Lecture 5 Worksheet

Chrysafis Vogiatzis

Every worksheet will work as follows.

1. You will be entered into a Zoom breakout session with other students in the class.
2. Read through the worksheet, discussing any questions with the other participants in your breakout session.
   - You can call me using the "Ask for help" button.
   - Keep in mind that I will be going through all rooms during the session so it might take me a while to get to you.
3. Answer each question (preferably in the order provided) to the best of your knowledge.
4. While collaboration between students in a breakout session is highly encouraged and expected, each student has to submit their own version.
5. You will have 24 hours (see Compass) to submit your work.

Worksheet 1: Basic discrete random variable questions

Consider a discrete random variable X with probability mass function

$$p(x) = \begin{cases} \frac{2x+2}{c}, & x = 0, 1, 2, 3, 4 \\ 0, & \text{otherwise.} \end{cases}$$

Problem 1: Probability mass functions

What is the value of c for which $p(x)$ is a valid probability mass function?

Answer to Problem 1.

Problem 2: Constructing cumulative distribution functions

Using the same probability mass function $p(x)$ that you were given in Problem 1 (replace the value for c that you calculated), what is the cumulative distribution function of discrete random variable X?

Answer to Problem 2.

Problem 3: Calculating probabilities

Answer the following questions:

- What is $P(1 \leq X \leq 3)$?
- What is $P(1 < X \leq 3)$?

Answer to Problem 3.

Worksheet 2: Deriving the binomial distribution pmf

In this part of the worksheet, we will use a small example to derive the formula for the probability mass function of the binomial distribution. [1]

A group of students is participating in a seminar class that requires each student to take a one-question exam in the end of the semester. The class has been historically difficult, and 70% of the students pass, while the remaining 30% fails. This semester, the class has 3 students.

[1] Hint: as a reminder, we should finally obtain that

$$P(X = x) = \binom{n}{x} p^x \cdot (1-p)^{n-x}.$$

Problem 4

This seminar class could end up with:

- All students pass.
- 2 students pass.
- 1 student passes.

- No students pass.

We would like to calculate the probability of each of them. Before that, though, enumerate all the possible scenarios we may have. [2] How many scenarios are there in total?

[2] Hint: assume that the students are given a grade of P or F, then you could have PPP (everyone passes) or PFP (the first and third students pass, the second one fails), among others.

Answer to Problem 4.

Problem 5

Assuming that the performance of one student does not affect the performance of any other student, what is the probability that every student passes and what is the probability that every student fails?

Answer to Problem 5.

- $P(PPP) =$
- $P(FFF) =$

Problem 6

Under the same independence assumption from Problem 5, what is the probability that the first student is the only one that passes? What is the probability that the second student is the only one that passes? And what is the probability that the third student is the only one that passes? How would that calculation change if instead of 3 students, we had n students?

Answer to Problem 6.

- $P(PFF) =$
- $P(FPF) =$
- $P(FFP) =$
- $P(\underbrace{PFF\ldots F}_{n \text{ students}}) =$

Problem 7

Based on your answers in Problem 6 and the scenarios in Problem 4, what is the probability that exactly one student passes?

Answer to Problem 7.

$P(\text{one student passes}) =$

Problem 8

Can we generalize the result in Problem 7? If we had n students, and we wanted to have exactly $x \leq n$ of them pass, how many probabilities would we have to sum up? [3]

Answer to Problem 8.

$P(x \text{ students pass}) =$

[3] Recall that in Problem 7, you had to add 3 of the probabilities, as there were 3 possible scenarios. In general, though, you would want to enumerate all possible *combinations* of x elements from a total of n elements...

Worksheet 3: Deriving the geometric distribution pmf

This will be much easier. In the questions that follow, you will be asked to derive the probability mass function for the geometric distribution.

Problem 9

A juggler repeats the same sequence of operations until they are all successful. The sequence is successful with probability $p = 0.1$. What is the probability that:

- they are successful in their first try?
- they are unsuccessful in their first try, but they are successful in the second one?
- they are unsuccessful in their first two tries, but they are successful in the third one?
- they are unsuccessful in their first $n - 1$ tries, but they are successful in the n-th one?

Let U be the event that the operations were unsuccessful and S the event that they were successful.

Answer to Problem 9.

- $P(S) =$
- $P(US) =$
- $P(UUS) =$
- $P(\underbrace{U \ldots US}_{n \text{ attempts}}) =$

Problem 10

Based on your answers in Problem 9, derive the probability mass function for the geometric distribution. If the probability of success is p and the probability of failure is $q = 1 - p$, what is the probability that it takes exactly x tries to get to the first success?

Answer to Problem 10.

Worksheet 4: Binomial, geometric, or hypergeometric?

Problem 11

A foundry has received an order for **5 castings**, made from precious metals. Each casting produced is of high quality (and hence can be sold to the customer) with probability 0.97. All castings are produced independently. What is the probability you schedule 6 castings for production and get more than or equal to 5 high quality castings to sell to the customer?

Answer to Problem 11.

Problem 12

The foundry from earlier has received the same order for **5 castings**. However, instead of producing new ones, they use a batch of older castings already produced. The batch contains 100 castings, 97 of whom are of high quality. They decide to pick 6 of them and give them to the customer. What is the probability that the customer more than or equal to 5 high quality castings in the sample?

Answer to Problem 12.

Problem 13

5% of all bits (a signal of 0 or 1) transmitted are sent in error (a 0 is sent instead of a 1, or vice versa). The message stops transmitting when the first bit is transmitted in error. Let X be the length of the message. What is the probability that $X = 5$? What is the probability that $X \leq 5$?

Answer to Problem 13.