Nov 30, 2020 (Due: 08:00 Dec 7, 2020)

1. Let x be a normalized eigenvector of a Hermitian matrix, corresponding to the eigenvalue λ. Suppose that $\tilde{x} = x + \delta x$ with $\|\delta x\| = \epsilon$. Show that

$$\frac{\tilde{x}^* A \tilde{x}}{\tilde{x}^* \tilde{x}} = \lambda + O(\epsilon^2).$$

Use some numerical experiments to verify this fact.

2. Can you construct a Hermitian positive definite matrix $A \in \mathbb{C}^{n\times n}$ such that it is impossible for the Lanczos algorithm to capture the two largest eigenvalues simultaneously for any starting vector?

3. Implement Lanczos algorithm for the symmetric eigenvalue problem. Test it with some examples of size several thousands. What do you observe for the convergence of Ritz pairs and the orthogonality of the Lanczos vectors?

4. Construct a low rank matrix and use random projection to compute its full rank decomposition.

Instead of random projection, you can also randomly select a square submatrix. What is the rank of the submatrix you typical obtain?

5. (optional) Implement Arnoldi algorithm for the unsymmetric eigenvalue problem. Test it with some examples of size several thousands. Which eigenvalues can you obtain?