习题四

4.2

$$E(X)=\int_{-\infty}^{+\infty}x\,p_X(x)\,dx=\int_{-1}^{0}\frac{x}{2}dx+\int_{0}^{2}\frac{x}{4}dx$$

$$=\frac{x^2}{4}\Big|_{-1}^{0}+\frac{x^2}{8}\Big|_{0}^{2}$$

$$=-\frac{1}{4}+\frac{4}{8}=\frac{1}{4}$$

$$E(Y)=E(X^2)=\int_{-\infty}^{+\infty}x^2p_X(x)\,dx=\int_{-1}^{0}\frac{x^2}{2}dx+\int_{0}^{2}\frac{x^2}{4}dx$$

$$=\frac{x^3}{6}\Big|_{-1}^{0}+\frac{x^3}{12}\Big|_{0}^{2}$$

$$=\frac{1}{6}+\frac{8}{12}=\frac{5}{6}$$

$$E(XY)=E(X^3)=\int_{-\infty}^{+\infty}x^3p_X(x)\,dx=\int_{-1}^{0}\frac{x^3}{2}dx+\int_{0}^{2}\frac{x^3}{4}dx$$

$$=\frac{x^4}{8}\Big|_{-1}^{0}+\frac{x^4}{16}\Big|_{0}^{2}$$

$$=-\frac{1}{8}+\frac{16}{16}=\frac{7}{8}$$

$$Cov(X,Y)=E(XY)-E(X)E(Y)=\frac{7}{8}-\frac{1}{4}\cdot\frac{5}{6}=\frac{16}{24}=\frac{2}{3}$$

4.3

$$X\sim E(\lambda)\Rightarrow D(X)=\frac{1}{\lambda^2}$$

$$P(X>\sqrt{D(X)})=P(X>\frac{1}{\lambda})=\int_{\frac{1}{\lambda}}^{+\infty}\lambda e^{-\lambda x}dx$$

$$=-e^{-\lambda x}\Big|_{\frac{1}{\lambda}}^{+\infty}$$

$$=e^{-1}$$

4.4

(1) X的概率分布如下：

X	1	2
P_X	$\frac{2}{3}$	$\frac{1}{3}$

Y的概率分布如下：

Y	1	2
P_Y	$\frac{2}{3}$	$\frac{1}{3}$

$\Rightarrow$ U的可能取值为 1, 2；V的可能取值为 1, 2；

~~$P(U=1)=P(X=1,Y=1)=P(X=1)P(Y=1)=\frac{2}{3}\cdot\frac{2}{3}=\frac{4}{9}$~~

$P(U=1,V=1)=P(X=1,Y=1)=P(X=1)P(Y=1)=\frac{4}{9}$；

$P(U=1,V=2)=0$

$P(U=2,V=1)=P(X=1,Y=2)+P(X=2,Y=1)=\frac{2}{3}\cdot\frac{1}{3}\cdot 2=\frac{4}{9}$.

$P(U=2,V=2)=P(X=2,Y=2)=\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{9}$.

即

$V \backslash U$	1	2	P_V
1	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{8}{9}$
2	0	$\frac{1}{9}$	$\frac{1}{9}$
P_U	$\frac{4}{9}$	$\frac{5}{9}$	

(2) $E(U)=P(U=1)+2P(U=2)=\frac{4}{9}+2\cdot\frac{5}{9}=\frac{14}{9}$.

$E(V)=P(V=1)+2P(V=2)=\frac{8}{9}+2\cdot\frac{1}{9}=\frac{10}{9}$.

(3) $E(UV)=1\cdot P(U=1,V=1)+2(P(U=1,V=2)+P(U=2,V=1))$
$+4\cdot P(U=2,V=2)$

$=\frac{4}{9}+2\cdot\frac{4}{9}+4\cdot\frac{1}{9}=\frac{16}{9}$.

$Cov(U,V)=E(UV)-E(U)E(V)=\frac{16}{9}-\frac{14}{9}\cdot\frac{10}{9}=\frac{4}{81}$.

4.5

(1) X、Y的联合分布律如下

$X \backslash Y$	-1	0	1	P_X
0	0.07	0.18	0.15	0.40
1	0.08	0.32	0.20	0.60
P_Y	0.15	0.50	0.35	

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) = P(X=1) = 0.6$$

$$E(Y) = -1 \cdot P(Y=-1) + 0 \cdot P(Y=0) + 1 \cdot P(Y=1)$$

$$= -0.15 + 0.35 = 0.2$$

$$E(X^2) = 0 \cdot P(X=0) + 1 \cdot P(X=1) = P(X=1) = 0.6$$

$$E(Y^2) = 0 \cdot P(Y=0) + 1 \cdot (P(Y=-1) + P(Y=1))$$

$$= P(Y=-1) + P(Y=1) = 0.5$$

$$E(XY) = -1 \cdot P(X=1, Y=-1) + 1 \cdot P(X=1, Y=1)$$

$$= -0.08 + 0.2 = 0.12$$

$$\cancel{\rho = \frac{cov(X,Y)}{\sqrt{\ }}}$$

$$cov(X,Y) = E(XY) - E(X)E(Y) = 0.12 - 0.2 \cdot 0.6 = 0.$$

$$D(X) = E(X^2) - E^2(X) = 0.6 - 0.6^2 = 0.24$$

$$D(Y) = E(Y^2) - E^2(Y) = 0.5 - 0.2^2 = 0.46$$

$$\rho = \frac{cov(X,Y)}{\sqrt{D(X)D(Y)}} = \frac{0}{0.24 \cdot 0.46} = 0.$$

(2) $E(X^2Y^2) = P(X=1, Y=-1) + P(X=1, Y=1) = 0.28$

$$cov(X^2, Y^2) = E(X^2Y^2) - E(X^2)E(Y^2)$$

$$= 0.28 - 0.6 \cdot 0.5$$

$$= -0.02$$

4.6

$$p(x,y)=\begin{cases}2, & 0<x<y<1,\\ 0, & 其他.\end{cases}$$

$$E(X)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x\,p(x,y)\,dx\,dy$$
$$=\int_0^1 dx\int_{1-x}^{1} \cancel{\ } 2x\,dy$$
$$=2\int_0^1 x^2\,dx$$
$$=2\cdot\frac{x^3}{3}\Big|_0^1$$
$$=\frac{2}{3}.$$

$$E(Y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\,p(x,y)\,dx\,dy$$
$$=\int_0^1 dy\int_{1-y}^{1} 2y\,dx$$
$$=\frac{2}{3}$$

$$E(X^2)=\int_0^1 dx\int_{1-x}^{1} 2x^2\,dy.$$
$$=2\int_0^1 x^3\,dx$$
$$=\frac{x^4}{2}\Big|_0^1$$
$$=\frac{1}{2}$$

$$E(Y^2)=\int_0^1 dy\int_{1-y}^{1} 2y^2\,dy$$
$$=\frac{1}{2}.$$

$$E(XY)=\int_0^1 dx\int_{1-x}^{1} 2xy\,dy.$$
$$=\cancel{2}\int_0^1 x\cdot y^2\Big|_{1-x}^{1}\,dx$$
$$=\int_0^1 2x^2-x^3\,dx$$
$$=\left(\frac{2}{3}x^3-\frac{x^4}{4}\right)\Big|_0^1$$
$$=\frac{5}{12}$$

$$\Rightarrow D(X)=E(X^2)-E^2(X)=\frac{1}{2}-\frac{4}{9}=\frac{1}{18}$$

$$D(Y)=\frac{1}{18}$$

$$\cancel{D}\,Cov(X,Y)=E(XY)-E(X)E(Y)=\frac{5}{12}-\frac{4}{9}=-\frac{1}{36}$$

$$D(U)=D(X+Y)=D(X)+D(Y)+2Cov(X,Y)$$
$$=\frac{1}{9}-\frac{1}{18}=\frac{1}{18}.$$

4.7

$E(X)=E(Y)=0$

$E(X^2)=E(Y^2)=2$

$\Rightarrow D(X)=E(X^2)-E^2(X)=2$

$D(Y)=E(Y^2)-E^2(Y)=2.$

由 $\rho=\dfrac{Cov(X,Y)}{\sqrt{D(X)D(Y)}}$

$\Rightarrow Cov(X,Y)=\rho\cdot\sqrt{D(X)D(Y)}=0.5\cdot 2=1.$

由 $Cov(X,Y)=E(XY)-E(X)E(Y)$

$\Rightarrow E(XY)=Cov(X,Y)+E(X)E(Y)=1.$

$$\Rightarrow E\left((3X+Y)^2\right)=E(9X^2+Y^2+6XY)$$
$$=9E(X^2)+E(Y^2)+6E(XY)$$
$$=9\cdot 2+2+6\cdot 1$$
$$=26.$$