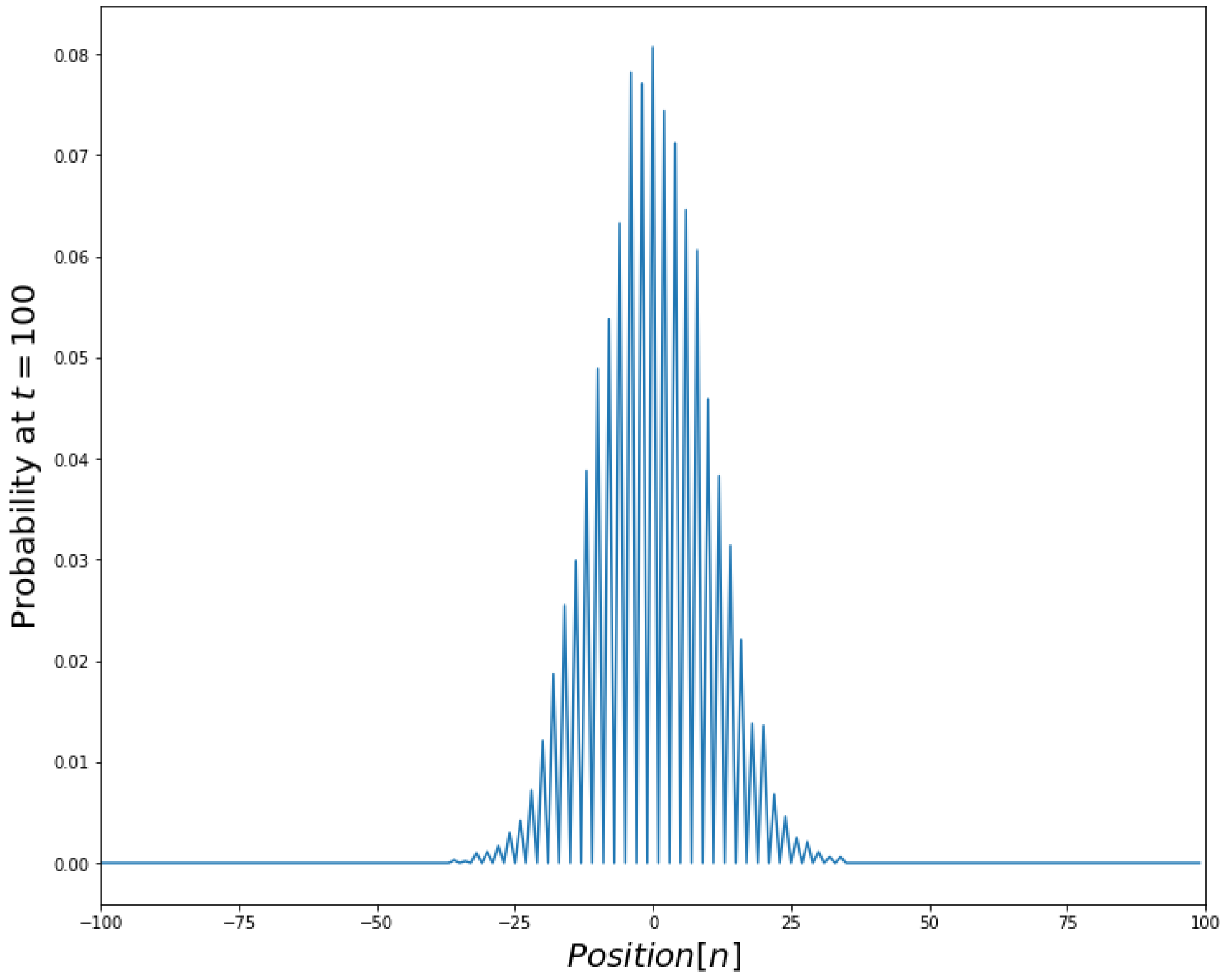
In [2]:
```
from numpy import *
from matplotlib.pyplot import *
import numpy as np
import random
from scipy import *
import scipy as sp
from scipy.linalg import expm, sinm, cosm
import scipy.integrate as integrate
```

Classical Random Walk

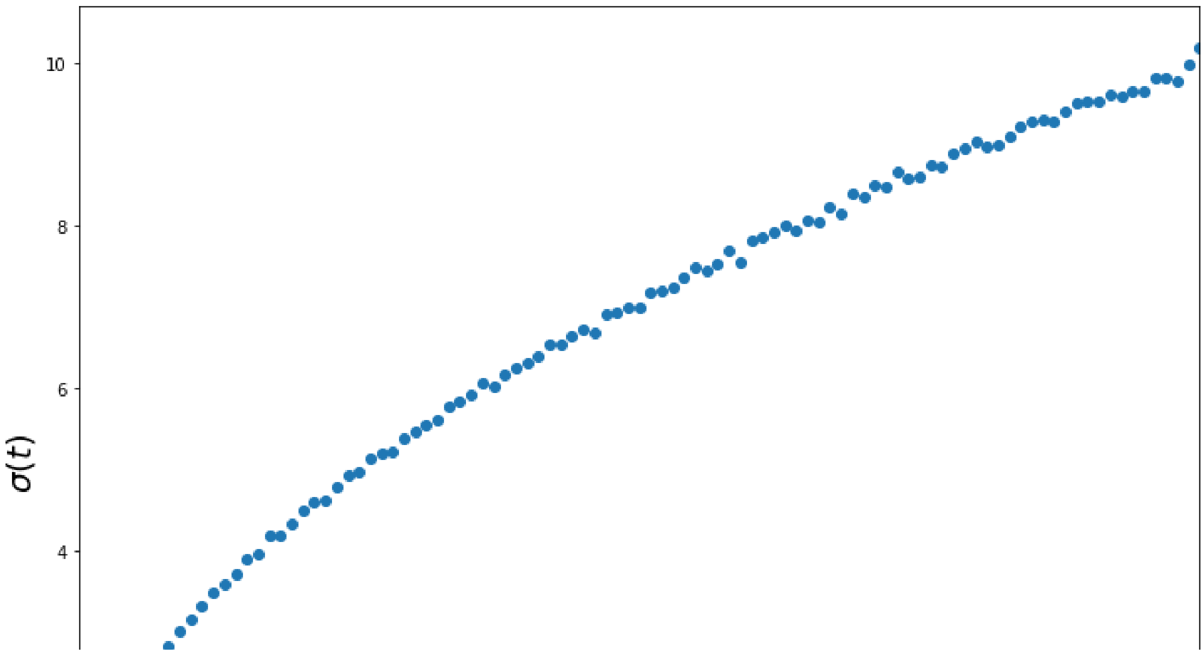
This code may be a bit slow. Wait for more than 1 mins please. I have not optimized it.

In [5]:
```
sigmaRW=[] #record the standart deviation of the Random Walk

particle_num=10000 #randomly propagate 10000 particles

for n in range(0,101):
    step = n
    particle=[0]*particle_num
    for i in range(step):
        for j in range(len(particle)):
            movement = [1,-1]
            particle[j]+=random.choice(movement) #randomly push the particle


    count = 0
    distribution = []
    for i in range(-100,100):
        for j in range(particle_num-1):
            if particle[j] == i:
                count+=1
        distribution.append(count)
        count = 0

    for i in range(len(distribution)):
        distribution[i]=distribution[i]/particle_num

    sigmaRW.append(std(particle))

x = []
y = distribution
for i in range(-100,100):
    x.append(i)
```

In [6]:
```
fig = figure(figsize=(12, 10))
ax = fig.add_subplot(111)
plot(x, y,'-')
#plot(x, y,'o')
xlim(-100, 100)
xlabel('$Position [n]$',size=20)
ylabel('Probability at $t = 100$',size=20)
show()
```



In [7]:
```
nstep =[]
for i in range(0,101):
    nstep.append(i)

fig = figure(figsize=(12, 10))
ax = fig.add_subplot(111)
#plot(nstep, sigma,'-')
plot(nstep, sigmaRW,'o')
xlim(0, 100)
xlabel('t',size=20)
ylabel('$\sigma(t)$',size=20)
#legend()
show()
```



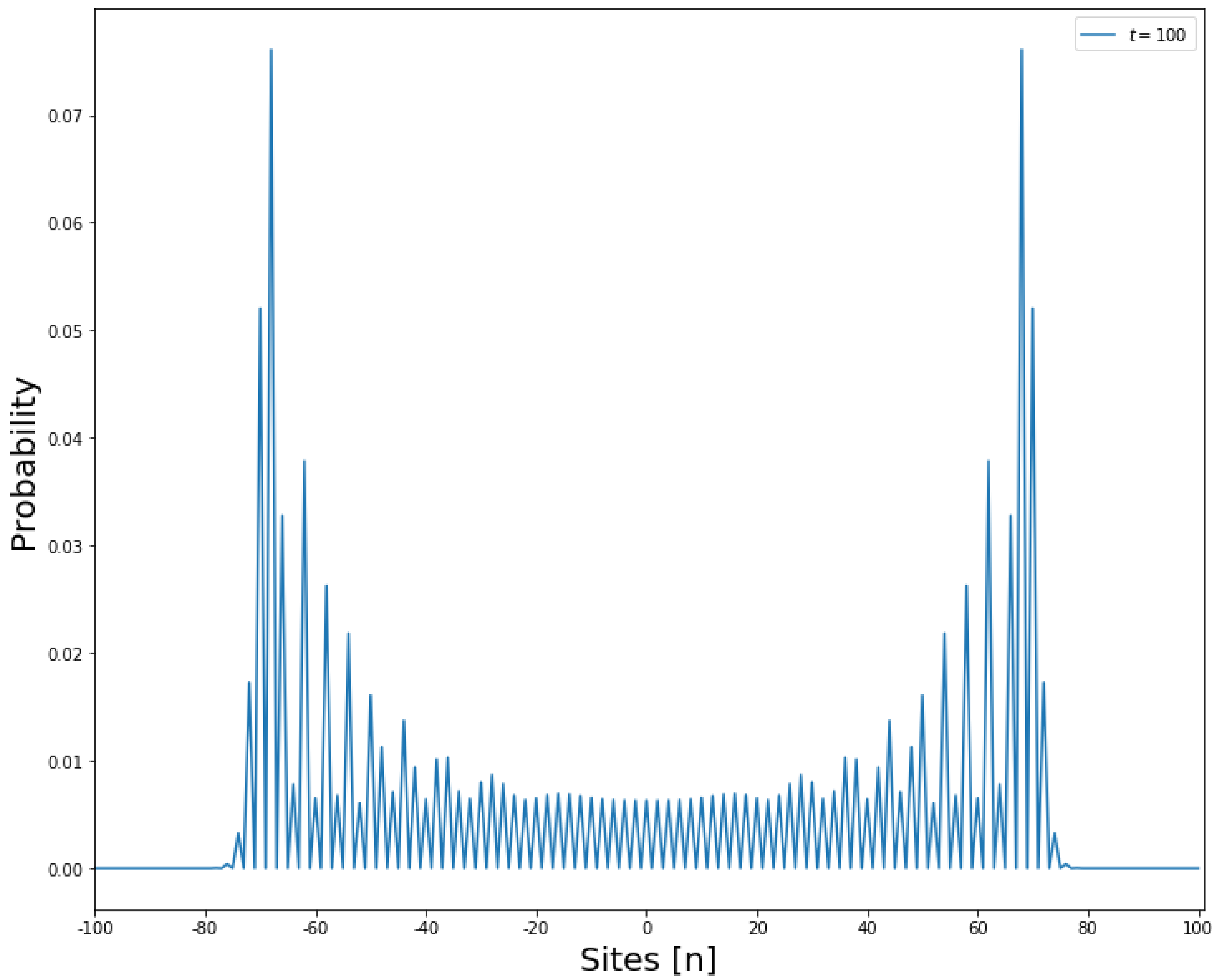
Quantum Random Walk

This code is modified based on the DTQW code on Susan Stepney's blog:https://susan-stepney.blogspot.com/2014/02/mathjax.html (https://susan-stepney.blogspot.com/2014/02/mathjax.html).

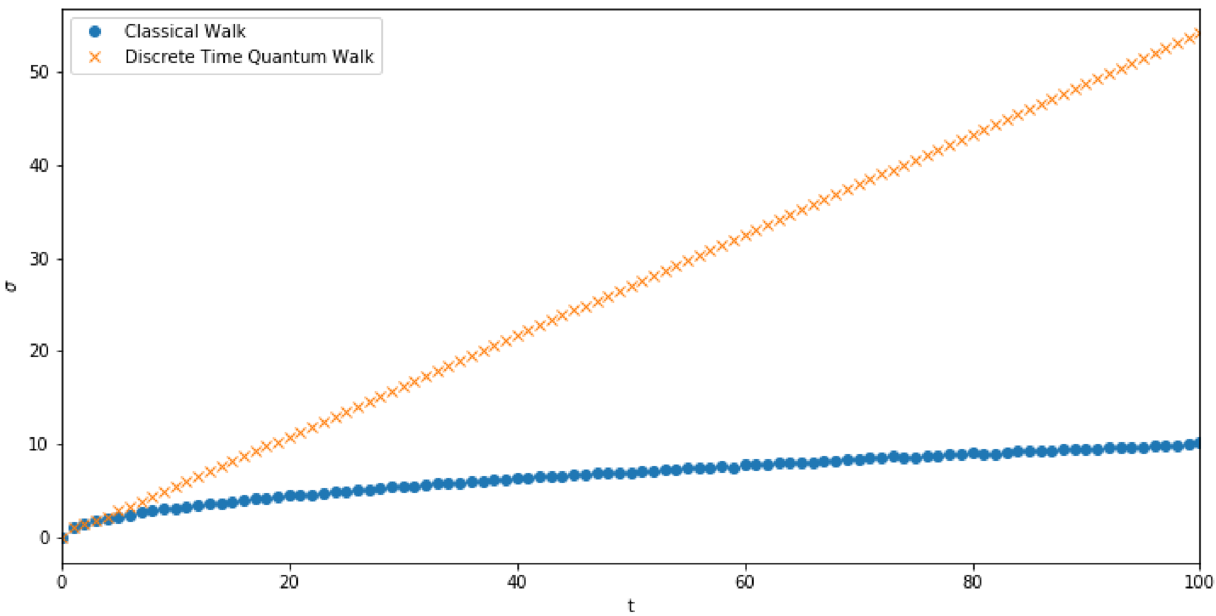
Symmetric DTQW

In [49]:
```
sigmaSQW=[] # will record the standard deviation of the symmetric DTQW latte

for n in range(0,101):

    N = n       # number of random steps
    P = 2*N+1     # number of positions

    spin0 = array([1, 0])  # |0>
    spin1 = array([0, 1])  # |1>

    C00 = outer(coin0, coin0)  # |0><0|
    C01 = outer(coin0, coin1)  # |0><1|
    C10 = outer(coin1, coin0)  # |1><0|
    C11 = outer(coin1, coin1)  # |1><1|


    C_hat = (C00 + C01 + C10 - C11)/sqrt(2.)

    ShiftPlus = roll(eye(P), 1, axis=0)#roll the matrix so that S(+)(1,0,0,(
    ShiftMinus = roll(eye(P), -1, axis=0)#roll the matrix so that S(-)(1,0,(
    S_hat = kron(ShiftPlus, C00) + kron(ShiftMinus, C11)#condition selectior

    U = S_hat.dot(kron(eye(P), C_hat))#Total unitary operator

    posn0 = zeros(P)
    posn0[N] = 1
    psi0 = kron(posn0,(coin0+coin1*1j)/sqrt(2.))

    psiN = linalg.matrix_power(U, N).dot(psi0)

    prob_sym_DTQW = empty(P)
    for k in range(P): #meaure the state after N step
        posn = zeros(P)
        posn[k] = 1
        M_hat_k = kron( outer(posn,posn), eye(2))
        proj = M_hat_k.dot(psiN)
        prob_sym_DTQW[k] = proj.dot(proj.conjugate()).real

    #Get the standard deviation of the Prob dist of the final state
    nlattice =[]
    for i in range(-n,n+1):
        nlattice.append(i)
    mean_nsquare = 0
    mean_n = 0
    for i in range(len(prob_sym_DTQW)):
        mean_nsquare += (nlattice[i]**2)*prob_sym_DTQW[i]
        mean_n += nlattice[i]*prob_sym_DTQW[i]

    sigmaSQW.append(sqrt(mean_nsquare-mean_n))
```

In [50]:
```
fig = figure(figsize=(12, 10))
ax = fig.add_subplot(111)

plot(arange(P), prob_sym_DTQW, label='$t=100$')
loc = range (0, P, P//10) #Location of ticks
xticks(loc)
xlim(0, P)
ax.set_xticklabels(range (-N, N+1, int(P / 10)))
xlabel('Sites [n]', size=20)
ylabel('Probability', size=20)
legend()
show()
```



In [10]:
```
fig = figure(figsize=(12, 6))
ax = fig.add_subplot(111)
plot(nstep, sigmaRW,'o',label='Classical Walk')
plot(nstep, sigmaSQW,'x', label = 'Discrete Time Quantum Walk')
xlim(0, 100)
xlabel('t')
ylabel('$\sigma$')
legend()
show()
```

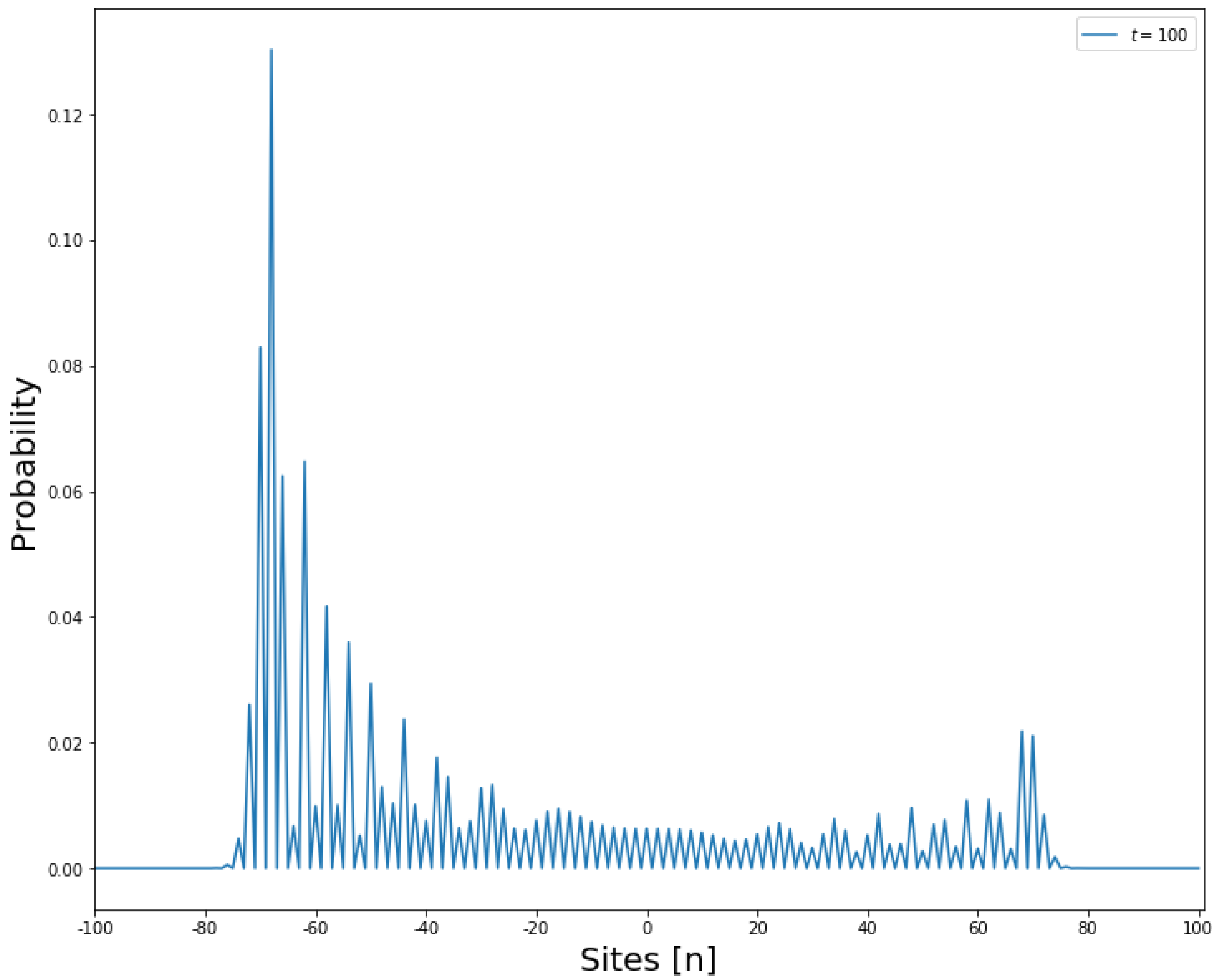


Asymmetric DTQW

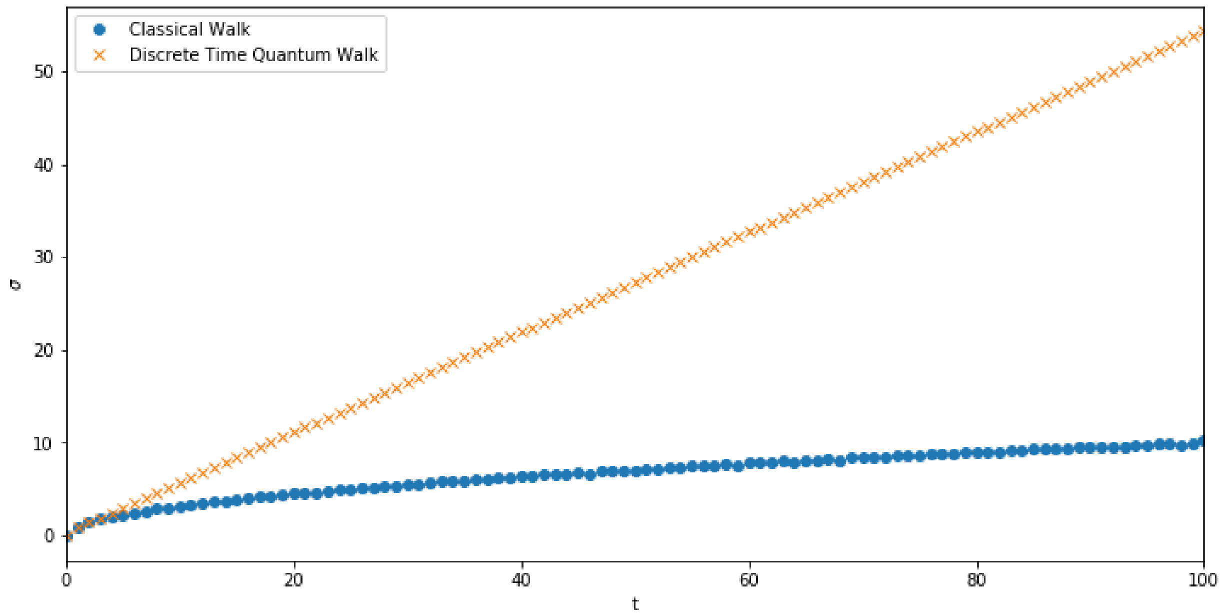
Left Skew

In [11]:
```
sigmaASQW=[]
for n in range(0,101):

    N = n      # number of random steps
    P = 2*N+1    # number of positions

    coin0 = array([1, 0])  # |0>
    coin1 = array([0, 1])  # |1>

    C00 = outer(coin0, coin0)  # |0><0|
    C01 = outer(coin0, coin1)  # |0><1|
    C10 = outer(coin1, coin0)  # |1><0|
    C11 = outer(coin1, coin1)  # |1><1|


    C_hat = (C00 + C01 + C10 - C11)/sqrt(2.)

    ShiftPlus = roll(eye(P), 1, axis=0)
    ShiftMinus = roll(eye(P), -1, axis=0)
    S_hat = kron(ShiftPlus, C00) + kron(ShiftMinus, C11)

    U = S_hat.dot(kron(eye(P), C_hat))

    posn0 = zeros(P)
    posn0[N] = 1
    psi0 = kron(posn0,coin1)

    psiN = linalg.matrix_power(U, N).dot(psi0)

    prob = empty(P)
    for k in range(P):
        posn = zeros(P)
        posn[k] = 1
        M_hat_k = kron( outer(posn,posn), eye(2))
        proj = M_hat_k.dot(psiN)
        prob[k] = proj.dot(proj.conjugate()).real

    nlattice =[]
    for i in range(-n,n+1):
        nlattice.append(i)
    mean_nsquare = 0
    mean_n = 0
    for i in range(len(prob)):
        mean_nsquare += (nlattice[i]**2)*prob[i]
        mean_n += nlattice[i]*prob[i]

    sigmaASQW.append(sqrt(mean_nsquare-mean_n))
```

In [12]:
```
fig = figure(figsize=(12, 10))
ax = fig.add_subplot(111)

plot(arange(P), prob, label='$t=100$')
loc = range (0, P, P//10) #Location of ticks
xticks(loc)
xlim(0, P)
ax.set_xticklabels(range (-N, N+1, int(P / 10)))
xlabel('Sites [n]', size=20)
ylabel('Probability', size=20)
legend()
show()
```

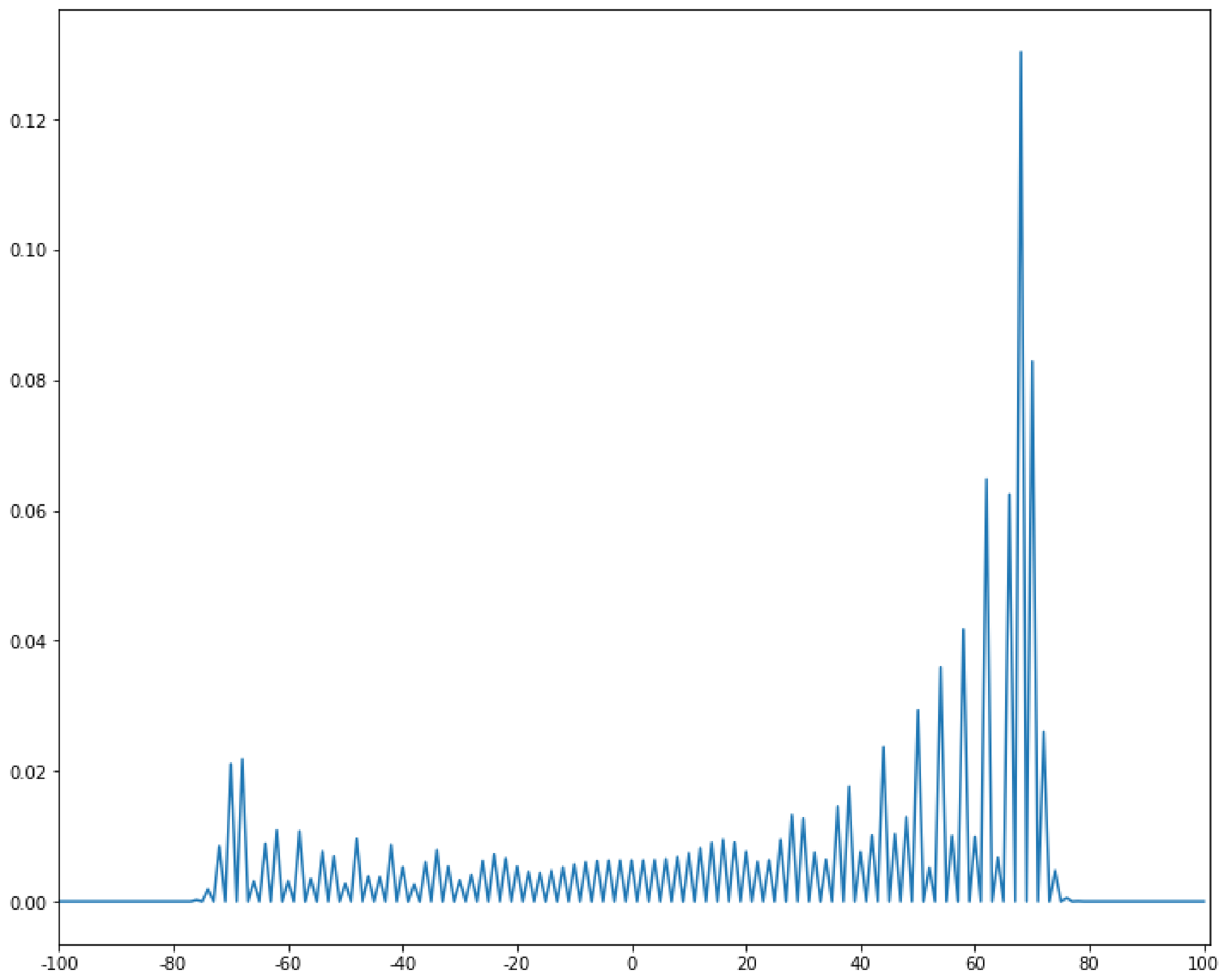


In [13]:
```
fig = figure(figsize=(12, 6))
ax = fig.add_subplot(111)
#plot(nstep, sigma,'-')
plot(nstep, sigmaRW,'o',label='Classical Walk')
plot(nstep, sigmaASQW,'x', label = 'Discrete Time Quantum Walk')
xlim(0, 100)
xlabel('t')
ylabel('$\sigma$')
legend()
show()
```



Right skew

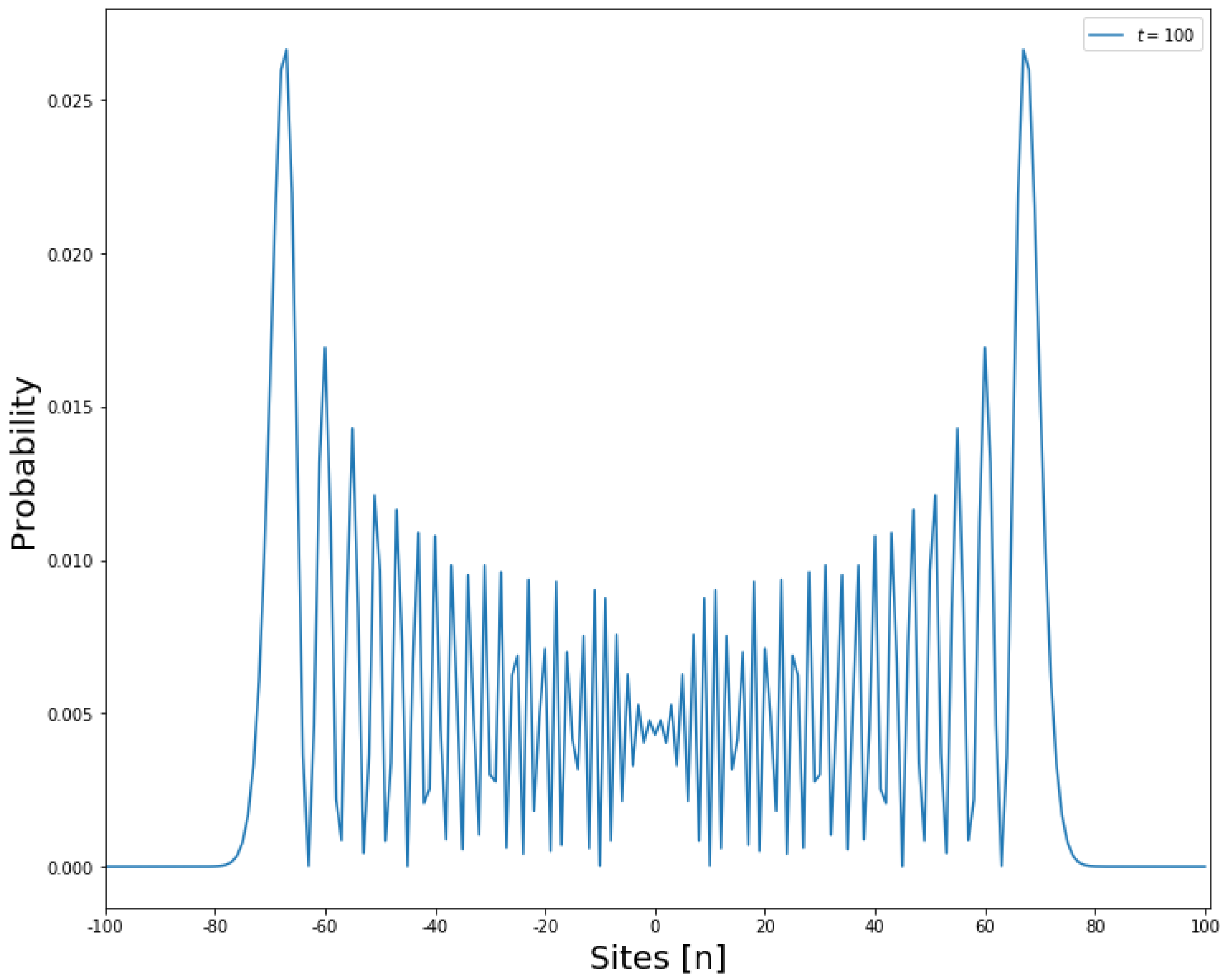
In [15]:
```
posn0 = zeros(P)
posn0[N] = 1     # array indexing starts from 0, so index N is the central p
psi0 = kron(posn0,coin0)

psiN = linalg.matrix_power(U, N).dot(psi0)

prob = empty(P)
for k in range(P):
    posn = zeros(P)
    posn[k] = 1
    M_hat_k = kron( outer(posn,posn), eye(2))
    proj = M_hat_k.dot(psiN)
    prob[k] = proj.dot(proj.conjugate()).real

fig = figure(figsize=(12, 10))
ax = fig.add_subplot(111)

plot(arange(P), prob)
loc = range (0, P, int(P/10)) #Location of ticks
xticks(loc)
xlim(0, P)
ax.set_xticklabels(range (-N, N+1, int(P / 10)))

show()
```



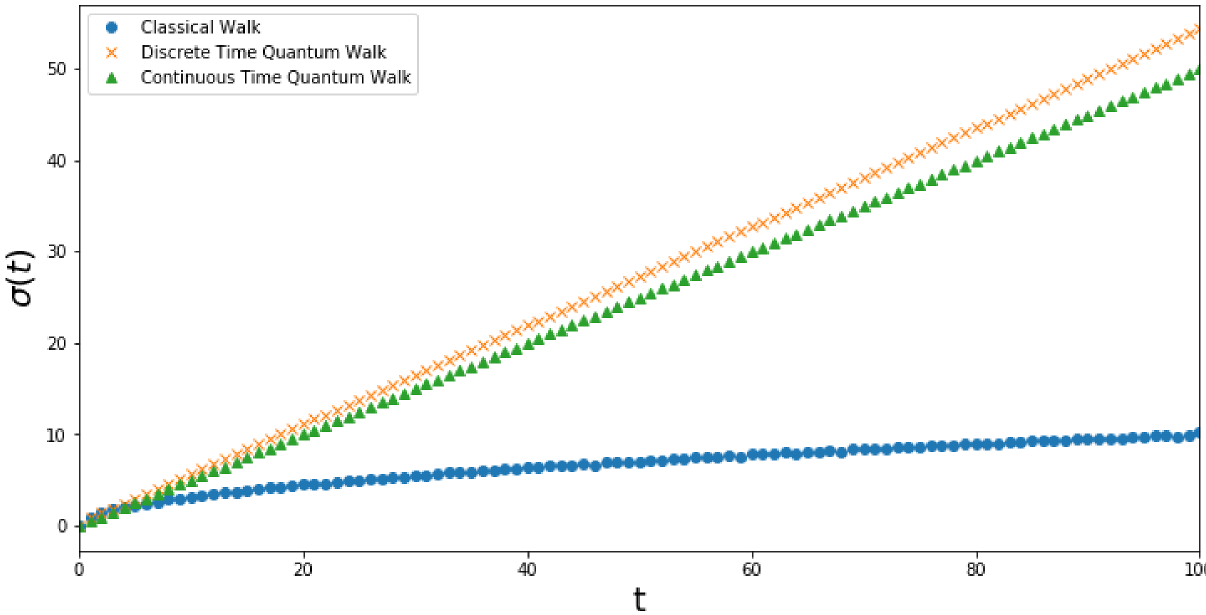
Continuous Time Quantum Walk

In [21]:
```
sigmaCTQW=[]
for n in range(0,101):
    t=n
    N = n
    P = 2*N+1
    gamma = 1/(2*np.sqrt(2))

    H = []

    for i in range(P):
        for j in range(P):
            if j==i:
                H.append(2*gamma)
            elif j==i+1 or j==i-1:
                H.append(-gamma)
            else:
                H.append(0)

    H=np.asarray(H)
    H=H.reshape(P,P)
    U = expm(-1.j*H*t)

    posn0 = zeros(P)
    posn0[N] = 1
    psi0=posn0

    psiN = U.dot(psi0)

    prob = empty(P)
    for k in range(P):
        posn = zeros(P)
        posn[k] = 1
        M_hat_k = outer(posn,posn)
        proj = M_hat_k.dot(psiN)
        prob[k] = proj.dot(proj.conjugate()).real

    nlattice =[]

    for i in range(-n,n+1):
        nlattice.append(i)
    mean_nsquare = 0
    mean_n = 0

    for i in range(len(prob)):
        mean_nsquare += (nlattice[i]**2)*prob[i]
        mean_n += nlattice[i]*prob[i]

    sigmaCTQW.append(sqrt(mean_nsquare-mean_n))
```

In [22]:
```
fig = figure(figsize=(12, 10))
ax = fig.add_subplot(111)

plot(arange(P), prob, label='$t=100$')
#plot(arange(P), prob, 'o')
loc = range (0, P, P//10) #Location of ticks
xticks(loc)
xlim(0, P)
ax.set_xticklabels(range (-N, N+1, int(P / 10)))
xlabel('Sites [n]', size=20)
ylabel('Probability', size=20)
legend()
show()
```

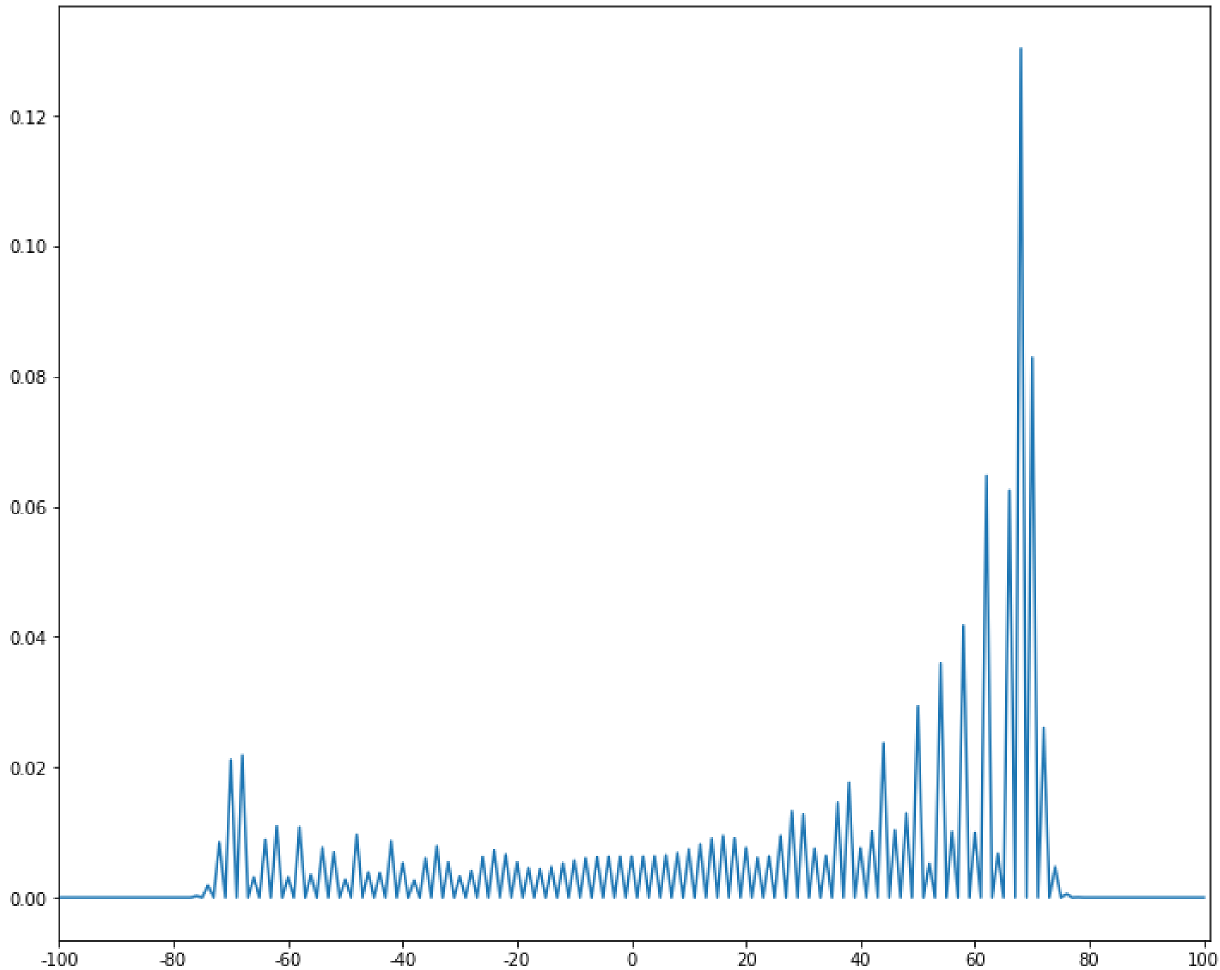


In [23]:
```
fig = figure(figsize=(12, 6))
ax = fig.add_subplot(111)
#plot(nstep, sigma,'-')
plot(nstep, sigmaRW,'o',label='Classical Walk')
plot(nstep, sigmaASQW,'x', label = 'Discrete Time Quantum Walk')
plot(nstep, sigmaCTQW,'^', label = 'Continuous Time Quantum Walk')
xlim(0, 100)
xlabel('t',size=20)
ylabel('$\sigma(t)$',size =20)
legend()
show()
```



Comparison between Analytical Solution and DTQW

In [31]:
```
for n in range(100,101):

    N = n      # number of random steps
    P = 2*N+1    # number of positions

    coin0 = array([1, 0])  # |0>
    coin1 = array([0, 1])  # |1>

    C00 = outer(coin0, coin0)  # |0><0|
    C01 = outer(coin0, coin1)  # |0><1|
    C10 = outer(coin1, coin0)  # |1><0|
    C11 = outer(coin1, coin1)  # |1><1|


    C_hat = (C00 + C01 + C10 - C11)/sqrt(2.)

    ShiftPlus = roll(eye(P), 1, axis=0)
    ShiftMinus = roll(eye(P), -1, axis=0)
    S_hat = kron(ShiftPlus, C00) + kron(ShiftMinus, C11)

    U = S_hat.dot(kron(eye(P), C_hat))

    posn0 = zeros(P)
    posn0[N] = 1
    psi0 = kron(posn0,coin0)

    psiN = linalg.matrix_power(U, N).dot(psi0)

    prob = empty(P)
    for k in range(P):
        posn = zeros(P)
        posn[k] = 1
        M_hat_k = kron( outer(posn,posn), eye(2))
        proj = M_hat_k.dot(psiN)
        prob[k] = proj.dot(proj.conjugate()).real


fig = figure(figsize=(12, 10))
ax = fig.add_subplot(111)

plot(arange(P), prob)
loc = range (0, P, P//10)
xticks(loc)
xlim(0, P)
ax.set_xticklabels(range (-N, N+1, int(P / 10)))

show()
```



In [32]:
```
def psi0_integrand(k, t, x):
    wk = arcsin(sin(k)/sqrt(2))
    return (1+cos(k)/sqrt(1+cos(k)**2))*exp(-1j*(wk*t-k*x))/(2*pi)

def psi1_integrand(k, t, x):
    wk = arcsin(sin(k)/sqrt(2))
    return (exp(1j*k)/sqrt(1+cos(k)**2))*exp(-1j*(wk*t-k*x))/(2*pi)
```

In [33]:
```
def complex_quadrature(func, a, b, **kwargs):
    def real_func(k,t,x):
        return sp.real(func(k,t,x))
    def imag_func(k,t,x):
        return sp.imag(func(k,t,x))
    real_integral = integrate.quad(real_func, a, b, limit=200, **kwargs)
    imag_integral = integrate.quad(imag_func, a, b, limit=200, **kwargs)
    return real_integral[0] + 1j*imag_integral[0]# real_integral[1:], imag_i
```

In [34]:
```
nlattice =[]
for i in range(-100,100+1):
    nlattice.append(i)
```

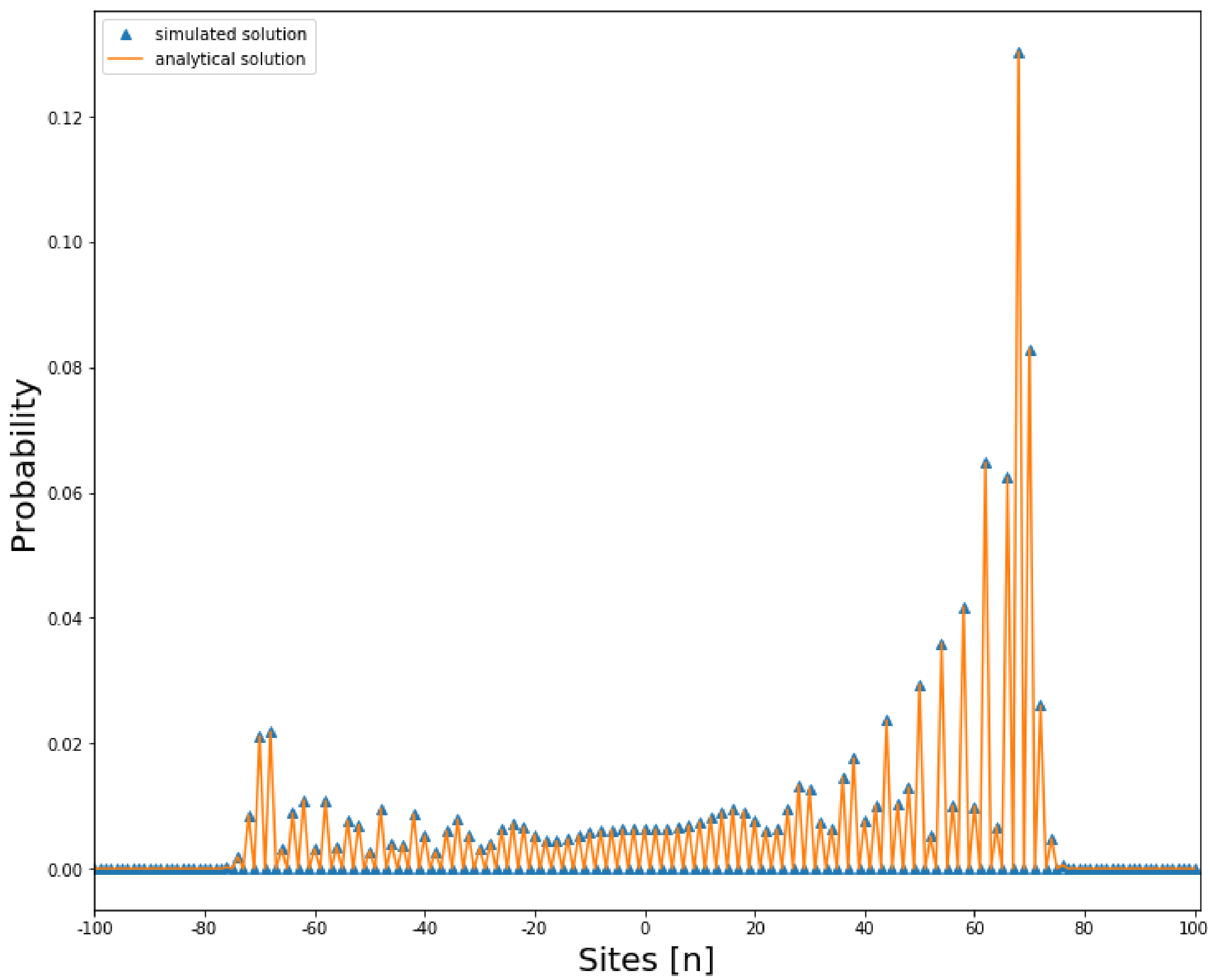
In [35]:
```
psi0=[]
psi1=[]
even = -100
for i in nlattice:
    t=100
    x=i
    if x%2 ==0:
        psi0.append(complex_quadrature(psi0_integrand, -pi,pi, args=(t, x)))
        psi1.append(complex_quadrature(psi1_integrand, -pi,pi, args=(t, x)))
    else:
        psi0.append(0)
        psi1.append(0)
```

In [36]:
```
psi_Analystical=[]
for i in range(len(psi0)):
    psi_Analystical.append(psi0[i].conjugate() *psi0[i] +psi1[i].conjugate()
```

In [37]:

```
fig = figure(figsize=(12, 10))
ax = fig.add_subplot(111)

N=100
plot(arange(P), prob,'^',label='simulated solution',)
plot(arange(P), psi_Analystical,label='analytical solution')
loc = range (0, P, P//10) #Location of ticks
xticks(loc)
xlim(0, P)
ax.set_xticklabels(range (-N, N+1, P // 10))
xlabel('Sites [n]', size=20)
ylabel('Probability', size=20)
legend()
show()
```

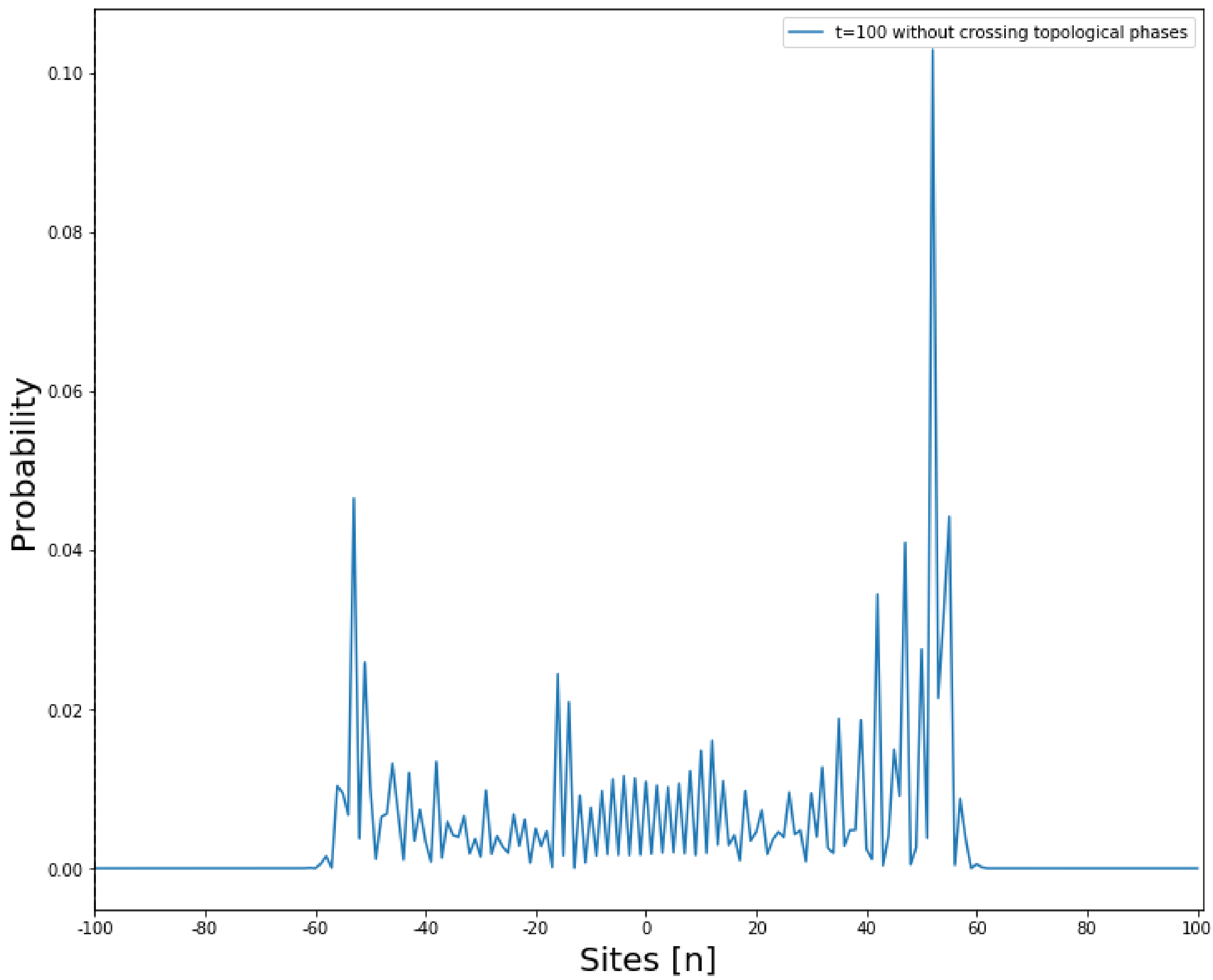
```
/Users/chenwu/anaconda3/lib/python3.6/site-packages/numpy/core/numeric.p
y:492: ComplexWarning: Casting complex values to real discards the imagin
ary part
  return array(a, dtype, copy=False, order=order)
```



Quantum Walk on 1D Topological Phase Transition

In [132]:
```
N = 100
P = 2*N+1

coin0 = array([1, 0])
coin1 = array([0, 1])

C00 = outer(coin0, coin0)
C01 = outer(coin0, coin1)
C10 = outer(coin1, coin0)
C11 = outer(coin1, coin1)

def Ry(theta):
    y_SU2 = cos(theta/2)*C00+sin(theta/2)*C10-sin(theta/2)*C01+cos(theta/2)*
    #array([[cos(theta/2), -sin(theta/2)],[sin(theta/2), cos(theta/2)]])
    return kron(eye(P),y_SU2)

ShiftPlus = roll(eye(P), 1, axis=0)
ShiftMinus = roll(eye(P), -1, axis=0)
T_up = kron(ShiftPlus, C00) + kron(eye(P), C11)
T_down = kron(ShiftMinus, C11) + kron(eye(P), C00)

def theta2(x,theta2Plus,theta2Minus):
    return (1/2)*(theta2Plus+theta2Minus)+(1/2)*(theta2Plus-theta2Minus)*tan

def U(x,theta2Plus,theta2Minus, theta1):
    return T_down.dot(Ry(theta2(x,theta2Plus,theta2Minus)).dot(T_up.dot(Ry(t

posn0 = zeros(P)
posn0[N] = 1
psi0 = kron(posn0, coin0)
psiN = psi0

x = -N
for i in range(N):
    x+=2
    psiN = U(x,1*pi/4,  3*pi/4, -pi/2).dot(psiN)#0.99*pi/2,  0

prob = empty(P)
for k in range(P):
    posn = zeros(P)
    posn[k] = 1
    M_hat_k = kron( outer(posn,posn), eye(2))
    proj = M_hat_k.dot(psiN)
    prob[k] = proj.dot(proj.conjugate()).real
```

In [133]:
```
fig = figure(figsize=(12, 10))
ax = fig.add_subplot(111)

plot(arange(P), prob, label='t=100 without crossing topological phases')
loc = range (0, P, P//10)
xticks(loc)
xlim(0, P)
axvline(x = 0,color='black',ls='dashed')
ax.set_xticklabels(range (-N, N+1, int(P / 10)))
xlabel('Sites [n]', size=20)
ylabel('Probability', size=20)
legend()
show()
```

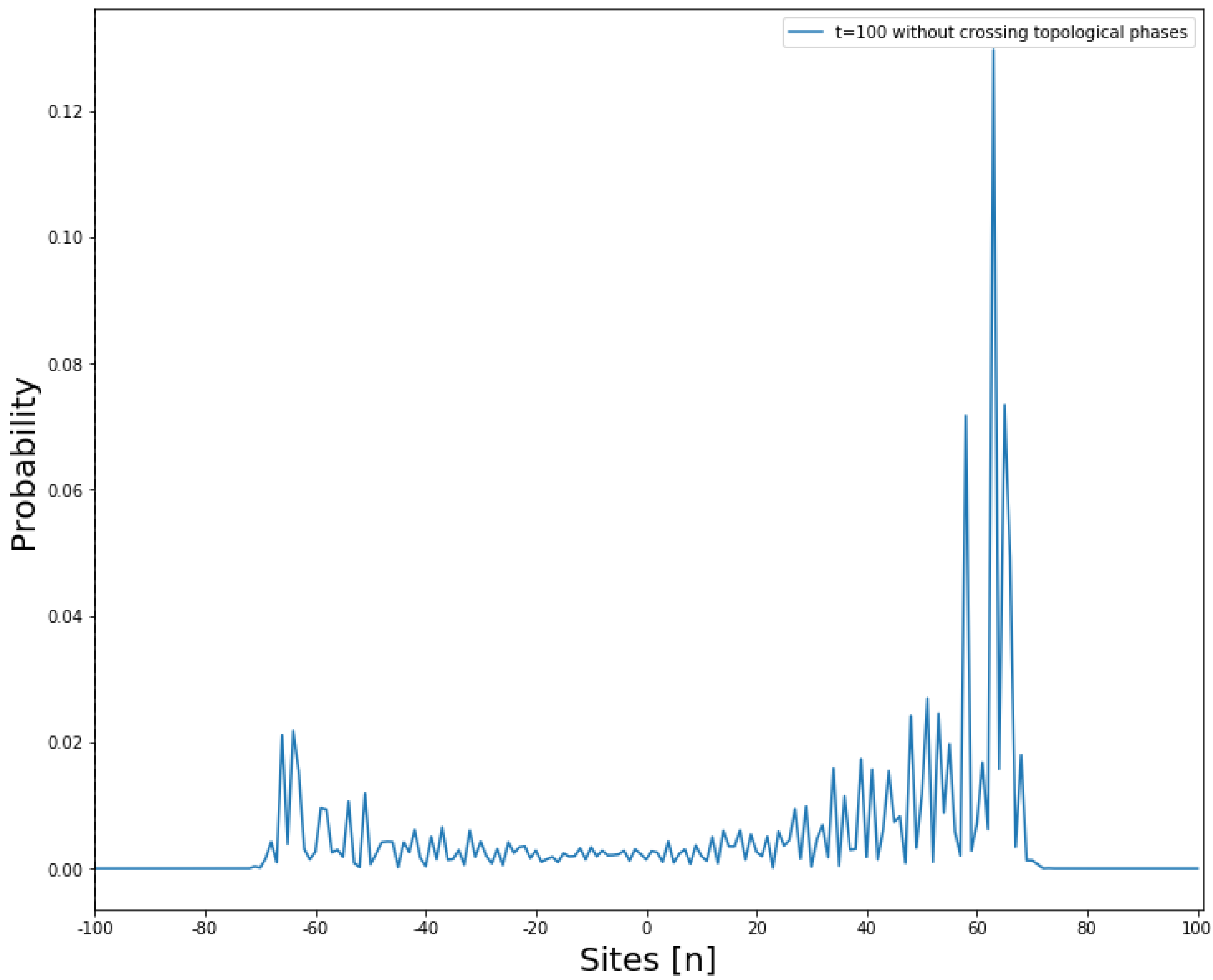


In [117]:
```
prob[100]
```

Out[117]: 2.8518530201539177e-13

In [136]:
```
N = 100
P = 2*N+1

coin0 = array([1, 0])
coin1 = array([0, 1])

C00 = outer(coin0, coin0)
C01 = outer(coin0, coin1)
C10 = outer(coin1, coin0)
C11 = outer(coin1, coin1)

def Ry(theta):
    y_SU2 = cos(theta/2)*C00+sin(theta/2)*C10-sin(theta/2)*C01+cos(theta/2)*
    #array([[cos(theta/2), -sin(theta/2)],[sin(theta/2), cos(theta/2)]])
    return kron(eye(P),y_SU2)

ShiftPlus = roll(eye(P), 1, axis=0)
ShiftMinus = roll(eye(P), -1, axis=0)
T_up = kron(ShiftPlus, C00) + kron(eye(P), C11)
T_down = kron(ShiftMinus, C11) + kron(eye(P), C00)

def theta2(x,theta2Plus,theta2Minus):
    return (1/2)*(theta2Plus+theta2Minus)+(1/2)*(theta2Plus-theta2Minus)*tar

def U(x,theta2Plus,theta2Minus, theta1):
    return T_down.dot(Ry(theta2(x,theta2Plus,theta2Minus)).dot(T_up.dot(Ry(t

posn0 = zeros(P)
posn0[N] = 1
psi0 = kron(posn0, coin0)
psiN = psi0

x = -N
for i in range(N):
    x+=2
    psiN = U(x,0,.99*pi/2, -pi/2).dot(psiN)

prob = empty(P)
for k in range(P):
    posn = zeros(P)
    posn[k] = 1
    M_hat_k = kron( outer(posn,posn), eye(2))
    proj = M_hat_k.dot(psiN)
    prob[k] = proj.dot(proj.conjugate()).real
```

In [137]:
```
fig = figure(figsize=(12, 10))
ax = fig.add_subplot(111)

plot(arange(P), prob, label='t=100 without crossing topological phases')
loc = range (0, P, P//10)
xticks(loc)
xlim(0, P)
axvline(x = 0,color='black',ls='dashed')
ax.set_xticklabels(range (-N, N+1, int(P / 10)))
xlabel('Sites [n]', size=20)
ylabel('Probability', size=20)
legend()
show()
```



In [138]:
```
prob[100]
```

Out[138]: 0.0014239696869836505

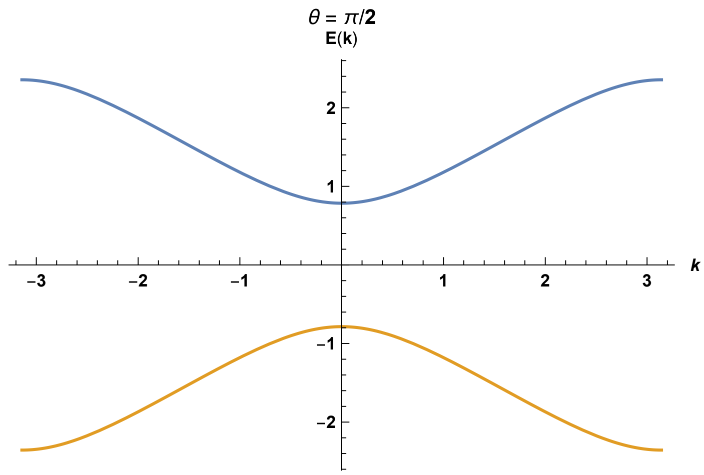
In [ ]:

In[150]:=
```
Clear["Global`*"]
```

In[151]:=
```
Ep[k_, θ_] = ArcCos[Cos[θ / 2] Cos[k]];
Em[k_, θ_] = -ArcCos[Cos[θ / 2] Cos[k]];
```

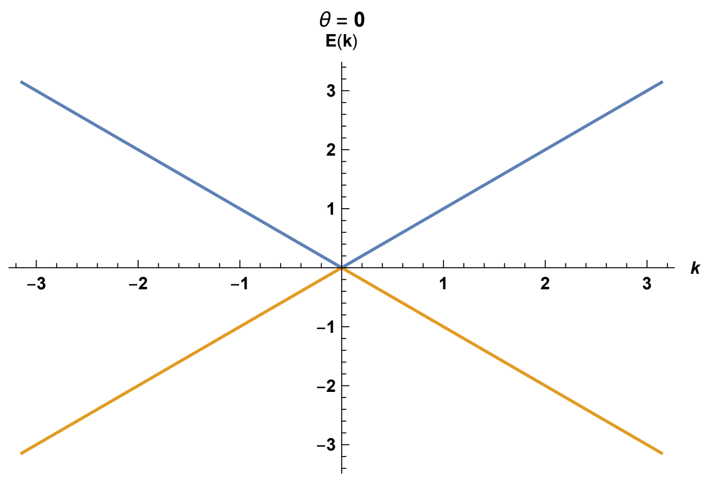
In[153]:=
```
Plot[{Ep[k, Pi / 2], Em[k, Pi / 2]}, {k, -Pi, Pi},
 AxesLabel → {k, "E(k)"}, PlotLabel → "θ = π/2"]
```

Out[153]=



In[154]:=
```
Plot[{Ep[k, 0], Em[k, 0]}, {k, -Pi, Pi}, AxesLabel → {k, "E(k)"}, PlotLabel → "θ = 0"]
```

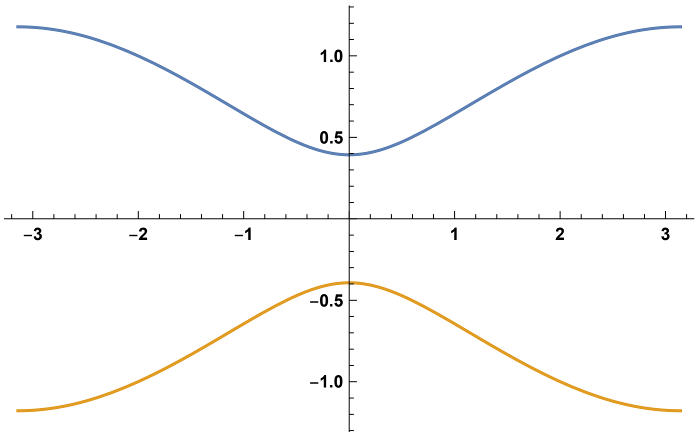
Out[154]=



In[155]:=
```
EE1[k_, θ1_, θ2_] = ArcCos[Cos[θ2 / 2] Cos[θ1 / 2] Cos[k] - Sin[θ1 / 2] Sin[θ2 / 2]];
EE2[k_, θ1_, θ2_] = -ArcCos[Cos[θ2 / 2] Cos[θ1 / 2] Cos[k] - Sin[θ1 / 2] Sin[θ2 / 2]];
```

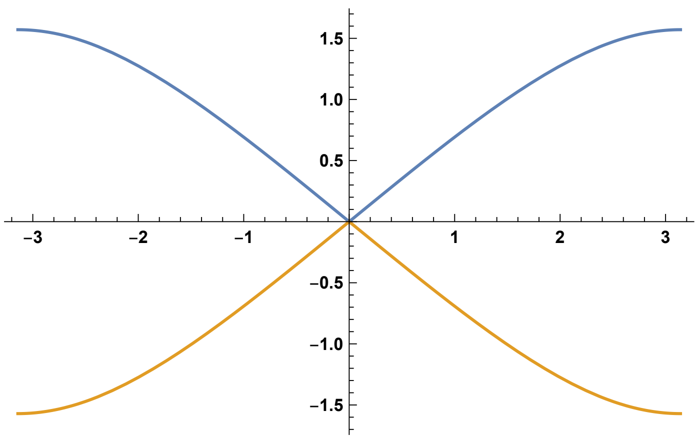
In[157]:=
```
Plot[{EE1[k, -Pi/2, 3 Pi/4], EE2[k, -Pi/2, 3 Pi/4]}, {k, -Pi, Pi}]
```

Out[157]=



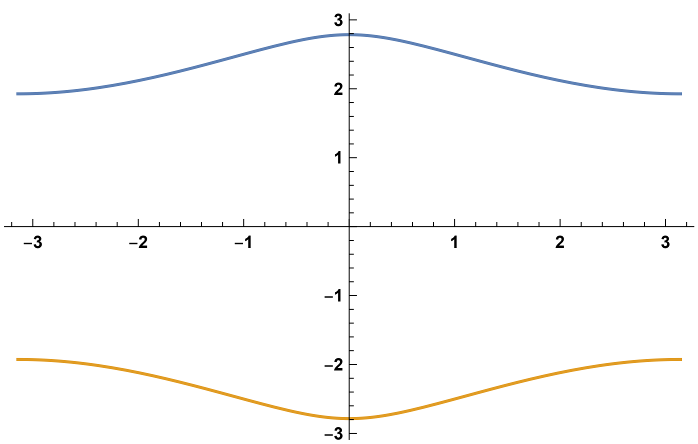
In[158]:=
```
Plot[{EE1[k, -Pi/2, 2 Pi/4], EE2[k, -Pi/2, 2 Pi/4]}, {k, -Pi, Pi}]
```

Out[158]=



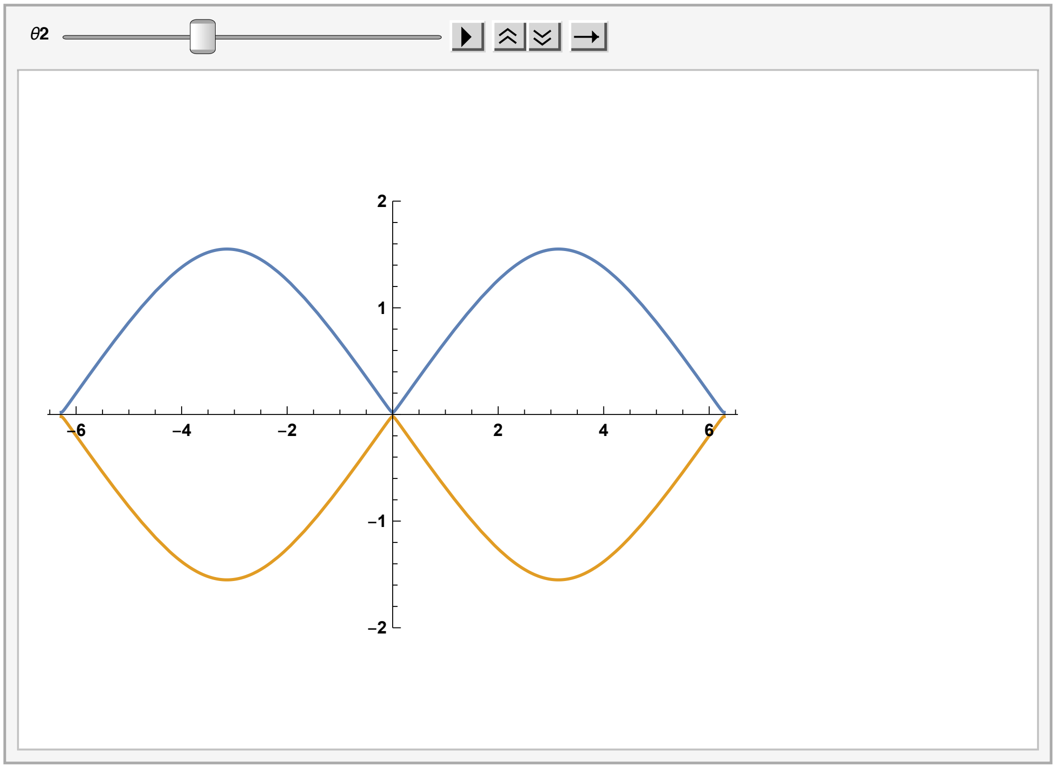
In[159]:=
```
Plot[{EE1[k, -Pi/2, -4], EE2[k, -Pi/2, -4]}, {k, -Pi, Pi}]
```

Out[159]=



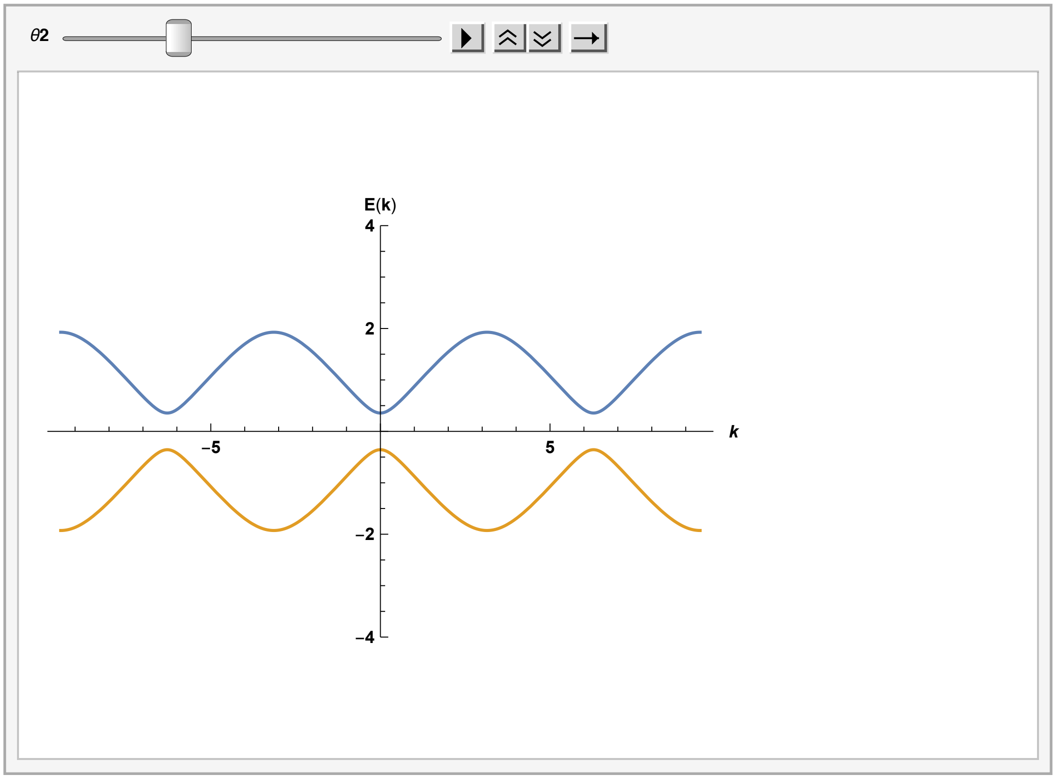
In[54]:=
```
Animate[Plot[{EE1[k, -Pi/2, θ2], EE2[k, -Pi/2, θ2]},
  {k, -2 Pi, 2 Pi}, PlotRange → {-2, 2}], {θ2, Pi/4, 4 Pi/4}]
```

Out[54]=



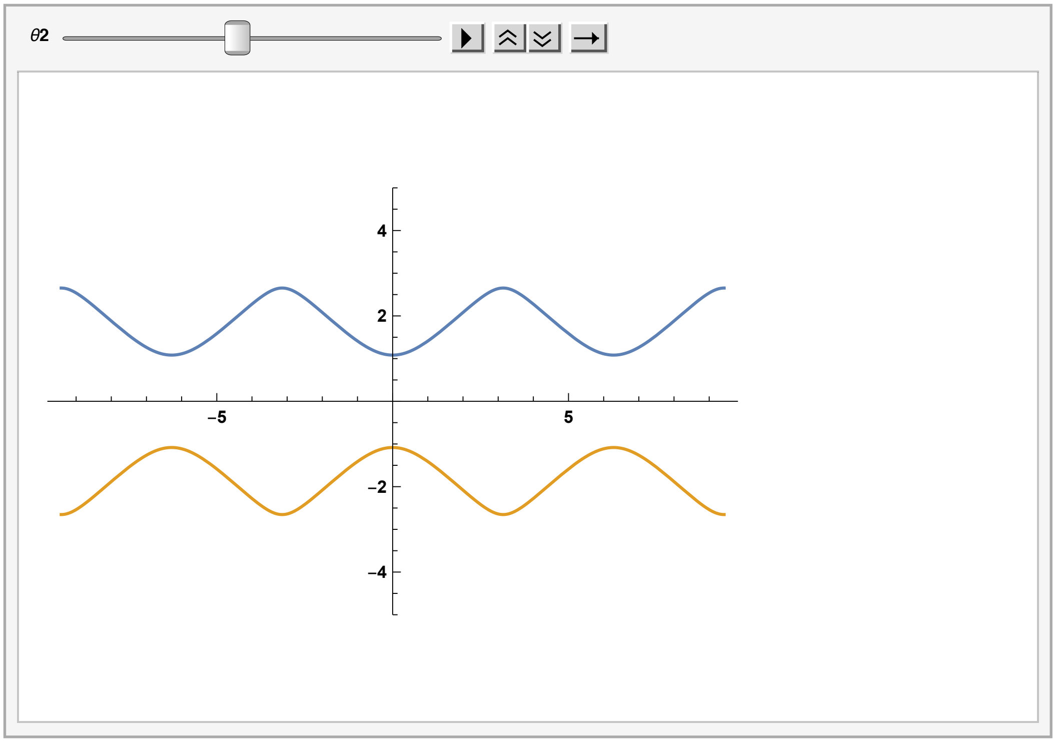
In[53]:=
```
Animate[Plot[{EE1[k, -Pi/2, θ2], EE2[k, -Pi/2, θ2]},
  {k, -3 Pi, 3 Pi}, PlotRange → {-4, 4}, AxesLabel → {k, "E(k)"}], {θ2, 0, Pi}]
```

Out[53]=



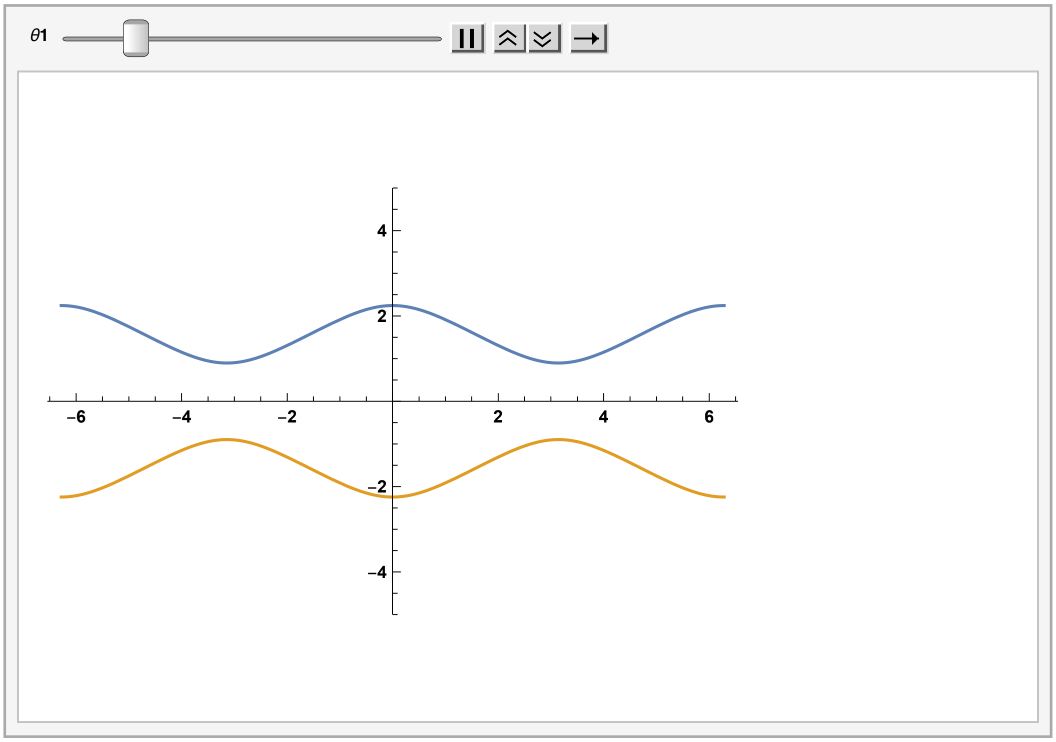
In[49]:=
```
Animate[Plot[{EE1[k, -π/2, θ2], EE2[k, -π/2, θ2]}, {k, -3 π, 3 π}, PlotRange → {-5, 5}],
 {θ2, -2 π, 2 π}, AnimationRunning → False, DisplayAllSteps → True]
```

Out[49]=



In[45]:=
```
Animate[Plot[{Ep[k, θ1], Em[k, θ1]}, {k, -2 Pi, 2 Pi}, PlotRange → {-5, 5}],
 {θ1, -2 Pi, 2 Pi}]
```

Out[45]=



1-D Topological simulation using split-step DTQW

The the code is provided by the author of Topological phenomena in quantum walks: elementary introduction to the physics of topological phases: Takuya Kitagawa

```
In[160]:= distribution[plotmax_, initialangle_, theta1m_, theta1p_, theta2_] :=
  Module[{n = plotmax, iangle = initialangle, thetam = theta1m, thetap = theta1p,
    theta2angle = theta2, t, i, boundarylength, Initialup, Initialdown,
    theta1, a, temp, d, rotation1, rotation2}, boundarylength = 0.01;
   Initialup = N[Cos[iangle]]; (* Not Needed Normalised*)
   Initialdown = N[ Sin[iangle]]; theta1 = Table[(thetam + thetap) / 2 +
       (thetap - thetam) / 2 * Tanh[(i - 2 n + 1 / 2) / boundarylength], {i, 4 n + 1}];
   (* at "l" step and "i" position, coin up "j" + down "k" *)
   a = Table[0, {l, n}, {i, 4 n + 1}, {k, 2}];
   temp = Table[0, {i, 4 n + 1}, {k, 2}]; d = Table[0, {i, 4 n + 1}, {k, 2}];
   rotation1 = N[Table[MatrixExp[-I PauliMatrix[2] theta1[[i]] / 2], {i, 4 n + 1}]];
   rotation2 = N[MatrixExp[-I PauliMatrix[2] theta2angle / 2]];
   (* Normalised Initial Coin Condition at 2n, zero step is t=1,
   edge is n+1*) a[[1, 2 n, 1]] = Initialup; a[[1, 2 n, 2]] = Initialdown;
 (* Time Evolution Step *) For[t = 1, t <= n - 1, t++,
    For[i = 1 + 2 n - 2 t, i <= 2 n + 2 t + 1, i++, (* Coin Flip *)
     d[[i, All]] = rotation1[[i, All, All]].a[[t, i, All]];];
    (* Shift Process with the normalization *) For[i = 1 + 2 n - 2 t,
     i <= 2 n + 2 t + 1, i++, temp[[i + 1, 1]] = d[[i, 1]]; temp[[i, 2]] = d[[i, 2]];];
    For[i = 1 + 2 n - 2 t, i <= 2 n + 2 t + 1, i++, (* Coin Flip *)
     d[[i, All]] = rotation2.temp[[i, All]];];
    (* Shift Process with the normalization *) For[i = 1 + 2 n - 2 t, i <= 2 n + 2 t + 1,
     i++, a[[t + 1, i, 1]] = d[[i, 1]]; a[[t + 1, i - 1, 2]] = d[[i, 2]];]; ];
   Table[{i - 2 n, Abs[a[[t, i, 1]]] ^ 2 + Abs[a[[t, i, 2]]] ^ 2},
    {t, 1, n - 1, 1}, {i, n, 3 n, 1}] ];
```

```
In[161]:= phasediagram[theta1m_, theta1p_, theta2_] :=
  Module[{thetam = theta1m, thetap = theta1p, theta2angle = theta2,
    pline1, pline2, pline3, pline4, pline5, pline6, theta2line, tx, dots},
   pline1 = Plot[x, {x, -2 π, 2 π}, PlotStyle -> {Red, Dotted, Thickness[.005]}];
   pline2 = Plot[2 π - x, {x, 0, 2 π}, PlotStyle -> {Red, Dotted, Thickness[.005]}];
   pline3 = Plot[-2 π - x, {x, -2 π, 0}, PlotStyle -> {Red, Dotted, Thickness[.005]}];
   pline4 = Plot[-x, {x, -2 π, 2 π}, PlotStyle -> {Black, Thickness[.005]}];
   pline5 = Plot[2 π + x, {x, -2 π, 0}, PlotStyle -> {Black, Thickness[.005]}]; pline6 =
    Plot[-2 π + x, {x, 0, 2 π}, PlotStyle -> {Black, Thickness[.005]}]; theta2line =
    Plot[theta2, {x, -2 π, 2 π}, PlotStyle -> {Black, Dotted, Thickness[.005]}];
   tx = Graphics[{GrayLevel[.8], Rotate[Rectangle[{π - π / Sqrt[2], -π / Sqrt[2]},
         {π + π / Sqrt[2], π / Sqrt[2]}], 45 Degree, {π, 0}], GrayLevel[.8], Rotate[
        Rectangle[{-π - π / Sqrt[2], -π / Sqrt[2]}, {-π + π / Sqrt[2], π / Sqrt[2]}], 45
         Degree, {-π, 0}], GrayLevel[.8], Polygon[{{-π, -π}, {-2 π, -2 π}, {0, -2 π}}],
      GrayLevel[.8], Polygon[{{π, -π}, {2 π, -2 π}, {0, -2 π}}], GrayLevel[.8],
      Polygon[{{π, π}, {2 π, 2 π}, {0, 2 π}}], GrayLevel[.8], Polygon[
       {{-π, π}, {-2 π, 2 π}, {0, 2 π}}], Text[Style["1", 30, Bold, Black], {π, 0}],
      Text[Style["1", 30, Bold, Black], {-π, 0}], Text[Style["1", 30, Bold, Black],
       {π, 3 π / 2}], Text[Style["1", 30, Bold, Black], {-π, 3 π / 2}],
      Text[Style["1", 30, Bold, Black], {π, -3 π / 2}], Text[Style["1", 30,
         Bold, Black], {-π, -3 π / 2}], Text[Style["0", 30, Bold, Black], {0, π}],
      Text[Style["0", 30, Bold, Black], {0, -π}], (*Text[Style["0",30,Bold,Black],
       {3π/2,π}],*)Text[Style["0", 30, Bold, Black], {-3 π / 2, π}],
      Text[Style["0", 30, Bold, Black], {-3 π / 2, -π}] }];
   dots = Graphics[{Green, Disk[{thetam, theta2}, π / 10], Blue, Disk[{thetap, theta2},
        π / 10], Text[Style["Left", Bold, Green], {thetam, theta2 + π / 5}],
      Text[Style["Right", Bold, Blue], {thetap, theta2 + π / 5}]}];
   Show[tx, pline1, pline2, pline3, pline4, pline5, pline6, theta2line,
    dots, PlotRange -> {{-2 π, 2 π}, {-2 π, 2 π}},
    PlotRangePadding -> 0, Axes -> False, Frame -> True,
    FrameTicks -> {{{-2 π, -π, 0, π, 2 π}, None}, {{-2 π, -π, 0, π, 2 π}, None}},
    AspectRatio -> 1, FrameLabel -> {{"second rotation", None},
      {"first rotation", "phase diagram (winding number)"}}] ];
```

```
In[162]:= Manipulate[GraphicsRow[
    {Show[Graphics[{Opacity[0.1, Green], Rectangle[{-plotmax, 0}, {0, 1.0}],
        Opacity[0.1, Blue], Rectangle[{0, 0}, {plotmax, 1.0}]}],
      ListPlot[distribution[plotmax, iniangle, thetam, thetap, theta2][[t + 1, All]],
       Filling -> Axis, FillingStyle -> Directive[Black, Thick],
       PlotRange -> {{-plotmax, plotmax}, {0, 1}}, PlotStyle -> PointSize[Medium],
       Joined -> True, Mesh -> All], AspectRatio -> 0.6,
      PlotRange -> {{-plotmax, plotmax}, {0, 1}}, PlotRangePadding -> 0,
      Axes -> True, Frame -> True, FrameTicks -> {{{0, 0.2, 0.4, 0.6, 0.8, 1.0}, None},
        {Table[(plotmax - 2) / 4 i - (plotmax - 2), {i, 0, 8}], None}},
      FrameLabel -> {{"probability", None}, {"sites", "probability distribution"}}],
     phasediagram[thetam, thetap, theta2]}, ImageSize -> {1000, 500}],
   {{t, 0, "steps"}, 0, plotmax - 2, 1, Appearance -> "Labeled"},
   {{plotmax, 20, "maximum number of steps"},
    {100 -> "20", 42 -> "40"}, ControlType -> RadioButton},
   {{iniangle, 0, "initial spin"}, {0 -> "up", π/2 -> "down"},
    ControlType -> RadioButton}, Delimiter,
   {{thetam, -3 * π/8, "first rotation θ1: left bulk"}, -2 π, 2 π, π/8},
   {{thetap, 9 * π/8, "first rotation θ1: right bulk"}, -2 π, 2 π, π/8},
   Delimiter,
   {{theta2, π/2, "second rotation θ2"}, -2 π, 2 π, π/8},
   AutorunSequencing -> {1},
   SaveDefinitions -> True]
```

Out[162]=

steps 18

maximum number of steps 20 40

initial spin up down

first rotation θ_1: left bulk

first rotation θ_1: right bulk

second rotation θ_2