SICP - Building Abstractions with Procedures

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1 Excercise 1.13

Prove that $Fib(n)$ is the closest integer to $\phi^5/\sqrt{5}$, where $\phi = (1+\sqrt{5})/2$. Let $\psi = (1-\sqrt{5})/2$. Use induction and the definition of the Fibonacci numbers to prove that $Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$.

First, we identify that ϕ is the *golden ratio* constant, which exhibit the following 2 properties:

$$\begin{aligned} \phi^2 &= \phi + 1 \\ \phi &= 1 + \frac{1}{\phi} \end{aligned} \tag{1.1}$$

When we rearrange the above like so: $\phi^2 - \phi - 1 = 0$ and solve the equation, we found two possible solutions:

$$\begin{aligned} \phi &= \frac{1+\sqrt{5}}{2} \\ \phi &= \frac{1-\sqrt{5}}{2} \end{aligned} \tag{1.2}$$

Thus, ϕ and ψ being solutions of the equation, they share the same properties listed above.

Now, we know that the first three values of the Fibonacci sequence are: $1, 1, 2$. Let us find out if $Fib(n) = \phi^n/\sqrt{5}$:

For $Fib(1)$

$$\begin{aligned}
Fib(1) &= \frac{\phi^1 - \psi^1}{\sqrt{5}} \\
&= \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \\
&= \frac{\frac{2+2\sqrt{5}-2+2\sqrt{5}}{4}}{\sqrt{5}} \\
&= \frac{\frac{4\sqrt{5}}{4}}{\sqrt{5}} \\
&= \frac{\sqrt{5}}{\sqrt{5}} \\
&= 1
\end{aligned} \tag{1.3}$$

For $Fib(2)$

$$\begin{aligned}
Fib(2) &= \frac{\phi^2 - \psi^2}{\sqrt{5}} \\
&= \frac{(\phi + 1) - (\psi + 1)}{\sqrt{5}} \\
&= \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \\
&= \frac{\frac{2+2\sqrt{5}-2+2\sqrt{5}}{4}}{\sqrt{5}} \\
&= \frac{\frac{4\sqrt{5}}{4}}{\sqrt{5}} \\
&= \frac{\sqrt{5}}{\sqrt{5}} \\
&= 1
\end{aligned} \tag{1.4}$$

For $Fib(3)$

$$\begin{aligned}
Fib(3) &= \frac{\phi^3 - \psi^3}{\sqrt{5}} \\
&= \frac{((\phi+1)\times\phi) - ((\psi+1)\times\psi)}{\sqrt{5}} \\
&= \frac{(\phi^2+\phi) - (\psi^2+\psi)}{\sqrt{5}} \\
&= \frac{(\phi+1+\phi) - (\psi+1+\psi)}{\sqrt{5}} \\
&= \frac{2\phi - 2\psi}{\sqrt{5}} \\
&= \frac{2\frac{1+\sqrt{5}}{2} - 2\frac{1-\sqrt{5}}{2}}{\sqrt{5}} \\
&= \frac{2\sqrt{5}}{\sqrt{5}} \\
&= \frac{10}{5} \\
&= 2
\end{aligned} \tag{1.5}$$

We found out that it holds for the first three n values, let us prove that it holds for any n value. First, consider that $Fib(n) = Fib(n-1) + Fib(n-2)$. Then:

$$\begin{aligned}
Fib(n) &= \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\phi^{n-2} - \psi^{n-2}}{\sqrt{5}} \\
&= \frac{\phi^n(\phi^{-1}+\phi^{-2}) - \psi^n(\psi^{-1}+\psi^{-2})}{\sqrt{5}} \\
&= \frac{\phi^n(\frac{1}{\phi}+\frac{1}{\phi^2}) - \psi^n(\frac{1}{\psi}+\frac{1}{\psi^2})}{\sqrt{5}} \\
&= \frac{\phi^n(\frac{\phi+1}{\phi^2}) - \psi^n(\frac{\psi+1}{\psi^2})}{\sqrt{5}} \\
&= \frac{\phi^n(\frac{\phi+1}{\phi+1}) - \psi^n(\frac{\psi+1}{\psi+1})}{\sqrt{5}} \\
&= \frac{\phi^n(1) - \psi^n(1)}{\sqrt{5}} \\
&= \frac{\phi^n - \psi^n}{\sqrt{5}}
\end{aligned} \tag{1.6}$$

Now, we want to prove that $Fib(n) = [\phi^n/\sqrt{5}]$, this means, we need to prove that $\frac{\psi^n}{\sqrt{5}} < 1/2$. Any value greater than 1/2 will affect the nearest integer result of the previous assumption:

We have that:

$$\psi = \frac{1-\sqrt{5}}{2} = -0.6180339... \tag{1.7}$$

Then, for $n \geq 1$:

$$\begin{aligned} \psi &< 0 \\ \psi^n &\leq \psi^2 \\ \psi^n &\leq 0.381966 \end{aligned} \tag{1.8}$$

Now, we check if $0.381966/\sqrt{5} < 1/2$:

$$\begin{aligned} \frac{0.381966}{\sqrt{5}} &< 1/2 \\ \frac{0.381966}{2.236} &< 0.5 \\ 0.17082 &< 0.5 \end{aligned} \tag{1.9}$$

Then we conclude that:

$$Fib(n) = \left[\frac{\phi^n}{\sqrt{5}}\right] \tag{1.10}$$