Lecture 1 Worksheet

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Every worksheet will work as follows.

1. You will be entered into a Zoom breakout session with other students in the class.
2. Read through the worksheet, discussing any questions with the other participants in your breakout session.
   - You can call me using the "Ask for help" button.
   - Keep in mind that I will be going through all rooms during the session so it might take me a while to get to you.
3. Answer each question (preferably in the order provided) to the best of your knowledge.
4. While collaboration between students in a breakout session is highly encouraged and expected, each student has to submit their own version.
5. You will have 24 hours (see Compass) to submit your work.

Worksheet 1: Playing with coins

Problem 1

A coin is tossed 4 times in a row and we mark whether it has come up Heads or Tails each time. Explain (in a sentence) why this is a random experiment. [1]

[1] Consider what happens if we repeat this process.

Answer to Problem 1.

Problem 2

In the experiment from Problem 1, what is the sample space? [2] What is the cardinality of the sample space?

[2] In general, there are multiple ways to define a sample space. Here, we may want to focus on each individual coin toss as it happens.

Answer to Problem 2.

$S =$

$|S| =$

Problem 3

Once again, consider the experiment from Problem 1. What is the cardinality of the following events:

1. Get the sequence Heads, Tails, Tails, Heads.
2. The third coin toss comes up Heads.
3. Get at least three Heads.

Answer to Problem 3.

1.

2.

3.

Problem 4

Consider the events in Problem 3. Are the first and the second events mutually exclusive? How about the first and the third events? Finally, what can you say about the second and the third events? Justify (in a sentence) your answer.

Answer to Problem 4.

1. "Get the sequence Heads, Tails, Tails, Heads" and "The third coin toss comes up Heads":

2. "Get the sequence Heads, Tails, Tails, Heads" and "Get at least three Heads":

3. "The third coin toss comes up Heads" and "Get at least three Heads":

Worksheet 2: An experimental design

Problem 5

An experiment happens year long in an environment where the temperature is always between 20 and 100 Fahrenheit – that is, the temperature belongs to $S = [32, 100]$. Define the events $A = [80, 100]$ (temperature is greater than or equal to 80 F), $B = [20, 40]$ and $C = [32, 85]$ (that is, the temperature is between 32 and 85 Fahrenheit). We say the experiment is successful when the temperature belongs to C, and we say event C has happened. In other cases, the experiment is unsuccessful. We also say that the experiment is hot when the temperature belongs to A, and we say that event A has happened. Similarly, we say that the experiment is cold when the temperature belongs to B, and we say that B has happened. [3]

[3] Be careful with inclusion and exclusion [,] and (,)

Answer to Problem 5.

1. What is $\overline{A}$ in English and mathematically?

2. What is $B \setminus A$ in English and mathematically?

3. What is $B \cap C$ in English and mathematically?

4. What is $(\overline{A} \cap B) \cup (\overline{A} \cap C)$ in English and mathematically?

5. Define the set of outcomes where an experiment that is both successful and in regular temperatures (i.e., neither hot nor cold) in set notation and mathematically.

Problem 6

Draw a Venn diagram with three events (A, B, C) to represent the events from Problem 5. [4]

[4] See the lecture notes for examples with three events.

Answer to Problem 6.

Worksheet 3: Set and cardinality properties

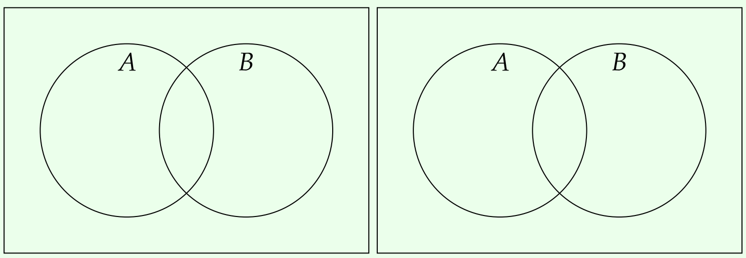
We saw many interesting properties in the first pre-lecture video and the accompanying slides. Now, it is time to see their derivations. To begin with, consider the first of the two DeMorgan's law we saw in the lecture notes:

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}.$$

Problem 7

In the two Venn diagrams below, mark the event $(A \cup B)$ and $\overline{(A \cup B)}$ (corresponding to the left hand side).

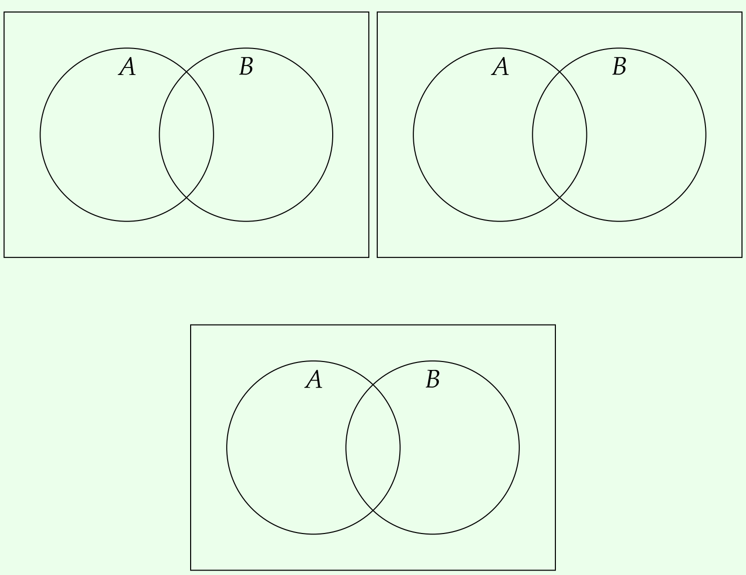
Answer to Problem 7.



Problem 8

Now, in the three Venn diagrams provided, mark the events $\overline{A}$, $\overline{B}$, and $\overline{A} \cap \overline{B}$ (corresponding to the right hand side).

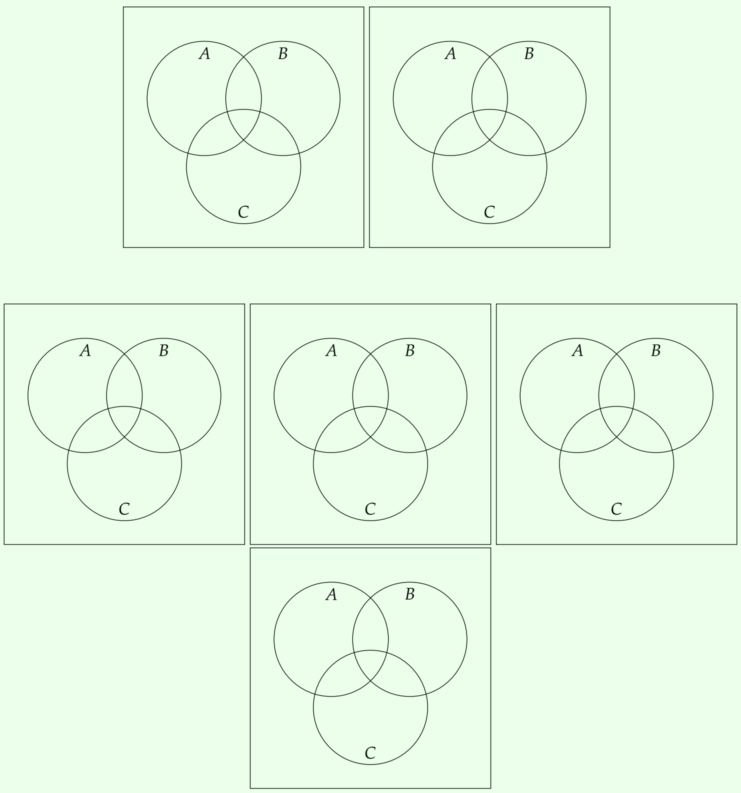
Answer to Problem 8.



Problem 9

Based on the previous constructive derivation you did, what can you say about the extension of this DeMorgan's law to more than 2 events? Specifically, what is $\overline{(A \cup B \cup C)}$? Use the diagrams below to do something similar to what you did in Problems 7 and 8.

Answer to Problem 9.



Problem 10

Consider two mutually exclusive events A and B. What can you say about the cardinality of $A \cap B$? What can you say about the cardinality of $A \cup B$?

Answer to Problem 10.

- $|A \cap B| =$
- $|A \cup B| =$

Problem 11

It is true that for two general sets A, B, we have that $|A \cup B| = |A| + |B| - |A \cap B|$. Let us construct a proof for the statement using your proof in Problem 10.

Consider an event X that is comprised of three mutually exclusive events X_1, X_2, X_3. What can you say about the cardinality of X and the cardinalities of X_1, X_2, X_3?

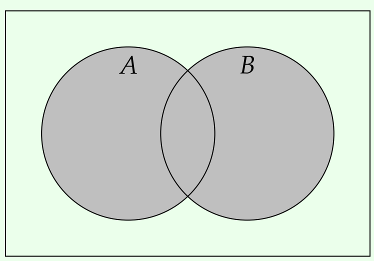
Answer to Problem 11.

$$|X| \quad \begin{matrix} < \\ \leq \\ = \\ \geq \\ > \end{matrix} \quad |X_1| + |X_2| + |X_3|$$

Problem 12

Based on your observation in Problem 11, can we think of $A \cup B$ as three mutually exclusive events? Check the following Venn diagram and mark the three mutually exclusive events. How are they described mathematically?

Answer to Problem 12.



Problem 13

Combine your observations from Problems 11 and 12 to derive that $|A \cup B| = |A| + |B| - |A \cap B|$.

Answer to Problem 13.