Oct 26, 2020 (Due: 08:00 Nov 2, 2020)

1. Let

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Design an algorithm to find an orthogonal matrix $Q \in \mathbb{R}^{2\times 2}$ such that

$$Q^\top A Q = \begin{bmatrix} b & * \\ 0 & a \end{bmatrix}.$$

What can you say about the entry "*"?
For general eigenvalue swapping problem, see Exercise 6.

2. Design an algorithm to compute all eigenvectors of a given real Schur form. You may assume that all eigenvalues are simple (i.e., of multiplicity one).

3. Let

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ c_2 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-1} & a_{n-1} & b_{n-1} \\ & & & c_n & a_n \end{bmatrix} \in \mathbb{R}^{n\times n},$$

with $b_i c_{i+1} \geq 0$ for $i \in \{1, 2, \dots, n-1\}$. Show that A is diagonalizable, and has real spectrum.

4. Let $A \in \mathbb{C}^{n\times n}$ be Hermitian. We can tridiagonalize A by unitary equivalences, i.e., there exists a unitary matrix $Q \in \mathbb{C}^{n\times n}$ such that $T = Q^* A Q$ is tridiagonal. When we need to diagonalize A, ideally we would like to impose additional properties on T to simplify the calculation. What kind of additional properties can we impose?

5. Implement the Jacobi diagonalization algorithm for real symmetric matrices. Visualize the performance for a few matrices with different sizes. Visualize the convergence history for one example. You are encouraged to make observations on *entry-wise* convergence.

6. (optional) Eigenvalue swapping in real Schur form:
Suppose that A_{11}, $A_{22} \in \mathbb{R}^{1\times 1} \cup \mathbb{R}^{2\times 2}$, and A_{12} is also real. Design an algorithm to find a real orthogonal matrix Q such that

$$Q^\top \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} Q = \begin{bmatrix} \tilde{A}_{22} & \tilde{A}_{12} \\ 0 & \tilde{A}_{11} \end{bmatrix},$$

and $\operatorname{spec}(A_{11}) = \operatorname{spec}(\tilde{A}_{11})$, $\operatorname{spec}(A_{22}) = \operatorname{spec}(\tilde{A}_{22})$. You may assume that 2×2 diagonal blocks have non-real eigenvalues.