UNIT 1 DIFFERENTIAL CALCULUS

Structure

1.0 INTRODUCTION

In this Unit, we shall define the concept of limit, continuity and differentiability.

1.1 OBJECTIVES

After studying this unit, you should be able to :

- define limit of a function;
- define continuity of a function; and
- define derivative of a function.

1.2 LIMITS AND CONTINUITY

We start by defining a function. Let A and B be two non empty sets. A function f from the set A to the set B is a rule that assigns to each element x of A a unique element y of B.

We call y the image of x under f and denote it by $f(x)$. The domain of f is the set A, and the co−domain of f is the set B. The range of f consists of all images of elements in A. We shall only work with functions whose domains and co–domains are subsets of real numbers.

Given functions f and g, their sum $f + g$, difference $f - g$, product $f.\ g$ and quotient f / g are defined by

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$

$$(f . g)(x) = f(x)\, g(x)$$
and $$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$$

For the functions $f + g$, $f - g$, $f.\,g$, the domain is defined to be intersections of the domains of f and g, and for $f\,/\,g$ the domain is the intersection excluding the points where $g(x) = 0$.

The composition of the function f with function g, denoted by fog, is defined by $(fog)\,(x) = f\,(g(x))$.

The domain of fog is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Limit of a Function

We now discuss intuitively what we mean by the limit of a function. Suppose a function f is defined on an open interval (α, β) except possibly at the point $a \in (\alpha, \beta)$ we say that

$f(x) \to L$ as $x \to a$

(read $f(x)$ approaches L as x approaches a), if $f(x)$ takes values very, very close to L, as x takes values very, very close to a, and if the difference between $f(x)$ and L can be made as small as we wish by taking x sufficiently close to but different from a.

As a mathematical short hand for $f(x) \to L$ as $x \to a$, we write

$$\lim_{x \to a} f(x) = L.$$

Example 1 : Evaluate $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

Solution : Let $f(x) = \dfrac{x^2 - 9}{x - 3}$. This function is defined for each x except for $x = 3$. This function is defined for each x except for $x = 3$. Let us calculate the value of f at $x = 3 + h$, where $h \neq 0$. We have

$$f\,(3 + h) = \frac{(3 + h)^2 - 9}{3 + h - 3} = \frac{9 + 6h + h^2 - 9}{h} = \frac{h(6 + h)}{h} = 6 + h$$

We now note that as x takes values which are very close to 3, that is, h takes values very close to 0, $f(3 + h)$ takes values which are very close to 6. Also, the difference between $f\,(3 + h)$ and 6 (which is equal to h) can be made as small as we wish by taking h sufficiently close to zero.

Thus,

$$\lim_{x \to 3} f(x) = 6$$

Properties of Limits

We now state some properties of limit (without proof) and use them to evaluate limits.

Theorem 1 : Let a be a real number and let $f(x) = g(x)$ for all $x \neq a$ in an open interval containing a. If the limit $g(x)$ as $x \to a$ exists, then the limit of $f(x)$ also exists, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

Theorem 2 : If c and x are two real numbers and n is a positive integer, then the following properties are true :

(1) $\lim_{x\to a} c = c$

(2) $\lim_{x\to a} x = a$

(3) $\lim_{x\to a} x^n = a^n$

Theorem 3 : Let c and a be two real numbers, n a positive integer, and let f and g be two functions whose limit exist as $x \to a$. Then the following results hold :

1. $\lim_{x\to a} [c\, f(x)] = c\left[\lim_{x\to a} f(x)\right]$

2 $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)]\,[\lim_{x\to a} g(x)]$

4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0,$

5. $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$

6. If $\lim_{x\to a} f(x) = f(a)$, then $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)}$

Example 2: Evaluate $\lim_{x\to 3} (4x^2 + 7)$

Solution : $\lim_{x\to 3} (4x^2 + 7) = \lim_{x\to 3} 4x^2 + \lim_{x\to 3} 7$

$= 4\lim_{x\to 3} x^2 + \lim_{x\to 2} 7$

$= 4(3)^2 + 7 = 4 \times 9 + 7$

$= 43$

Note : If $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.
If $q(x)$ is also a polynomial and $q(a) \neq 0$, then

$$\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

Example 3: Evaluate the following limits :

(i) $\lim_{x\to 2} [(x- 1)2 + 6]$ (ii) $\lim_{x\to 0} \frac{ax+b}{cx+d}$ $(d \neq 0)$

(iii) $\lim_{x\to 2} \frac{x^2+5x+7}{x^2+8}$ (iv) $\lim_{x\to -1} \sqrt{x+17}$

Solution: (i) $\lim_{x\to 2}$ $[(x-1)^2+6] = (2-1)^2+6 = 1+6 = 7$

(ii) Since $\lim_{x\to 0} cx + d = d \neq 0$,

$$\lim_{x\to 0} \frac{ax+b}{cx+d} = \frac{a(0)+b}{c(0)+d} = \frac{b}{d}$$

(iii) Since $\lim_{x\to 3}$ $(x^2+8) = 3^2+8 = 17 \neq 0$,

$$\therefore \lim_{x\to 3} \frac{x^2+5x+7}{x^2+8} = \frac{3^2+5(3)+7}{3^2+8} = \frac{31}{17}$$

(iv) Since $\lim_{x\to -1}$ $x + 17 = -1 + 17 = 16$, we have

$$\lim_{x\to -1} \sqrt{x+17} = \sqrt{16} = 4$$

Example 4: Evaluate the following limits.

(i) $\lim_{x\to 5} \frac{x^2-7x+10}{x-5}$ (ii) $\lim_{x\to 1} \frac{x^2-1}{x-1}$

(iii) $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$ (iv) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Solution : (i) Here, $\lim_{x\to 5} (x-5) = 0$. So direct substitution will not work.

We can proceed by cancelling the common factor $(x-5)$ in numerator and denominator and using theorem 1, as shown below :

$$\lim_{x\to 5} \frac{x^2-7x+10}{x-5} = \lim_{x\to 5} \frac{(x-2)(x-5)}{(x-5)}$$

$$= \lim_{x\to 5} (x-2), \quad \text{for } x \neq 5$$

$$= 5 - 2 = 3$$

(ii) Since $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$ for $x \neq 1$,

therefore by theorem 1, we have

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} (x+1) = 1+1 = 2.$$

(iii) Once again we see that direct substitution fails because itsleads to indeterminate form $\frac{0}{0}$. In this case, rationalising the numerator helps as follows. For $x \neq 0$,

$$\frac{\sqrt{x+2}-\sqrt{2}}{x} = \left(\frac{\sqrt{x+2}-\sqrt{2}}{x}\right)\left(\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\right)$$

$$= \left(\frac{x+2-2}{\sqrt{x+2}+\sqrt{2}}\right) = \frac{x}{x(\sqrt{x+2}+\sqrt{2}\,)}$$

$$= \frac{1}{\sqrt{x+2}+\sqrt{2}}$$

Therefore, by Theorem 1, we have

$$\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}} = \lim_{x\to 0}\frac{1}{\sqrt{0+2}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

(iv) For $x \neq 0$, we have

$$\frac{\sqrt{1+x}\ -\ \sqrt{1-x}}{x} = \left(\frac{\sqrt{1+x}\ -\ \sqrt{1-x}}{x}\right)\left(\frac{\sqrt{1+x}\ +\ \sqrt{1-x}}{\sqrt{1+x}\ +\ \sqrt{1-x}}\right)$$

$$= \frac{2x}{x\ \sqrt{1+x}\ -\ \sqrt{1-x}} = \frac{2}{\sqrt{1+x}\ +\ \sqrt{1-x}}$$

$\therefore$ by theorem 1, we have

$$\lim_{x\to 0}\frac{\sqrt{1+x}\ -\ \sqrt{1-x}}{x} = \lim_{x\to 0}\frac{2}{\sqrt{1+x}\ +\ \sqrt{1-x}} = \frac{2}{\sqrt{1+0}\ +\ \sqrt{1-0}} = \frac{2}{2} = 1$$

An important limit

Example 5 : Prove that $\lim\limits_{x\to 0}\frac{x^n\ -\ a^n}{x-a} = na^{n-1}$ where n is positive integer

Solution : We know that

$$x^n - a^n = (x-a)\,(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots\ldots\ldots\ldots.\ xa^{n-2} + a^{n-1})$$

Therefore, for $x \neq a$, we get

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots\ldots\ldots\ldots.\ xa^{n-2} + a^{n-1}$$

Hence by Theorem 1, we get

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = \lim_{x\to a}\,(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots\ldots\ldots\ldots.\ xa^{n-2} + a^{n-1})$$

$$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \ldots\ldots\ldots\ldots.\ aa^{n-2} + a^{n-1}$$

$$= n\,a^{n-1}$$

Note : The above limit is valid for negative integer n, and in general for any rational index n provided $a > 0$. The above formula can be directly used to evaluate limits.

Example 6 : Evaluate $\lim_{x\to 3}\frac{x^3-27}{x^2-9}$

Solution : $\lim_{x\to 3}\frac{x^3-27}{x^2-9} = \lim_{x\to 3}\frac{x^3-3^3}{x^2-3^2}$

$$= \lim_{x\to 3}\frac{\frac{x^3-3^3}{x-3}}{\frac{x^2-3^2}{x-3}}$$

$$= \frac{3.3^{3-1}}{2.3^{2-1}} \quad \left(\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right)$$

$$= \frac{27}{6} = \frac{9}{2}$$

One–sided Limits

Definition : Let f be a function defiend on an open interval $(a-h, a+h)$ $(h > 0)$. A number L is said to be the **Left Hand Limit (L.H.L.)** of f at a if $f(x)$ takes values very close to L as x takes values very close to a on the left of a $(x \neq a)$. We then write

$$\lim_{x\to a} f(x) = L$$

We similarly define L to be the **Right Hand Limit** if $f(x)$ takes values close to L as x takes values close to a on the right of a and write $\lim_{x\to a+} f(x) = L$

Note that $\lim_{x\to a} f(x)$ exists and is equal to L if and only if $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$ both exist and are equal to L.

$$\lim_{x\to a-} f(x) = L = \lim_{x\to a+} f(x) = \lim_{x\to a} f(x)$$

Example 7 : Show that $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Solution : Let $f(x)=\frac{|x|}{x}$, $x \neq 0$.

Since $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

$\therefore f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

So, $\lim_{x\to 0+} f(x) = \lim_{x\to 0}(1) = 1$ and

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0}(-1) = -1$$

Thus $\lim_{x\to 0} f(x)$ does not exist.

Continuous Functions

Definition : A function f is said to be **continuous** at $x = a$ if the following three conditions are met :

(1) $f(a)$ is defined

(2) $\lim_{x \to a} f(x)$ *exists*

(3) $\lim_{x \to a} f(x) = f(a)$

Example 8: Show that $f(x) = |x|$ is continuous at $x = 0$

Solution: Recall that

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

To show that f is continuous at $x = 0$, it is sufficient to show that

$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} (x) = f(0)$ and

We have

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0+} f(0 - h) = \lim_{h \to 0+} f(-h)$$

$$= \lim_{h \to 0+} -(-h)$$

$$= \lim_{h \to 0+} h = 0$$

and $\lim_{x \to 0+} f(x) = \lim_{h \to 0+} f(0 + h) = \lim_{h \to 0+} f(h)$

$$= \lim_{h \to 0+} (h) = 0.$$

Thus, $\lim_{h \to 0-} f(x) = \lim_{h \to 0+} f(x) = 0$

Also, $f(0) = 0$

Therefore, $\lim_{x \to 0+} f(x) = 0 = f(0)$

Hence, f is continuous at $x = 0$.

Example 9: Check the continuity of f at the indicated point

(*i*) $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0$

$$(ii) \quad f(x) = \begin{cases} ,\\ \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 2 & x = 1 \end{cases} \qquad \text{at} \quad x = 1$$

Solution : (i) We have already seen in Example 7 that $\lim\limits_{x\to 0}\dfrac{|x|}{x}$ does not exist.

Hence, f is not continous at $x = 0$

(ii) Here, $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$

$$= \lim_{x\to 1}(x+1)\measuredangle$$

$$= 2$$

Also, $f(1) = 2$

$$\therefore \lim_{x\to 1} f(x) = f(1)$$

Hence, f is continuous at $x = 1$**.**

Definition : A function is said to be **continuous on an open interval** (a,b) if it is continuous at each point of the interval. A function which is continuous on the entire real line $(-\infty,\infty)$ is said to be **everywhere continuous.**

Algebra of Continuous Functions

Theorum : Let c be a real number and let f and g be continuous at $x = a$. Then the functions cf, $f+g$, $f-g$, fg are also continuous at $x = a$. The functions $\frac{1}{g}$ and $\frac{f}{g}$ are continuous provided $g(a) \neq 0$.

Remark: It must be noted that polynomial functions, rational functions, trigonometric functions, exponential and logarithmic function are continuous in their domains.

Example 10 : Find the points of discontinuity of the following functions :

(i) $$f(x) = \begin{cases} x^2 & if\ x > 0 \\ x+3 & x \leq 0 \end{cases}$$

(ii) $$f(x) = \begin{cases} x & if\ x \neq 0 \\ 1 & x = 0 \end{cases}$$

Solution : (i) Since x^2 and $x + 3$ are polynomial functions, and polynomial functions are continuous at each point in R, f is continuous at each x € R except possibly at $x = 0$. For $x = 0$, we have

$$\lim_{x\to 0-} f(x) = \lim_{h\to 0+} f(0-h) = \lim_{h\to 0+}(-h+3) = 0+3 = 3$$

$$\text{and} \lim_{x\to 0+} f(x) = \lim_{h\to 0+} f(0+h) = \lim_{h\to 0+} f(h) = \lim_{h\to 0+} h^2 = 0.$$

Therefore, since $\lim\limits_{x\to 0-} f(x) f \lim\limits_{x\to 0+} f(x), f$ is not continuous $at\ x = 0$

(ii) Since, polynomial functions are continuous at each point of R, f is also continuous at each x € R except possibly at x = 0. At this point, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x = 0 \neq f(0).$$

Thus, f is not continuous at x = 0

Check Your Progress – 1

1. Evaluate the following limits:

(i) $\lim_{x\to 2}(3x^3 + 2x + 1)$ (ii) $\lim_{x\to 2}\frac{x-2}{x+2}$

(iii) $\lim_{x\to 2}\frac{x^2 - 5x + 2}{x - 1}$ (iv) $\lim_{x\to 2}\sqrt[3]{3x^2 - 19}$

2. Evaluate the following limits:

(i) $\lim_{x\to 2}\frac{x^2 - 4}{x + 2}$ (ii) $\lim_{x\to 5}\frac{\sqrt{x-1} - 2}{x - 5}$

3. Evaluate the following limits:

(i) $\lim_{x\to a}\frac{x^{7/6} - a^{7/6}}{x^{3/5} - a^{3/5}}$ $(a > 0)$

(ii) $\lim_{x\to a}\frac{x^m - a^m}{x^n - a^n}$ $(m, n \text{ are rational numbers}, a > 0)$

4. Check the continuity of f at the indicated point where

$f(x) = \begin{cases} 2 - x & if\ x < 0 \\ x + x & if\ x \geq 0 \end{cases}$ at x =0

5. For what value of constant k the function f is continuous at x = 5 ?

$$f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5} & if\ x \neq 5 \\ k & if\ x = 5 \end{cases}$$

1.3 DERIVATIVE OF A FUNCTION

Definition: A function f is said to be **differentiable** at x if and only if

$$\lim_{\Delta x\to 0}\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

exists. If this limit exsits, it is called the derivative of f at x and is denoted by

$f^1(x)$ or $\frac{dy}{dx}$.

i.e., $\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} = f^1(x)$

A function is said to be **differentiable on an open interval I** if it is differentiable at each point of I.

Example 11: Differentiate $f(x) = x^2$ by using the definition.

Solution : We first find the difference quotient as follows :

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \frac{x^2 + 2\,x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \frac{\Delta x(2x+\Delta x)}{\Delta x}$$

$$= 2x + \Delta x$$

It follows that

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} (2x+\Delta x) = 2x$$

Remark: It can be easily proved that if f is differentiable at a point x, then f is continuous at x. Thus, if f is not continuous at x, then f is not differentiable at x.

Some differentiation Rules

We now develop several "rules" that allow us to calculate derivatives without the direct use of limit definition.

Theorem 1 (Constant Rule). The derivative of a constant is zero. That is,

$$\frac{d}{dx}[c] = 0$$

where c is a real number.

Proof : Let $f(x) = c$ then

$$\frac{d}{dx}[c] = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c-c}{\Delta x} = 0$$

Theorem 2 : (Scalar Multiple Rule). If f is differentiable function and c is a real number, then

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Proof : By definiton

$$\frac{d}{dx}[cf(x)] = \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - c\,f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \mathrm{c}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right] = cf'(x)$$

Theorem 3 : (Sum and Difference Rule). If f and g are two differentiable functions, then

$$\textbf{Sum Rule } \frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\textbf{Difference Rule } \frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

Proof : We have

$$\frac{d}{dx}[f(x) + g(x)] = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) + g(x+\Delta x) - [f(x) + g(x)]}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= f'(x) + g'(x)$$

We can similarly prove the difference rule.

Theorum 4 : (Product Rule). If f and g are two differentiable functions, then

$$\frac{d}{dx}[f(x)\, g(x)] = f(x) + g'(x) + f'(x) + g(x)$$

Proof: We have $\frac{d}{dx}[f(x)\, g(x)] = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)\, g(x+\Delta x) - f(x)\, g(x)}{\Delta x}\right]$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)\, (g(x+\Delta x) - f(x+\Delta x)\, g(x) + f(x+\Delta x) g(x) - f(x) g(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)\, (g(x+\Delta x) - g(x))}{\Delta x} + g(x) \frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x)}{\Delta x}\right]\left[\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}\right] + g(x)\left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

(using the product and scalar multiple rules of limits). Now, since f is differentiable at x, it is also continuous at x.

$$\therefore \lim_{\Delta x \to 0} f(x+\Delta x) = f(x)$$

Thus $\frac{d}{dx}[f(x)\, g(x)] = f(x) g'(x) + g(x)\, f'(x)$

Theorem 5 : (Power Rule) If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

For n =1, we have

$$\frac{d}{dx}(x^n) = \frac{d}{dx}(x) = \lim_{\Delta x \to 0} f(x)\frac{x+\Delta x - x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$$

$$= 1 = 1x^0 = nx^{n-1}.$$

If $n > 1$, then the binomial expansion produces

$$\frac{d}{dx}(x^n) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{{}^nC_0\, x^n + {}^nC_1\, x^{n-1} + {}^nC_2\, x^{n-2}(\Delta x)^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots {}^nC_n\, (\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + \ldots\ldots\ldots\ldots\ldots\ldots + (\Delta x)^{n-1}\right]$$

$$= nx^{n-1}.$$

Theorem 6: (Reciprocal Rule). If f is differentiable function such that $f(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \frac{-f'(x)}{[f(x)]^2}$$

Proof $$\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{1}{f(x+\Delta x)} - \frac{1}{f(x)}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x) - f(x+\Delta x)}{f(x+\Delta x)f(x)}\right]$$

$$= \lim_{\Delta x \to 0} \left[-\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)\right]\left[\left(\frac{1}{f(x+\Delta x)f(x)}\right)\right]$$

$$= -f'(x).\frac{1}{f(x)f(x)}$$ ($\because \lim_{\Delta x \to 0} (f(x+\Delta x) = f(x)$
as f being diff. at x is continuous at x)

$$= \frac{-f'(x)}{[f(x)]^2}$$

Theorem 7 : (Quotient Rule) : If f and g are two differentiable function such that $g(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Proof: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left[f(x)\frac{1}{g(x)}\right]$

$$= \frac{1}{g(x)}\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}\left[\frac{1}{g(x)}\right] \text{ [Product Rule]}$$

$$= \frac{1}{g(x)} + f'(x) + f(x)[\frac{-g'(x)}{[g(x)]^2}]$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Remark : The power rule can be extended for any integer. Indeed, if $n = 0$, we have

$$\frac{d}{dx}(x^{n)=} \frac{d}{dx}(1) = 0 = 0x^{-1} \qquad x \neq 0,$$

and if n is a negative integer, then by using reciprocal rule we can prove

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Thus we have

$$\frac{d}{dx}(x^n) = nx^{n-1}, \text{for any integer } n.$$

Example 2 : Find the derivatives of the following function .

(i) $y = 2x^5 - 3x$ (ii) $y = \frac{1}{x^2 + 3}$

(iii) $y = \frac{x}{x+2}$ (iv) $y = \frac{x^2}{x^2 - 5}$

Solution : (i) $\frac{dy}{dx} = \frac{d}{dx}(2x^5 - 3x)$

$$= 2\frac{d}{dx}(x^5) - 3\frac{d}{dx}(x)$$

$$= 2.(5x^4) - 3.1$$

$$= 10x^4 - 3$$

(ii) $\frac{dy}{dx} = \frac{-\frac{d}{dx}[x^2+3]}{[x^2+3]^2}$ [using reciprocal rule]

$$= \frac{-2x}{(x^2+3)^2}$$

(iii) $$\frac{dy}{dx} = \frac{(x+2)\frac{d}{dx}(x) - x\frac{d}{dx}(x+2)}{(x+2)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{(x+2).1 - x.1}{(x+2)^2}$$

$$= \frac{2}{(x+2)^2}$$

(iv) $$\frac{dy}{dx} = \frac{(x^2-5)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(x^2-5)}{(x^2-5)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{(x^2-5)(2x\quad) - x^2(2x)}{(x^2-5)^2}$$

$$= \frac{2x^3 - 10x - 2x^3}{(x^2-5)^2} = \frac{-10x}{(x^2-5)^2}$$

Derivative of Exponential and Logarithmic Functions

To find the derivatives of the natural exponential function e^x and the natural logarithmic function lnx, we need the following limits.

(1) $$\lim_{\Delta x \to 0} \frac{e^x - 1}{x} = 1$$

(2) $$\lim_{\Delta x \to 0} \frac{\ln(1+x)}{x} = 1$$

Theorem 8 : The derivative of the natural exponential function is given by

$$\frac{d}{dx}(e^x) = e^x \qquad (x \in \mathrm{R})$$

Proof : By definition

$$\frac{d}{dx}(e^x) = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x}$$

$$= e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x(1)$$

$$= e^x$$

Theorum 9 : The derivative of the natural logarithmic function is given by

$$\frac{d}{dx}(lnx) = \frac{1}{x} \quad (x > 0)$$

Proof : By definition

$$\frac{d}{dx}(lnx) = \lim_{\Delta x \to 0} \frac{ln(x+\Delta x) - lnx}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} ln \frac{(x+\Delta x)}{x} \left(\because lna - lnb = ln\frac{a}{b}\right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} ln\left(1 + \frac{\Delta x}{x}\right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\Delta x/x}$$

$$= \frac{1}{x} \lim_{\Delta x \to 0} \frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\Delta x/x} = \frac{1}{x}(1) = \frac{1}{x}$$

Corollary : If $a > 0$ and $a \neq 1$, then the derivative of the general logarithemic function is

$$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$$

Proof : We know that

$$\log_a x = (lnx)(\log_a e)$$

$$\Rightarrow \frac{d}{dx}(\log_a x) = \frac{d}{dx}[(logx)(\log_a e)]$$

$$= \log_a e \frac{d}{dx}(logx)$$

$$= \frac{1}{x}(\log_a e)$$

Remark : Similar to the proof of theorem, we can prove that if $a > 0$, and $a \neq 1$, then the derivative of the general exponential function is

$$\frac{d}{dx}(a^x) = a^x lna \quad (x \in R)$$

Example 13 : Find the derivative of the following functions.

(i) $x^2 e^x$ (ii) $\frac{lnx}{x}$

(iii) $\frac{e^x}{x^2+3}$ (iv) $5^x lnx$

Solution : (i) Using the product rule

$$\frac{d}{dx}\,(x^2e^x) = \frac{d}{dx}\,(x^2)e^x + x^2\frac{d}{dx}\,(e^x)$$

$$= 2x\, e^x + x^2e^x = (2x + \ x^2)\ e^x$$

(i) Using the quotient rule, we have

$$\frac{d}{dx}\,\frac{lnx}{x} = \frac{x\frac{d}{dx}(lnx) - \ lnx\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\,.\frac{1}{x} - \ (lnx)(1)}{x^2}$$

$$= \ \frac{1 - lnx}{x^2}$$

(ii) Using the quotient rule, we have

$$\frac{d}{dx}\left(\frac{e^x}{x^2+3}\right) = \frac{(x^2+3)\frac{d}{dx}\ (e^x) - \ e^x\frac{d}{dx}(x^2+3)}{(x^2+3)^2}$$

$$= \frac{(x^2+3)\ (e^x) - \ e^x(2x)}{(x^2+3)^2}$$

$$= \frac{(x^2-2x+3)e^x}{(x^2+3)^2}$$

(iii) Using the product rule, we have

$$\frac{d}{dx}(5^x lnx) = \ \frac{d}{dx}(5^x)lnx + \ 5^x\frac{d}{dx}(lnx)$$

$$= \ (5^x lnx5) + \ 5^x\left(\frac{1}{x}\right)$$

$$= \ 5^x(ln5)lnx + \ \left(\frac{5^x}{x}\right).$$

Check Your Progress – 2

1. Find the derivative of each of the following functions.

(i) $y \ = \ x^5 - \ 3x^4 + \ 2\,x - 1$ (ii) $y = \ \dfrac{2x-1}{\pi^2}$

(iii) $\dfrac{3x+5}{2x+7}$ (iv) $y \ = \ \dfrac{x^3-4}{x^3}$

2. Find the derivative of each of the following functions.

(i) $e^x \ln x$ (ii) $\frac{e^x}{x^2}$

(iii) $\frac{\ln x}{x^2}$ (iv) $2^x + x^2 + 2^2$

(v) $\frac{e^x}{x^2+7}$ (vi) $5^x e^x$

3. Using the limit $\lim_{x\to 0} \frac{a^x-1}{x} = lna,$ prove that $\frac{d}{dx}(a^x) = a^x\, lna,$ where $a>0$ and $a\neq 1$.

1.4 THE CHAIN RULE

We now discuss one of the most powerful rules in differential calculus, the chain rule, which deals with composite of functions.

Theorem 10: If $y = f(u)$ is differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

or, equivalently, $\frac{d}{dx}[f(gx))] = f'(g(x))g'(x).$

Proof : Let $(F(x) = f(g(x))$. We have to show that for $x = c$,

$$F'(c) = f'(g((c))g'(c).$$

An important consideration in this proof is the behaviour of g as x approaches c. A problem occurs if there are values of x other than c such that $g(x) = g(c)$. However, in this proof we shall assume that $g(x) \neq g(c)$ for values of x other than c. Thus, we can multiply and divide by the same (non– zero) quantity $g(x) - g(c)$. Note that as g is differentiable, it is continuous and it follows that $g(x) \to g(c)$ as $x \to$ c.

$$F'(c) = \lim_{x\to c} \frac{f(g(x)) - f(g(c))}{x-c}$$

$$= \lim_{x\to c}\left[\frac{f(g(x)) - f(g(c))}{g(x)-g(c)}\,\frac{g(x)-g(c)}{x-c}\right] \qquad [\because g(x) \neq g(c)]$$

$$= \lim_{x\to c}\frac{f(g(x)) - f(g(c))}{g(x)-g(c)}\lim_{x\to c}\frac{g(x)-g(c)}{x\to c}$$

$$= f'(g(c)g'(c)$$

Remark : We can extend the chain rule for more than two functions. For example, if $F(x) = f[g(h(x))]$, then

$$F'(c) = f'[g(h(c))]g'(h(c))h'(c).$$

In other words

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

Example 14 : Find the derivatives of the following functions.

(i) $y = (x^2+1)^3$ (ii) $y = e^{x^2}$

(iii) $y = ln\,(2x^2 + e^x)$ (iv) $y = (x + lnx)^2$

Solution : (i) Put $x^2 +1 = u$

Then $y = u^3$ where $u = x^2 + 1$

$$\therefore \quad \frac{dy}{du} = 3u^2 \qquad \frac{du}{dx} = 2x$$

Then by the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (3u^2)(2x)$$

$$= 6x\,(x^2+1)^2$$

(ii) In this case we take $x^2 = u$, so that $y = e^u$

Then by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (e^u)(2x) = 2xe^{x^2}.$$

(iii) Take $u = 2x^2 + e^x$, so that $y = lnu$.

Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \frac{1}{u}\,(4x + e^x)$$

$$= \frac{4x + e^x}{2x^2 + e^x}$$

(iv) Take $u = x + \ln x$, so that $y = u^2$

Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (2u)\left(1 + \frac{1}{x}\right)$$

$$= 2(x + \ln x)\left(1 + \frac{1}{x}\right)$$

We will now extend the power rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

to real exponents. We will do this in two stages – first to rational exponents and then to real exponents. We shall use the chain rule.

Theorem 11 : For rational values of n

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof : Let $n = {}^{p}/_{q}$, where p, q are integers and $q > 0$. Then $nq = p$ is an integer.
Let $u = x^n$ and consider the equation.

$$(x^n)^q = x^{nq} = x^p \ \ or\ u^q = x^p \ \ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now differentiate (1) using the chain rule on the left and the power rule (for integers) on the right to obtain

$$qu^{q-1}\frac{du}{dx} = p\,x^{p-1}$$

$$\Rightarrow \frac{d}{dx}(x^n) = \frac{px^{p-1}}{qu^{q-1}}$$

But $u^{q-1} = {}^{u^q}/_{u} = {}^{x^p}/_{x^n}$, because $u = x^n$. Thus

$$\frac{d}{dx}(x^n) = \frac{px^{p-1}}{q\,{}^{x^p}/_{x^n}} = nx^{p-1+n-p} = nx^{n-1}$$

Theorem 12 : For a real number n

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof : Recall that if n is real, then by definition

$$x^n = e^{nlnx}$$

Now put $u = nlnx$, so that $x^n = e^u$. Then by the chain rule

$$\frac{d}{dx}(x^n) = \frac{d}{du}(e^u)\frac{du}{dx} = (e^u)\frac{d}{dx}(nlnx) = (e^{nlnx})\left(\frac{n}{x}\right)$$

$$= \frac{nx^n}{x} = nx^{n-1}$$

Example 15 : Find the derivative of each of the following functions:

(i) $y = (x^2+2)^{2/3}$ (ii) $y = e^{\sqrt{x}}$

(iii) $y = \ln(1+\sqrt{1+x^2})$ (iv) $y = x^2\,e^{x^2}$

Solution : (i) Putting $u = x^2 + 2$, we have

$$\frac{dy}{dx} = \frac{2}{3}(x^2+2)^{\frac{2}{3}-1}\frac{d}{dx}(x^2+2)$$

$$= \frac{2}{3}(x^2+2)^{-1/3}(2x)$$

$$= \frac{4x}{3(x^2+1)^{1/3}}$$

(ii) Putting $u = \sqrt{x}$, we have

$$\frac{dy}{dx} = e^{\sqrt{x}} \frac{d}{dx}\left(\sqrt{x}\right) = e^{\sqrt{x}} \frac{1}{\sqrt[2]{x}} = \frac{e^{\sqrt{x}}}{\sqrt[2]{x}}$$

(iii) $$\frac{dy}{dx} = \frac{1}{1+\sqrt{1+x^2}} \frac{d}{dx}\left(1+\sqrt{1+x^2}\right)$$

$$= \frac{1}{1+\sqrt{1+x^2}} \frac{1}{2\sqrt{1+x^2}} \frac{d}{dx}(1+x^2)$$

$$= \left(\frac{1}{1+\sqrt{1+x^2}}\right)\left(\frac{1}{2\sqrt{1+x^2}}\right)(2x)$$

$$= \left(\frac{x}{(1+\sqrt{1+x^2})\sqrt{1+x^2}}\right)$$

(iv) $$\frac{dy}{dx} = \frac{d}{dx}(x^2)\, e^{x^2} + x^2 \frac{d}{dx}\left(e^{x^2}\right)$$

$$= 2x\, e^{x^2} + x^2 e^{x^2} \frac{d}{dx}(x^2)$$

$$= 2x\, e^{x^2} + x^2 e^{x^2}(2x)$$

$$= 2x\, e^{x^2}(1+x^2)$$

Example 16 : Find the derivatives of following functions :

(i) $y = \ln\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$ (ii) $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(iii) $y = \sqrt[3]{x(x+1)(x+2)}$

Solution : (i) Rewriting the argument of the log, we have

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\ \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$$

$$= \frac{(\sqrt{1+x}-\sqrt{1-x})^2}{(1+x)-(1-x)}$$

$$= \frac{(1+x)+(1-x)-\sqrt[2]{1+x}\ \sqrt{1-x}}{2x}$$

$$= \frac{2-\sqrt[2]{1-x^2}}{2x} = \frac{1-\sqrt{1-x^2}}{x}$$

Therefore, $y = ln\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$$= ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)$$

$$= ln\ (1-\sqrt{1-x^2}) - lnx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-\sqrt{1-x^2}}\frac{d}{dx}\left(1-(1-x^2)^{-1/2}\right) - \frac{1}{x}$$

$$= \left[\frac{1}{1-\sqrt{1-x^2}}\left\{\frac{d}{dx}0 - \frac{1}{2}(1-x^2)^{-1/2}(-2x)\right\} - \frac{1}{x}\right]$$

$$= \frac{1}{1-\sqrt{1-x^2}}\frac{x}{\sqrt{1-x^2}} - \frac{1}{x}$$

$$= \frac{x^2 - [\sqrt{1-x^2}(1-\sqrt{1-x^2})]}{x\sqrt{1-x^2}(1-\sqrt{1-x^2})}$$

$$= \frac{x^2 - \sqrt{1-x^2} + (1-x^2)}{x\sqrt{1-x^2}(1-\sqrt{1-x^2})}$$

$$= \frac{1-\sqrt{1-x^2}}{x\sqrt{1-x^2}(1-\sqrt{1-x^2})} = \frac{1}{x\sqrt{1-x^2}}$$

(i) One can apply the quotient rule in this case. However, we will avoid it by rewriting the given expression.

$$Y = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} = \frac{e^{2x}+1}{e^{2x}-1} = \frac{e^{2x}-1+2}{e^{2x}-1}$$

$$= 1 + \frac{2}{e^{2x}-1} = 1+2\,(e^{2x}-1)^{-1}$$

$$\Rightarrow \frac{dy}{dx} = 0 + 2(-1)(e^{2x}-1)^{-2}\frac{d}{dx}(e^{2x}-1)$$

$$= \frac{-2}{(e^{2x}-1)^2}(2e^{2x}) = \frac{-4e^{2x}}{(e^{2x}-1)^2}$$

(ii) We have $y = [x(x+1)(x+2)]^{1/3}$

So, $\dfrac{dy}{dx} = \dfrac{1}{3}[x(x+1)(x+2)]^{\frac{1}{3}-1}\dfrac{d}{dx}[x(x+1)(x+2)]$ (chain Rule)

$$= \frac{1}{3}[x(x+1)(x+2)]^{-\frac{2}{3}}\frac{d}{dx}[x(x+1)(x+2)] \text{ (product rule)}$$

$$= \frac{1}{3}[x(x+1)(x+2)]^{-\frac{2}{3}}\frac{d}{dx}[(x+1)(x+2) + x(x+2) + x(x+1)]$$

$$= \frac{1}{3}[x(x+1)(x+2)]^{-\frac{2}{3}}\ x(x+1)(x+2)\left[\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}\right]$$

$$= \frac{1}{3}[x(x+1)(x+2)]^{1/3}\ \left[\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}\right]$$

Check Your Progress 3

1. Find the derivatives of each of the following functions :

   (i) $y = (x^3+x)^{3/2}$ (ii) $y = ln\left(\frac{x^2}{2}\right)$

   (iii) $y = e^{(x^2+2x)}$ (iv) $y = ln\ (x+\sqrt{x})$

2. Find $\frac{dy}{dx}$ $where$

   (i) $y = \frac{1-e^x}{e^{2x}}$ (ii) $y = \frac{x}{\sqrt{x^2-1}}$

   (iii) $y = 2^{x/lnx}$

3. Differentiate each of the following functions :

   (i) $y = ln\left[e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right]$ (ii) $y = \sqrt{\frac{1-x}{1+x}}$

   (iii) $y = \frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1-\sqrt{x^2-1}}}$

1.5 DIFFERENTIATION OF PARAMETRIC FORMS

Suppose *x* and *y* are given as functions of another variable *t*. We call *t*, the variable in which *x* and *y* are expressed as parameter. In this case, we find $\frac{dy}{dx}$ as follows :

Let *x* = *f(t)* and y = *g(t)*,where *f* and *g* are differentiable functions of *t* and $f'(t) \neq 0$ ∀ t. Let Δx and Δy be the increments and *x* and *y* respectively, corresponding to the increment Δt in *t*. That is $\Delta x = f(t+\Delta t) - f(t)$ and $\Delta y = g(t+\Delta t) - g(t)$

Since $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

and $\Delta x \to 0$ as $\Delta t \to 0,$ we can write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{g(t+\Delta t) - g(t)}{f(t+\Delta t) - f(t)}$$

Dividing both the numerator and denominator by Δt, we can use the differentiability of *f* and *g* to conclude that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\left[\frac{g(t+\Delta t) - g(t)}{\Delta t}\right]}{\left[\frac{f(t+\Delta t) - f(t)}{\Delta t}\right]}$$

$$= \frac{g'(t)}{f'(t)} = \frac{dy/dt}{dx/dt}$$

Example 17 : Find $\frac{dy}{dx}$ when

(a) $x = at^2$, $y = 2at$ (b) $= ct$, $y = {}^{c}/_{t}$

(c) $x = lnt, y = {}^{1}/_{t}$ (d) $y = \frac{3at}{1+t^2}$

Solution: (a) We have

$$\frac{dy}{dx} = \frac{d}{dt}[at^2] = 2at$$

and $\frac{dy}{dt} = \frac{d}{dt}[2at] = 2a$

$$so, \frac{dy}{dx} = \frac{{}^{dy}/_{dt}}{{}^{dx}/_{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

(b) We have

$$\frac{dy}{dx} = \frac{d}{dt}[ct] = c, and\ \frac{dy}{dx} = \frac{d}{dt}\left[\frac{c}{t}\right] = \frac{d}{dt}[ct^{-1}]$$

$$= [c(-1)t^{-2}] = \frac{c}{t^2}$$

since, $\frac{dy}{dx} = \frac{{}^{dy}/_{dt}}{{}^{dx}/_{dt}}$, we get

$$\frac{dy}{dx} = \frac{-c/t^2}{c} = -\frac{1}{t^2}$$

(c) We have $\frac{dx}{dt} = \frac{1}{t}$ and $\frac{dy}{dt} = -\frac{1}{t^2}$

$$\therefore \frac{dy}{dx} = \frac{{}^{dy}/_{dt}}{{}^{dx}/_{dt}} = (-1)\ \frac{1/t}{1/t^2} = -\frac{1}{t}$$

(d) We have

$$\frac{dx}{dt} = \frac{d}{dt}\left[\frac{3at}{1+t^2}\right]$$

$$= 3a \frac{(1+t^2)\frac{dx}{dt} - t\ \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= 3a \frac{(1+t^2)(1) - t\ (2t)}{(1+t^2)^2}$$

$$= 3a \frac{(1-t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{d}{dt}\left[\frac{3at^2}{(1+t^2)}\right] \text{ and}$$

$$= 3a \frac{(1+t^2)\frac{d}{dt}(t^2) - (t^2)\ \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= 3a \frac{(1+t^2)(2t) - (t^2)\ (2t)}{(1+t^2)^2}$$

$$= \frac{6at}{(1+t^2)^2}$$

Since, $\quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

we get

$$\frac{dy}{dx} = \frac{6at/(1+t^2)^2}{3a(1-t^2)\ \big/(1+t^2)^2}$$

$$= \frac{2t}{1-t^2}$$

Second Order Derivatives

Let *y = f(x)* be a function. If *f* is a differentiable function, then its derivative is a function. If the derivative is itself differentiable we can differentiate it and get another function called the second derivative. The second derivative is denoted by y or $f\ (x)$ or $\dfrac{d^2y}{dx^2}$

Thus

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Example 18 : If $y = \dfrac{lnx}{x}$, show that $\dfrac{d^2y}{dx^2} = \dfrac{2lnx - 3}{x^3}$

Solution : we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{lnx}{x}\right] = \frac{d}{dx}\left[x^{-1}lnx\right]$$

$= \frac{d}{dx}(x^{-1})lnx + \ x^{-1}\frac{d}{dx}\ (lnx)$ (product rule)

$= \ (-\ 1)x^{-2}lnx + \ x^{-1}\frac{1}{x}$

$= x^{-2}$ [1 – ln$x$]

Differentiating both sides with respect to x, we get

$$\frac{d^2y}{dx^2} = \ \frac{d}{dx}\ [x^{-2}][1 - lnx] + \ x^{-2}\frac{d}{dx}[1 - lnx]$$

$$= (-2)\ x^{-3}(1 - lnx) + \ x^{-2}\left(0 - \frac{1}{x}\right)$$

$$= -2x^{-3}(1 - lnx) + \ x^{-3}$$

$$= -x^{-3}(2-2\ln x + 1)$$

$$= \ \frac{2lnx - 3}{x^3}$$

Example 19 : If y = $ae^{mx} + \ be^{-mx}$, show that $\frac{d^2y}{dx^2} = \ m^2y$

Solution : We have $y = ae^{mx} + \ be^{-mx}$

Differentiating both sides with respect to x, we get $\frac{dy}{dx} = \ \frac{d}{dx}\ (ae^{mx} + \ be^{-mx})$
$= \ ame^{mx} - \ bme^{-mx}$

$$\Rightarrow \frac{d^2y}{dx^2} = \ \frac{d}{dx}(ame^{mx} - \ bme^{-mx})$$

$$= am^2e^{mx} - \ bm(-m)e^{-mx}$$

$$= am^2e^{mx} + \ bm^2e^{-mx}$$

$$= m^2(ae^{mx} + \ be^{-mx})$$

$$= m^2y$$

Example 20 : If $y = ln\ (x + \sqrt{x^2 + 1}),$ prove that

$$(x^2 + 1)\ \frac{d^2y}{dx^2} + x\ \frac{dy}{dx} = 0$$

Solution : We have $y = ln\ (x + \sqrt{x^2 + 1})$

Differentiating both sides, we get

$$\frac{dy}{dx} = \ \frac{1}{x + \sqrt{x^2 + 1}}\frac{d}{dx}\left[x + \ (x^2 + 1)^{\frac{1}{2}}\right] \quad \text{(chain rule)}$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{1}{2}\ (x^2+1)^{-\frac{1}{2}}(2x)\right]$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left[1+\ \frac{x}{\sqrt{x^2+1}}\right]$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right]$$

$$= \frac{1}{\sqrt{x^2+1}}\ =\ (x^2+1)^{-\frac{1}{2}}$$

$$\therefore \frac{d^2y}{dx^2} = \left(-\frac{1}{2}\right)(x^2+1)^{-\frac{3}{2}}\frac{d}{dx}[(x^2+1)]$$

$$= \ -\frac{1}{2}\frac{1}{(x^2+1)^{\frac{3}{2}}}\ \ 2x\ =\ \frac{-x}{(x^2+1)^{\frac{3}{2}}}$$

Now, $(x^2+1)\ \dfrac{d^2y}{dx^2}+\ x\dfrac{dy}{dx}$

$$= \ (x^2+1)\left[\frac{-x}{(x^2+1)^{\frac{3}{2}}}\right]+\ x\frac{1}{\sqrt{x^2+1}}$$

$$= \ -\frac{x}{\sqrt{x^2+1}}+\frac{x}{\sqrt{x^2+1}} = 0$$

Thus, $(x^2+1)\ \dfrac{d^2y}{dx^2}+\ x\ \dfrac{dy}{dx} = 0$

Check Your Progress – 4

1. Find $\frac{dy}{dx}$ when

1. Find $\dfrac{dy}{dx}$ when

   (a) $x = \dfrac{1}{2}(e^{\theta}-e^{-\theta})$ and $y = \dfrac{1}{2}(e^{\theta}-e^{-\theta})$

   (b) $x = a\left(t-\dfrac{1}{t}\right)$ and $y = a\left(t+\dfrac{1}{t}\right)$

   (c) $x = \dfrac{a(1-t^2)}{(1+t^2)}$ and y = $\dfrac{2bt}{1+t^2}$

2. If $y = \sqrt{1\ +x^2}$, find $\frac{d^2y}{dx^2}$

3. If $y = ln\ (\sqrt{x-1}\ +\sqrt{x+1})$, prove that

   $(x^2-\ 1)\dfrac{d^2y}{dx^2}+\ x\dfrac{dy}{dx}-y = 0$

4. If $y = ax + \frac{b}{x}$, show that $\frac{xd^2y}{dx^2} + \frac{xdy}{dx} - y = 0$

1.6 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. (i) $\lim_{x\to 3} (3\,x^3 + 2x + 1) = 3\,.(2)^3 + 2(2) + 1 = 29$

(ii) $\lim_{x\to 2} \frac{x-2}{x+2} = \frac{2-2}{2+2} = \frac{0}{4} = 0$

(iii) $\lim_{x\to 2} \frac{x^2 - 5x + 2}{x - 1} = \frac{2^2 - 5(2) + 2}{2 - 1} = -2$

(iv) $\lim_{x\to 3} \sqrt[3]{3x^2 - 19} = \sqrt[3]{3(3)^2 - 19} = \sqrt[3]{27 - 19} = \sqrt[3]{8} = 2$

2. (i) $\lim_{x\to 2} \frac{x^2 - 4}{x + 2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x+2} = \lim_{x\to -2} (x - 2) = -2 - 2 = -4$

(ii) $\lim_{x\to 5} \frac{\sqrt{x-1} - 2}{x - 5} = \lim_{x\to 5} \left[\left(\frac{\sqrt{x-1} - 2}{x-5}\right)\left(\frac{\sqrt{x-1} + 2}{\sqrt{x-1} + 2}\right)\right]$

$$= \lim_{x\to 5} \frac{(x-1) - 4}{(x-5)(\sqrt{x-1} + 2)}$$

$$= \lim_{x\to 5} \frac{(x-5)}{(x-5)(\sqrt{x-1} + 2)}$$

$$= \lim_{x\to 5} \frac{1}{(\sqrt{x-1} + 2)} = \lim_{x\to 5} \frac{1}{(\sqrt{5-1} + 2)} = \frac{1}{4}$$

3. (i) $\lim_{x\to a} \frac{x^{7/6} - a^{7/6}}{x^{3/5} - a^{3/5}} = \lim_{x\to a} \frac{\frac{x^{7/6} - a^{7/6}}{x - a}}{\frac{x^{3/5} - a^{3/5}}{x - a}}$

$$= \frac{\lim_{x\to a} \frac{x^{7/6} - a^{7/6}}{x - a}}{\lim_{x\to a} \frac{x^{3/5} - a^{3/5}}{x - a}} = \frac{(7/6)}{(3/5)} \frac{a^{7/6-1}}{a^{3/5-1}} \left(\because \lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right)$$

$$= \frac{35}{18} \frac{a^{1/6}}{a^{-2/5}} = \frac{35}{18} a^{\frac{17}{30}}$$

(ii) $\lim\limits_{x\to a} \dfrac{x^m - a^m}{x^n - a^n} \quad = \lim\limits_{x\to a} \dfrac{(x^m - a^m)/(x-a)}{(x^n - a^n)/(x-a)}$

$$= \frac{\lim\limits_{x\to a} \dfrac{x^m - a^m}{x-a}}{\lim\limits_{x\to a} \dfrac{x^n - a^n}{x-a}}$$

$$= \frac{ma^{m-1}}{na^{n-1}} = \frac{m}{n}\, a^{m-n}$$

4. We have

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0^+} f(0-\text{h}) = \lim_{h\to 0^+} f(-\text{h})$$

$$= \lim_{h\to 0^+} [2-(-h)] = \lim_{h\to 0^+}(2+h) = 2$$

and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0^+} f(0+h) = \lim\limits_{h\to 0^+}(f(h)) = \lim\limits_{h\to 0^+}(2+h) = 2$

Thus, $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = 2 \quad \Rightarrow \lim\limits_{x\to 0^+} f(x) = 2$

Also, $f(0) = 2 + 0 = 2$

$\therefore \quad \lim\limits_{x\to 0} f(x) = f(0)$

Hence, f is continuous at $x = 0$.

5. For f to be continuous at $x = 5$, we must have

$$f(5) = \lim_{x\to 5} f(x)$$

$$\Rightarrow k = \lim_{x\to 5} \frac{x^2-25}{x-5} = \lim_{x\to 5} \frac{(x-5)(x+5)}{x-5}$$

So, $k = \lim\limits_{x\to 5}(x+5) = 5 + 5 = 10$

Thus, $k = 10$

Check Your Progress 2

1. (i) $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^5 - 3x^4 + 2x - 1) = 5x^4 - 12x^3 + 2$

(ii) $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{2x-1}{\pi^2} = \dfrac{1}{\pi^2}\dfrac{d}{dx}(2x-1) = \dfrac{2}{\pi^2}$

(iii) $\dfrac{dy}{dx} = \dfrac{(2x+7)\dfrac{d}{dx}(3x+5) - (3x+5)\dfrac{d}{dx}(2x+7)}{(2x+7)^2}$ (Quotient Rule)

$$= \frac{(2x+7).3 - (3x+5).2}{(2x+7)^2}$$

$$= \frac{11}{(2x+7)^2}$$

(iv) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3-4}{x^3}\right) = \frac{(x^3)\frac{d}{dx}(x^3-4) - (x^3-4)\frac{d}{dx}(x^3)}{(x^3)^2}$

$$= \frac{x^3(3x^2) - (x^3-4)(3x^2)}{x^6}$$

$$= \frac{4x^2}{x^6} = \frac{4}{x^4}$$

2. (i) $\frac{d}{dx}(e^x lnx) = \frac{d}{dx}(e^x)lnx + e^x \frac{d}{dx} lnx$

$$= (e^x lnx) + \frac{e^x}{x} = e^x(lnx + \frac{1}{x})$$

(ii) $\frac{d}{dx}\left(\frac{e^x}{x^2}\right) = \frac{x^2\frac{d}{dx}(e^x) - (e^x)\frac{d}{dx}(x^2)}{x^4} = \frac{e^x(x-2)}{x^3}$

(iii) $\frac{d}{dx}\left(\frac{lnx}{x^3}\right) = \frac{x^3\frac{d}{dx}(lnx) - (lnx)\frac{d}{dx}(x^3)}{(x^3)^2}$

$$= \frac{x^3\frac{1}{x} - (lnx)\frac{d}{dx}(3x^2)}{x^6}$$

$$= \frac{x^2(1-3lnx)}{x^6} = \frac{1-3lnx}{x^4}$$

(iv) $\frac{d}{dx}(2^x + x^2 + 2^2) = \frac{d}{dx}(2^x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(2^2)$

$$= 2^x ln2 + 2\,x + 0$$

$$= 2^x ln2 + 2\,x$$

(v) $\frac{d}{dx}\left(\frac{e^x}{x^2+7}\right) = \frac{(x^2+7)\frac{d}{dx}(e^x) - (e^x)\frac{d}{dx}(x^2+7)}{(x^2+7)^2}$

$$= \frac{(x^2+7)e^x - e^x(2x)}{(x^2+7)^2}$$

$$= \frac{e^x[\,x^2 - 2x + 7]}{(x^2+7)^2}$$

(vi) $\frac{d}{dx}(5^x e^x) = \frac{d}{dx}(5^x)e^x + 5^x \frac{d}{dx}(e^x)$

$= 5^x \ln 5 \; e^x + 5^x \; e^x$

$= 5^x e^x (ln5 + 1)$

3. $\frac{d}{dx}(a^x) = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{a^x(a^{\Delta x} - 1)}{\Delta x}$

$= a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$

$= a^x lna$ (using the given limit)

Check Your Progress – 3

1. (i) $\frac{dy}{dx} = \frac{3}{2}(x^3 + x)^{\frac{3}{2}-1} \frac{d}{dx}(x^3 + x)$

$= \frac{3}{2}(x^3 + x)^{1/2}(3x^2 + 1)$

(ii) $\frac{dy}{dx} = \frac{1}{(x^2/2)} \frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{2}{x^2}\left(\frac{2x}{2}\right) = \frac{2}{x}$

(iii) $\frac{dy}{dx} = e^{(x^2+2x)} \frac{d}{dx}(x^2 + 2x) = e^{(x^2+2x)}(2x + 2)$

$= 2(x + 1)e^{(x^2+2x)}$

(iv) $\frac{dy}{dx} = \frac{1}{x + \sqrt{x}} \frac{d}{dx}(x + \sqrt{x}) = \frac{1}{x + \sqrt{x}}\left(1 + \frac{1}{2\sqrt{x}}\right) = \frac{2\sqrt{x} + 1}{2\sqrt{x}(x + \sqrt{x})}$

2. (i) $\frac{dy}{dx} = \frac{\frac{d}{dx}(1 - e^x)e^{2x} - (1 - e^x)\frac{d}{dx}(e^{2x})}{(e^{2x})^2}$

$= \frac{e^{2x}(-e^x) - (1 - e^x)(2e^{2x})}{e^{4x}} = \frac{e^x - 2}{e^{2x}}$

(ii) $$\frac{dy}{dx} = \frac{\sqrt{(x^2-1}\frac{d}{dx}(x) - x\frac{d}{dx}\sqrt{(x^2-1}}{(\sqrt{(x^2-1)}^2}$$

$$= \frac{\sqrt{(x^2-1} - x\left(\frac{1}{2\sqrt{(x^2-1)}}\right)2x}{x^2-1}$$

$$= \frac{(x^2-1) - x^2}{(x^2-1)\sqrt{x^2-1}} = \frac{-1}{(x^2-1)^{3/2}}$$

(iii) $$\frac{dy}{dx} = 2x^{x/ln2}ln2\ \frac{d}{dx}\left(\frac{x}{lnx}\right)$$

$$= 2x^{x/lnx}\left[\frac{1.\,lnx - x.\frac{1}{x}}{(lnx)^2}\right]$$

$$= \frac{2x^{x/lnx}ln2(lnx-1)}{(lnx)^2}$$

3. (i) Rewriting the given expression, we have

$$y = ln\left[e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right]$$

$$= In\ e^x + ln\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}} \qquad [ln(ab) = lna + lnb]$$

$$= xlne\ \frac{3}{4} + ln\left(\frac{x-2}{x+2}\right) \qquad [lna^x = xlna]$$

$= x + \frac{3}{4}$ [$ln(x-2) - ln(x+2)$] [ln(e) = 1 and $ln(a/b) = lna - lnb$]

$$\Rightarrow \frac{dy}{dx} = 1' + \frac{3}{4}\left[\frac{1}{x-2} - \frac{1}{x+2}\right]$$

$$= 1 + \frac{3}{4}\left[\frac{(x+2)-(x-2)}{(x-2)(x+2)}\right]$$

$$= 1 + \frac{3}{4}\left[\frac{x+2-x+2}{x^2-4}\right]$$

$$= 1 + \frac{3}{x^2-4}$$

$$= \frac{x^2-4+3}{x^2-4} = \frac{x^2-1}{x^2-4}$$

(ii) $y = \left(\frac{1-x}{1+x}\right)^{1/2}$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}-1} \frac{d}{dx}\left(\frac{1-x}{1+x}\right) \quad \text{(Chain Rule)}$$

$$= \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\ \frac{-2}{(1+x)^2}$$

$$= \frac{-1}{(1+x)^2}\sqrt{\frac{1+x}{1-x}}$$

(iii) Rewriting the given expression, we have

$$y = \frac{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}} = \frac{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}}\ \frac{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}}$$

$$= \frac{\left(\sqrt{(x^2+1)}+\sqrt{(x^2-1)}\right)^2}{(x^2+1)-(x^2-1)} = \frac{(x^2+1)+(x^2-1)+\sqrt[2]{(x^2+1)(x^2-1)}}{2}$$

$$= \frac{2x^2+\sqrt[2]{x^4-1}}{2} = x^2+(x^4-1)^{1/2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}[((x^4-1)^{\frac{1}{2}}]$$

$$= 2x + \frac{1}{2}(x^4-1)^{-\frac{1}{2}}\frac{d}{dx}(x^4-1)$$

$$= 2x + \frac{1}{\sqrt[2]{(x^4-1)}}(4x^3)$$

$$= 2x + \frac{2x^3}{\sqrt{(x^4-1)}}$$

Check Your Progress 4

1. (a) $\frac{dx}{d\theta} = (e^{\theta}+e^{-\theta})/2$

$$\frac{dx}{d\theta} = (e^{\theta}-e^{-\theta})/2$$

$$\therefore \frac{dx}{dy} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{1}{2}(e^{\theta} - e^{-\theta})}{\frac{1}{2}(e^{\theta} + e^{-\theta})} = \frac{x}{y}$$

(b) $\quad \dfrac{dx}{dt} = a\left(1 + \dfrac{1}{t^2}\right), \quad \dfrac{dy}{dt} = b\left(1 - \dfrac{1}{t^2}\right)$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b(1 - \frac{1}{t^2})}{a(1 + \frac{1}{t^2}),} = \frac{b(t^2 - 1)}{a(t^2 + 1)}$$

(c) $\quad \dfrac{dx}{dt} = \dfrac{a(1 + t^2)(-2t) - a(1 - t^2)\,(2t)}{(1 + t^2)^2}$ (Quotient Rule)

$$= \frac{a[-2t - 2t^3 - 2t + 2t^3]}{(1 + t^2)^2}$$

$$= \frac{-4at}{(1 + t^2)^2}$$

$$\frac{dy}{dt} = 2b\,\frac{(1 - t^2)(1) - t(2t)}{(1 + t^2)^2}$$

$$= \frac{2b(1 - t^2)}{-4at}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b(1 - t^2)}{-4at} = \frac{-b(t^2 - 1)}{(1 + t^2)^2}$$

2. $\quad \dfrac{dy}{dx} = \dfrac{d}{dx}[(1 + x^2)] = \dfrac{1}{2}(1 + x^2)^{\frac{1}{2}-1}\dfrac{d}{dx}(1 + x^2)$

$$= \frac{1}{2}(1 + x^2)^{\frac{1}{2}-1}(2x)$$

$$= x(1 + x^2)^{-1/2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[x(1 + x^2)^{-\frac{1}{2}}\right] = \frac{d}{dx}(x)(1 + x^2)^{-\frac{1}{2}} + x\frac{d}{dx}(1 + x^2)^{-\frac{1}{2}}$$

$$= 1.\,(1 + x^2)^{-\frac{1}{2}} + x\left[-\frac{1}{2}(1 + x^2)^{-\frac{1}{2}-1}(2x)\right]$$

$$= (1 + x^2)^{-\frac{1}{2}} - x^2(1 + x^2)^{-3/2}$$

$$= (1 + x^2)^{-\frac{1}{2}}\left[1 - \frac{x^2}{1 + x^2}\right] = \frac{(1 + x^2)^{-\frac{1}{2}}}{1 + x^2} = \frac{1}{(1 + x^2)^{3/2}}$$

3. We have y = ln ($\sqrt{x-1} + \sqrt{x+1}$)

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x-1}+\sqrt{x+1}} \frac{d}{dx}(\sqrt{x-1} + \sqrt{x+1}) \quad \text{(Chain Rule)}$$

$$= \frac{1}{\sqrt{x-1}+\sqrt{x+1}} \left(\frac{1}{\sqrt[2]{x-1}} + \frac{1}{\sqrt[2]{x-1}}\right)$$

$$= \frac{(\sqrt{x-1}+\sqrt{x+1})}{2(\sqrt{x-1}+\sqrt{x+1})\sqrt{x-1}\sqrt{x+1}}$$

$$= \frac{1}{\sqrt[2]{x^2-1}} = \frac{1}{2}(x^2-1)^{-1/2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2}\frac{d}{dx}\left[(x^2-1)^{-\frac{1}{2}}\right]$$

$$= -\frac{1}{4}[(x^2-1)^{-\frac{1}{2}-1}(2x)] \quad \text{(Chain Rule)}$$

$$= -\frac{1}{2}x(x^2-1)^{-3/2}$$

$$\therefore (x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = (x^2-1)\left[-\frac{1}{2}x(x^2-1)^{-3/2}\right] + x\left[\frac{1}{2}(x^2-1)^{-1/2}\right]$$

$$= -\frac{1}{2}x(x^2-1)^{-1/2} + \frac{1}{2}x(x^2-1)^{-1/2} = 0.$$

4. When have y = $ax + \frac{b}{x}$

$$\therefore \frac{dy}{dx} = a - \frac{b}{x^2} \text{ and } \frac{d^2y}{dx^2} = \frac{d}{dx}(a - bx^{-2}) = 2bx^{-3} = \frac{2b}{x^3}$$

$$\therefore x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2\left(\frac{2b}{x^3}\right) + x\left(a - \frac{b}{x^2}\right) - \left(ax + \frac{b}{x}\right)$$

$$= \frac{2b}{x} + ax - \frac{b}{x} - ax - \frac{b}{x}$$

$$= 0$$

1.7 SUMMARY

In **section 1.2** of the unit, to begin with, the concept of limit of a function is defined. Then, some properties of limits are stated. Next, the concept of one-sided limit is defined. Then, the concept of continuity of a function is defined. Each of these concepts is illustrated with a number of examples.

In **section 1.3**, the concepts of differentiability of a function at a point and in an open interval are defined. Then, a number of rules for finding derivatives of simple functions are derived. In **section 1.4,** chain rule of differentiation is derived and is explained with a number of examples. In **section 1.5**, the concept of differentiation of parametric forms is defined followed by the definition of the concept of second order derivative. Each of these concepts is explained with a number of suitable examples.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 1.6**.