Nov 9, 2020 (Due: 08:00 Nov 16, 2020)

1. Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. Let r_k be the residual vector at kth iterate when applying SD to the linear system $Ax = b$. Show that if $r_{k+1} = 0$, then r_k is an eigenvector of A.

2. Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. Let x_k be the approximate solution at kth iterate when applying SD to the linear system $Ax = b$. Show that

$$f(x_{k+1}) \le \bigl(1 - \kappa^{-1}\bigr) f(x_k),$$

where $f(x) = x^\top Ax - 2b^\top x$ and $\kappa = \|A\|_2\|A^{-1}\|_2$.

3. Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. Show that when applying CG to solve $Ax = b$, CG converges within k iterates if A has k distinct eigenvalues.

4. Implement steepest descent (SD) and conjugate gradient (CG) methods. Use them to solve a Poisson equation over $[0,1]\times[0,1]$ with Dirichlet boundary condition and a reasonably smooth right-hand side. Note that the 2D Laplacian can be discretized as

$$A = T \otimes I_n + I_n \otimes T,$$

where

$$T = (n+1)^2 \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^n.$$

Visualize the solution and plot the convergence history.