Lecture 8 Worksheet

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Every worksheet will work as follows.

1. You will be entered into a Zoom breakout session with other students in the class.
2. Read through the worksheet, discussing any questions with the other participants in your breakout session.
   - You can call me using the "Ask for help" button.
   - Keep in mind that I will be going through all rooms during the session so it might take me a while to get to you.
3. Answer each question (preferably in the order provided) to the best of your knowledge.
4. While collaboration between students in a breakout session is highly encouraged and expected, each student has to submit their own version.
5. You will have 24 hours (see Compass) to submit your work.

Worksheet 1: Exponential, Poisson, and Erlang

A job requires 4 steps to be completed. Each step requires time that is exponentially distributed with a rate of 1 completion every 3 minutes. Steps are completed sequentially. Answer the following questions.

Problem 1

What is the probability the 2nd step is completed within 4 minutes? [1]

Answer to Problem 1.

[1] Is the time in which a step is completed Poisson, exponential, or Gamma? Based on your answer, does it matter if you were asked about any other step or would your answer stay the same?

Problem 2

What is the probability there are exactly 3 steps that are completed in the first 10 minutes? [2]

Answer to Problem 2.

[2] First, calculate the rate of completed steps in 10 minutes; then decide if you are using Poisson, exponential, or Erlang..

Problem 3

What is the probability that the 1st job (all 4 steps, one after the other) is completed in the first 10 minutes? [3]

[3] Feel free to use an online calculator for your integral.

Answer to Problem 3.

Worksheet 2: Normal distribution

In this part of the worksheet, we turn our focus to the normal distribution.

Problem 4: Converting to z values

Let's practice with converting to the proper z values. Let X be a normally distributed random variable with $\mu = 10, \sigma^2 = 4$.

Answer to Problem 4.

- $X = 12 \implies z =$
- $X = 8 \implies z =$
- $X = 4 \implies z =$

Problem 5: Simple normal distribution probabilities

For the previous random variable $X \sim \mathcal{N}(10, 4)$, find the probabilities. Use the z values you calculated earlier. [4]

[4] A z-table as described in the lecture notes is provided in the last page of the worksheet. Also recall that $\Phi(-z) = 1 - \Phi(z)$ due to symmetry.

Answer to Problem 5.

- $P(X \leq 12) =$
- $P(X \geq 4) =$
- $P(4 \leq X \leq 12) =$

Problem 6: Interesting probabilities

As we saw in class, the normal distribution is centered at μ. [5] A follow-up question would be to find the range of values centered at μ that satisfy a certain probability. Let's see an example here: what is the probability that X is within 1 unit from its mean, that is what is the probability that X is between 9 and 11? How about 2 units from its mean?

[5] So, following the previous random variable X, it would be centered at 10.

Answer to Problem 6.

- $P(9 \leq X \leq 11) =$
- $P(8 \leq X \leq 12) =$

Worksheet 3: Normal distribution revisited

Problem 7

Let's now focus on the opposite problem. What should the range be (centered at μ) so that the probability of the range is 50%? In essence, what should a be in order for $P(\mu - a \leq X \leq \mu + a) = 0.5$? Remember that earlier we calculated two range probabilities:

1. $P(9 \leq X \leq 11) = 0.383$.
2. $P(8 \leq X \leq 12) = 0.6826$.

Based on that, we should anticipate a to fall somewhere above 1 unit but below 2 units. But how big should it be exactly? Recall that we assume that $X \sim \mathcal{N}(10, 4)$. [6]

[6] $\mu = 10, \sigma^2 = 4 \implies \sigma = 2$.

Answer to Problem 7.

Problem 8

For $X \sim \mathcal{N}(10, 4)$, we want to see how big the range should be in order to have a probability equal to 95% that X falls in that range. In essence, we are now interested in $P(\mu - a \leq X \leq \mu + a) = 0.95$. Graphically:

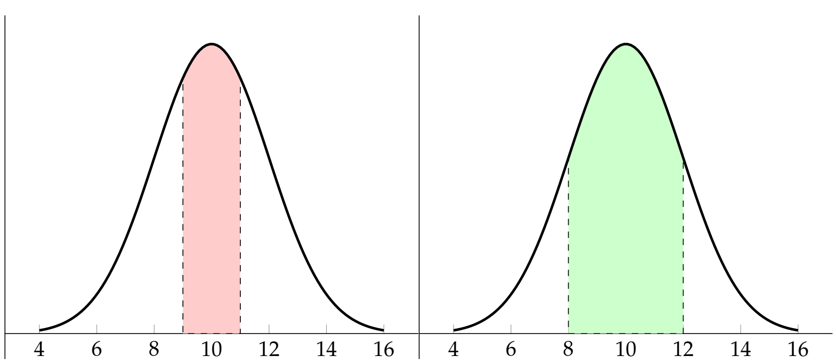
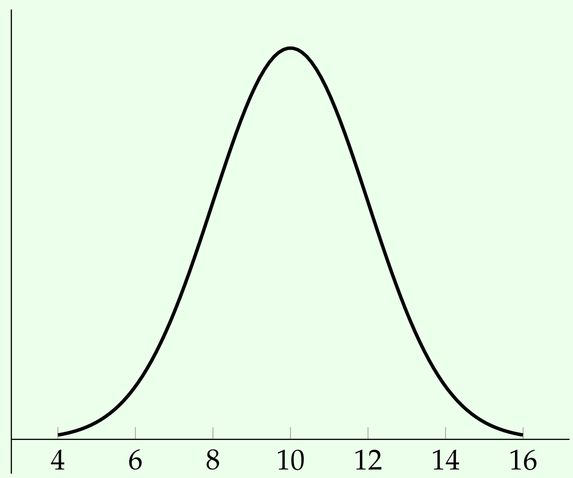


Figure 1: What should a be for $P(\mu - a \leq X \leq \mu + a) = 0.95$? Here we show in red the area for $P(\mu - 1 \leq X \leq \mu + 1) = 0.383$ and in green the area for $P(\mu - 2 \leq X \leq \mu + 2) = 0.6826$.

Answer to Problem 8.

After you have found a, try to draw the resultant area!



Problem 9

We may have started observing that a does not depend on μ all that much. Instead it depends on σ. For example, seeing as we may write $P(\mu - a \leq X \leq \mu + a) = 0.5$ as a probability of z as follows:

1. $z_1 = \frac{\mu-\alpha-\mu}{\sigma} = -\frac{\alpha}{\sigma}$
2. $z_2 = \frac{\mu+\alpha-\mu}{\sigma} = \frac{\alpha}{\sigma}$.
3. Note that $z_1 = -z_2$.

Based on that: $P(\mu - a \leq X \leq \mu + a) = P(z_1 \leq Z \leq z_2)$. Now recall that $P(z_1 \leq Z \leq z_2) = \Phi(z_2) - \Phi(z_1) = \Phi(z_2) - \Phi(-z_2) = \Phi(z_2) - (1 - \Phi(z_2)) = 2\Phi(z_2) = 2\Phi(\frac{\alpha}{\sigma}) - 1$.

With that in mind, answer the following three questions. Try to answer them generally, not only for $X \sim \mathcal{N}(10, 4)$.

Answer to Problem 9.

- $P(\mu - \sigma \leq X \leq \mu + \sigma) =$
- $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) =$
- $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) =$

Based on our answers, we have the following realization: σ is very important. Any normally distributed random variable probability can be expressed as a "distance" in terms of "σ". In essence:

$$\boxed{X = \mu + z\sigma \implies F(X) = \Phi(z).}$$

Worksheet 4: Contrasting exponentials

Back to the exponential distribution. Consider two exponentially distributed random variables X_1, X_2 with rates λ_1, λ_2.

Problem 10

What is the probability of $X_1 > X_2$, given that $X_2 = x$? [7]

Answer to Problem 10.

$$P(X_1 > X_2 | X_2 = x) = P(X_1 > x) =$$

[7] Can't we say that $P(X_1 > X_2 | X_2 = x)$ is simply $P(X_1 > x)$?

Problem 11

What is the probability of $X_1 > X_2$, in general? Recall the total probability law? [8] We can apply this to continuous distributions, too! We cannot sum here, but we may integrate. Let X_1 be random variable distributed with pdf $f(\cdot)$ and X_2 be a random variable distributed with pdf $g(\cdot)$, then:

$$\boxed{P(X_1 > X_2) = \int_{-\infty}^{+\infty} P(X_1 > X_2 | X_2 = x) g(x) dx.}$$

[8] For an event B, and m mutually exclusive and collectively exhaustive events A_i, $i = 1, \ldots, m$, then we have $P(B) = \sum_{i=1}^{m} P(B|A_i) \cdot P(A_i)$.

Use this to answer the following question:

Answer to Problem 11.

$P(X_1 > X_2) =$

From this last part, we see that for two exponentially distributed random variables X_1, X_2 with rates λ_1, λ_2, respectively, we have:

$$\boxed{P(X_1 \leq X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}}$$

Problem 12

Two employees are using a website to place an order with a supplier at exactly the same time. The first person is more tech savvy and completes an order with rate 1 order every 3 minutes. The second person is just starting the job and learning, so they are a little slower and complete an order with rate 1 order every 5 minutes. Both times are exponentially distributed. What is the probability that the second person completes the order faster than or equal to the time the first person takes to complete an order?

Answer to Problem 12.

NORMAL CUMULATIVE DISTRIBUTION FUNCTION ($\Phi(z)$)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000