QUESTION1: Use Cramer's Rule to solve the system of linear equation given below 2x – y + 3z = 0; x + 5y – 7z = 0; x – 6y + 10z = 0

Solution

Evaluate Δ first. We have

$$\Delta=\begin{bmatrix}2 & -1 & 3\\1 & 5 & -7\\1 & -6 & 10\end{bmatrix}$$

Apply $R_1 \longrightarrow R_1-2R_2$, $R_2 \longrightarrow R_2-R_3$

$$=\begin{bmatrix}0 & -11 & 17\\0 & 11 & -17\\1 & -6 & 10\end{bmatrix}$$

$\Delta=0$ [because R_1 and R_2 are proportional]

Therefore, the given system of linear homogeneous equations has an infinite number of solutions.

Now, Rewrite the equation

$2x-y=-3z$

$x+5y=7z$

$$\Delta=\begin{vmatrix}2 & -1\\1 & 5\end{vmatrix}$$

$\Delta=10+1=11$

$\Delta\neq 0$

Now we have,

$$\Delta x=\begin{vmatrix}-3z & -1\\7z & 5\end{vmatrix}$$

$=-15z+7z=-8z$

$$\Delta x=\begin{vmatrix}2 & -3z\\1 & 7z\end{vmatrix}$$

$=14z-(-3z)=17z$

BY using Cramer's Rule

$$x=\frac{\Delta x}{\Delta}=\frac{-8z}{11}$$

$$y=\frac{\Delta y}{\Delta}=\frac{17z}{11}$$

QUESTION2:Find the inverse of A= $\begin{bmatrix}1 & 2 & 5\\2 & 3 & 1\\-1 & 1 & 1\end{bmatrix}$ **and verify that $A^{-1}A=I_3$**

Ans

$$|A|=1\begin{vmatrix}3 & 1\\1 & 1\end{vmatrix}-2\begin{vmatrix}2 & 1\\-1 & 1\end{vmatrix}+5\begin{vmatrix}2 & 3\\-1 & 1\end{vmatrix}$$

= 1(3-1)-2(2+1)+5(2+3)

= 1×2-2×3+5×5

= 21

Since $|A| \neq 0, A\ is\ invertible.$

Evaluating the cofactors of the elements

$A_{11}= (-1)^{1+1}\begin{vmatrix}3 & 1\\1 & 1\end{vmatrix} = 2$ $A_{12}= (-1)^{1+2}\begin{vmatrix}2 & 1\\-1 & 1\end{vmatrix} = -3$ $A_{13}= (-1)^{1+3}\begin{vmatrix}2 & 3\\-1 & 1\end{vmatrix} = 5$

$A_{21}= (-1)^{2+1}\begin{vmatrix}2 & 5\\1 & 1\end{vmatrix} = 3$ $A_{22}= (-1)^{2+2}\begin{vmatrix}1 & 5\\-1 & 1\end{vmatrix} = 6$ $A_{23}= (-1)^{2+3}\begin{vmatrix}1 & 2\\-1 & 1\end{vmatrix} = -3$

$A_{31}= (-1)^{3+1}\begin{vmatrix}2 & 5\\3 & 1\end{vmatrix} = -13$ $A_{32}= (-1)^{3+2}\begin{vmatrix}1 & 5\\2 & 1\end{vmatrix} = 9$ $A_{33}= (-1)^{3+3}\begin{vmatrix}1 & 2\\2 & 3\end{vmatrix} = -1$

$$\text{AdjA}=\begin{bmatrix}2 & -3 & 5\\3 & 6 & -3\\-13 & 9 & -1\end{bmatrix}^{-1} = \begin{bmatrix}2 & 3 & -13\\-3 & 6 & 9\\5 & -3 & -1\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}adjA=\frac{1}{21}\begin{bmatrix}2 & 3 & -13\\-3 & 6 & 9\\5 & -3 & -1\end{bmatrix}$$

$$=\begin{bmatrix}2/21 & 3/21 & -13/21\\-3/21 & 6/21 & 9/21\\5/21 & -3/21 & -1/21\end{bmatrix}$$

$$A^{-1}A=\begin{bmatrix}2/21 & 3/21 & -13/21\\-3/21 & 6/21 & 9/21\\5/21 & -3/21 & -1/21\end{bmatrix}\begin{bmatrix}1 & 2 & 5\\2 & 3 & 1\\-1 & 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\frac{2}{21}+\frac{6}{21}+\frac{13}{21} & \frac{4}{21}+\frac{9}{21}+\frac{13}{21} & \frac{10}{21}+\frac{3}{21}-\frac{13}{21}\\-\frac{3}{21}+\frac{12}{21}-\frac{9}{21} & -\frac{6}{21}+\frac{18}{21}+\frac{9}{21} & -\frac{15}{21}+\frac{6}{21}+\frac{9}{21}\\\frac{5}{21}-\frac{6}{21}+\frac{1}{21} & \frac{10}{21}-\frac{9}{21}-\frac{1}{21} & \frac{25}{21}-\frac{3}{21}-\frac{1}{21}\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

QUESTION3: Solve the system of equation using matrix method 2x – y + 3z = 5; 3x + 2y – z = 7; 4x + 5y – 5z = 9

Ans

We can rewrite the above system of equations as the single matrix equation AX =0, where

$$A=\begin{bmatrix}2 & -1 & 3\\3 & 2 & -1\\4 & 5 & -5\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } B=\begin{bmatrix}5\\7\\9\end{bmatrix}$$

$$|A| = 2\begin{vmatrix}2 & -1\\5 & -5\end{vmatrix} + 1\begin{vmatrix}3 & -1\\4 & -5\end{vmatrix} + 3\begin{vmatrix}3 & 2\\4 & 5\end{vmatrix}$$

2(-10+5)+1(-15+4)+3(15-8)

-10-11+21=0

Here, |A|=0

A is s singular matrix.

Thus, AX = B has an infinite number of solutions. To find these solutions, we write $2x - y = 5 - 3z$, $3x + 2y = 7 + z$ or as a single matrix equation

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5-3z \\ 7+z \end{bmatrix}$$

Here, |A| = 7 ≠ 0

Since |A| ≠ 0, A is an invertible matrix Now,

$A_{11}=2$ $A_{12}=-3$ $A_{21}=1$ $A_{22}=2$

adj A $=\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

Therefore, from $X = A^{-1}B$, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 5-3z \\ 7+z \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 10-6z+7+z \\ -15+9z-21+2z \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 17-5z \\ -15+9z+14+2z \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 17-6z \\ -1+11z \end{bmatrix}$$

$X=\frac{17-6z}{7}$ $y=\frac{-1+11z}{7}$

Question4:

$$\begin{pmatrix} 2 & 5 & -3 & -4 \\ 4 & 7 & -4 & -3 \\ 6 & 9 & -5 & 2 \\ 0 & -9 & 6 & 5 \end{pmatrix}$$

Let A= $\begin{pmatrix} 2 & 5 & -3 & -4 \\ 4 & 7 & -4 & -3 \\ 6 & 9 & -5 & 2 \\ 0 & -9 & 6 & 5 \end{pmatrix}$

Applying $R_2 \rightarrow R_2 - 2R1$ and $R3 \rightarrow R_3 - 3R_1$

$= \begin{pmatrix} 2 & 5 & -3 & -4 \\ 0 & -3 & 2 & 5 \\ 0 & -6 & 4 & 14 \\ 0 & -9 & 6 & 15 \end{pmatrix}$

$= \begin{pmatrix} 2 & 5 & -3 & -4 \\ 0 & -3 & 2 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$|B|=0$

So,

$= \begin{pmatrix} 2 & 5 & -4 \\ 0 & -3 & 5 \\ 0 & 0 & 4 \end{pmatrix}$ is a square sub matrix

$|A|$ =2(---3 X 4 – 0 X 5)

= $-24 \neq 0$

Rank of this matrix is 3

QUESTION5: Show that x(x + 1) (2x + 1) is a multiple of 6 for every natural number x.

Solution : Let $P_n$denote the statement $X(X+1)(2X+1)$ is a multiple of 6.

When X=1, P_n becomes 1(1 + 1) ((2)(1) + 1) = (1)(2)(3) = 6 is a multiple of 6.

This shows that the result is true for n = 1.

Assume that P_k is true for some $k \in$ **N.**

That is assume that $k(k + 1)(2k + 1)$ is a mutliple of 6.

Let $k(k + 1)(2k + 1) = 6m$ for some $m \in$ **N.**

We now show that the truth of P_k implies the truth of P_{k+1}, where P_{k+1} is $(k + 1)(k + 2)[2(k + 1) + 1] = (k + 1)(k + 2)(2k + 3)$ is a multiple of 6.

We have $(k + 1)(k + 2)(2k + 3) = (k + 1)(k + 2)[(2k + 1) + 2]$

$= (k+ 1)[k(2k + 1) + 2(2k+1) + 4)]$

$= (k+ 1)[k(2k + 1) + 6(k + 1)]$

$= k(k + 1)(2k + 1) + 6(k + 1)^2 = 6m + 6(k + 1)^2$

$= 6[m + (k + 1)^2]$

Thus $(k + 1)(k + 2)(2k + 3)$ is multiple of 6.

QUESTION6: Find the sum of the series $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2$

Solution-

Let t_r denote the rth term of $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2$, then

$t_r = (2r - 1)^2 = 4r^2 - 4r + 1$

$\sum_{r=1}^{n} t_r = \sum_{r=1}^{n} 4r^2 - 4r + 1 = 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1), \sum_{r=1}^{n} r = \frac{1}{2} n(n+1)$ and $\sum_{r=1}^{n} 1 = n$

$\sum_{r=1}^{n} t_r = 4\{\frac{1}{6} n(n+1)(2n+1) - 4\left\{\frac{1}{2n(n+1)}\right\} + 2$

$= 2/3n(n + 1)(2n + 1) - 2n(n + 1) + n.$

We now take $1/3n$ common from each on the right side, so that

$\sum_{r=1}^{n} t_r = 1/3\, n[2(n + 1)(2n + 1) - 6(n + 1) + 3)]$

$= 1/3\, n[2(2n2 + 2n + n + 1) - (6n + 6) + 3]$

$= 1/3\, n[(4n2 + 6n + 2 - 6n - 6 + 3) = 1/3\, n(4n^2 - 1)$

QUESTION7: If 1, ω, ω^2 are three cube roots of unity. Show that :
$(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})=49$

Ans-

Since ,

$\omega^{10} = (\omega^3)^3\omega$

$\omega^{11} = (\omega^3)^3\omega^2$

Thus,

$(2-\omega)(2-\omega^2)(2-(\omega^3)^3\omega)(2-(\omega^3)^3\omega^2)$

$(2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$

$[(2-\omega)(2-\omega^2)]^2$
$[4-2\omega^2-2\omega+\omega^3]^2$
$[4-2(\omega^2+\omega)+\omega^3]^2$
$[4-2(-1)+1]^2$
$[4+2+1]^2$
49=RHS

QUESTION9: Solve the inequality $\frac{2}{|x-3|} > 5$ and graph its solution.

Solution

The domain of the inequality is$\{ x \mid x \neq 3)$

For $x \neq |x-3| > 0$

Thus, the given inequality

$= \frac{2}{|x-3|} > 5$

$= 2 > 5|x-3|$

$= |x-3| < \frac{2}{5}$

$= \frac{-2}{5} < x-3 < \frac{2}{5}$

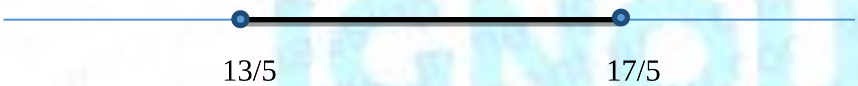
$= 3-\frac{2}{5} < x-3+3 < \frac{2}{5}+3$

$= \frac{13}{5} < x < \frac{17}{5}$

the solution set of the inequality is

$\{x|13/5 < x < 17/5\} = (13/5, 17/5)$

The graph of this set is



QUESTION 10:Show that f(x) = | x | is continuous at x = 0.

Solution

$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

To show that f is continuous at $x = 0$, it is sufficient to show that

$\lim_{x\to0^+} f(x) = \lim_{x\to0^+}(x) = f(0)$and

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-}(0-h) = f(-h)$

$= \lim_{h\to0^+} -(-h)$

$= \lim_{h\to0^+}(h) = 0$

And $\lim_{h\to0^+} f(x) = \lim_{x\to0^+}(0+h) = f(h)$

$= \lim_{h\to0^+}(h)$

$= \lim_{h\to0^+}(h) = 0$

Thus,

$\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = 0$

F(0)=0

Hence, f is continuous at $x = 0$.

QUESTION11: Find derivative of the following (i) x^2e^x(ii) ln x/x

Solution

(i) x^2e^x

Using product rule,

$\frac{d}{dx}(x^2e^x) = \frac{d}{dx}(x)^2e^x + x^2\frac{d}{dx}e^x$

$=2xe^x + x^2e^x$

$= (2x + x^2)e^x$

(ii)ln x/x

Using the quotient rule,

$$\frac{d}{dx}\frac{lnx}{x} = \frac{x\frac{d}{dx}(lnx) - lnx\frac{d}{dx}(x)}{x^2}$$

$= \frac{x.\frac{1}{x} - lnx\ (1)}{x^2}$

$= \frac{1 - lnx}{x^2}$

QUESTION 12:If Y = ln (x +$\sqrt{x^2+1}$), Prove that (x^2+ 1)$\frac{d^2y}{dx^2}$ + x$\frac{dy}{dx}$ = 0

Solution

Y= ln (x +$\sqrt{x^2+1}$)

$\frac{dy}{dx} = \frac{d}{dx}$ ln (x $+\sqrt{x^2+1}$

$= \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}\left[x + (x^2+1)^{\frac{1}{2}}\right]$ [By using chain Rule]

$= \frac{1}{x+\sqrt{x^2+1}}\left[1 + \frac{1}{2}(x^2+1)^{\frac{1}{2}}2x\right]$

$=\frac{1}{x+\sqrt{x^2+1}}\left[1 + \frac{x}{\sqrt{x^2+1}}\right]$

$=\frac{1}{x+\sqrt{x^2+1}}\left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right]$

$= \frac{1}{\sqrt{x^2+1}} = (x^2+1)^{-1/2}$

Now Second Derivative we need to calculate

$\frac{d^2y}{dx^2} = (x^2+1)^{-1/2}$

$\left(-\frac{1}{2}\right)(x^2+1)^{\frac{-3}{2}}\frac{d}{dx}[x^2+1]$

$-\frac{1}{2}\frac{1}{(x^2+1)^{\frac{3}{2}}}2x = -\frac{x}{(x^2+1)^{\frac{3}{2}}}$

Now, (x^2+ 1)$\frac{d^2y}{dx^2}$ + x$\frac{dy}{dx}$

=(x^2+ 1)$\frac{-x}{(x^2+1)^{\frac{3}{2}}} + (x^2+1)^{-1/2}$

=(x^2+ 1)$\frac{-x}{(x^2+1)^{\frac{3}{2}}} + x\frac{1}{\sqrt{x^2+1}}$

$= -\frac{x}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}}$=0

Thus,

(x^2+ 1)$\frac{d^2y}{dx^2}$ + x$\frac{dy}{dx}$ = 0

QUESTION 13 If a camphor ball evaporates at a rate proportional to its surface area 4πr^2 .Show that its radius decreases at a constant rate.

Solution

Let *r* be the radius *V* be the volume of the camphor ball at time *t*. Then

V=$\frac{4}{3}\pi r^3$

We are given that the camphor ball evaporates at a rate proportional to its surface area. This means that the rate of decrease of volume V of the camphor ball is proportional to $4\pi r^2$

So that, $\frac{dv}{dt} = -k(4\pi r^2)$

where $k > 0$ is a constant.

$\frac{dv}{dt} = (4\pi r^2)\frac{dr}{dt}$

$= (4\pi r^2)\frac{dr}{dt} = -k4\pi r^2$

$= \frac{dr}{dt} = -k$

[Negative sign has been introduced to show that the volume is decreasing.]

QUESTION 14: Determine the intervals in which the function f(x) = e $^{1/x}$. (x ≠ 0) is increasing or decreasing .

Solution

We have, for $x \neq 0$ $f'(x) = e^{1/x}\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}e^{\frac{1}{x}}$

As $e^{\frac{1}{x}} > 0$ and $x^2 >= 0$ $\forall\, x \neq 0$ we get $f'(x) < 0$ $\forall\, x \neq 0$.

Thus, $f(x)$ decreases on $(-\infty, 0)$ $(0, \infty)$

QUESTION 15: Find local maximum and local minimum values for f(x) = x^3– 6x^2+ 9x +1. (x€R).

Solution

Thus, $f'(x) = 3x^2 - 12x + 9 = 3(x^2-4x+3) = 3(x-3)(x-1)$.

To obtain critical number of f, we set $f'(x) = 0$ this yields $x = 1, 3$.

Therefore, the critical number of f are $x = 1, 3$.

Now $f''(x) = 6x - 12 = 6(x - 2)$

We have $f''(1) = 6(1 - 2) = -6 < 0$ and $f''(3) = 6(3 - 2) = 6 > 0$.

Using the second derivative test, we see that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = 3$.

The value of local maximum at $x = 1$ is

$f(1) = 1 - 6 + 9 + 1 = 5$

and the value of local minimum at $x = 3$ is

$f(3) = 3^3 - 6(3)^2 + 9(3) + 1 = 27 - 54 + 27 + 1 = 1$.

QUESTION 16: Evaluate the integral

(ii) $I = \int x^3 (\log x)^2\, dx$

Sol: See Block-3, Unit-3, Example-24, Page 93.

QUESTION 17: Find the area bounded by curve

Solution: Block -3, Unit-4, Example-13, Page-115

QUESTION 18. Find the length of the curve Y = 2x + 3

Solution: Block -3, Unit-4, Example-17,Page-117

QUESTION 19: Prove that the straight line joining the mid points of two non-parallel sides of a trapezium is parallel to the parallel sides and half of their sum.

Sol: See Block-4,Unit-1,Page20, Example-11

QUESTION 20: Find maximum values of 5 x + 2y, subject to the following constraints. – 2x – 3y ≤ – 6 ; x – 2y ≤ 2 ; 6x + 4y ≤ 24 ; – 3x + 2y ≤ 3; x ≥ 0, y ≥ 0.

Sol: See IGNOU Block-4, Page-84, Unit-4, Example-1