Lecture 11 Worksheet

Chrysafis Vogiatzis

Every worksheet will work as follows.

1. You will be entered into a Zoom breakout session with other students in the class.
2. Read through the worksheet, discussing any questions with the other participants in your breakout session.
   - You can call me using the "Ask for help" button.
   - Keep in mind that I will be going through all rooms during the session so it might take me a while to get to you.
3. Answer each question (preferably in the order provided) to the best of your knowledge.
4. While collaboration between students in a breakout session is highly encouraged and expected, each student has to submit their own version.
5. You will have 24 hours (see Compass) to submit your work.

Worksheet 1: Jointly distributed discrete random variables

In the next four questions, we focus on a pair of *discrete* random variables X and Y, distributed with joint probability mass function

$$f_{XY}(x,y) = c \cdot \frac{x+1}{y}.$$

At this point, it is useful to remind ourselves that the probability mass function values *are actual probabilities*, since we are discussing about discrete random variables.

Problem 1: Joint probability mass functions

Assume that $X = \{1, 2, 3\}$ and $Y = \{10, 20\}$, that is X is allowed to take value 1, 2, or 3, and Y is allowed to be equal to either 10 or 20. [1]

[1] Remember the first axiom of mass functions for discrete random variables: summing over all possible values should give us 1.

Answer to Problem 1.

Problem 2: Table form

We may construct a table! Based on your answer in Problem 1, we may collect the different probability mass function values in tabular form. Fill the table below with the actual probabilities for each pair of values.

Answer to Problem 2.

		Y	
		10	20
X	1		
	2		
	3		

Verify once again that indeed the summation all of them is equal to 1, just to be sure.

Problem 3: Marginal probabilities

Where may we find the *marginal probabilities* for X or Y alone? In the table form we saw earlier, they are found by summing over a row or a column, depending on which one we are looking for! In this case, what is:

a) $P(X = 1)$?

b) $P(Y = 20)$?

Answer to Problem 3.

$P(X = 1) =$

$P(Y = 20) =$

Problem 4: Conditional probabilities

Where may we find the *conditional probabilities* for X given $Y = y$ or for Y given $X = x$? In the table form we saw earlier, they are found by focusing on one column or row, and then dividing the probability we are looking for over the summation of all elements in that column or row. [2] In our example, what is:

[2] Without a table, we would calculate a conditional probability by dividing appropriately. For example, the probability of getting $P(X = x|Y = y)$ could be found by $\frac{f_{XY}(x,y)}{f_Y(y)}$.

- $P(X = 1|Y = 20)$?
- $P(Y = 20|X = 1)$?

Answer to Problem 4.

$$P(X = 1|Y = 20) =$$

$$P(Y = 20|X = 1) =$$

Of course, this table form is valuable; but it is also limited to smaller sample spaces. What happens when we are dealing with a huge number of cases? In that case, we need to resort to the actual formulations for each and every one of our probability calculations. Let's see that in the next worksheet.

Worksheet 2: Discrete, but infinite

Consider two discrete random variables X and Y that take on integer values $X \geq 0$ and $1 \leq Y \leq 5$. That is, X could be 3, 107, or 0, and Y could be equal to 1, 2, 3, 4, or 5. Their joint probability mass function is given by:

$$f_{XY}(x, y) = e^{-y} \cdot \frac{y^x}{5 \cdot x!}.$$

As a side note, remember that

$$\boxed{\sum_{i=0}^{\infty} e^{-\alpha} \cdot \frac{\alpha^i}{i!} = 1 \text{ for any } \alpha > 0.}$$

Problem 5: Using the joint pmf

Let's start easy. What is the probability that both X and Y are equal to 1? That is, what is $P(X = 1 \cap Y = 1)$? [3]

[3] This can also be written as $P(X = 1, Y = 1)$. Recall that (because this is a pmf for a discrete random variable) this can be found as simply the value of the pmf for the given X and Y.

Answer to Problem 5.

Problem 6: Deriving a marginal distribution

What is the probability that $Y = 1$, regardless of what X is? That is, what is $P(Y = 1)$? [4]

[4] Remember! To find the marginal distribution of one discrete random variable, sum over the other variable! In our case:

$$f_Y(y) = P(Y = y) = \sum_{x=0}^{\infty} f_{XY}(x, y).$$

Answer to Problem 6.

After answering this, what is the probability that $Y = y$, for any value y? Is it always $\frac{1}{5} = 20\%$?

Problem 7: Deriving a conditional distribution

What is the probability that $X \geq 1$ given that $Y = 1$? That is, what is $P(X \geq 1 | Y = 1)$? [5]

[5] You will probably get something similar to what we had seen earlier in the semester...

Answer to Problem 7.

Worksheet 3: Jointly distributed continuous random variables

Let X and Y be two random variables that are allowed to take any value between 0 and 1: that is, $0 \leq X \leq 1$ and $0 \leq Y \leq 1$. We further assume that they are jointly distributed with probability density function:

$$f_{XY}(x, y) = \frac{12}{11}\left(x^2 + y^2 + xy\right).$$

Problem 8: Calculating a probability

Recall that with continuous random variables, we need to integrate properly to calculate a probability. With that in mind, what is the probability that $0.3 \leq X \leq 0.7$ and $Y \geq 0.75$? [6]

Answer to Problem 8.

[6] You'll need to do a double integration. I'll get you started:

$$\int_{0.3}^{0.7}\int_{0.75}^{1} \ldots\ldots$$

Problem 9: Deriving a marginal distribution

Again, like you did earlier, derive the two marginal distributions. However, remember, that we are no longer summing! In the continuous space, we integrate. With that in mind, what is the marginal distribution of X? What is the marginal distribution of Y? [7]

Answer to Problem 9.

[7] In our case, because X and Y are only allowed to be between 0 and 1, we have (for Y, but it is very similar for X):

$$f_Y(y) = \int_{x=0}^{1} f_{XY}(x, y)dx.$$

Problem 10: Deriving the conditional distribution

What is $P(X \leq 0.5|Y = 1)$? [8]

Answer to Problem 10.

Think about your approach. Could you have calculated

$$P(X \leq 0.5|Y = y)?$$

Problem 11: Final touch

This is a little tougher. What is $P(X \leq Y)$? [9]

Answer to Problem 11.

[8] For the conditional distribution, apart from the fact we are integrating instead of summing, we follow exactly the same logic as for discrete random variables. Remember to divide appropriately!

[9] Think! You have $P(X \leq x|Y = y)$. You also can calculate $P(X \leq y|Y = y)$! Isn't this the last probability equal to $P(X \leq y)$?