Sep 28, 2020 (Due: 08:00 Oct 12, 2020)

1. It can be show that Gaussian elimination with partial pivoting is numerically stable for solving nonsingular tridiagonal linear systems, in the sense that the growth factor is always $O(1)$. Give a concrete upper bound on the growth factor.

2. Find the exact Cholesky factor of the $n \times n$ positive definite matrix

$$\begin{bmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ & & & 1 & 2 \end{bmatrix}.$$

3. Consider the following systems of linear equations:

$$\begin{bmatrix} 8 & 1 & & & \\ 6 & 8 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 6 & 8 & 1 \\ & & & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ \vdots \\ 15 \\ 14 \end{bmatrix},$$

$$\begin{bmatrix} 6 & 1 & & & \\ 8 & 6 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 8 & 6 & 1 \\ & & & 8 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \\ \vdots \\ 15 \\ 14 \end{bmatrix}.$$

Note that the exact solutions are known. Use Gaussian elimination, with and without partial pivoting, to solve these linear systems. Visualize the forward errors as well as the residuals for a number of different values of n (e.g., for $n = 10$, 20, $\ldots$, 1000). Explain your observations.

4. Find an example of linear system such that Jacobi method converges while Gauss–Seidel method diverges. Justify your claim.

5. Find an example of positive definite linear system such that Gauss–Seidel method converges while Jacobi method diverges. Justify your claim.

6. Let $B \in \mathbb{C}^{n\times n}$, $g \in \mathbb{C}^n$. Show that the iterative scheme

$$x^{(k+1)} = Bx^{(k)} + g$$

converges in at most n steps if $\rho(B) = 0$.

7. (optional) Numerically solve the 2D Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with boundary conditions

$$u(0, y) = u(1, y) = u(x, 1) = 0, \qquad u(x, 0) = 0, \qquad (x,\, y \in [0, 1]).$$

Use Jaboci method or Gauss–Seidel method to solve the discretized system. Visualize the solution and the convergence history.