Lecture 9 Worksheet

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Every worksheet will work as follows.

1. You will be entered into a Zoom breakout session with other students in the class.
2. Read through the worksheet, discussing any questions with the other participants in your breakout session.
   - You can call me using the "Ask for help" button.
   - Keep in mind that I will be going through all rooms during the session so it might take me a while to get to you.
3. Answer each question (preferably in the order provided) to the best of your knowledge.
4. While collaboration between students in a breakout session is highly encouraged and expected, each student has to submit their own version.
5. You will have 24 hours (see Compass) to submit your work.

Worksheet 1: Basic expectation and variance properties

First, let's practice some basic expectation and variance properties. Answer the following questions with True or False. If False, correct the statement. [1]

[1] For these statements, consult with pages 3-4 and 8-9 of the notes.

Problem 1: Expectation 1

Answer to Problem 1.

$$E\left[3X+2\right] = 3E\left[X\right]$$

a) True

b) False

Problem 2: Expectation 2

Answer to Problem 2.

$$E\left[3X^2\right] = 3E\left[X^2\right]$$

a) True

b) False

Problem 3: Variance 1

Answer to Problem 3.

$$Var\,[3X+2] = 9Var\,[X]+4$$

a) True b) False

Problem 4: Variance 2

For the next one, you may assume that X and Y are independent random variables.

Answer to Problem 4.

$$Var\,[3X+Y] = 9Var\,[X]+Var\,[Y]$$

a) True b) False

Had we not been told that X and Y are independent, then the statement would have been False; alas, we have yet to know what the correct version is. Stay tuned: we will soon discuss this (in Lecture 12).

Worksheet 2: Using the definitions to rate TENET

A movie magazine decides to allow its reviewers to provide a rating ranging from 1 to 4 stars.

Problem 5: Integer stars

Assume that the number of stars is represented as a **discrete random variable** X with pmf equal to $p(x) = \frac{x^2}{30}$ for $x = 1, 2, 3, 4$. What is the expected number of stars for any movie (that is, what is the expectation of X)? What is the variance of X?

Answer to Problem 5.

$E[X] =$

$Var[X] =$

Problem 6: Real stars

Now, assume that the number of stars a reviewer can give to a movie can be represented as a **continuous random variable** X with pdf equal to $f(x) = \frac{x^2}{21}$ for $1 \leq x \leq 4$ [2]. What is the expected value of X in this case? How about the variance of X?

[2] This means that a reviewer may opt to give a movie 3.5 stars, while another may decide to give out 2.78554 stars.

Answer to Problem 6.

$E[X] =$

$Var[X] =$

Worksheet 3: The law of total expectation

Problem 7: A simple case

Let us begin with something simple. An experiment is successful 90% of the time (and failed the remaining time). We perform 10 experiments. How many should we expect to be successful? [3]

[3] Think about what distribution this could be modeled as. Then, you may use the expectation formula for that specific distribution!

Answer to Problem 7.

Problem 8: External conditions

Let us complicate this slightly. Once again we perform 10 experiments, where an experiment can be successful or failed. However, the success probability depends on some external conditions. If the conditions are good, the probability of success is 95%; in average conditions, the probability becomes 90%; in bad conditions, the probability is lower at 75%. [4]

[4] Think of it like that: if the conditions are good, then the expected number of successes would be $0.95 \cdot 10 = 9.5$; if the conditions are average, then the expected number of successes would be $0.90 \cdot 10 = 9$; finally, if the conditions are bad, then the expectation becomes $0.75 \cdot 10 = 7.5$. Could we multiply each expectation with its respective probability? Are we allowed to do that?

Assuming conditions are equally probable (that is, good/average/bad conditions appear $\frac{1}{3}$ of the time), what is the expected number of successful experiments now?

Answer to Problem 8.

Problem 9: Generalizing the result

Can we generalize the previous result? What if we had m different possible conditions, each appearing with probability $\pi_i, i = 1, \ldots, m$ and each leading to probability of success p_i? How many experiments should we expect to be successful if we perform $n = 10$ experiments?

Answer to Problem 9.

Based on your answers so far, we observe that if we can partition the space in m mutually exclusive and collectively exhaustive events A_i each with probability of appearing equal to $P(A_i)$ [5], then the expected value of random variable X can be found by:

$$\boxed{E\left[X\right] = \sum_{i=1}^{m} E\left[X|A_i\right] \cdot P(A_i)}$$

How do you think this should look like for continuous random variables? [6]

[5] Look at this! This is also the setup for the law of total probability (see Lecture 4).

[6] Recall during the previous lecture we saw that summations become integrations, and probabilities become probability distribution functions..

Problem 10

Consider a continuous random variable with pdf $f(x) = \frac{1}{2}\left(1 + \theta \cdot x\right)$ for $-1 \leq x \leq 1$, where θ is uniformly distributed between 0 and 1. What is the expected value of X?

Answer to Problem 10.

The law of total expectation applies to continuous random variables, too. Consider X, Y as continuous random variables, such that we know $E[X|Y=y]$. Also assume that Y has pdf $g(y)$. Then, we have:

$$\boxed{E[X] = \int_{-\infty}^{+\infty} E[X|Y=y] \cdot g(y) dy}$$

Worksheet 4: A printer replacement policy

A company has bought a new printer which is supposed to have lifetime that is **exponentially distributed** with an expected lifetime at 2 years. [7] Answer the following questions.

[7] Usually you were provided rates for exponential distributions: however, we may now equivalently provide the expectation.

Problem 11

What is the probability that the printer still works after 2 years?

Answer to Problem 11.

Problem 12

The company is starting a new policy. They will replace the printer either when it breaks down (recall that it breaks down in time that is exponentially distributed like in Problem 9) or when it becomes 2 years old, whichever comes first. What is the expected lifetime of every printer the company buys? [8]

[8] An equivalent question: every how often does the company buy a new printer?

Answer to Problem 12.

Worksheet 5: Expectations of functions

This is optional. I will provide answers to this question over the weekend. That said, if you do submit them, I will consider it for extra credit in the first exam.

Problem 13

Consider again the printer from Worksheet 4. The speed with which the printer works (and prints documents) is a function of its age. When the printer is x years old, its speed is given by $g(x) = \frac{\sqrt{x+1}}{x+1}$ velocity units. For example, when it is just bought, and the printer is 0 years old, its speed is equal to 1 velocity unit; on the other hand, after 2 years its speed drops to $\sqrt{3}/3$ velocity units. [9]

[9] If it helps ground the question better, you may think of 1 velocity unit as the biggest speed there is, and 0 velocity units as the lowest speed there is.

What is the expected speed of a printer if the company implements the policy of replacing the printer when it breaks down or when it becomes 2 years old, whichever comes first?

Answer to Problem 13.