Lecture 12 Worksheet

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Every worksheet will work as follows.

1. You will be entered into a Zoom breakout session with other students in the class.
2. Read through the worksheet, discussing any questions with the other participants in your breakout session.
   - You can call me using the “Ask for help” button.
   - Keep in mind that I will be going through all rooms during the session so it might take me a while to get to you.
3. Answer each question (preferably in the order provided) to the best of your knowledge.
4. While collaboration between students in a breakout session is highly encouraged and expected, each student has to submit their own version.
5. You will have 24 hours (see Compass) to submit your work.

Worksheet 1: Jointly distributed discrete random variables

During our last worksheet, we saw two discrete random variables $X = \{1, 2, 3\}$ and $Y = \{10, 20\}$ that were jointly distributed with probability mass function:

$$f_{XY}(x, y) = \frac{20}{27} \cdot \frac{x+1}{y}.$$

Problem 1: Expectations and variances

What is the expectation of X and what is the variance of X? [1]

Answer to Problem 1.

[1] Recall that the expectation of one of two jointly distributed random variables can be found by properly summing (if discrete, as is the case here) or integrating (when continuous) its **marginal distribution**.

Problem 2: Conditional expectations and variances

What is the expectation of X and what is the variance of X *given that* $Y = 10$? [2]

[2] The previous hint still applies! However, we now replace the marginal with the **conditional distribution**.

Answer to Problem 2.

Problem 3: Independent?

Can we make the claim that X and Y are two independent random variables? Why/Why not?

Answer to Problem 3.

Problem 4: Covariance

Based on your answer in Problem 3, what is the covariance? What is the correlation? [3]

[3] Recall that two independent random variables have **zero** covariance and, consequently, **no** correlation.

Answer to Problem 4.

$\sigma_{XY} = Cov\left[X, Y\right] =$

$\rho_{XY} = Corr\left[X, Y\right] =$

Worksheet 2: Jointly distributed continuous random variables

Consider two jointly distributed *continuous* random variables $0 \leq X \leq 2$ and $0 \leq Y \leq 1$ with joint probability density function equal to:

$$f_{XY}(x, y) = \frac{3}{4}x^3y^2.$$

Answer the following questions.

Problem 5: Marginal distributions

Let's repeat what we had done during our previous lecture. What are the marginal distributions of X and Y? [4]

[4] As a reminder, the marginal distribution of X will be a function of x and the marginal distribution of Y will be a function of y.

Answer to Problem 5.

Problem 6: Getting the expectations

What are the expectations of X and Y? Don't forget that they are defined over *different domains*! [5]

[5] X is defined over $[0, 2]$, whereas Y is defined over the range $[0, 1]$.

Answer to Problem 6.

Problem 7: Independent?

Are X and Y independent? Why/Why not?

Answer to Problem 7.

Worksheet 3: When X and Y restrict each other

Assume that random variables X and Y are jointly distributed with probability density function $f_{XY}(x, y) = \frac{1}{4}(x+y)$ defined over $0 \leq X \leq Y \leq 2$. Note how random variable X always takes values that are at most as big as the value of random variable Y. [6] Answer the following questions.

Problem 8: Independent?

X and Y are not independent; this is clear from their definition, as knowing the one restricts the values the other one may take. What is the covariance of X and Y then? [7]

Answer to Problem 8.

[6] If you are wondering how this is a valid pdf, we may show that the double integration is equal to 1. Be **very careful** with how you are integrating this. Following are the two correct ways (for an incorrect way, look at the notes!):

$$\int_0^2 \int_0^y \frac{1}{4}(x+y)\, dxdy = 1$$

$$\int_0^2 \int_x^2 \frac{1}{4}(x+y)\, dydx = 1$$

[7] You will need to calculate a lot of things. Namely you will need:

1. the marginal distributions $f_X(x), f_Y(y)$;
2. the expectation $E[X], E[Y]$;
3. the expectation of function $X \cdot Y$: $E[X \cdot Y]$;
4. finally, you'll get

$$Cov[X, Y] = E[X \cdot Y] - E[X] \cdot E[Y].$$

Problem 9: A small conditional pdf

Assume that we are given that $y = 1$. What is the probability that X is smaller than or equal to $\frac{1}{2}$?

Answer to Problem 9.

Problem 10: A tougher one

What is the probability that $X \leq \frac{Y}{2}$?

Answer to Problem 10.