Dec 7, 2020 (Due: 08:00 Dec 14, 2020)

1. Let A, $B \in \mathbb{C}^{n\times n}$. Suppose that $\exp(At)\exp(Bt) = \exp\bigl((A+B)t\bigr)$ holds for all $t \in \mathbb{C}$. Show that $AB = BA$.

2. Let $A \in \mathbb{C}^{n\times n}$ with $\rho(A) < 1$. Show that

$$\biggl\|\sum_{k=0}^{N-1}\frac{A^k}{k!} - \exp(A)\biggr\|_2 \leq \frac{\|A\|_2^N}{N!}\cdot\frac{1}{1-\|A\|_2/(N+1)}.$$

3. Simplify the expression

$$\sum_{k=0}^{N}(u+v\omega)^* f(A)(u+v\omega),$$

where $\omega = \exp(2\pi\mathrm{i}/N)$.

4. Implement the Lanczos algorithm for computing the matrix functional $v^* f(A)v$, where A is Hermitian. Test it with some matrices of size several thousands and some sufficiently smooth functions. Plot the convergence history.

5. (optional) Read the papers "Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later" by C. Moler and C. F. Van Loan, and "The scaling and squaring method for the matrix exponential revisited" by N. J. Higham.