Oct 19, 2020 (Due: 08:00 Oct 26, 2020)

1. Prove the real Schur decomposition: Let $A \in \mathbb{R}^{n\times n}$. Then there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that $Q^\top AQ$ is quasi-upper triangular (i.e., block triangular with block size 1×1 and 2×2).

2. Investigate the behavior of the power method applied to the following matrices:

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad B = \begin{bmatrix} \lambda & 1 \\ 0 & -\lambda \end{bmatrix},$$

where $\lambda \in \mathbb{C}$ is a given constant.

3. Let $A \in \mathbb{C}^{n\times n}$, $x \in \mathbb{C}^n$. Suppose that $X = [x, Ax, \dots, A^{n-1}x]$ is nonsingular. Show that $X^{-1}AX$ is upper Hessenberg.

4. Let $A_0 \in \mathbb{C}^{n\times n}$, μ_0, μ_1, …, $\mu_m \in \mathbb{C}$. Define A_1, A_2, …, A_{m+1} by

$$A_k - \mu_k I = Q_k R_k, \qquad A_{k+1} = R_k Q_k + \mu_k I,$$

for $k \in \{0, 1, \dots, m\}$, where Q_k's are unitary matrices. Show that

$$(A_0 - \mu_0 I)(A_0 - \mu_1 I) \cdots (A_0 - \mu_m I) = (Q_0 Q_1 \cdots Q_m)(R_m \cdots R_1 R_0).$$

5. Read the paper “From Random Polygon to Ellipse: An Eigenanalysis” by A. N. Elmachtoub and C. F. Van Loan (available on eLearning). Reproduce the experiments in this paper.

6. (optional) Implement inverse iteration and Rayleigh quotient iteration. Apply them to some examples and observe the performance and convergence rate.