Oct 12, 2020 (Due: 08:00 Oct 26, 2020)

1. Given x, $y \in \mathbb{R}^n$. Find a rotation matrix Q such that the columns of $[x, y]Q$ are orthogonal to each other.

2. Suppose that a tall-skinny matrix $A \in \mathbb{R}^{m\times n}$ is upper bidiagonal (i.e., $a_{i,j} \neq 0$ only if $i - j \in \{0, -1\}$). Design an algorithm based on Givens rotation to solve the ridge regression problem

$$\min \|A - xb\|_2^2 + \lambda \|x\|_2^2,$$

where λ is a given positive number.

3. Implement QR factorization through CGS/MGS/CGS2/MGS2. Construct some well-conditioned (e.g., $\kappa(A) \approx 10^1$), mildly ill-conditioned (e.g., $\kappa(A) \approx 10^6$), and extremely ill-conditioned (e.g., $\kappa(A) \approx 10^{16}$) examples to test these orthogonalization subprograms. Compute the (relative) residual norm $\|A - QR\|_F / \|A\|_F$, and visualize the orthogonality $|Q^\top Q - I|$ for each case. What do you observe?

4. Consider the constrained least squares problem

$$\min_{Cx=d} \|Ax - b\|_2.$$

In the lecture we use LQ factorization to eliminate the linear constraint $Cx = d$ and reduce the problem to a standard least squares one. Design an algorithm based on Gaussian elimination to eliminate the linear constraint.

5. The least squares problem $\min\|Ax - b\|_2$ is equivalent to an augmented linear system

$$\begin{bmatrix} I & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix},$$

which is Hermitian and indefinite. Similar augmented linear systems exist for the constrained least squares problem

$$\min_{Cx=d} \|Ax - b\|_2.$$

Can you figure it out?

6. (optional) Construct two linear least squares problems mildly ill-conditioned (e.g., $\kappa(A) \approx 10^6$) and extremely ill-conditioned (e.g., $\kappa(A) \approx 10^{16}$) matrices. Use CGS, MGS (both without reorthogonalization), and Householder QR to solve these least squares problems. Compare the quality of the solutions.

(When constructing these examples, you are suggested to carefully choose the right-hand-sides such that the exact solutions are known.)