Nov 23, 2020 (Due: 08:00 Nov 30, 2020)

1. Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. Let x_k be the approximate solution at kth iterate when applying CG to the linear system $Ax = b$. Show that

$$\|x_k - x_*\|_2 \le 2\sqrt{\kappa}\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \|x_0 - x_*\|_2,$$

where $x_* = A^{-1}b$ and $\kappa = \|A\|_2\|A^{-1}\|_2$.

2. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, and $b \in \mathbb{R}^n$. Suppose that $\mathcal{V} \subset \mathbb{R}^n$ is a subspace. Show that

$$x_0 = \arg\min_{x\in\mathcal{V}} \|x - A^{-1}b\|_A$$

if and only if $b - Ax_0 \in \mathcal{V}^\perp$. (The orthogonal complement is defined using the standard inner product.)

3. Implement the preconditioned CG (PCG) algorithm. Use an artificial example to test the convergence of CG, with and without preconditioning.

One possible way to construct an artificial example for preconditioning is as follows: Create a random lower bidiagonal matrix L_0 with $\max_{i,j}|L_0(i,j)| = 1$. Create another lower triangular matrix L by perturbing L_0 a little bit, and filling a few lower triangular entries with elements in $[-0.1, 0.1]$. Then you can test (P)CG with $A = LL^\top$ and $M = L_0L_0^\top$. You are certainly free to try other examples.

4. Suppose that $A \in \mathbb{R}^{m\times n}$ is sparse and tall-skinny (i.e., $m \gg n$). Design an iterative algorithm solve the ridge regression problem

$$\min \|Ax - b\|_2^2 + \lambda\|x\|_2^2,$$

where $b \in \mathbb{R}^m$ and $\lambda \in (0, +\infty)$ are given.

5. (optional) Let $p(x) = \sum_{k=0}^n a_k x^k$ be a real polynomial such that

$$\max_{-1\le x\le 1} |p(x)| \le 1.$$

Show that $|a_n| \le 2^{n-1}$.