Exercise 2.

Given :

$$Y_i = \beta x_i + e_i$$

$$e_i \sim N(0, \sigma^2).$$

a) show that the least squares estimate of the slope is given by.

$$\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}$$

Answer :

Least squares estimate

$$Q = \sum (Y_i - \beta X_i)^2$$

Derivating the equation by β.

$$\frac{dQ}{d\beta} = -2 \sum x_i (y_i - \beta x_i).$$

As the errors are independent and normally distributed with constant variance.

$$-2 \sum x_i (y_i - \hat{\beta} x_i) = 0.$$

$$\sum x_i (y_i - \hat{\beta} x_i) = 0.$$

$$\sum x_i y_i - \hat{\beta} \sum x_i^2 = 0.$$

$$\sum x_i y_i = \hat{\beta} \sum x_i^2 .$$

$$\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}$$

b). Show that $E(\hat{\beta}) = \beta$

Answer :

$$E(\hat{\beta} \mid x) = E\left\{ \frac{\sum x_i y_i}{\sum x_i^2} \,\middle|\, x_i \right\}.$$

$$= \frac{\sum x_i \beta x_i}{\sum x_i^2}$$

$$E(\hat{\beta}). = \beta$$

since $\sum \bar{X}(Y_i - \bar{Y}) = \bar{X} \sum (Y_i - \bar{Y}) = 0$

$$E(\hat{\beta}_1 \mid X) = E\left\{ \frac{\sum x_i (y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \,\middle|\, x \right\}.$$

$$= \frac{\sum (x_i - \bar{x}) \{ E(y_i \mid x) - E(\bar{y} \mid x)}{\sum (x_i - \bar{x})^2}.$$

$$= \frac{\sum (x_i - \bar{x}) \{(\beta x_i + \alpha - \beta \bar{x})}{\sum (x_i - \bar{x})^2}.$$

$$= \frac{\Sigma (x_i - \bar{x}) \beta (x_i - \bar{x})}{\Sigma (x_i - \bar{x})^2}$$

$$= \beta$$

$$E(\hat{\beta}) = \beta.$$

This shows that $\hat{\beta} = b$ is unbiased for β

c). Show that $Var(\hat{\beta}) = \dfrac{\sigma^2}{\Sigma x_i^2}.$

Answer:

$$\hat{\sigma}^2 = MSE = \frac{\Sigma e_i^2}{n-1} = \frac{SSE}{n-1}.$$

Hence, $E(\hat{\sigma}^2) = E(MSE) = \sigma^2.$

$$\hat{\sigma} = S_e = \sqrt{\frac{\Sigma (y_i - (\beta x_i))^2}{n-1}}$$

$$= \sqrt{\frac{\Sigma e_i^2}{n-1}}$$

$$= \sqrt{\frac{SSE}{n-1}} = \sqrt{MSE}$$

$n-1$ is used as there is only

one estimator β.