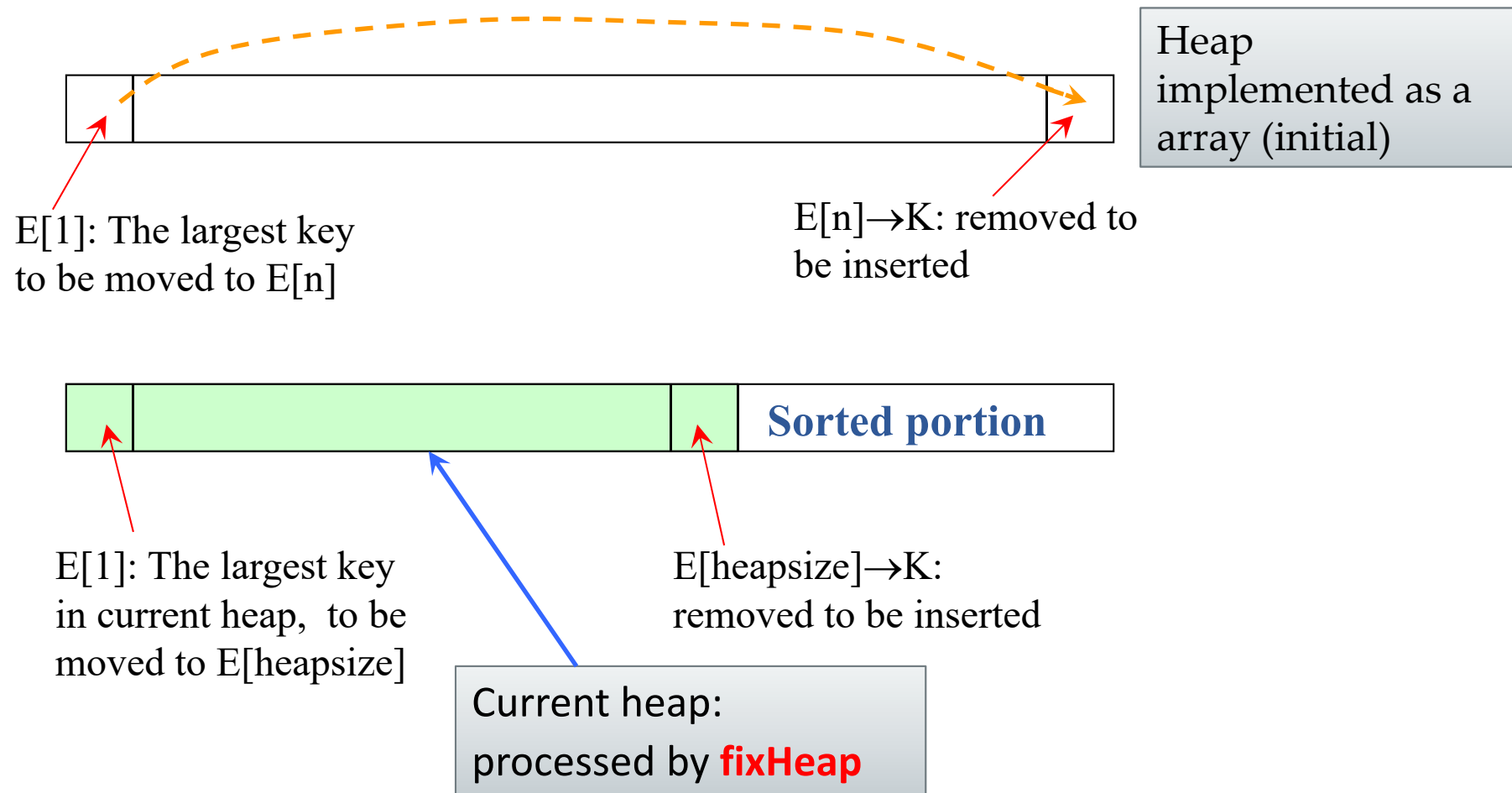nodes = [0 1 2 2 4 4 4 1 8 8 10 10]


```
for i=2:1:treeSize
    nodeArray(i)=floor(i/2);
end
```

HeapSort: In-Space Implementation

Heap implemented as a array (initial)

E[1]: The largest key to be moved to E[n]

E[n]→K: removed to be inserted

Sorted portion

E[1]: The largest key in current heap, to be moved to E[heapsize]

E[heapsize]→K: removed to be inserted

Current heap: processed by **fixHeap**

FixHeap: Using Array

- **Void fixHeap(Element[ ] E, int heapSize, int root, Element K)**
- **int left=2*root; right=2*root+1;**
- **if (left>heapSize) E[root]=K;** *//Root is a leaf.*
- **else**
- **int largerSubHeap;** *//Right or Left to filter down.*
- **if (left==heapSize) largerSubHeap=left;** *// No right SubHeap.*
- **else if (E[left].key>E[right].key) largerSubHeap=left;**
- **else largerSubHeap=right;**
- **if (K.key≥E[largerSubHeap].key) E[root]=K;**
- **else E[root]=E[largerSubHeap];** **//vacant filtering down one level.**
- **fixHeap(E, heapSize, largerSubHeap, K);**
- **return**

Heapsort: the Algorithm

- **Input: E, an unsorted array with n(>0) elements, indexed from 1**
- **Sorted E, in nondecreasing order**
- Procedure:

> *"array version"*

```
void heapSort(Element[] E, int n)
   int heapsize
   constructHeap(E,n,root)
   for (heapsize=n; heapsize≥2; heapsize--;)
       Element curMax=E[1];
       Element K=E[heapsize];
       fixHeap(E,heapsize-1,1,K);
       E[heapsize]=curMax;
   return;
```

Worst Case Analysis of Heapsort

- **We have:** $W(n) = W_{cons}(n) + \sum_{k=1}^{n-1} W_{fix}(k)$

- **It has been shown that:** $W_{cons}(n) \in \Theta(n)$ and $W_{fix}(k) \leq 2\log k$

- **Recall that:**

$$2\sum_{k=1}^{n-1} \lceil \log k \rceil \leq 2\int_1^n \log e \ln x\, dx = 2\log e(n\ln n - n) = 2(n\log n - 1.443n)$$

- **So, W(n) ≤ 2*n*lg*n* + Θ(*n*), that is W(n)∈Θ(*n*log*n*)**

> Coefficient doubles that of mergeSort approximately

HeapSort: the Right Choice

- **For heapSort, $W(n) \in \Theta(n\log n)$**
- **Of course, $A(n) \in \Theta(n\log n)$**
- **More good news: HeapSort is an in-space algorithm** (using iteration instead of recursion)

- **It will be more competitive *if only* the coefficient of the leading term can be decreased to 1**

Number of Comparisons in fixHeap

Procedure:

fixHeap(H,K)

    if (H is a leaf) insert K in root(H);

    else

        Set **_largerSubHeap_**;

        if (K.key$\geq$root(largerSubHeap).key) insert K in root(H)

        else

            insert root(largerSubHeap) in root(H);

            fixHeap(largerSubHeap, K);

    return

2 comparisons are done in filtering down for one level.

A One-Comparison-per-Level Fixing
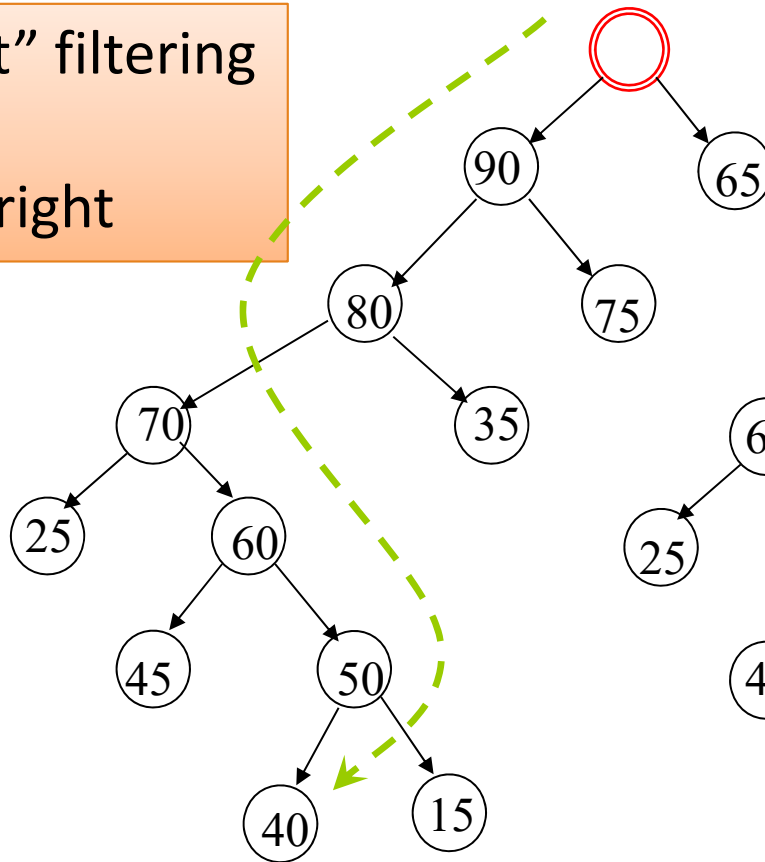
- **Bubble-Up Heap Algorithm:**

  void bubbleUpHeap(Element []E, **int** root, Element K, **int** vacant)

  if (vacant==root) E[vacant]=K;

  else

      int parent=vacant/2;

      if (K.key≤E[parent].key)

          E[vacant]=K

      else

          E[vacant]=E[parent];

          bubbleUpHeap(E,root,K,parent);

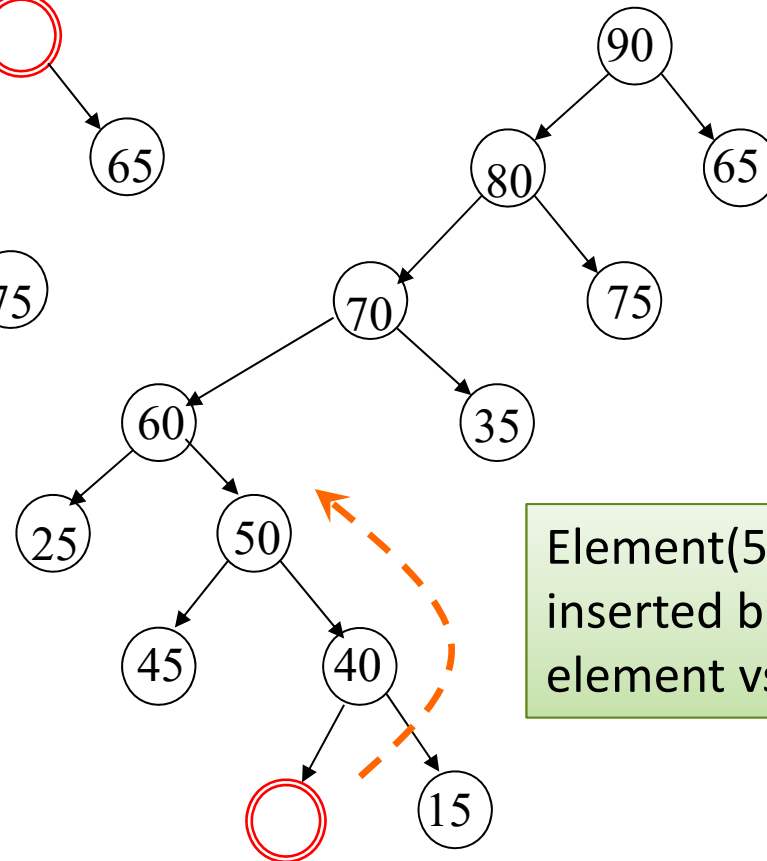Bubbling up from vacant through to the root, recursively

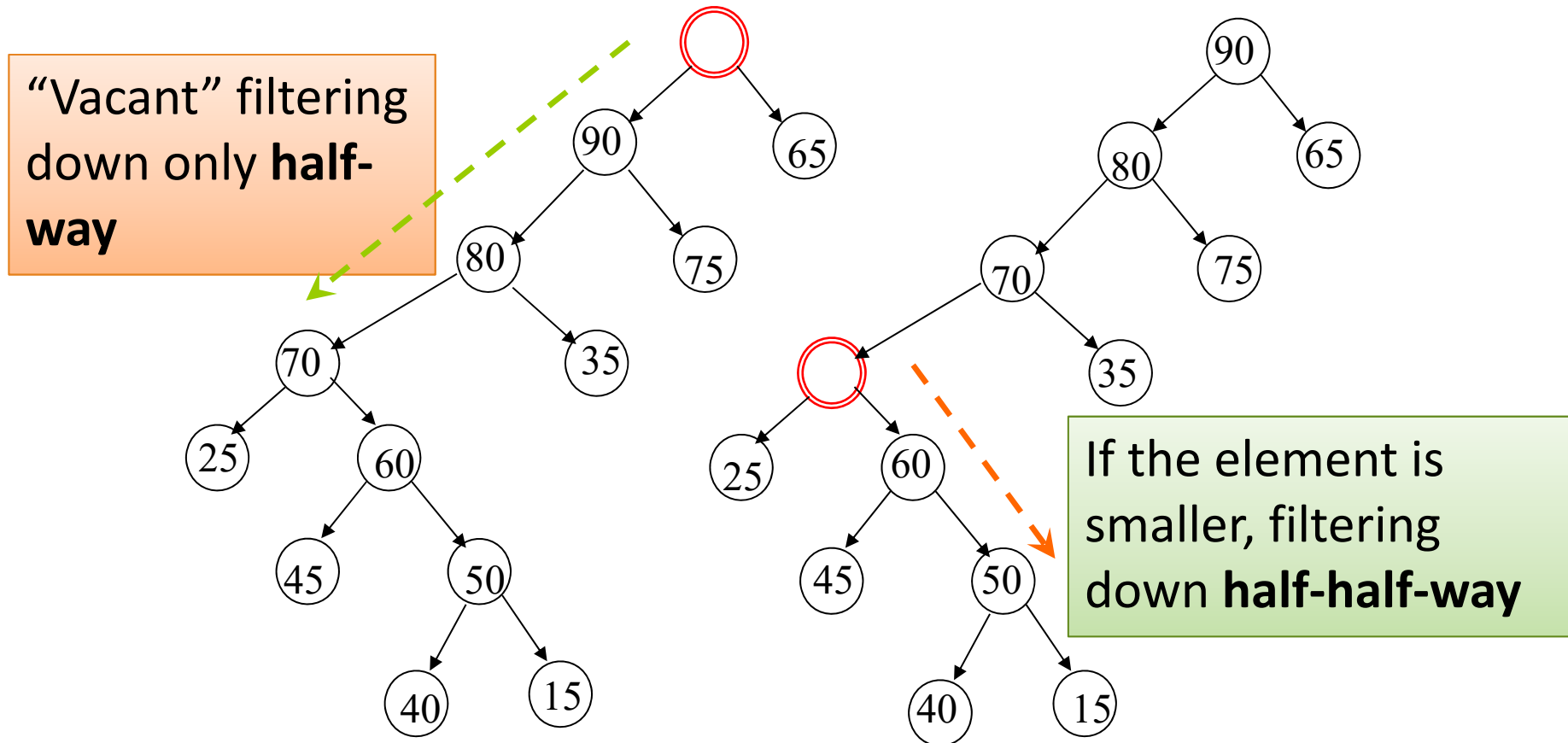# *Risky* FixHeap

"Vacant" filtering down:
left vs. right

In fact, the "risk" is no more than "no improvement"

Element(55) to be inserted bubling up: element vs. parent

Improvement by Divide-and-Conquer

"Vacant" filtering down only **half-way**

If the element is smaller, filtering down **half-half-way**

The bubbling up will not beyond last vacStop

Depth Bounded Filtering Down

```
int promote(Element [ ] E, int hStop, int vacant, int h)
    int vacStop;
    if (h≤hStop) vacStop=vacant;
    else if (E[2*vacant].key≤E[2*vacant+1].key)
        E[vacant]=E[2*vacant+1];
        vacStop=promote(E, hStop, 2*vacan+1, h-1);
    else
        E[vacant]=E[2*vacant];
        vacStop=promote(E, hStop, 2*vacant, h-1);
    return vacStop
```

Depth Bound

FixHeap Using Divide-and-Conquer

void fixHeapFast(Element [ ] E, Element K, int vacant, int h)
   //h=⌈lg(n+1)/2⌉in uppermost call
   if (h≤1) *Process heap of height 0 or 1*;
   else
      int hStop=h/2;
      int vacStop=promote(E, hStop, vacant, h);
      int vacParent=vacStop/2;
      if (E[vacParent].key≤K.key)
         E[vacStop]=E[vacParent];
         bubbleUpHeap(E, vacant, K, vacParent);
      else
         fixHeapFast(E, K, vacStop, hStop)

Number of Comparisons in Accelerated FixHeap

- Moving the vacant one level up or down need one comparison exactly in promote or bubbleUpHeap.
- In a cycle, t calls of promote and 1 call of bubbleUpHeap are executed at most. So, the number of comparisons in promote and bubbleUpHeap calls are:

$$\sum_{k=1}^{t} \left\lceil \frac{h}{2^k} \right\rceil + \left\lceil \frac{h}{2^t} \right\rceil = h = \log(n+1)$$

- At most, $\lg(h)$ checks for reverse direction are executed. So, the number of comparisons in a cycle is at most $h + \log(h)$

- So, for accelerated heapSort: W(n)=$n$log$n$+Θ($n$loglog$n$)

Recursion Equation of Accelerated heapSort

- **The recurrence equation about h, which is about $\log(n+1)$**
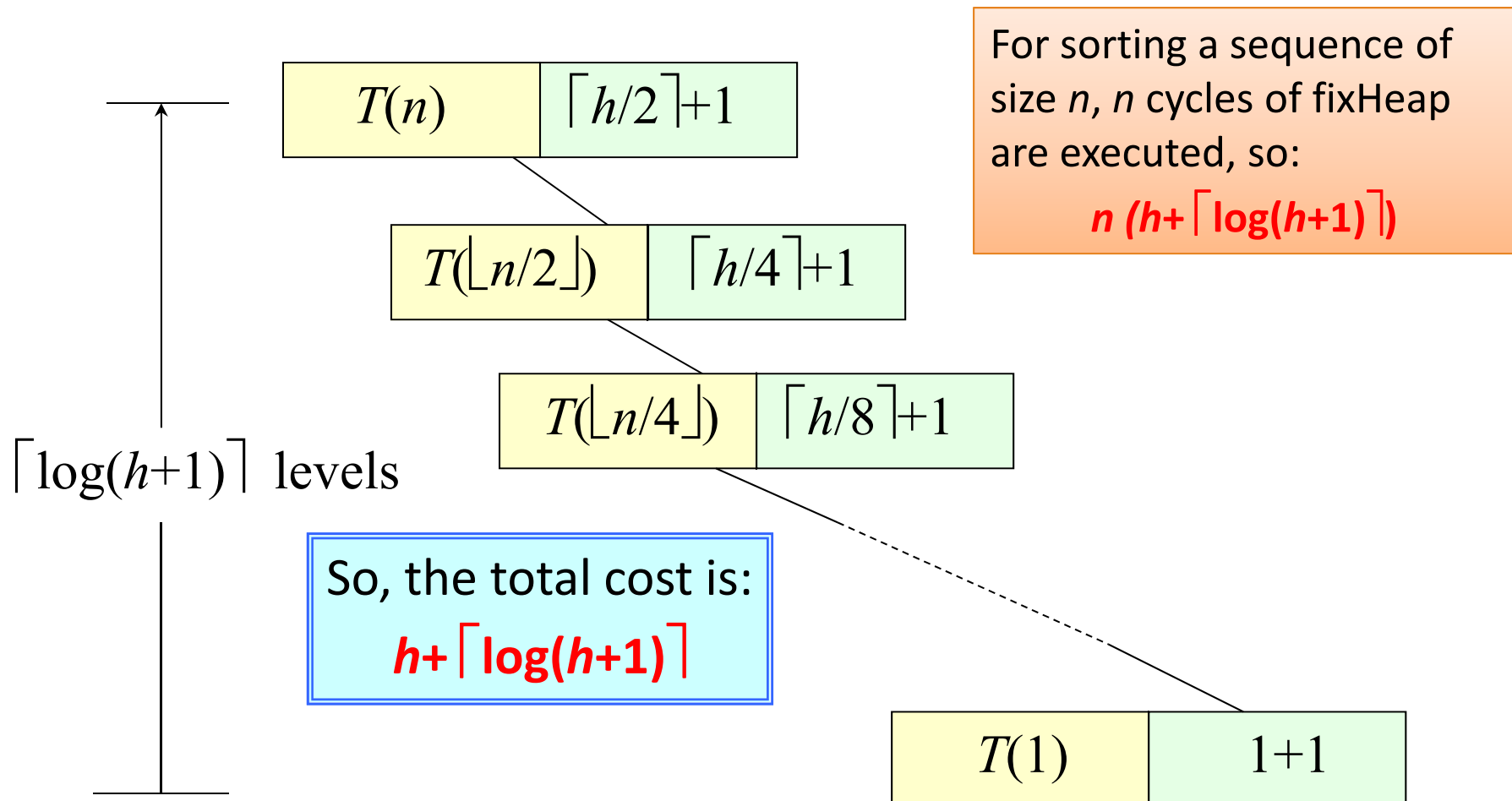
$$\begin{cases} T(1) = 2 \\ T(h) = \left\lceil \dfrac{h}{2} \right\rceil + \max\left(\left\lceil \dfrac{h}{2} \right\rceil, 1 + T\left(\left\lfloor \dfrac{h}{2} \right\rfloor \right) \right) \end{cases}$$

- **Assuming T(h)≥h, then:**

$$\begin{cases} T(1) = 2 \\ T(h) = \left\lceil \dfrac{h}{2} \right\rceil + 1 + T\left(\left\lfloor \dfrac{h}{2} \right\rfloor \right) \end{cases}$$

Solving the Recurrence Equation by Recursive Tree

$T(n)$ | $\lceil h/2 \rceil + 1$

$T(\lfloor n/2 \rfloor)$ | $\lceil h/4 \rceil + 1$

$T(\lfloor n/4 \rfloor)$ | $\lceil h/8 \rceil + 1$

$T(1)$ | $1+1$

$\lceil \log(h+1) \rceil$ levels

For sorting a sequence of size n, n cycles of fixHeap are executed, so:

$$n\ (h + \lceil \log(h+1) \rceil)$$

So, the total cost is:

$$h + \lceil \log(h+1) \rceil$$

Inductive Proof

$$\begin{cases} T(1) = 2 \\ T(h) = \left\lceil \dfrac{h}{2} \right\rceil + 1 + T\left(\left\lfloor \dfrac{h}{2} \right\rfloor\right) \end{cases}$$

- **The recurrence equation for fixHeapFast:**
- **Proving the following solution by induction:**

$$T(\mathbf{h}) = \mathbf{h} + \left\lceil \log(h+1) \right\rceil$$

o According to the recurrence equation:

$$T(h+1) = \left\lceil (h+1)/2 \right\rceil + 1 + T(\left\lfloor (h+1)/2 \right\rfloor)$$

o Applying the inductive assumption to the last term:

$$T(h+1) = \left\lceil (h+1)/2 \right\rceil + 1 + \left\lfloor (h+1)/2 \right\rfloor + \left\lceil \log(\left\lfloor (h+1)/2 \right\rfloor + 1) \right\rceil$$

(It can be proved that for any positive integer:

$$\left\lceil \log(\left\lfloor (h)/2 \right\rfloor + 1) \right\rceil + 1 = \left\lceil \log(h+1) \right\rceil)$$
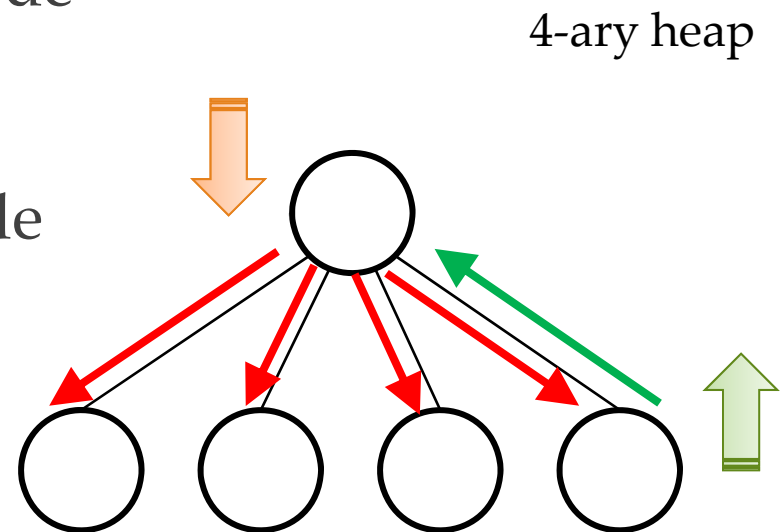
The sum is $h+1$

W(n)=$n$log$n$+Θ($n$loglog$n$) for Accelerated HeapSort

Generalization of a Heap

- **d-ary heap**
  - Structure / partial order
- **How to choose "d"?**
  - Top-down: fix the parent node
    - Cost: d comparisons in the worst case
  - Bottom-up: fix the child node
    - Cos: always 1

4-ary heap

Not only for Sorting

- **Eg1: how to find the kth max element?**
  - The cost should be f(k)
- **Eg2: how to find the first k elements**
  - In sorted order?
- **Eg3: how to merge k sorted lists?**
- **Eg4: how to find the median dynamically?**
- **…**

Thank you!

Q & A

Yu Huang

yuhuang@nju.edu.cn

http://cs.nju.edu.cn/yuhuang