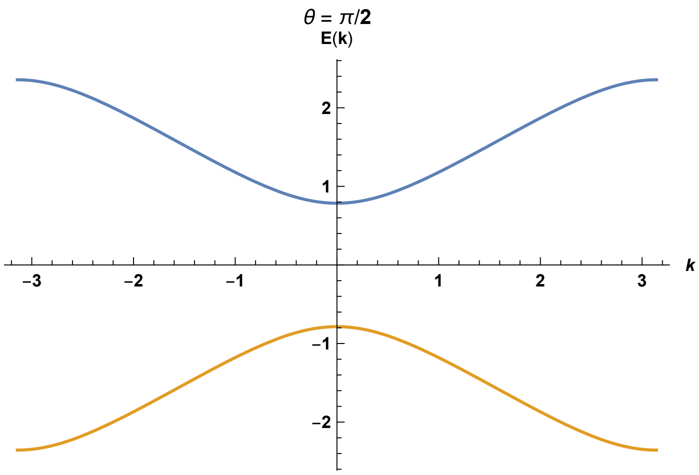
In[150]:=
```
Clear["Global`*"]
```

In[151]:=
```
Ep[k_, θ_] = ArcCos[Cos[θ / 2] Cos[k]];
Em[k_, θ_] = -ArcCos[Cos[θ / 2] Cos[k]];
```

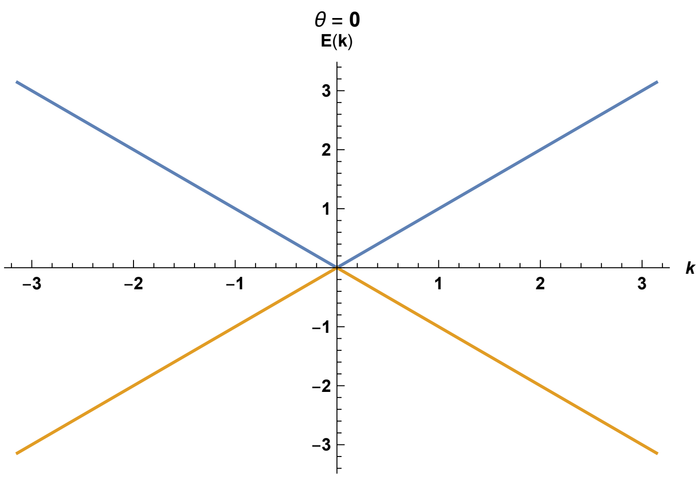
In[153]:=
```
Plot[{Ep[k, Pi / 2], Em[k, Pi / 2]}, {k, -Pi, Pi},
 AxesLabel → {k, "E(k)"}, PlotLabel → "θ = π/2"]
```

Out[153]=



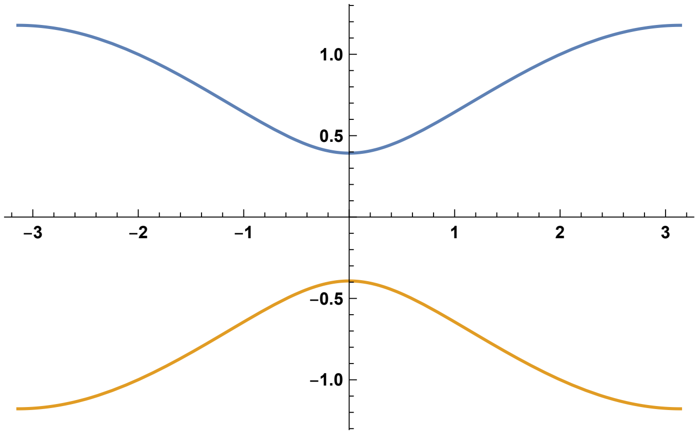
In[154]:=
```
Plot[{Ep[k, 0], Em[k, 0]}, {k, -Pi, Pi}, AxesLabel → {k, "E(k)"}, PlotLabel → "θ = 0"]
```

Out[154]=

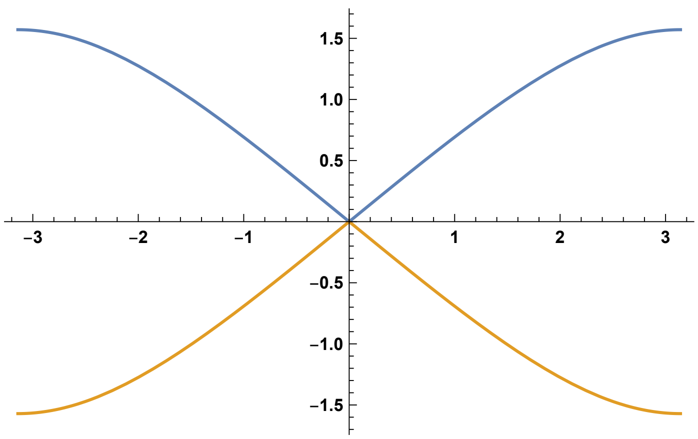


In[155]:=
```
EE1[k_, θ1_, θ2_] = ArcCos[Cos[θ2 / 2] Cos[θ1 / 2] Cos[k] - Sin[θ1 / 2] Sin[θ2 / 2]];
EE2[k_, θ1_, θ2_] = -ArcCos[Cos[θ2 / 2] Cos[θ1 / 2] Cos[k] - Sin[θ1 / 2] Sin[θ2 / 2]];
```

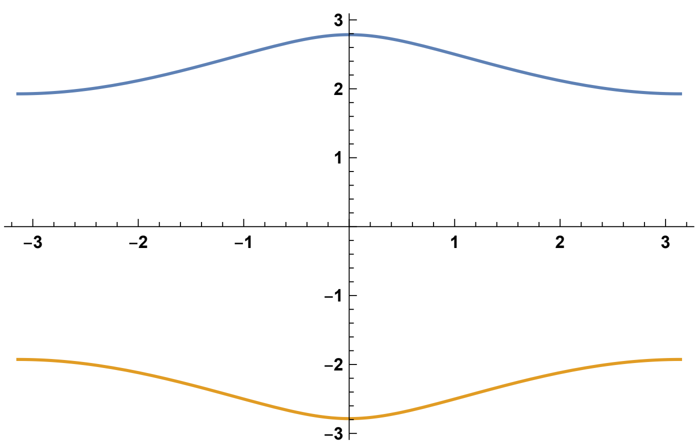
In[157]:=
```
Plot[{EE1[k, -Pi/2, 3 Pi/4], EE2[k, -Pi/2, 3 Pi/4]}, {k, -Pi, Pi}]
```

Out[157]=


In[158]:=
```
Plot[{EE1[k, -Pi/2, 2 Pi/4], EE2[k, -Pi/2, 2 Pi/4]}, {k, -Pi, Pi}]
```

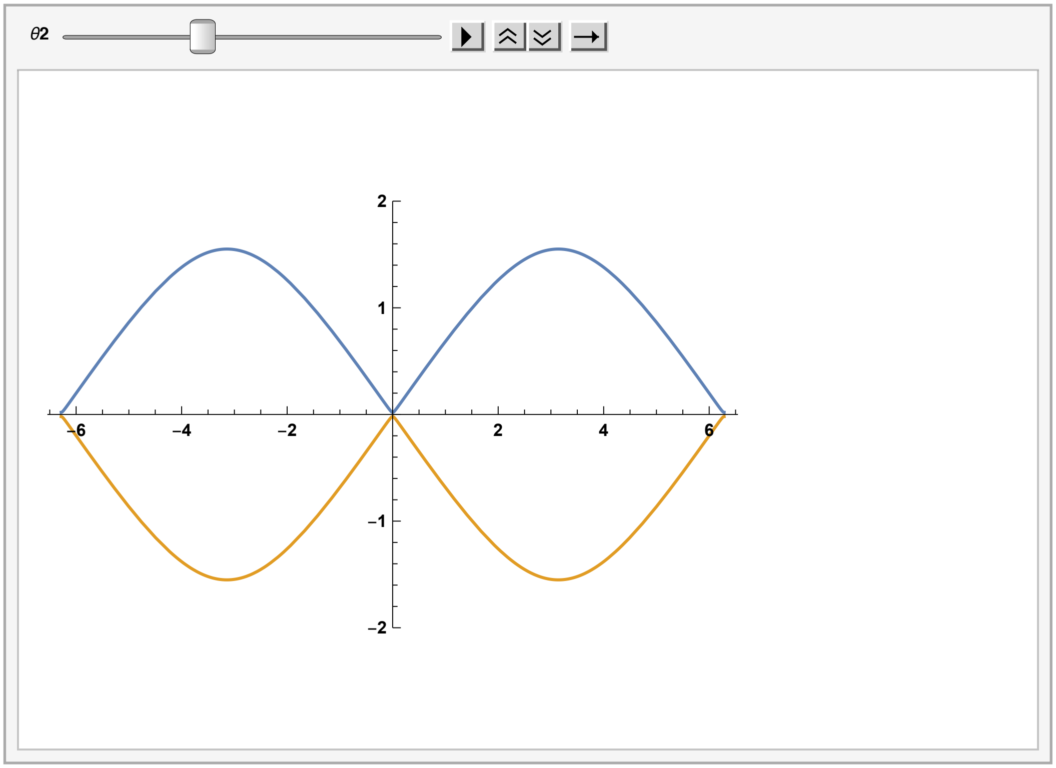
Out[158]=


In[159]:=
```
Plot[{EE1[k, -Pi/2, -4], EE2[k, -Pi/2, -4]}, {k, -Pi, Pi}]
```

Out[159]=


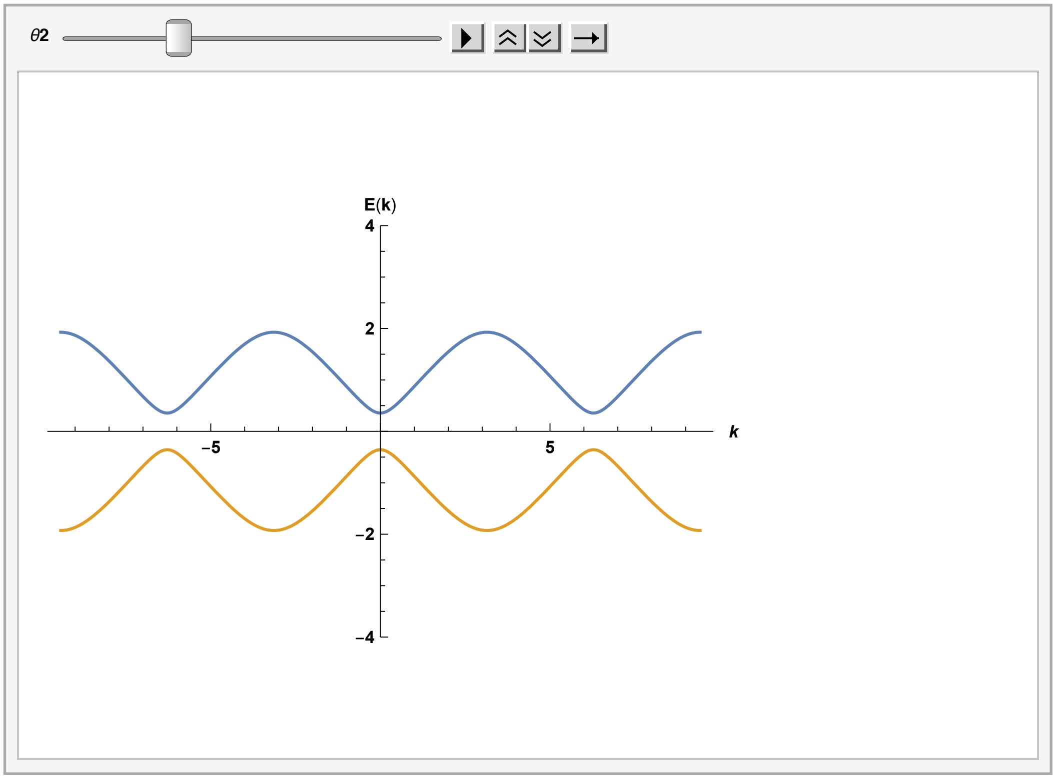
In[54]:=
```
Animate[Plot[{EE1[k, -Pi/2, θ2], EE2[k, -Pi/2, θ2]},
  {k, -2 Pi, 2 Pi}, PlotRange → {-2, 2}], {θ2, Pi/4, 4 Pi/4}]
```

Out[54]=



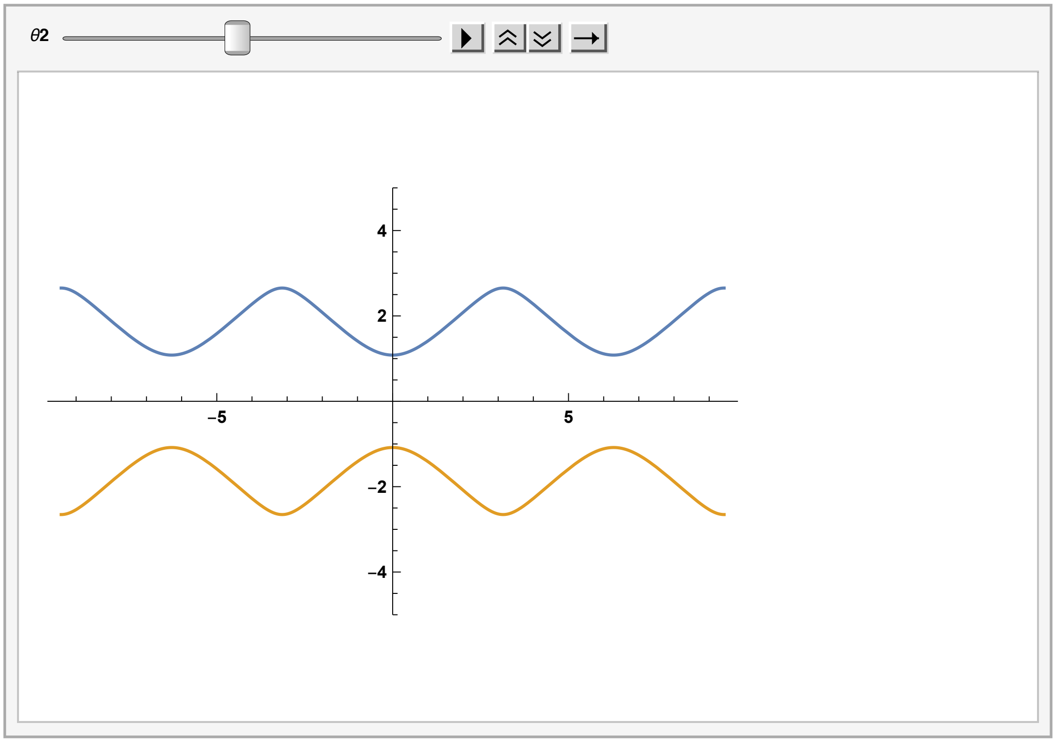
In[53]:=
```
Animate[Plot[{EE1[k, -Pi/2, θ2], EE2[k, -Pi/2, θ2]},
  {k, -3 Pi, 3 Pi}, PlotRange → {-4, 4}, AxesLabel → {k, "E(k)"}], {θ2, 0, Pi}]
```

Out[53]=

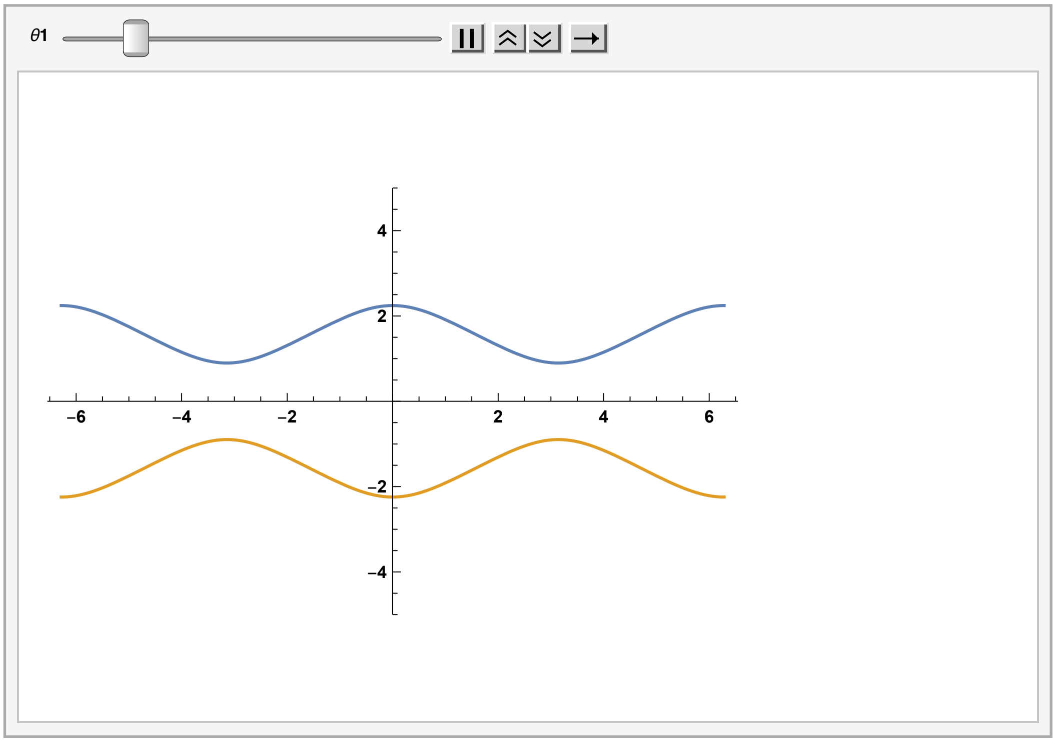


In[49]:=
```
Animate[Plot[{EE1[k, -π/2, θ2], EE2[k, -π/2, θ2]}, {k, -3 π, 3 π}, PlotRange → {-5, 5}],
 {θ2, -2 π, 2 π}, AnimationRunning → False, DisplayAllSteps → True]
```

Out[49]=



In[45]:=
```
Animate[Plot[{Ep[k, θ1], Em[k, θ1]}, {k, -2 Pi, 2 Pi}, PlotRange → {-5, 5}],
 {θ1, -2 Pi, 2 Pi}]
```

Out[45]=


1-D Topological simulation using split-step DTQW

The the code is provided by the author of Topological phenomena in quantum walks: elementary introduction to the physics of topological phases: Takuya Kitagawa

```
In[160]:= distribution[plotmax_, initialangle_, theta1m_, theta1p_, theta2_] :=
   Module[{n = plotmax, iangle = initialangle, thetam = theta1m, thetap = theta1p,
     theta2angle = theta2, t, i, boundarylength, Initialup, Initialdown,
     theta1, a, temp, d, rotation1, rotation2}, boundarylength = 0.01;
    Initialup = N[Cos[iangle]]; (* Not Needed Normalised*)
    Initialdown = N[ Sin[iangle]]; theta1 = Table[(thetam + thetap) / 2 +
        (thetap - thetam) / 2 * Tanh[(i - 2 n + 1 / 2) / boundarylength], {i, 4 n + 1}];
    (* at "l" step and "i" position, coin up "j" + down "k" *)
    a = Table[0, {l, n}, {i, 4 n + 1}, {k, 2}];
    temp = Table[0, {i, 4 n + 1}, {k, 2}]; d = Table[0, {i, 4 n + 1}, {k, 2}];
    rotation1 = N[Table[MatrixExp[-I PauliMatrix[2] theta1[[i]] / 2], {i, 4 n + 1}]];
    rotation2 = N[MatrixExp[-I PauliMatrix[2] theta2angle / 2]];
    (* Normalised Initial Coin Condition at 2n, zero step is t=1,
    edge is n+1*) a[[1, 2 n, 1]] = Initialup; a[[1, 2 n, 2]] = Initialdown;
  (* Time Evolution Step *) For[t = 1, t <= n - 1, t++,
     For[i = 1 + 2 n - 2 t, i <= 2 n + 2 t + 1, i++, (* Coin Flip *)
      d[[i, All]] = rotation1[[i, All, All]].a[[t, i, All]];];
     (* Shift Process with the normalization *) For[i = 1 + 2 n - 2 t,
      i <= 2 n + 2 t + 1, i++, temp[[i + 1, 1]] = d[[i, 1]]; temp[[i, 2]] = d[[i, 2]];];
     For[i = 1 + 2 n - 2 t, i <= 2 n + 2 t + 1, i++, (* Coin Flip *)
      d[[i, All]] = rotation2.temp[[i, All]];];
     (* Shift Process with the normalization *) For[i = 1 + 2 n - 2 t, i <= 2 n + 2 t + 1,
      i++, a[[t + 1, i, 1]] = d[[i, 1]]; a[[t + 1, i - 1, 2]] = d[[i, 2]];]; ];
    Table[{i - 2 n, Abs[a[[t, i, 1]]] ^ 2 + Abs[a[[t, i, 2]]] ^ 2},
     {t, 1, n - 1, 1}, {i, n, 3 n, 1}] ];
```

```
In[161]:= phasediagram[theta1m_, theta1p_, theta2_] :=
  Module[{thetam = theta1m, thetap = theta1p, theta2angle = theta2,
    pline1, pline2, pline3, pline4, pline5, pline6, theta2line, tx, dots},
   pline1 = Plot[x, {x, -2 π, 2 π}, PlotStyle -> {Red, Dotted, Thickness[.005]}];
   pline2 = Plot[2 π - x, {x, 0, 2 π}, PlotStyle -> {Red, Dotted, Thickness[.005]}];
   pline3 = Plot[-2 π - x, {x, -2 π, 0}, PlotStyle -> {Red, Dotted, Thickness[.005]}];
   pline4 = Plot[-x, {x, -2 π, 2 π}, PlotStyle -> {Black, Thickness[.005]}];
   pline5 = Plot[2 π + x, {x, -2 π, 0}, PlotStyle -> {Black, Thickness[.005]}]; pline6 =
    Plot[-2 π + x, {x, 0, 2 π}, PlotStyle -> {Black, Thickness[.005]}]; theta2line =
    Plot[theta2, {x, -2 π, 2 π}, PlotStyle -> {Black, Dotted, Thickness[.005]}];
   tx = Graphics[{GrayLevel[.8], Rotate[Rectangle[{π - π / Sqrt[2], -π / Sqrt[2]},
         {π + π / Sqrt[2], π / Sqrt[2]}], 45 Degree, {π, 0}], GrayLevel[.8], Rotate[
        Rectangle[{-π - π / Sqrt[2], -π / Sqrt[2]}, {-π + π / Sqrt[2], π / Sqrt[2]}], 45
         Degree, {-π, 0}], GrayLevel[.8], Polygon[{{-π, -π}, {-2 π, -2 π}, {0, -2 π}}],
      GrayLevel[.8], Polygon[{{π, -π}, {2 π, -2 π}, {0, -2 π}}], GrayLevel[.8],
      Polygon[{{π, π}, {2 π, 2 π}, {0, 2 π}}], GrayLevel[.8], Polygon[
       {{-π, π}, {-2 π, 2 π}, {0, 2 π}}], Text[Style["1", 30, Bold, Black], {π, 0}],
      Text[Style["1", 30, Bold, Black], {-π, 0}], Text[Style["1", 30, Bold, Black],
       {π, 3 π / 2}], Text[Style["1", 30, Bold, Black], {-π, 3 π / 2}],
      Text[Style["1", 30, Bold, Black], {π, -3 π / 2}], Text[Style["1", 30,
        Bold, Black], {-π, -3 π / 2}], Text[Style["0", 30, Bold, Black], {0, π}],
      Text[Style["0", 30, Bold, Black], {0, -π}], (*Text[Style["0",30,Bold,Black],
       {3π/2,π}],*)Text[Style["0", 30, Bold, Black], {-3 π / 2, π}],
      Text[Style["0", 30, Bold, Black], {-3 π / 2, -π}] }];
   dots = Graphics[{Green, Disk[{thetam, theta2}, π / 10], Blue, Disk[{thetap, theta2},
       π / 10], Text[Style["Left", Bold, Green], {thetam, theta2 + π / 5}],
      Text[Style["Right", Bold, Blue], {thetap, theta2 + π / 5}]}];
   Show[tx, pline1, pline2, pline3, pline4, pline5, pline6, theta2line,
    dots, PlotRange -> {{-2 π, 2 π}, {-2 π, 2 π}},
    PlotRangePadding -> 0, Axes -> False, Frame -> True,
    FrameTicks -> {{{-2 π, -π, 0, π, 2 π}, None}, {{-2 π, -π, 0, π, 2 π}, None}},
    AspectRatio -> 1, FrameLabel -> {{"second rotation", None},
      {"first rotation", "phase diagram (winding number)"}}] ];
```

```
In[162]:= Manipulate[GraphicsRow[
    {Show[Graphics[{Opacity[0.1, Green], Rectangle[{-plotmax, 0}, {0, 1.0}],
         Opacity[0.1, Blue], Rectangle[{0, 0}, {plotmax, 1.0}]}],
       ListPlot[distribution[plotmax, iniangle, thetam, thetap, theta2][[t + 1, All]],
        Filling -> Axis, FillingStyle -> Directive[Black, Thick],
        PlotRange -> {{-plotmax, plotmax}, {0, 1}}, PlotStyle -> PointSize[Medium],
        Joined -> True, Mesh -> All], AspectRatio -> 0.6,
       PlotRange -> {{-plotmax, plotmax}, {0, 1}}, PlotRangePadding -> 0,
       Axes -> True, Frame -> True, FrameTicks -> {{{0, 0.2, 0.4, 0.6, 0.8, 1.0}, None},
         {Table[(plotmax - 2) / 4 i - (plotmax - 2), {i, 0, 8}], None}},
       FrameLabel -> {{"probability", None}, {"sites", "probability distribution"}}],
     phasediagram[thetam, thetap, theta2]}, ImageSize -> {1000, 500}],
   {{t, 0, "steps"}, 0, plotmax - 2, 1, Appearance -> "Labeled"},
   {{plotmax, 20, "maximum number of steps"},
    {100 -> "20", 42 -> "40"}, ControlType -> RadioButton},
   {{iniangle, 0, "initial spin"}, {0 -> "up", π / 2 -> "down"},
    ControlType -> RadioButton}, Delimiter,
   {{thetam, -3 * π / 8, "first rotation θ1: left bulk"}, -2 π, 2 π, π / 8},
   {{thetap, 9 * π / 8, "first rotation θ1: right bulk"}, -2 π, 2 π, π / 8},
   Delimiter,
   {{theta2, π / 2, "second rotation θ2"}, -2 π, 2 π, π / 8},
   AutorunSequencing -> {1},
   SaveDefinitions -> True]
```

steps 18

maximum number of steps 20 40

initial spin up down

first rotation θ_1: left bulk

first rotation θ_1: right bulk

second rotation θ_2

Out[162]=