Chapter 8

Section 8.1

8.1.**1** $\vec{e}_1, \vec{e}_2$ is an orthonormal eigenbasis.

8.1.**2** $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ is an orthonormal eigenbasis.

8.1.**3** $\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}-1\\2\end{bmatrix}$ is an orthonormal eigenbasis.

8.1.**4** $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\2\end{bmatrix}$ is an orthonormal eigenbasis.

8.1.**5** Eigenvalues $-1, -1, 2$

Choose $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}$ in E_{-1} and $\vec{v}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ in E_2 and let $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$.

8.1.**6** $\frac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}, \frac{1}{3}\begin{bmatrix}2\\-1\\-2\end{bmatrix}, \frac{1}{3}\begin{bmatrix}1\\-2\\2\end{bmatrix}$ is an orthonormal eigenbasis.

8.1.**7** $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ is an orthonormal eigenbasis, so $S = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$ and $D = \begin{bmatrix}5 & 0\\0 & 1\end{bmatrix}$.

8.1.**8** $\frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix}-1\\3\end{bmatrix}$ is an orthonormal eigenbasis, with $\lambda_1 = 4$ and $\lambda_2 = -6$, so $S = \frac{1}{\sqrt{10}}\begin{bmatrix}3 & -1\\1 & 3\end{bmatrix}$ and $D = \begin{bmatrix}4 & 0\\0 & -6\end{bmatrix}$.

8.1.**9** $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ is an orthonormal eigenbasis, with $\lambda_1 = 3$, $\lambda_2 = -3$, and $\lambda_3 = 2$, so

$S = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1 & 0\\0 & 0 & \sqrt{2}\\1 & 1 & 0\end{bmatrix}$ and $D = \begin{bmatrix}3 & 0 & 0\\0 & -3 & 0\\0 & 0 & 2\end{bmatrix}$.

8.1.**10** $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 9$.

$\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\\0\end{bmatrix}$ is in E_0 and $\vec{v}_2 = \frac{1}{3}\begin{bmatrix}1\\-2\\2\end{bmatrix}$ is in E_9.

Let $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 \\ -4 \\ -5 \end{bmatrix}$; then $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ is an orthonormal eigenbasis.

$$S = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{3} & \frac{2}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{3} & -\frac{4}{3\sqrt{5}} \\ 0 & \frac{2}{3} & -\frac{\sqrt{5}}{3} \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

8.1.**11** $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an orthonormal eigenbasis, with $\lambda_1 = 2$, $\lambda_2 = 0$, and $\lambda_3 = 1$, so $S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

8.1.**12** a $E_1 = \text{span} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $E_{-1} = (E_1)^\perp$. An orthonormal eigenbasis is $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

b Use Theorem 7.4.1: $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

c $A = SBS^{-1} = \begin{bmatrix} -0.6 & 0 & 0.8 \\ 0 & -1 & 0 \\ 0.8 & 0 & 0.6 \end{bmatrix}$, where $S = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{bmatrix}$.

8.1.**13** Yes; if $\vec{v}$ is an eigenvector of A with eigenvalue λ, then $\vec{v} = I_3\vec{v} = A^2\vec{v} = \lambda^2\vec{v}$, so that $\lambda^2 = 1$ and $\lambda = 1$ or $\lambda = -1$. Since A is symmetric, E_1 and E_{-1} will be orthogonal complements, so that A represents the reflection about E_1.

8.1.**14** Let S be as in Example 3. Then $S^{-1}AS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

a. This matrix is $2A$ so that $S^{-1}(2A)S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

b. This is $A - 3I_3$, so that $S^{-1}(A - 3I_3)S = S^{-1}AS - 3I_3 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

c. This is $\frac{1}{2}(A - I_3)$, so that $S^{-1}\left(\frac{1}{2}(A - I_3)\right)S = \frac{1}{2}(S^{-1}AS - I_3) = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

8.1.**15** Yes, if $A\vec{v} = \lambda\vec{v}$, then $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$, so that an orthonormal eigenbasis for A is also an orthonormal eigenbasis for A^{-1} (with reciprocal eigenvalues).

8.1.**16** a $\ker(A)$ is four-dimensional, so that the eigenvalue 0 has multiplicity 4, and the remaining eigenvalue is $\mathrm{tr}(A) = 5$.

b $B = A + 2I_5$, so that the eigenvalues are 2, 2, 2, 2, 7.

c $\det(B) = 2^4 \cdot 7 = 112$ (product of eigenvalues)

8.1.**17** If A is the $n \times n$ matrix with all 1's, then the eigenvalues of A are 0 (with multiplicity $n-1$) and n. Now $B = qA + (p-q)I_n$, so that the eigenvalues of B are $p - q$ (with multiplicity $n-1$) and $qn + p - q$. Thus $\det(B) = (p-q)^{n-1}(qn + p - q)$.

8.1.**18** By Theorem 6.3.6, the volume is $|\det A| = \sqrt{\det(A^TA)}$. Now $\vec{v}_i \cdot \vec{v}_j = \|\vec{v}_i\|\|\vec{v}_j\|\cos(\theta) = \frac{1}{2}$, so that A^TA has all 1's on the diagonal and $\frac{1}{2}$'s outside. By Exercise 17 $\left(\text{with } p = 1 \text{ and } q = \frac{1}{2}\right)$, $\det(A^TA) = \left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}n + \frac{1}{2}\right) = \left(\frac{1}{2}\right)^n(n+1)$, so that the volume is $\sqrt{\det(A^TA)} = \left(\frac{1}{2}\right)^{n/2}\sqrt{n+1}$.

8.1.**19** Let $L(\vec{x}) = A\vec{x}$. Then A^TA is symmetric, since $(A^TA)^T = A^T(A^T)^T = A^TA$, so that there is an orthonormal eigenbasis $\vec{v}_1, \ldots, \vec{v}_m$ for A^TA. Then the vectors $A\vec{v}_1, \ldots, A\vec{v}_m$ are orthogonal, since $A\vec{v}_i \cdot A\vec{v}_j = (A\vec{v}_i)^TA\vec{v}_j = \vec{v}_i^TA^TA\vec{v}_j = \vec{v}_i \cdot (A^TA\vec{v}_j) = \vec{v}_i \cdot (\lambda_j\vec{v}_j) = \lambda_j(\vec{v}_i \cdot \vec{v}_j) = 0$ if $i \neq j$.

8.1.**20** By Exercise 19, there is an orthonormal basis $\vec{v}_1, \ldots, \vec{v}_m$ of $\mathbb{R}^m$ such that $T(\vec{v}_1), \ldots, T(\vec{v}_m)$ are orthogonal. Suppose that $T(\vec{v}_1), \ldots, T(\vec{v}_r)$ are nonzero and $T(\vec{v}_{r+1}), \ldots, T(\vec{v}_m)$ are zero. Then let $\vec{w}_i = \frac{1}{\|T(\vec{v}_i)\|}T(\vec{v}_i)$ for $i = 1, \ldots, r$ and choose an orthonormal basis $\vec{w}_{r+1}, \ldots, \vec{w}_n$ of $[\mathrm{span}(\vec{w}_1, \ldots, \vec{w}_r)]^\perp$. Then $\vec{w}_1, \ldots, \vec{w}_n$ does the job.

8.1.**21** For each eigenvalue there are two unit eigenvectors: $\pm\vec{v}_1$, $\pm\vec{v}_2$, and $\pm\vec{v}_3$. We have 6 choices for the first column of S, 4 choices remaining for the second column, and 2 for the third.

Answer: $6 \cdot 4 \cdot 2 = 48$.

8.1.**22** a If we let $k = 2$ then A is symmetric and therefore (orthogonally) diagonalizable.

b If we let $k = 0$ then 0 is the only eigenvalue (but $A \neq 0$), so that A fails to be diagonalizable.

8.1.**23** The eigenvalues are real (by Theorem 8.1.3), so that the only possible eigenvalues are ± 1. Since A is symmetric, E_1 and E_{-1} are orthogonal complements. Thus A represents a *reflection* about E_1.

8.1.**24** Note that A is symmetric and orthogonal, so that the eigenvalues are 1 and -1 (see Exercise 23).

$E_1 = \mathrm{span}\left(\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right)$ and $E_{-1} = \mathrm{span}\left(\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\right)$, so that

$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$ is an orthonormal eigenbasis.

8.1.**25** Note that A is symmetric an orthogonal, so that the eigenvalues of A are 1 and -1.

$$E_1 = \text{span}\left(\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}\right), E_{-1} = \text{span}\left(\begin{bmatrix}1\\0\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-1\\0\end{bmatrix}\right)$$

The columns of S must form an eigenbasis for $A : S = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&1&0\\0&1&0&0&1\\0&0&\sqrt{2}&0&0\\0&1&0&0&-1\\1&0&0&-1&0\end{bmatrix}$ is one possible choice.

8.1.**26** Since J_n is both orthogonal and symmetric, the eigenvalues are 1 and -1. If n is even, then both have multiplicity $\frac{n}{2}$ (as in Exercise 24). If n is odd, then the multiplicities are $\frac{n+1}{2}$ for 1 and $\frac{n-1}{2}$ for -1 (as in Exercise 25). One way to see this is to observe that $\text{tr}(J_n)$ is 0 for even n, and 1 for odd n (recall that the trace is the sum of the eigenvalues).

8.1.**27** If n is even, then this matrix is $J_n + I_n$, for the J_n introduced in Exercise 26, so that the eigenvalues are 0 and 2, with multiplicity $\frac{n}{2}$ each. E_2 is the span of all $\vec{e}_i + \vec{e}_{n+1-i}$, for $i = 1, \ldots, \frac{n}{2}$, and E_0 is spanned by all $\vec{e}_i - \vec{e}_{n+1-i}$. If n is odd, then E_2 is spanned by all $\vec{e}_i + \vec{e}_{n+1-i}$, for $i = 1, \ldots, \frac{n-1}{2}$; E_0 is spanned by all $\vec{e}_i - \vec{e}_{n+1-i}$, for $i = 1, \ldots, \frac{n-1}{2}$, and E_1 is spanned by $\vec{e}_{\frac{n+1}{2}}$.

8.1.**28** For $\lambda \neq 0$

$$f_A(\lambda) = \det\left[\begin{array}{cccc|c}-\lambda & & & 0 & 1\\ & -\lambda & & & 1\\ & & \ddots & & \vdots\\ 0 & & & -\lambda & 1\\ 1 & 1 & \cdots & 1 & 1-\lambda\end{array}\right] = \frac{1}{\lambda}\det\left[\begin{array}{cccc|c}-\lambda & & & 0 & 1\\ & -\lambda & & & 1\\ & & \ddots & & \vdots\\ & 0 & & -\lambda & 1\\ \lambda & \lambda & \cdots & \lambda & \lambda-\lambda^2\end{array}\right]$$

$$= \frac{1}{\lambda}\det\left[\begin{array}{cccc|c}-\lambda & & & 0 & 1\\ & -\lambda & & & 1\\ & & \ddots & & \vdots\\ & 0 & & -\lambda & 1\\ 0 & 0 & \cdots & 0 & -\lambda^2+\lambda+12\end{array}\right]$$

$= -\lambda^{11}(\lambda^2 - \lambda - 12) = -\lambda^{11}(\lambda - 4)(\lambda + 3)$

Eigenvalues are 0 (with multiplicity 11), 4 and -3.

Eigenvalues for 0 are $\vec{e}_1 - \vec{e}_i (i = 2, \ldots, 12)$,

$$E_4 = \text{span}\begin{bmatrix}1\\1\\\vdots\\1\\4\end{bmatrix} \text{(12 ones)}, E_{-3} = \text{span}\begin{bmatrix}1\\1\\\vdots\\1\\-3\end{bmatrix} \text{(12 ones)}$$

so

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -3 \end{bmatrix}$$

diagonalizes A, and $D = S^{-1}AS$ will have all zeros as entries except $d_{12,\ 12} = 4$ and $d_{13,\ 13} = -3$.

8.1.**29** By Theorem 5.4.1 $(\text{im } A)^{\perp} = \ker(A^T) = \ker(A)$, so that $\vec{v}$ is orthogonal to $\vec{w}$.

8.1.**30** The columns $\vec{v}, \vec{v}_2, \dots, \vec{v}_n$ of R form an orthogonal eigenbasis for $A = \vec{v}\,\vec{v}^T$, with eigenvalues $1, 0, 0, \dots, 0(n-1$ zeros), since

$A\vec{v} = \vec{v}v^T\vec{v} = \vec{v}(\vec{v}\cdot\vec{v}) = \vec{v}$, (since $\vec{v}\cdot\vec{v} = 1$) and $A\vec{v}_i = \vec{v}\,v^T\,\vec{v}_i = \vec{v}(\vec{v}\cdot\vec{v}_i) = \vec{0}$ (since $\vec{v}\cdot\vec{v}_i = 0$).

Therefore we can let $S = R$, and $D = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$

8.1.**31** True; A is diagonalizable, that is, A is similar to a diagonal matrix D; then A^2 is similar to D^2. Now $\text{rank}(D) = \text{rank}(D^2)$ is the number of nonzero entries on the diagonal of D (and D^2). Since similar matrices have the same rank (by Theorem 7.3.6b) we can conclude that $\text{rank}(A) = \text{rank}(D) = \text{rank}(D^2) = \text{rank}(A^2)$.

8.1.**32** By Exercise 17, $\det(A) = (1-q)^{n-1}(qn+1-q)$. A is invertible if $\det(A) \neq 0$, that is, if $q \neq 1$ and $q \neq \frac{1}{1-n}$.

8.1.**33** The angles must add up to 2π, so $\theta = \frac{2\pi}{3} = 120°$. (See Figure 8.1.)

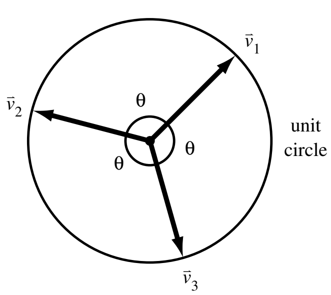

Figure 8.1: for Problem 8.1.33.

Algebraically, we can see this as follows: let $A = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]$, a 2×3 matrix.

Then $A^TA = \begin{bmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{bmatrix}$ is a noninvertible 3×3 matrix, so that $\cos\theta = \frac{1}{1-3} = -\frac{1}{2}$, by Exercise 32, and $\theta = \frac{2\pi}{3} = 120°$.

8.1.**34** Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ be such vectors. Form $A = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4]$, a 3×4 matrix.

Then $A^TA = \begin{bmatrix} 1 & \cos\theta & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta & \cos\theta \\ \cos\theta & \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & \cos\theta & 1 \end{bmatrix}$ is noninvertible, so that $\cos\theta = \frac{1}{1-4} = -\frac{1}{3}$, by Exercise 32, and $\theta = \arccos\left(-\frac{1}{3}\right) \approx 109.5°$. See Figure 8.2.

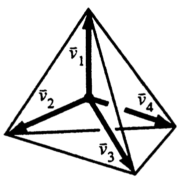

Figure 8.2: for Problem 8.1.34.

The tips of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ form a regular tetrahedron.

8.1.**35** Let $\vec{v}_1, \ldots, \vec{v}_{n+1}$ be these vectors. Form $A = [\vec{v}_1 \quad \cdots \quad \vec{v}_{n+1}]$, an $n \times (n+1)$ matrix.

Then $A^TA = \begin{bmatrix} 1 & \cos\theta & \cdots & \cos\theta \\ \cos\theta & 1 & \cdots & \cos\theta \\ \vdots & & \ddots & \\ \cos\theta & & \cdots & 1 \end{bmatrix}$ is a noninvertible $(n+1) \times (n+1)$ matrix with 1's on the diagonal and $\cos\theta$ outside, so that $\cos\theta = \frac{1}{1-n}$, by Exercise 32, and $\theta = \arccos\left(\frac{1}{1-n}\right)$.

8.1.**36** If $\vec{v}$ is an eigenvector with eigenvalue λ, then $\lambda\vec{v} = A\vec{v} = A^2\vec{v} = \lambda^2\vec{v}$, so that $\lambda = \lambda^2$ and therefore $\lambda = 0$ or $\lambda = 1$. Since A is symmetric, E_0 and E_1 are orthogonal complements, so that A represents the orthogonal projection onto E_1.

8.1.**37** In Example 4 we see that the image of the unit circle is an ellipse with semi-axes 2 and 3. Thus $||A\vec{u}||$ takes all values in the interval $[2, 3]$.

8.1.**38** The spectral theorem tells us that there exists an orthonormal eigenbasis $\vec{v}_1, \vec{v}_2$ for A, with associated eigenvalues -2 and 3. Consider a unit vector $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$ in $\mathbb{R}^2$, with $c_1^2 + c_2^2 = 1$. Then $\vec{u} \cdot A\vec{u} = (c_1\vec{v}_1 + c_2\vec{v}_2) \cdot (-2c_1\vec{v}_1 + 3c_2\vec{v}_2) = -2c_1^2 + 3c_2^2$, which takes all values on the interval $[-2, 3]$ since $-2 = -2c_1^2 - 2c_2^2 \leq -2c_1^2 + 3c_2^2 \leq 3c_1^2 + 3c_2^2 = 3$.

8.1.**39** The spectral theorem tells us that there exists an orthonormal eigenbasis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ for A, with associated eigenvalues -2, 3 and 4. Consider a unit vector $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ in $\mathbb{R}^3$, with $c_1^2 + c_2^2 + c_3^2 = 1$. Then

$\vec{u} \cdot A\vec{u} = (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) \cdot (-2c_1\vec{v}_1 + 3c_2\vec{v}_2 + 4c_3\vec{v}_3) = -2c_1^2 + 3c_2^2 + 4c_3^2$, which takes all values on the interval $[-2, 4]$ since $-2 = -2c_1^2 - 2c_2^2 - 2c_3^2 \leq -2c_1^2 + 3c_2^2 + 4c_3^2 \leq 4c_1^2 + 4c_2^2 + 4c_3^2 = 4$.

8.1.**40** Using the terminology introduced in Exercise 8.1.39, we have
$\|A\vec{u}\| = \|-2c_1\vec{v}_1 + 3c_2\vec{v}_2 + 4c_3\vec{v}_3\| = \sqrt{4c_1^2 + 9c_2^2 + 16c_3^2}$, which takes all values on the interval $[2, 4]$. Geometrically, the image of the unit sphere under A is the ellipsoid with semi-axes 2, 3, and 4.

8.1.**41** The spectral theorem tells us that there exists an orthogonal matrix S such that $S^{-1}AS = D$ is diagonal. Let D_1 be the diagonal matrix such that $D_1^3 = D$; the diagonal entries of D_1 are the cube roots of those of D. Now $B = SD_1S^{-1}$ does the job, since $B^3 = \left(SD_1S^{-1}\right)^3 = SD_1^3S^{-1} = SDS^{-1} = A$.

8.1.**42** We will use the strategy outlined in Exercise 8.1.41. An orthogonal matrix that diagonalizes $A = \frac{1}{5}\begin{bmatrix} 12 & 14 \\ 14 & 33 \end{bmatrix}$ is $S = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, with $S^{-1}AS = D = \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix}$. Now $D_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = SD_1S^{-1} = \frac{1}{5}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$.

8.1.**43** There is an orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \vec{v}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with associated eigenvalues -9, -9, 24. We are looking for a nonzero vector $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ such that $\vec{v} \cdot A\vec{v} = (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) \cdot (-9c_1\vec{v}_1 - 9c_2\vec{v}_2 + 24c_3\vec{v}_3) = -9c_1^2 - 9c_2^2 + 24c_3^2 = 0$ or $-3c_1^2 - 3c_2^2 + 8c_3^2 = 0$. One possible solution is $c_1 = \sqrt{8} = 2\sqrt{2},\ c_2 = 0,\ c_3 = \sqrt{3}$, so that $\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

8.1.**44** Use Exercise 8.1.43 as a guide. Consider an orthonormal eigenbasis $\vec{v}_1, ..., \vec{v}_n$ for A, with associated eigenvalues $\lambda_1 \leq \lambda_2 \leq .... \leq \lambda_n$, listed in ascending order. If $\vec{v} = c_1\vec{v}_1 + ... + c_n\vec{v}_n$ is any nonzero vector in $\mathbb{R}^n$, then $\vec{v} \cdot A\vec{v} = (c_1\vec{v}_1 + ... + c_n\vec{v}_n) \cdot (\lambda_1 c_1\vec{v}_1 + ... + \lambda_n c_n\vec{v}_n) = \lambda_1 c_1^2 + ... + \lambda_n c_n^2$. If all the eigenvalues are positive, then $\vec{v} \cdot A\vec{v}$ will be positive. Likewise, if all the eigenvalues are negative, then $\vec{v} \cdot A\vec{v}$ will be negative. However, if A has positive as well as negative eigenvalues, meaning that $\lambda_1 < 0 < \lambda_n$ (as in Example 8.1.43) , then there exist nonzero vectors $\vec{v}$ with $\vec{v} \cdot A\vec{v} = 0$, for example, $\vec{v} = \sqrt{\lambda_n}\vec{v}_1 + \sqrt{-\lambda_1}\vec{v}_n$.

8.1.**45** a If $S^{-1}AS$ is upper triangular then the first column of S is an eigenvector of A. Therefore, any matrix without real eigenvectors fails to be triangulizable over $\mathbb{R}$, for example, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

b Proof by induction on n: For an $n \times n$ matrix A we can choose a complex invertible $n \times n$ matrix P whose first column is an eigenvector for A. Then $P^{-1}AP = \begin{bmatrix} \lambda & \vec{v} \\ 0 & B \end{bmatrix}$. B is triangulizable, by induction hypothesis, that is, there is an invertible $(n-1) \times (n-1)$ matrix Q such that $Q^{-1}BQ = T$ is upper triangular.
Now let $R = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$. Then $R^{-1}\begin{bmatrix} \lambda & \vec{v} \\ 0 & B \end{bmatrix}R = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix}\begin{bmatrix} \lambda & \vec{v} \\ 0 & B \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \lambda & \vec{v}Q \\ 0 & T \end{bmatrix}$ is upper triangular.
$R^{-1}\begin{bmatrix} \lambda & \vec{v} \\ 0 & B \end{bmatrix}R = R^{-1}P^{-1}APR = S^{-1}AS$, where $S = PR$, proving our claim.

8.1.**46** a By definition of an upper triangular matrix, $\vec{e}_1$ is in ker U, $\vec{e}_2$ is in $\ker(U^2), \ldots, \vec{e}_n$ is in $\ker(U^n)$, so that all $\vec{x}$ in $\mathbb{C}^n$ are in $\ker(U^n)$, that is, $U^n = 0$.