Chapter 6

Section 6.1

6.1.1 Fails to be invertible; since $\det\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = 6 - 6 = 0$.

6.1.2 Invertible; since $\det\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = 10 - 12 = -2$.

6.1.3 Invertible; since $\det\begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix} = 33 - 35 = -2$.

6.1.4 Fails to be invertible; since $\det\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = 8 - 8 = 0$.

6.1.5 Invertible; since $\det\begin{bmatrix} 2 & 5 & 7 \\ 0 & 11 & 7 \\ 0 & 0 & 5 \end{bmatrix} = 2 \cdot 11 \cdot 5 + 0 + 0 - 0 - 0 - 0 = 110$.

6.1.6 Invertible; since $\det\begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} = 6 \cdot 4 \cdot 1 + 0 + 0 - 0 - 0 - 0 = 24$.

6.1.7 This matrix is clearly not invertible, so the determinant must be zero.

6.1.8 This matrix fails to be invertible, since the $\det(A) = 0$.

6.1.9 Invertible; since $\det\begin{bmatrix} 0 & 1 & 2 \\ 7 & 8 & 3 \\ 6 & 5 & 4 \end{bmatrix} = 0 + 3 \cdot 6 + 2 \cdot 7 \cdot 5 - 7 \cdot 4 - 2 \cdot 8 \cdot 6 = -36$.

6.1.10 Invertible; since $\det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = 1 \cdot 2 \cdot 6 + 1 \cdot 3 \cdot 1 + 1 \cdot 1 \cdot 3 - 3 \cdot 3 \cdot 1 - 2 \cdot 1 \cdot 1 - 6 \cdot 1 \cdot 1 = 1$.

6.1.11 $\det\begin{bmatrix} k & 2 \\ 3 & 4 \end{bmatrix} \neq 0$ when $4k \neq 6$, or $k \neq \frac{3}{2}$.

6.1.12 $\det\begin{bmatrix} 1 & k \\ k & 4 \end{bmatrix} \neq 0$ when $k^2 \neq 4$, or $k \neq 2, -2$.

6.1.13 $\det\begin{bmatrix} k & 3 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{bmatrix} = 8k$, so $k \neq 0$ will ensure that this matrix is invertible.

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6.1.14 $\det\begin{bmatrix}4&0&0\\3&k&0\\2&1&0\end{bmatrix}=0$, so the matrix will never be invertible, no matter which k is chosen.

6.1.15 $\det\begin{bmatrix}0&k&1\\2&3&4\\5&6&7\end{bmatrix}=6k-3$. This matrix is invertible when $k\neq\frac{1}{2}$.

6.1.16 $\det\begin{bmatrix}1&2&3\\4&k&5\\6&7&8\end{bmatrix}=60+84+8k-18k-35-64=45-10k$. So this matrix is invertible when $k\neq 4.5$.

6.1.17 $\det\begin{bmatrix}1&1&1\\1&k&-1\\1&k^2&1\end{bmatrix}=2k^2-2=2(k^2-1)=2(k-1)(k+1)$. So k cannot be 1 or -1.

6.1.18 $\det\begin{bmatrix}0&1&k\\3&2k&5\\9&7&5\end{bmatrix}=30+21k-18k^2=-3(k-2)(6k+5)$. So k cannot be 2 or $-\frac{5}{6}$.

6.1.19 $\det\begin{bmatrix}1&1&k\\1&k&k\\k&k&k\end{bmatrix}=-k^3+2k^2-k=-k(k-1)^2$. So k cannot be 0 or 1.

6.1.20 $\det\begin{bmatrix}1&k&1\\1&k+1&k+2\\1&k+2&2k+4\end{bmatrix}=(k+1)(2k+4)+k(k+2)+(k+2)-(k+1)-k(2k+4)-(k+2)(k+2)=(k+1)(3k+6)-(3k^2+9k+5)=1$. Thus, A will always be invertible, no matter the value of k, meaning that k can have any value.

6.1.21 $\det\begin{bmatrix}k&1&1\\1&k&1\\1&1&k\end{bmatrix}=k^3-3k+2=(k-1)^2(k+2)$. So k cannot be -2 or 1.

6.1.22 $\det\begin{bmatrix}\cos k&1&-\sin k\\0&2&0\\\sin k&0&\cos k\end{bmatrix}=2\cos^2 k+2\sin^2 k=2$. So k can have any value.

6.1.23 $\det(A-\lambda I_2)=\det\begin{bmatrix}1-\lambda&2\\0&4-\lambda\end{bmatrix}=(1-\lambda)(4-\lambda)=0$ if λ is 1 or 4.

6.1.24 $\det(A-\lambda I_2)=\det\begin{bmatrix}2-\lambda&0\\1&0-\lambda\end{bmatrix}=(2-\lambda)(-\lambda)=0$ if λ is 2 or 0.

6.1.25 $\det(A-\lambda I_2)=\det\begin{bmatrix}4-\lambda&2\\4&6-\lambda\end{bmatrix}=(4-\lambda)(6-\lambda)-8=(\lambda-8)(\lambda-2)=0$ if λ is 2 or 8.

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6.1.26 $\det(A-\lambda I_2) = \det\begin{bmatrix} 4-\lambda & 2 \\ 2 & 7-\lambda \end{bmatrix} = (4-\lambda)(7-\lambda)-4 = (\lambda-8)(\lambda-3) = 0$ if λ is 3 or 8.

6.1.27 $A-\lambda I_3$ is a lower triangular matrix with the diagonal entries $(2-\lambda), (3-\lambda)$ and $(4-\lambda)$. Now, $\det(A-\lambda I_3) = (2-\lambda)(3-\lambda)(4-\lambda) = 0$ if λ is 2, 3 or 4.

6.1.28 $A-\lambda I_3$ is an upper triangular matrix with the diagonal entries $(2-\lambda), (3-\lambda)$ and $(5-\lambda)$. Now, $\det(A-\lambda I_3) = (2-\lambda)(3-\lambda)(5-\lambda) = 0$ if λ is 2, 3 or 5.

6.1.29 $\det(A-\lambda I_3) = \det\begin{bmatrix} 3-\lambda & 5 & 6 \\ 0 & 4-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{bmatrix} = (3-\lambda)(\lambda-8)(\lambda-3) = 0$ if λ is 3 or 8.

6.1.30 $\det(A-\lambda I_3) = \det\begin{bmatrix} 4-\lambda & 2 & 0 \\ 4 & 6-\lambda & 0 \\ 5 & 2 & 3-\lambda \end{bmatrix} = (4-\lambda)(6-\lambda)(3-\lambda) - 8(3-\lambda)$

$= (3-\lambda)(8-\lambda)(2-\lambda) = 0$ if λ is 3, 8 or 2.

6.1.31 This matrix is upper triangular, so the determinant is the product of the diagonal entries, which is 24.

6.1.32 This matrix is upper triangular, so the determinant is the product of the diagonal entries, which is 210.

6.1.33 The determinant of this block matrix is $\det\begin{bmatrix} 1 & 2 \\ 8 & 7 \end{bmatrix}\det\begin{bmatrix} 2 & 3 \\ 7 & 5 \end{bmatrix} = (7-16)(10-21) = 99$,by Theorem 6.1.5.

6.1.34 The determinant of this block matrix is $\det\begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix}\det\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = (24-15)(3-8) = -45$, by Theorem 6.1.5.

6.1.35 There are two patterns with a nonzero product, $(a_{12}, a_{23}, a_{31}, a_{44}) = (3, 2, 6, 4)$, with two inversions, and $(a_{12}, a_{23}, a_{34}, a_{41}) = (3, 2, 3, 7)$, with 3 inversions. Thus $\det A = 3\cdot 2\cdot 6\cdot 4 - 3\cdot 2\cdot 3\cdot 7 = 18$.

6.1.36 There is one pattern with a nonzero product, containing all the 1's, with six inversions. Thus $\det A = 1$.

6.1.37 The determinant of this block matrix is
$\det\begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix}\det\begin{bmatrix} 5 & 6 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = (35-24)(5\cdot 1\cdot 1) = 55$, by Theorem 6.1.5.

6.1.38 The determinant of this block matrix is
$\det\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ 2 & 1 & 2 \end{bmatrix}\det\begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} = (2\cdot 4\cdot 2 + 3\cdot 3\cdot 1 - 1\cdot 4\cdot 1 - 2\cdot 3\cdot 2)(6\cdot 6 - 5\cdot 5) = 99$, by Theorem 6.1.5.

6.1.39 There is only one pattern with a nonzero product, containing all the nonzero entries of the matrix, with eight inversions. Thus $\det A = 1\cdot 2\cdot 4\cdot 3\cdot 5 = 120$.

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6.1.**40** There is only one pattern with a nonzero product, containing all the nonzero entries of the matrix, with seven inversions. Thus $\det A = -3 \cdot 2 \cdot 4 \cdot 1 \cdot 5 = -120$.

6.1.**41** There are two patterns with a nonzero product, $(a_{15}, a_{24}, a_{32}, a_{41}, a_{53}) = (2, 2, 3, 2, 3)$, with eight inversions, and $(a_{13}, a_{24}, a_{32}, a_{41}, a_{55}) = (1, 2, 3, 2, 4)$, with five inversions. Thus $\det A = 2 \cdot 2 \cdot 3 \cdot 2 \cdot 3 - 1 \cdot 2 \cdot 3 \cdot 2 \cdot 4 = 24$.

6.1.**42** There is only one pattern with a nonzero product, $(a_{13}, a_{24}, a_{32}, a_{45}, a_{51}) = (2, 2, 9, 5, 3)$, with six inversions. Thus $\det A = 2 \cdot 2 \cdot 9 \cdot 5 \cdot 3 = 540$.

6.1.**43** For each pattern P in A, consider the corresponding pattern P_{opp} in $-A$, with all the n entries being opposites. Then $\text{prod}\,(P_{opp}) = (-1)^n \text{prod}(P)$ and $sgn\,(P_{opp}) = sgn\,(P)$, so that $\det(-A) = (-1)^n \det A$.

6.1.**44** For each pattern P in A, consider the corresponding pattern P_m in kA, with all the n entries being multiplied by the scalar k. Then $\text{prod}\,(P_m) = k^n \text{prod}(P)$ and $sgn\,(P_m) = sgn\,(P)$, so that $\det(kA) = k^n \det A$.

6.1.**45** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A^T) = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - cb = \det(A)$. It turns out that $\det(A^T) = \det(A)$.

6.1.**46** Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$. If $a_1a_4 - a_2a_3 \neq 0$, then $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{bmatrix}$.

By Exercise 44, $\det(A^{-1}) = \left(\frac{1}{\det(A)}\right)^2 (a_1a_4 - a_2a_3) = \left(\frac{1}{\det(A)}\right)^2 \cdot \det(A)$ so $\det(A^{-1}) = \frac{1}{\det(A)}$.

6.1.**47** We have $\det(A) = (ah - cf)k + bef + cdg - aeg - bdh$. Thus matrix A is invertible for all k if (and only if) the coefficient $(ah - cf)$ of k is 0, while the sum $bef + cdg - aeg - bdh$ is nonzero. A numerical example is $a = c = d = f = h = g = 1$ and $b = e = 2$, but there are infinitely many other solutions as well.

6.1.**48** Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so $\det(A) = \det(B) = \det(C) = \det(D) = 0$ hence $\det(A)\det(D) - \det(B)\det(C) = 0$ but $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = -1$.

6.1.**49** The kernel of T consists of all vectors $\vec{x}$ such that the matrix $[\vec{x}\ \vec{v}\ \vec{w}]$ fails to be invertible. This is the case if $\vec{x}$ is a linear combination of $\vec{v}$ and $\vec{w}$ as discussed on Pages 249 and 250. Thus $\ker(T) = \text{span}(\vec{v}, \vec{w})$. The image of T isn't $\{0\}$, since $T(\vec{v} \times \vec{w}) \neq 0$, for example. Being a subspace of $\mathbb{R}$, the image must be all of $\mathbb{R}$.

6.1.**50** Theorem 6.1.1 tells us that $\det[\vec{u}\ \vec{v}\ \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w}) = \|\vec{u}\| \cos(\theta) \|\vec{v} \times \vec{w}\| = \|\vec{u}\| \cos(\theta) \|\vec{v}\| \sin(\alpha) \|\vec{w}\| = \cos(\theta) \sin(\alpha)$, where θ is the angle enclosed by vectors $\vec{u}$ and $\vec{v} \times \vec{w}$, and α is the angle between $\vec{v}$ and $\vec{w}$. Thus $\det[\vec{u}\ \vec{v}\ \vec{w}]$ can be any number on the closed interval $[-1, 1]$.

6.1.**51** Let a_{ii} be the first entry on the diagonal that fails to belong to the pattern. The pattern must contain an entry in the i^{th} row to the right of a_{ii}, above the diagonal, and also an entry in the i^{th} column below a_{ii}, below the diagonal.

6.1.**52** By Definition 6.1.1, we have $\det \begin{bmatrix} \vec{v} \times \vec{w} & \vec{v} & \vec{w} \end{bmatrix} = (\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w}) = \|\vec{v} \times \vec{w}\|^2$.

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6.1.**53** There is one pattern with a nonzero product, containing all the 1's. We have n^2 inversions, since each of the 1's in the lower left block forms an inversion with each of the 1's in the upper right block. Thus $\det A = (-1)^{n^2} = (-1)^n$.

6.1.**54** The pattern containing all the 1000's has 4 inversions so it contributes $(1000)^5 = 10^{15}$ to the determinant. There are $5! - 1 = 119$ other patterns with at most 3 entries being 1000, the others being ≤ 9. Thus the product associated with each of those patterns is less than$(1000)^3\,(10)^2 = 10^{11}$. Now $\det A > 10^{15} - 119 \cdot 10^{11} > 0$.

6.1.**55** By Exercise 2.4.93, a square matrix admits an LU factorization if (and only if) all its principal submatrices are invertible. Now

$A^{(1)} = [7]$, $A^{(2)} = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$, $A^{(3)} = A = \begin{bmatrix} 7 & 4 & 2 \\ 5 & 3 & 1 \\ 3 & 1 & 4 \end{bmatrix}$,

with $\det(A^{(1)}) = 7, \det(A^{(2)}) = 1$, $\det(A^{(3)}) = 1$.

Since all principal submatrices turn out to be invertible, the matrix A does indeed admit an LU factorization.

6.1.**56** There is only one pattern with a nonzero product, containing all the 1's. The number of inversions is $(n-1)+(n-2)+...+2+1 = \sum_{k=1}^{n-1} k = \frac{(n-1)n}{2}$. This number is even if either n or $n-1$ is divisible by 4, that is, for $n = 4, 5, 8, 9, 12, 13, ...$

a. $\det M_4 = \det M_5 = 1$, $\quad \det M_2 = \det M_3 = \det M_6 = \det M_7 = -1$.

b. $\det M_n = (-1)^{n(n-1)/2}$

6.1.**57** In a permutation matrix P, there is only one pattern with a nonzero product, containing all the 1's. Depending on the number of inversions in that pattern, we have $\det P = 1$ or $\det P = -1$.

6.1.**58** a If a, b, c, d are distinct prime numbers, then $ad \neq bc$, since the prime factorization of a positive integer is unique. Thus $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$: No matrix of the required form exists.

b We are looking for a noninvertible matrix $A = [\vec{u}\ \vec{v}\ \vec{w}]$ whose entries are nine distinct prime numbers. The last column vector, $\vec{w}$, must be redundant; to keep things simple, we will make $\vec{w} = \vec{u} + 2\vec{v}$. Now we have to pick six distinct prime entries for the first two columns, $\vec{u}$ and $\vec{v}$, such that the entries of $\vec{w} = \vec{u} + 2\vec{v}$ are prime as well. This can be done in many different ways; one solution is $A = \begin{bmatrix} 7 & 2 & 11 \\ 17 & 3 & 23 \\ 19 & 5 & 29 \end{bmatrix}$.

6.1.**59** $F \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} = ab + cd$

a. Yes, F is linear in both columns. To prove linearity in the second column, observe that $ab + cd$ is a linear combination of the variables b and d, with the constant coefficients a and c. An analogous argument proves linearity in the first column.

b. No, since $ab + cd$ fails to be a linear combination of a and b.

c. No, $F \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix} = F \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} = \vec{v} \cdot \vec{w}$: swapping the columns leaves F unchanged.

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6.1.**60** The functions in parts a, b, and d are linear in both columns; the functions in parts a, c, and d are linear in both rows; and $F(A) = -\det A$ in part d is alternating on the columns. For example, the function $F(A) = cd$ in part b is linear in the first column since cd is a linear combination of the entries a and c in the first column. However, $F(A) = cd$ fails to be linear in the second row since cd fails to be a linear combination of the entries c and d in the second row. Furthermore, $F\begin{bmatrix} a & b \\ c & d \end{bmatrix} = F\begin{bmatrix} b & a \\ d & c \end{bmatrix} = cd$, showing that F fails to be alternating on the columns.

6.1.**61** The function $F(A) = bfg$ is linear in all three columns and in all three rows since the product bfg contains exactly one factor from each row and from each column; it is the product associated with a pattern. For example, F is linear in the second row since bfg is a scalar multiple of f and thus a linear combination of the entries d, e, f in the second row. F fails to be alternating; for example, $F\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = 1$ but $F\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = 0$ after swapping the first two columns.

6.1.**62** If $A = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$, then $D(A) = D\begin{bmatrix} a & a \\ c & c \end{bmatrix} \underbrace{=}_{\substack{\textit{swap}\\ \textit{columns}}} -D\begin{bmatrix} a & a \\ c & c \end{bmatrix} = -D(A)$, so that $D(A) = 0$.

6.1.**63** $D\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \underbrace{=}_{\textit{Step 1}} D\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} + D\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \underbrace{=}_{\textit{Step 2}} ab\ D\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + ad\ D\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{=}_{\textit{Step 3}} ad$. In Step 1, we write $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \end{bmatrix}$ and use linearity in the second column. In Step 2, we write $\begin{bmatrix} a \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ etc. and use linearity in both columns. In Step 3, we use Exercise 62 for the first summand and the given property $D(I_2) = 1$ for the second summand.

6.1.**64** Writing $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$ and repeatedly using linearity in the columns, we find

$$D\begin{bmatrix} a & b \\ c & d \end{bmatrix} = D\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + D\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \underbrace{=}_{\textit{Step 2}} ad + D\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} + D\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$$

$$= ad + bc\ D\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + cd\ D\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \underbrace{=}_{\textit{Step 4}} ad - bc\ D\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = ad - bc = \det A$$

See the analogous computations in Exercise 63. In Step 2, we are using the result $D\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = ad$ from Exercise 63. In Step 4, we swap the columns of the matrix in the second summand and we apply Exercise 62 to the third summand.

6.1.**65** Freely using the linearity and alternating properties in the columns, and omitting terms with two equal columns (see Exercise 62), we find

$$D\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\ D\begin{bmatrix} 1 & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix} + d\ D\begin{bmatrix} 0 & b & c \\ 1 & e & f \\ 0 & h & i \end{bmatrix} + g\ D\begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 1 & h & i \end{bmatrix}$$

$$= ae\ D\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & i \end{bmatrix} + ah\ D\begin{bmatrix} 1 & 0 & c \\ 0 & 0 & f \\ 0 & 1 & i \end{bmatrix} + db\ D\begin{bmatrix} 0 & 1 & c \\ 1 & 0 & f \\ 0 & 0 & i \end{bmatrix} + dh\ D\begin{bmatrix} 0 & 0 & c \\ 1 & 0 & f \\ 0 & 1 & i \end{bmatrix}$$

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$$+gb\,D\begin{bmatrix}0&1&c\\0&0&f\\1&0&i\end{bmatrix}+ge\,D\begin{bmatrix}0&0&c\\0&1&f\\1&0&i\end{bmatrix}=aei\,D\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+ahf\,D\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

$$dbi\,D\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}+dhc\,D\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}+gbf\,D\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}+gec\,D\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$
$$=aei-ahf-dbi+dhc+gbf-gec=\det A$$

In the penultimate step, we perform up to two column swaps on each of the six matrices to make them I_3, recalling that $D(I_3)=1$.

6.1.**66** a. Proceeding as in Exercises 63 and 64, we can see that $F\begin{bmatrix}a&b\\c&d\end{bmatrix}=ab\,F\begin{bmatrix}1&1\\0&0\end{bmatrix}+ad\,F\begin{bmatrix}1&0\\0&1\end{bmatrix}+cb\,F\begin{bmatrix}0&1\\1&0\end{bmatrix}+cd\,F\begin{bmatrix}0&0\\1&1\end{bmatrix}$ for all functions F in V. Thus the functions ab, ad, cb, and cd span V . Apply these functions to the matrices $\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $\begin{bmatrix}0&1\\1&0\end{bmatrix}$, and $\begin{bmatrix}0&0\\1&1\end{bmatrix}$ to see that the four functions are linearly independent. Thus the functions ab, ad, cb, and cd form a basis of V, and $\dim V=4$.

b. In Exercise 64 we see that the formula $F\begin{bmatrix}a&b\\c&d\end{bmatrix}=F\begin{bmatrix}1&0\\0&1\end{bmatrix}(ad-bc)=F\begin{bmatrix}1&0\\0&1\end{bmatrix}(\det A)$ holds for all functions F in W. Thus W is one-dimensional, with $\det A$ as a basis.

Section 6.2

6.2.**1** $A=\begin{bmatrix}1&1&1\\1&3&3\\2&2&5\end{bmatrix}\begin{matrix}\\-I\\-2I\end{matrix}\rightarrow B=\begin{bmatrix}1&1&1\\0&2&2\\0&0&3\end{bmatrix}$. Now $\det(A)=\det(B)=6$, by Algorithm 6.2.5b.

6.2.**2** $A=\begin{bmatrix}1&2&3\\1&6&8\\-2&-4&0\end{bmatrix}\begin{matrix}\\-I\\+2I\end{matrix}\rightarrow B=\begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}$. Now $\det(A)=\det(B)=24$, by Algorithm 6.2.5b.

6.2.**3** $A=\begin{bmatrix}1&3&2&4\\1&6&4&8\\1&3&0&0\\2&6&4&12\end{bmatrix}\begin{matrix}\\-I\\-I\\-2I\end{matrix}\rightarrow B=\begin{bmatrix}1&3&2&4\\0&3&2&4\\0&0&-2&-4\\0&0&0&4\end{bmatrix}$. Now $\det(A)=\det(B)=-24$, by Algorithm 6.2.5b.

6.2.**4** $A=\begin{bmatrix}1&-1&2&-2\\-1&2&1&6\\2&1&14&10\\-2&6&10&33\end{bmatrix}\begin{matrix}\\+I\\-2I\\+2I\end{matrix}\rightarrow$

$$\begin{bmatrix}1&-1&2&-2\\0&1&3&4\\0&3&10&14\\0&4&14&29\end{bmatrix}\begin{matrix}\\\\-3II\\-4II\end{matrix}\rightarrow$$

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$$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 13 \end{bmatrix} \begin{matrix} \\ \\ \\ -2III \end{matrix} \to B = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

. Now $\det(A) = \det(B) = 9$, by Algorithm 6.2.5b.

6.2.**5** After three row swaps, we end up with $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Now, by Algorithm 6.2.5b, $\det(A) = (-1)^3 \det(B) = -24$.

6.2.**6** $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix} \begin{matrix} \\ -I \\ -I \\ -I \end{matrix} \to$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & -2 & 7 & -9 \end{bmatrix} \begin{matrix} \text{swap:} \\ II \leftrightarrow III \end{matrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{bmatrix} \begin{matrix} \\ \div -2 \\ \\ \\ \end{matrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{bmatrix} \begin{matrix} \\ \\ \\ +2II \end{matrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{bmatrix} \begin{matrix} \\ \\ \\ -2III \end{matrix} \to \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

$\det(A) = -\frac{1}{2}(-1)\det(B) = -72$, by Algorithm 6.2.5b.

6.2.**7** After two row swaps, we end up with an upper triangular matrix B with all 1's along the diagonal. Now $\det(A) = (-1)^2 \det(B) = 1$, by Algorithm 6.2.5b.

6.2.**8** After four row swaps, we end up with an upper triangular matrix B with all 1's along the diagonal, except for a 2 in the bottom right corner. Now $\det(A) = (-1)^4 \det(B) = 2$, by Algorithm 6.2.5b.

6.2.**9** If we subtract the first row from every other row, then we have an upper triangular matrix B, with diagonal entries 1, 1, 2, 3 and 4. Then $\det(A) = \det(B) = 24$ by Algorithm 6.2.5b.

6.2.**10** $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix} \begin{matrix} \\ -I \\ -I \\ -I \\ -I \end{matrix} \to$

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$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 3 & 9 & 19 & 34 \\ 0 & 4 & 14 & 34 & 69 \end{bmatrix} \begin{matrix} \\ \\ -2II \\ -3II \\ -4II \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 10 & 22 \\ 0 & 0 & 6 & 22 & 53 \end{bmatrix} \begin{matrix} \\ \\ \\ -3III \\ -6III \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 & 17 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ -4IV \end{matrix} \rightarrow B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now $\det(A) = \det(B) = 1$ by Algorithm 6.2.5b.

6.2.**11** By Theorem 6.2.3a, the desired determinant is $(-9)(8) = -72$.

6.2.**12** By Theorem 6.2.3b, the desired determinant is -8.

6.2.**13** By Theorem 6.2.3b, applied twice, since there are two row swaps, the desired determinant is $(-1)(-1)(8) = 8$.

6.2.**14** By Theorem 6.2.3c, the desired determinant is 8.

6.2.**15** By Theorem 6.2.3c, the desired determinant is 8.

6.2.**16** This determinant is 0, since the first row is twice the last.

6.2.**17** The standard matrix of T is $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}$, so that $\det(T) = \det(A) = 8$.

6.2.**18** The standard matrix of T is $A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -12 \\ 0 & 0 & 9 \end{bmatrix}$, so that $\det(T) = \det(A) = 27$.

6.2.**19** The standard matrix of T is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so that $\det(T) = \det(A) = -1$.

6.2.**20** The standard matrix of L is $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, so that $\det(L) = \det(M) = -1$.

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6.2.**21** The standard matrix of T is $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, so that $\det(T) = \det(A) = 1$.

6.2.**22** Using Exercises 19 and 21 as a guide, we observe that the standard matrix A of T is diagonal, of size $(n+1) \times (n+1)$, with diagonal entries $(-1)^0, (-1)^1, (-1)^2, \ldots, (-1)^n$. Thus $\det(T) = \det(A) = (-1)^{1+2+\cdots+n} = (-1)^{n(n+1)/2}$.

6.2.**23** Consider the matrix M of T with respect to a basis consisting of $n(n+1)/2$ symmetric matrices and $n(n-1)/2$ skew-symmetric matrices (see Exercises 54 and 55 or Section 5.3). Matrix M will be diagonal, with $n(n+1)/2$ entries 1 and $n(n-1)/2$ entries -1 on the diagonal. Thus, $\det(T) = \det(M) = (-1)^{n(n-1)/2}$.

6.2.**24** The standard matrix of T is $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$, so that $\det(T) = \det(A) = 13$.

6.2.**25** The standard matrix of T is $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$, so that $\det(T) = \det(A) = 16$.

6.2.**26** The matrix of T with respect to the basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is $A = \begin{bmatrix} 2 & 4 & 0 \\ 2 & 4 & 2 \\ 0 & 4 & 6 \end{bmatrix}$, so that $\det(T) = \det(A) = -16$.

6.2.**27** The matrix of T with respect to the basis $\cos(x), \sin(x)$ is $A = \begin{bmatrix} -b & a \\ -a & -b \end{bmatrix}$, so that $\det(T) = \det(A) = a^2 + b^2$.

6.2.**28** The matrix of T with respect to the basis $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ is $A = \begin{bmatrix} -6 & -10 \\ 5 & 6 \end{bmatrix}$, so that $\det(T) = \det(A) = 14$.

6.2.**29** Expand down the first column, realizing that all but the first contribution are zero, since $a_{21} = 0$ and A_{i1} has two equal rows for all $i > 2$. Therefore, $\det(P_n) = \det(P_{n-1})$.

Since $\det(P_1) = 1$, we can conclude that $\det(P_n) = 1$, for all n.

6.2.**30** a $f(t) = \det \begin{bmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{bmatrix} = (ab^2 - a^2b) + (a^2 - b^2)t + (b - a)t^2$ so $f(t)$ is a quadratic function of t. The coefficient of t^2 is $(b - a)$.

b In the cases $t = a$ and $t = b$ the matrix has two identical columns. It follows that $f(t) = k(t - a)(t - b)$ with $k =$ coefficient of $t^2 = (b - a)$.

c The matrix is invertible for the values of t for which $f(t) \neq 0$, i.e., for $t \neq a, t \neq b$.

6.2.**31** a If $n = 1$, then $A = \begin{bmatrix} 1 & 1 \\ a_0 & a_1 \end{bmatrix}$, so $\det(A) = a_1 - a_0$ (and the product formula holds).

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b Expanding the given determinant down the right-most column, we see that the coefficient k of t^n is the $n-1$ Vandermonde determinant which we assume is

$$\prod_{n-1\geq i>j}(a_i-a_j).$$

Now $f(a_0)=f(a_1)=\cdots=f(a_{n-1})=0$, since in each case the given matrix has two identical columns, hence its determinant equals zero. Therefore

$$f(t)=\left(\prod_{n-1\geq i>j}(a_i-a_j)\right)(t-a_0)(t-a_1)\cdots(t-a_{n-1})$$

and

$$\det(A)=f(a_n)=\prod_{n\geq i>j}(a_i-a_j),$$

as required.

6.2.**32** By Exercise 31, we need to compute $\prod_{i>j}(a_i-a_j)$ where $a_0=1,a_1=2,a_2=3,a_3=4,a_4=5$ so

$\prod_{i>j}(a_i-a_j)=(2-1)(3-1)(3-2)(4-1)(4-2)(4-3)(5-1)(5-2)(5-3)(5-4)=288.$

6.2.**33** Think of the i^{th} column of the given matrix as $a_i\begin{bmatrix}1\\a_i\\a_i^2\\\vdots\\a_i^{n-1}\end{bmatrix}$, so, by Theorem 6.2.3a, the determinant can be written as $(a_1a_2\cdots a_n)\det\begin{bmatrix}1&1&\cdots&1\\a_1&a_2&\cdots&a_n\\a_1^2&a_2^2&\cdots&a_n^2\\\vdots&\vdots&&\vdots\\a_1^{n-1}&a_2^{n-1}&\cdots&a_n^{n-1}\end{bmatrix}$. The new determinant is a Vandermonde determinant (see Exercise 6.2.31), and we get

$$\prod_{i=1}^{n}a_i\prod_{i>j}(a_i-a_j).$$

6.2.**34** a The hint pretty much gives it away. Since the columns of matrix $\begin{bmatrix}B\\-I_n\end{bmatrix}$ are in the kernel of $[\,I_n\quad M\,]$, we have $[\,I_n\quad M\,]\begin{bmatrix}B\\-I_n\end{bmatrix}=B-M=0$, and $M=B$, as claimed.

b If $B=A^{-1}$ we get $\text{rref}[A\vdots I_n]=[I_n\vdots A^{-1}]$ which tells us how to compute A^{-1} (see Theorem 2.4.5).

6.2.**35** $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ must satisfy $\det\begin{bmatrix}1&1&1\\x_1&a_1&b_1\\x_2&a_2&b_2\end{bmatrix}=0$, i.e., must satisfy the linear equation

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$(a_1b_2 - a_2b_1) - x_1(b_2 - a_2) + x_2(b_1 - a_1) = 0$.

We can see that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ satisfy this equation, since the matrix has two identical columns in these cases.

6.2.**36** Expanding down the first column we see that the equation has the form $A - Bx_1 + Cx_2 - D(x_1^2 + x_2^2) = 0$. If $D \neq 0$ this equation defines a circle; otherwise it is a line. From Exercise 35 we know that $D = 0$ if and only if the three given points $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ are collinear. Note that the circle or line runs through the three given points.

6.2.**37** Applying Theorem 6.2.6 to the equation $AA^{-1} = I_n$ we see that $\det(A)\det(A^{-1}) = 1$. The only way the product of the two integers $\det(A)$ and $\det(A^{-1})$ can be 1 is that they are both 1 or both -1. Therefore, $\det(A) = 1$ or $\det(A) = -1$.

6.2.**38** $\det(A^TA) = \det(A^T)\det(A) = [\det(A)]^2 = 9$

$\uparrow$ $\uparrow$

Theorem 6.2.6 Theorem 6.2.1

6.2.**39** $\det(A^TA) = \det(A^T)\det(A) = [\det(A)]^2 > 0$

$\uparrow$ $\uparrow$

Theorem 6.2.6 Theorem 6.2.1

6.2.**40** By Exercise 38, $\det(A^TA) = [\det(A)]^2$. Since A is orthogonal, $A^TA = I_n$ so that $1 = \det(I_n) = \det\,(A^TA) = [\det(A)]^2$ and $\det(A) = \pm 1$.

6.2.**41** $\det(A) = \det(A^T) = \det(-A) = (-1)^n(\det A) = -\det(A)$, so that $\det(A) = 0$. We have used Theorems 6.2.1 and 6.2.3a.

6.2.**42** $\det\left(A^TA\right) \underbrace{=}_{\text{Step 1}} \det\left((QR)^T QR\right) = \det\left(R^TQ^TQR\right) \underbrace{=}_{\text{Step 3}} \det\left(R^TI_mR\right)$

$= \det\left(R^TR\right) \underbrace{=}_{\text{Step 5}} \det\left(R^T\right)\det(R) \underbrace{=}_{\text{Step 6}} (\det R)^2 \underbrace{=}_{\text{Step 7}} \left(\prod_{i=1}^m r_{ii}\right)^2 > 0.$

In Step 1 we are using the definition of matrix A. Equation 3 holds since the column vectors of matrix Q are orthonormal. In Steps 5 and 6 we are using Theorems 6.2.6 and 6.2.1, respectively. Finally, Equation 7 holds since matrix R is triangular.

6.2.**43** $\det(A^TA) = \det\left(\begin{bmatrix} \vec{v}^T \\ \vec{w}^T \end{bmatrix} [\vec{v}\ \vec{w}]\right) = \det\begin{bmatrix} \vec{v}\cdot\vec{v} & \vec{v}\cdot\vec{w} \\ \vec{v}\cdot\vec{w} & \vec{w}\cdot\vec{w} \end{bmatrix} = \det\begin{bmatrix} \|\vec{v}\|^2 & \vec{v}\cdot\vec{w} \\ \vec{v}\cdot\vec{w} & \|\vec{w}\|^2 \end{bmatrix}$

$= \|\vec{v}\|^2\|\vec{w}\|^2 - (\vec{v}\cdot\vec{w})^2 \geq 0$ by the Cauchy-Schwarz inequality (Theorem 5.1.11).

6.2.**44** a We claim that $\vec{v}_2\times\vec{v}_3\times\cdots\times\vec{v}_n \neq \vec{0}$ if and only if the vectors $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent. If the vectors $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, then we can find a basis $\vec{x}, \vec{v}_2, \ldots, \vec{v}_n$ of $\mathbb{R}^n$ (any vector $\vec{x}$ that is not in span $(\vec{v}_2, \ldots, \vec{v}_n)$ will do). Then $\vec{x}\cdot(\vec{v}_2\times\cdots\times\vec{v}_n) = \det[\vec{x}\,\vec{v}_2\cdots\vec{v}_n] \neq 0$, so that $\vec{v}_2\times\cdots\times\vec{v}_n \neq \vec{0}$. Conversely, suppose that

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$\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \neq 0$; say the ith component of this vector is nonzero. Then $0 \neq \vec{e}_i \cdot (\vec{v}_2 \times \cdots \times \vec{v}_n) = \det[\vec{e}_i \; \vec{v}_2 \cdots \vec{v}_n]$, so that the vectors $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent (being columns of an invertible matrix).

b ith component of $\vec{e}_2 \times \vec{e}_3 \times \cdots \times \vec{e}_n = \det \begin{bmatrix} 1 & 1 & & 1 \\ \vec{e}_i & \vec{e}_2 & \cdots & \vec{e}_n \\ 1 & 1 & & 1 \end{bmatrix} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$

so $\vec{e}_2 \times \vec{e}_3 \times \cdots \times \vec{e}_n = \vec{e}_1$.

c $\vec{v}_i \cdot (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) = \det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{v}_i & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix} = 0$

for any $2 \leq i \leq n$ since the above matrix has two identical columns.

d Compare the ith components of the two vectors:

$\det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{e}_i & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix}$ and $\det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{e}_i & \vec{v}_3 & \vec{v}_2 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix}$.

The two determinants differ by a factor of -1 by Theorem 6.2.3b, so that
$\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n = -\vec{v}_3 \times \vec{v}_2 \times \cdots \times \vec{v}_n$.

e $\det[\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \; \vec{v}_2 \; \vec{v}_3 \cdots \vec{v}_n] = (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) \cdot (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) = \|\vec{v}_2 \times \cdots \times \vec{v}_n\|^2$

f In Definition 6.1.1 we saw that the "old" cross product satisfies the defining equation of the "new" cross product: $\vec{x} \cdot (\vec{v}_2 \times \vec{v}_3) = \det[\,\vec{x} \; \vec{v}_2 \; \vec{v}_3\,]$.

6.2.**45** $f(x)$ is a linear function, so $f'(x)$ is the coefficient of x (the slope). Expanding down the first column, we see that the coefficient of x is $-\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} = -24$, so $f'(x) = -24$.

6.2.**46** a $\det \begin{bmatrix} a & 3 & d \\ b & 3 & e \\ c & 3 & f \end{bmatrix} = 3\det \begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 21$. We have used Theorem 6.2.3.

b $\det \begin{bmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{bmatrix} = \det \begin{bmatrix} a & 2(1)+1 & d \\ b & 2(2)+1 & e \\ c & 2(3)+1 & f \end{bmatrix} = \det \begin{bmatrix} a & 2(1) & d \\ b & 2(2) & e \\ c & 2(3) & f \end{bmatrix} + \det \begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 2\det \begin{bmatrix} a & 1 & d \\ b & 2 & e \\ c & 3 & f \end{bmatrix} + \det \begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} =$ $2 \cdot 11 + 7 = 29$. We have used Theorem 6.2.2 and Theorem 6.2.3a.

6.2.**47** Yes! For example, $T \begin{bmatrix} x & b \\ y & d \end{bmatrix} = dx + by$ is given by the matrix $[d \; b]$, so that T is linear in the first column.

6.2.**48** A vector $\vec{x}$ is in the kernel of T if (and only if) $\vec{x}$ is redundant in the list $\vec{v}_1, \vec{v}_2, ..., \vec{v}_{n-1}, \vec{x}$, meaning that $\vec{x}$ is in the span of the vectors $\vec{v}_1, ..., \vec{v}_{n-1}$. Thus

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$$\ker T = \text{span}(\vec{v}_1, ..., \vec{v}_{n-1}), \quad \dim(\ker T) = n - 1, \quad \dim(\text{Im}T) = 1, \text{ and } \text{Im}T = \mathbb{R}.$$

The equation $\dim(\text{Im}T) = 1$ follows from the rank-nullity theorem.

6.2.**49** For example, we can start with an upper triangular matrix B with $\det(B) = 13$, such as $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 13 \end{bmatrix}$. Adding the first row of B to both the second and the third to make all entries nonzero, we end up with $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 14 \end{bmatrix}$. Note that $\det(A) = \det(B) = 13$.

6.2.**50** There are many ways to do this problem; here is one possible approach:

Subtracting the second to last row from the last, we can make the last row into

$[0 \quad 0 \quad \cdots \quad 0 \quad 1]$.

Now expanding along the last row we see that $\det(M_n) = \det(M_{n-1})$.

Since $\det(M_1) = 1$ we can conclude that $\det(M_n) = 1$ for all n.

6.2.**51** Notice that it takes n row swaps (swap row i with $n + i$ for each i between 1 and n) to turn A into I_{2n}. So, $\det(A) = (-1)^n \det(I_{2n}) = (-1)^n$.

6.2.**52** a We build B column-by-column:

$$\begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} d \\ -c \end{bmatrix} & \begin{bmatrix} -b \\ a \end{bmatrix} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

b $\det(A) = ad - bc = \det(B)$. The two determinants are equal.

c $BA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da - bc & 0 \\ 0 & -cb + ad \end{bmatrix} = (ad - bc)I_2$.

$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & -cb + da \end{bmatrix} = (ad - bc)I_2$ also.

d Any vector $\vec{u}$ in the image of A will be of the form $c_1\begin{bmatrix} a \\ c \end{bmatrix} + c_2\begin{bmatrix} b \\ d \end{bmatrix}$. We note that $B\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} ad - bc \\ -ca + ac \end{bmatrix} = \vec{0}$. The same is true of $B\begin{bmatrix} b \\ d \end{bmatrix}$. Thus, anything in the image of A will be in the kernel of B. Since both matrices have a rank of 1, the dimensions of the kernel and image of each will be exactly 1. So, it must be that $\text{im}(A) = \ker(B)$.

Also, any vector $\vec{u}$ in the image of B will be of the form $c_1\begin{bmatrix} d \\ -c \end{bmatrix} + c_2\begin{bmatrix} -b \\ a \end{bmatrix}$. However, we see that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} -ab + ba \\ -bc + ad \end{bmatrix} = \vec{0}$. The same is true for $A\begin{bmatrix} d \\ -c \end{bmatrix}$. Thus, by the same reasoning as above, the image of B will equal the kernel of A.

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e $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}B$.

6.2.**53** a See Exercise 37.

b If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ has integer entries.

6.2.**54** $f(t) = (\det(A+tB))^2 - 1 = \left(\det\begin{bmatrix} a_1+tb_1 & a_2+tb_2 \\ a_3+tb_3 & a_4+tb_4 \end{bmatrix}\right)^2 - 1$ assuming $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$.

Then the determinant above is a polynomial of degree ≤ 2 so its square is a polynomial of degree ≤ 4. Hence $f(t)$ is a polynomial of degree ≤ 4.

Since $A, A+B, A+2B, A+3B, A+4B$ are invertible and their inverses have integer entries, by Exercise 53a, it follows that their determinants are ± 1. Hence $f(0) = f(1) = f(2) = f(3) = f(4) = 0$.

Since f is a polynomial of degree ≤ 4 with at least 5 roots, it follows that $f(t) = 0$ for all t, in particular for $t = 5$, so $\det(A+5B) = \pm 1$. Hence $A + 5B$ is an invertible 2×2 matrix whose inverse has integer entries by Exercise 53b.

6.2.**55** We start out with a preliminary remark: If a square matrix A has two equal rows, then $D(A) = 0$. Indeed, if we swap the two equal rows and call the resulting matrix B, then $B = A$, so that $D(A) = D(B) = -D(A)$, by property b, and $D(A) = 0$ as claimed.

Next we need to understand how the elementary row operations affect D. Properties a and b tell us about how row multiplications and row swaps, but we still need to think about row additions.

We will show that if B is obtained from A by adding k times the i^{th} row to the j^{th}, then $D(B) = D(A)$. Let's label the row vectors of A by $\vec{v}_1, \ldots \vec{v}_n$. By linearity of D in the j^{th} row (property c) we have

$$D(B) = D\left(\begin{bmatrix} & \vdots & \\ — & \vec{v}_i & — \\ & \vdots & \\ — & \vec{v}_j + k\vec{v}_i & — \\ & \vdots & \end{bmatrix}\right) = D(A) + kD\left(\begin{bmatrix} & \vdots & \\ — & \vec{v}_i & — \\ & \vdots & \\ — & \vec{v}_i & — \\ & \vdots & \end{bmatrix}\right) = D(A).$$

Note that in the last step we have used the preliminary remark. Now, using the terminology introduced on Page 265, we can write $D(A) = (-1)^s k_1 k_2 \cdots k_r D(\text{rref } A)$.

Next we observe that $D(\text{rref } A) = \det(\text{rref } A)$ for all square matrices A. Indeed, if A is invertible, then $\text{rref}(A) = I_n$, and $D(I_n) = 1 = \det(I_n)$ by property c of function D. If A fails to be invertible, then $D(\text{rref } A) = 0 = \det(\text{rref } A)$ by linearity in the last row.

It follows that $D(A) = (-1)^s k_1 k_2 \cdots k_r D(\text{rref } A) = (-1)^s k_1 k_2 \cdots k_r \det(\text{rref } A) = \det(A)$ for all square matrices, as claimed.

6.2.**56** a We show first that D is linear in the ith row.

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$$D\begin{bmatrix}\vec{v}_1\\ \vdots\\ \vec{x}\\ \vdots\\ \vec{v}_n\end{bmatrix} = \frac{1}{\det M}\det\begin{bmatrix}\vec{v}_1 M\\ \vdots\\ \vec{x}M\\ \vdots\\ \vec{v}_n M\end{bmatrix}$$
$$\uparrow \qquad\qquad \uparrow$$
$$A \qquad\qquad AM$$

The entries in the ith row of AM are linear combinations of the components x_i of the vector $\vec{x}$, while the other entries of AM are constants. Therefore, $\det(AM)$ is a linear combination of the x_i (expand along the i^{th} row).

Since $\frac{1}{\det M}$ is a constant, we have $D\begin{bmatrix}\vec{v}_1\\ \vdots\\ \vec{x}\\ \vdots\\ \vec{v}_n\end{bmatrix} = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ for some constants c_i, as claimed.

c The property $D(I_n) = 1$ is obvious.

It now follows from Exercise 41 that $\det(A) = D(A) = \frac{\det(AM)}{\det(M)}$ and therefore $\det(AM) = \det(A)\det(M)$.

6.2.**57** Note that matrix A_1 is invertible, since $\det(A_1) \neq 0$. Now

$T\begin{bmatrix}\vec{y}\\ \vec{x}\end{bmatrix} = [A_1 \ A_2]\begin{bmatrix}\vec{y}\\ \vec{x}\end{bmatrix} = A_1\vec{y} + A_2\vec{x} = \vec{0}$ when $A_1\vec{y} = -A_2\vec{x}$, or,

$\vec{y} = -A_1^{-1}A_2\vec{x}$. This shows that for every $\vec{x}$ there is a unique $\vec{y}$ (that is, $\vec{y}$ is a function of $\vec{x}$); furthermore, this function is linear, with matrix $M = -A_1^{-1}A_2$.

6.2.**58** Using the approach of Exercise 57, we have $A_1 = \begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix}, A_2 = \begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$,

and $M = -A_1^{-1}A_2 = \begin{bmatrix}1 & -8\\ -1 & 3\end{bmatrix}$. The function is $\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & -8\\ -1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$.

Alternatively, we can solve the linear system

$$\begin{aligned} y_1 + 2y_2 + x_1 + 2x_2 &= 0\\ 3y_1 + 7y_2 + 4x_1 + 3x_2 &= 0 \end{aligned}$$

Gaussian Elimination gives

$$\begin{aligned} y_1 - x_1 + 8x_2 &= 0\\ y_2 + x_1 - 3x_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} y_1 &= x_1 - 8x_2\\ y_2 &= -x_1 + 3x_2 \end{aligned}$$

6.2.**59** $\det\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = 0 = \det\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, but these matrices fail to be similar.

6.2.**60** We argue using induction on n. The base case ($n = 2$) is discussed in the text. Now we assume that B is obtained from the $n \times n$ matrix A by adding k times the p^{th} row to the q^{th} row.

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We will evaluate the determinant of B by expanding across the i^{th} row (where i is neither p nor q).

$$\det(B) = \Sigma_{j=1}^{n}(-1)^{i+j} b_{ij} \det(B_{ij})$$

$$= \Sigma_{j=1}^{n}(-1)^{i+j} a_{ij} \det(B_{ij}) = \Sigma_{j=1}^{n}(-1)^{i+j} a_{ij} \det(A_{ij}) = \det(A)$$

Note that the $(n-1)\times(n-1)$ matrix B_{ij} is obtained from A_{ij} by adding k times some row to another row. Now, $\det(B_{ij}) = \det(A_{ij})$ by induction hypothesis.

6.2.**61** We follow the hint: $\begin{bmatrix} I_n & 0 \\ -C & A \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ -CA+AC & -CB+AD \end{bmatrix}$

$= \begin{bmatrix} A & B \\ 0 & AD-CB \end{bmatrix}$. So, $\det\left(\begin{bmatrix} I_n & 0 \\ -C & A \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A)\det(AD-CB)$.

Thus, $\det(I_n)\det(A)\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A)\det(AD-CB)$, which leads to

$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(AD-CB)$, since $\det(A) \neq 0$.

6.2.**62** a We compute $\begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & -CA^{-1}B+D \end{bmatrix}$. Since the matrix $\begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix}$ is invertible (its determinant is 1),the product $\begin{bmatrix} A & B \\ 0 & -CA^{-1}B+D \end{bmatrix}$ will have the same rank as $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, namely, n. With A being invertible, this implies that $-CA^{-1}B+D=0$, or $CA^{-1}B = D$, as claimed.

b Take determinants on both sides of the equation $D = CA^{-1}B$ from part (a) to find that $\det(D) = \det(C)(\det A)^{-1}\det(B), or \det(A)\det(D) - \det(B)\det(C) = 0$, proving the claim.

6.2.**63** Let M_n be the number of multiplications required to compute the determinant of an $n\times n$ matrix by Laplace expansion. We will use induction on n to prove that $M_n > n!$, for $n \geq 3$.

In the lowest applicable case, $n = 3$, we can check that $M_3 = 9$ and $3! = 6$.

Now let's do the induction step. If A is an $n\times n$ matrix, then $\det(A) = a_{11}\det(A_{11}) + \cdots + (-1)^{n+1}a_{n1}\det(A_{n1})$, by Theorem 6.2.10. We need to compute n determinants of $(n-1)\times(n-1)$ matrices, and then do n extra multiplications $a_{i1}\det(A_{i1})$, so that $M_n = nM_{n-1} + n$. If $n > 3$, then $M_{n-1} > (n-1)!$, by induction hypothesis, so that $M_n > n(n-1)! + n > n!$, as claimed.

6.2.**64** To compute $\det(A)$ for an $n\times n$ matrix A by Laplace expansion, $\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + \cdots + (-1)^{n+1}a_{n1}\det(A_{n1})$, we first need to compute the n minors, which requires nL_{n-1} operations; then we compute the n products $a_{i1}\det(A_{i1})$; and finally we have to do $n-1$ additions. Altogether,

$L_n = nL_{n-1} + n + (n-1) = nL_{n-1} + 2n - 1$.

Now we can prove the formula $\frac{L_n}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} - \frac{1}{n!}$ by induction on n.

For $n = 2$, the formula gives $\frac{L_2}{2} = 1 + 1 - \frac{1}{2}$, or $L_2 = 3$, which is correct: We have to perform 2 multiplications and 1 addition to compute the determinant of a 2×2 matrix.

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For an $n \times n$ matrix A we can use the recursive formula derived above to see that $\frac{L_n}{n!} = \frac{nL_{n-1}+2n-1}{n!} = \frac{L_{n-1}}{(n-1)!} + \frac{2}{(n-1)!} - \frac{1}{n!}$. Applying the induction hypothesis to the first summand, we find that $\frac{L_n}{n!} = \frac{L_{n-1}}{(n-1)!} + \frac{2}{(n-1)!} - \frac{1}{n!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{(n-2)!}-\frac{1}{(n-1)!}+\frac{2}{(n-1)!}-\frac{1}{n!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{(n-1)!}-\frac{1}{n!}$, as claimed.

Now recall from the theory of Taylor series in calculus that $e = e^1 = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{(n-1)!}+\frac{1}{n!}+\cdots$. Thus $L_n = (1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{(n-1)!}-\frac{1}{n!})n! < en!$, as claimed.

6.2.**65** a. Using Laplace expansion along the first row, we find $d_n = \det(M_n) = \det(M_{n-1}) - \det\begin{bmatrix} -1 & * \\ 0 & M_{n-2} \end{bmatrix} = \det(M_{n-1}) + \det(M_{n-2}) = d_{n-1} + d_{n-2}$.
b. $d_1 = 1,\ d_2 = 2,\ d_3 = 3,\ d_4 = 5, ..., d_{10} = 89$
c. Since $d_n > 0$ for all positive integers n, the matrix M_n is always invertible.

6.2.**66** a. Using Laplace expansion along the first row, we find $d_n = \det(M_n) = \det(M_{n-1}) - \det\begin{bmatrix} 1 & * \\ 0 & M_{n-2} \end{bmatrix} = \det(M_{n-1}) - \det(M_{n-2}) = d_{n-1} - d_{n-2}$.
b. $d_1 = 1,\ d_2 = 0,\ d_3 = -1,\ d_4 = -1,\ d_5 = 0,\ d_6 = 1,\ d_7 = 1,\ d_8 = 0$.
c. Since $d_4 = -d_1$ and $d_5 = -d_2$, the formula $d_{n+3} = -d_n$ holds for all positive integers n. (Give a formal proof by induction.) Now $d_{n+6} = -d_{n+3} = -(-d_n) = d_n$ for all positive integers n, meaning that the sequence d_n has a period of six.

6.2.**67** Let k be the number of pattern entries to the left and above a_{ij}. Then the number of pattern entries to the right and above a_{ij} is $i-1-k$, since there are $i-1$ rows above a_{ij}, each of which contains exactly one pattern entry. Likewise, the number of pattern entries to the left and below a_{ij} is $j-1-k$. Now the number of inversions involving a_{ij} is $(i-1-k)+(j-1-k) = i+j-2-2k$. It follows that $(-1)^{\#(\text{inversions involving } a_{ij})}$ $= (-1)^{i+j-2-2k} = (-1)^{i+j}$. This means that the number of inversions involving a_{ij} is even if (and only if) $i+j$ is even, as claimed.

6.2.**68** Using Exercise 67 and the terminology introduced in the proof of Theorem 6.2.10, we have
$sgnP = (-1)^{\#(\text{inversions in } P)} = (-1)^{\#(\text{inversions involving } a_{ij})} (-1)^{\#(\text{inversions not involving } a_{ij})}$

$$= (-1)^{i+j} (-1)^{\#(\text{inversions in } P_{ij})} = (-1)^{i+j} sgn(P_{ij}).$$

6.2.**69** a. The integers 0, 1, 2, 4, 5, 8, 9, and 10 are in G.

b. Note that $(a^2+b^2)(c^2+d^2) = \det\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \det\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \det\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix}\right)$

$= \det\begin{bmatrix} ac-bd & -ad-bc \\ bc+ad & -bd+ac \end{bmatrix} = (ac-bd)^2 + (ad+bc)^2$, proving that G is closed under multiplication.

c. Using part b, we find $8,501,956 = (31^2+41^2)(37^2+43^2) = (31 \cdot 37 - 41 \cdot 43)^2 + (31 \cdot 43 + 41 \cdot 37)^2 = (-616)^2 + 2,850^2 = 616^2 + 2,850^2$. Another solution is $8,501,956 = 184^2 + 2,910^2$.

6.2.**70** a.

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
0	1	1	2	3	5	8	13	21

b. In the base case, $n = 1$, we have $A^1 = A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix}$.

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Now the induction step: $A^{n+1} = A^n A = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{n+1}+f_n & f_{n+1} \\ f_n+f_{n-1} & f_n \end{bmatrix} = \begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix}$

c. Using part b, we find $f_{n+1}f_{n-1} - f_n^2 = \det(A^n) = (\det A)^n = (-1)^n$, as claimed.

Section 6.3

6.3.1 By Theorem 2.4.10, the area equals $|\det \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}| = |-50| = 50$.

6.3.2 By Theorem 2.4.10, Area $= \frac{1}{2}\left[\det \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}\right] = \frac{1}{2}|-50| = 25$

6.3.3 Area of triangle $= \frac{1}{2}|\det \begin{bmatrix} 6 & 1 \\ -2 & 4 \end{bmatrix}| = 13$ (See Figure 6.1.)

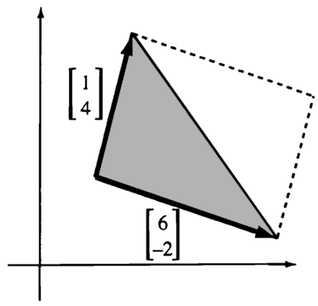

Figure 6.1: for Problem 6.3.3.

6.3.4 Note that area of triangle $= \frac{1}{2}\left[\det \begin{bmatrix} b_1-a_1 & c_1-a_1 \\ b_2-a_2 & c_2-a_2 \end{bmatrix}\right]$. (See Figure 6.2.)

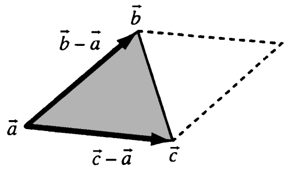

Figure 6.2: for Problem 6.3.4.

On the other hand, $\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} a_1 & b_1-a_1 & c_1-a_1 \\ a_2 & b_2-a_2 & c_2-a_2 \\ 1 & 0 & 0 \end{bmatrix}$, by subtracting the first column from the second and third.

This, in turn, equals $\det \begin{bmatrix} b_1-a_1 & c_1-a_1 \\ b_2-a_2 & c_2-a_2 \end{bmatrix}$, by expanding across the bottom row.

Therefore, area of triangle $= \frac{1}{2}\left[\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix}\right]$.

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6.3.5 The volume of the tetrahedron T_0 defined by $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is $\frac{1}{3}(\text{base})(\text{height}) = \frac{1}{6}$.

Here we are using the formula for the volume of a pyramid. (See Figure 6.3.)

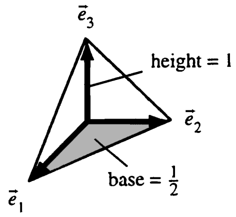

Figure 6.3: for Problem 6.3.5.

The tetrahedron T defined by $\vec{v}_1, \vec{v}_2, \vec{v}_3$ can be obtained by applying the linear transformation with matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ to T_0.

Now we have $\text{vol}(T) = |\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]|\text{vol}(T_0) = \frac{1}{6}|\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]| = \frac{1}{6}V(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

↑ ↑

Theorem 6.3.7 and Page 283 Theorem 6.3.4 and Page 280

6.3.6 From Exercise 5 we know that volume of tetrahedron $= \frac{1}{6}\left[\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix}\right]$, and Exercise 4 tells us that

area of triangle $= \frac{1}{2}\left[\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix}\right]$, so that area of tetrahedron $= \frac{1}{3}$ (area of triangle).

We can see this result more directly if we think of the tetrahedron as an inverted pyramid whose base is the triangle and whose height is 1. (See Figure 6.4.)

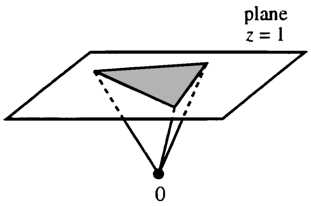

Figure 6.4: for Problem 6.3.6.

The three vertices of the shaded triangle are $\begin{bmatrix} a_1 \\ a_2 \\ 1 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \\ 1 \end{bmatrix}$.

6.3.7 Area $= \frac{1}{2}\left[\det\begin{bmatrix} 10 & -2 \\ 11 & 13 \end{bmatrix}\right] + \frac{1}{2}\left[\det\begin{bmatrix} 8 & 10 \\ 2 & 11 \end{bmatrix}\right] = 110$. (See Figure 6.5.)

6.3.8 We need to show that both sides of the equation in Theorem 6.3.3 give zero.

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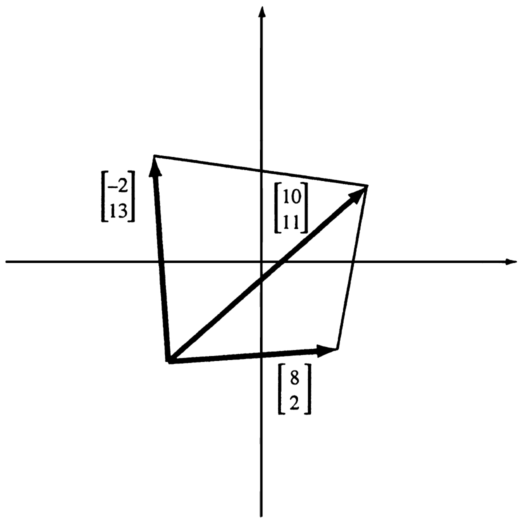

Figure 6.5: for Problem 6.3.7.

$|\det(A)| = 0$ since A is not invertible. On the other hand, since A is not invertible, the $\vec{v}_i$ will be linearly dependent, i.e., one of the $\vec{v}_i$ will be redundant. This implies that $\vec{v}_i^{\parallel} = \vec{v}_i$ and $\vec{v}_i^{\perp} = \vec{0}$, so that the right-hand side of the equation is 0, as claimed.

6.3.9 Using linearity in the second column, we find that $\det\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \det\begin{bmatrix} \vec{v}_1 & \left(\vec{v}_2^{\parallel} + \vec{v}_2^{\perp}\right) \end{bmatrix} = \det\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2^{\parallel} \end{bmatrix}}_{\text{noninvertible}}$ $+ \det\begin{bmatrix} \vec{v}_1 & \vec{v}_2^{\perp} \end{bmatrix} = \det\begin{bmatrix} \vec{v}_1 & \vec{v}_2^{\perp} \end{bmatrix}$. Thus the two determinants are equal.

6.3.10 $|\det(A)| \leq \|\vec{v}_1\|\|\vec{v}_2\|\cdots\|\vec{v}_n\|$ since $|\det(A)| = \|\vec{v}_1\|\|\vec{v}_2^{\perp}\|\cdots\|\vec{v}_n^{\perp}\|$ and $\|\vec{v}_i\| \geq \|\vec{v}_i^{\perp}\|$. The equality holds if $\|\vec{v}_i\| = \|\vec{v}_i^{\perp}\|$ for all i, that is, if the $\vec{v}_i$'s are mutually perpendicular.

6.3.11 The matrix of the transformation T with respect to the basis $\vec{v}_1, \vec{v}_2$ is $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, so that $\det(A) = \det(B) = 12$, by Theorem 6.2.7.

6.3.12 Denote the columns by $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$. By Theorem 6.3.3 and Exercise 6.3.8 we know that $|\det(A)| \leq \|\vec{v}_1\|\|\vec{v}_2\|\ \|\vec{v}_3\|\|\vec{v}_4\|$; equality holds if the columns are orthogonal. Since the entries of the $\vec{v}_i$ are 0, 1, and -1, we have $\|\vec{v}_i\| \leq \sqrt{1+1+1+1} = 2$. Therefore, $|\det A| \leq 16$.

To build an example where $\det(A) = 16$ we want all 1's and -1's as entries, and the columns need to be orthogonal. A little experimentation produces $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$ (there are other solutions). Note that we need to *check* that $\det(A) = 16$ (and not -16).

6.3.13 By Theorem 6.3.6, the desired 2-volume is

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$$\sqrt{\det\left(\begin{bmatrix}1&1&1&1\\1&2&3&4\end{bmatrix}\begin{bmatrix}1&1\\1&2\\1&3\\1&4\end{bmatrix}\right)} = \sqrt{\det\begin{bmatrix}4&10\\10&30\end{bmatrix}} = \sqrt{20}.$$

6.3.**14** By Theorem 6.3.6, the desired 3-volume is

$$\sqrt{\det\left(\begin{bmatrix}1&0&0&0\\1&1&1&1\\1&2&3&4\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&2\\0&1&3\\0&1&4\end{bmatrix}\right)} = \sqrt{\det\begin{bmatrix}1&1&1\\1&4&10\\1&10&30\end{bmatrix}} = \sqrt{6}.$$

6.3.**15** If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly dependent and if $A = [\vec{v}_1 \cdots \vec{v}_m]$, then $\det(A^T A) = 0$ since $A^T A$ and A have equal and nonzero kernels (by Theorem 5.4.2), hence $A^T A$ fails to be invertible.

On the other hand, since the $\vec{v}_i$ are linearly dependent, at least one of them will be redundant. For such a redundant $\vec{v}_i$, we will have $\vec{v}_i = \vec{v}_i^{\parallel}$ and $\vec{v}_i^{\perp} = \vec{0}$, so that $V(\vec{v}_1, \ldots, \vec{v}_m) = 0$, by Definition 6.3.5. This discussion shows that $V(\vec{v}_1, \ldots, \vec{v}_m) = 0 = \sqrt{\det(A^T A)}$ if the vectors $\vec{v}_1, \ldots, \vec{v}_m$ are linearly dependent.

6.3.**16** False

If T is given by $A = 2I_3$ then $|\det(A)| = 8$. But if Ω is the square defined by $\vec{e}_1, \vec{e}_2$ in $\mathbb{R}^3$ (of area 1), then $T(\Omega)$ is the square defined by $2\vec{e}_1, 2\vec{e}_2$ and the area of $T(\Omega)$ is 4.

6.3.**17** a Let $\vec{w} = \vec{v}_1 \times \vec{v}_2 \times \vec{v}_3$. Note that $\vec{w}$ is orthogonal to $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$, by Exercise 6.2.44c. Then $V(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}) = V(\vec{v}_1, \vec{v}_2, \vec{v}_3)\|\vec{w}^{\perp}\| = V(\vec{v}_1, \vec{v}_2, \vec{v}_3)\|\vec{w}\|$.

$\uparrow$

by Definition 6.3.5.

b By Exercise 6.2.44e,

$$V(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1 \times \vec{v}_2 \times \vec{v}_3) = |\det[\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_1 \times \vec{v}_2 \times \vec{v}_3]|$$
$$= |\det[\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3 \quad \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]| = \|\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3\|^2.$$

c By parts a and b, $V(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \|\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3\|$. If the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent, then both sides of the equation are 0, by Exercise 15 and Exercise 6.2.44a.

6.3.**18** a (See Figure 6.6.) $\begin{bmatrix}p&0\\0&q\end{bmatrix}\begin{bmatrix}\cos(t)\\\sin(t)\end{bmatrix} = \begin{bmatrix}p\cdot\cos(t)\\q\cdot\sin(t)\end{bmatrix}$, the ellipse with semi-axis $\pm\begin{bmatrix}p\\0\end{bmatrix}$ and $\pm\begin{bmatrix}0\\q\end{bmatrix}$.

(area of the ellipse) $= |\det(A)|$(area of the unit circle) $= pq\pi$

b By Theorem 6.3.7, $|\det(A)| = \dfrac{\text{area of the ellipse}}{\text{area of the unit circle}} = \frac{ab\pi}{\pi} = ab$ so $|\det(A)| = ab$.

c The unit circle consists of all vectors of the form $\vec{x} = \cos(t)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} + \sin(t)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$; its image is the ellipse consisting of all vectors

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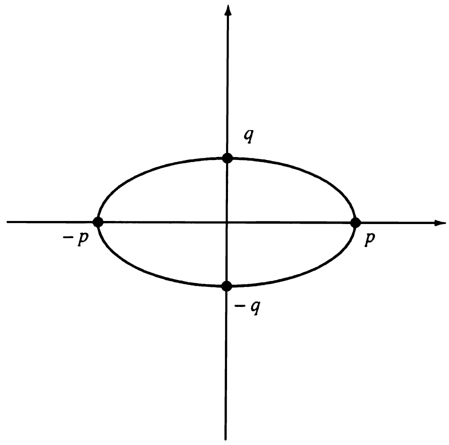

Figure 6.6: for Problem 6.3.18a.

$$T(\vec{x}) = \cos(t)\underbrace{2\sqrt{2}\begin{bmatrix}1\\1\end{bmatrix}}_{\text{semi-major axis}} + \sin(t)\underbrace{\sqrt{2}\begin{bmatrix}1\\-1\end{bmatrix}}_{\text{semi-minor axis}}. \text{ (See Figure 6.7.)}$$

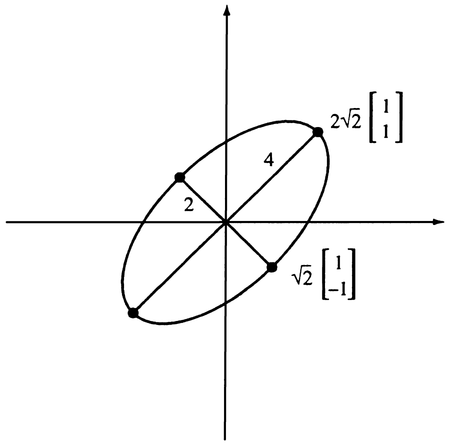

Figure 6.7: for Problem 6.3.18c.

6.3.**19** $\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \vec{v}_1\cdot(\vec{v}_2\times\vec{v}_3) = \|\vec{v}_1\|\|\vec{v}_2\times\vec{v}_3\|\cos\theta$ where θ is the angle between $\vec{v}_1$ and $\vec{v}_2\times\vec{v}_3$ so $\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] > 0$ if and only if $\cos\theta > 0$, i.e., if and only if θ is acute $(0 \le \theta \le \frac{\pi}{2})$. (See Figure 6.8.)

6.3.**20** By Exercise 19, $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ constitute a positively oriented basis if and only if $\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] > 0$. Assume that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is such a basis. We want to show that $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ is positively oriented if and only if $\det(A) > 0$. We have $\det[A\vec{v}_1\ A\vec{v}_2\ A\vec{v}_3] = \det(A[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]) = \det(A)\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ so since $\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] > 0$ by assumption, $\det[A\vec{v}_1\ A\vec{v}_2\ A\vec{v}_3] > 0$ if and only if $\det(A) > 0$. Hence A is orientation preserving if and only if $\det(A) > 0$.

6.3.**21** a Reverses

Consider $\vec{v}_2$ and $\vec{v}_3$ in the plane (not parallel), and let $\vec{v}_1 = \vec{v}_2\times\vec{v}_3$; then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a positively oriented basis,

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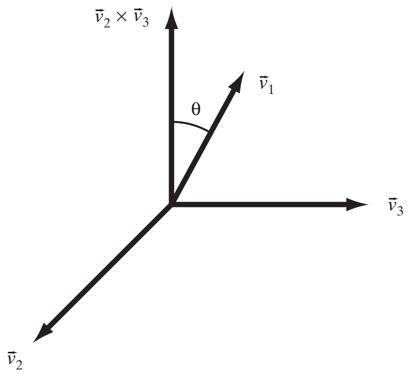

Figure 6.8: for Problem 6.3.19.

but $T(\vec{v}_1) = -\vec{v}_1, T(\vec{v}_2) = \vec{v}_2, T(\vec{v}_3) = \vec{v}_3$ is negatively oriented.

b Preserves

Consider $\vec{v}_2$ and $\vec{v}_3$ orthogonal to the line (not parallel), and let $\vec{v}_1 = \vec{v}_2 \times \vec{v}_3$; then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a positively oriented basis, and $T(\vec{v}_1) = \vec{v}_1, T(\vec{v}_2) = -\vec{v}_2, T(\vec{v}_3) = -\vec{v}_3$ is positively oriented as well.

c Reverses

The standard basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is positively oriented, but $T(\vec{e}_1) = -\vec{e}_1, T(\vec{e}_2) = -\vec{e}_2, T(\vec{e}_3) = -\vec{e}_3$ is negatively oriented.

6.3.**22** Here $A = \begin{bmatrix} 3 & 7 \\ 4 & 11 \end{bmatrix}$, $\det(A) = 5$, $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, so by Theorem 6.3.8

$x = \frac{\det\begin{bmatrix} 1 & 7 \\ 3 & 11 \end{bmatrix}}{5} = -2$, $y = \frac{\det\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}}{5} = 1$.

6.3.**23** Here $A = \begin{bmatrix} 5 & -3 \\ -6 & 7 \end{bmatrix}, \det(A) = 17, \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so by Theorem 6.3.8

$$x_1 = \frac{\det\begin{bmatrix} 1 & -3 \\ 0 & 7 \end{bmatrix}}{17} = \frac{7}{17}, x_2 = \frac{\det\begin{bmatrix} 5 & 1 \\ -6 & 0 \end{bmatrix}}{17} = \frac{6}{17}.$$

6.3.**24** Here $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix}$, $\det(A) = 146$, $\vec{b} = \begin{bmatrix} 8 \\ 3 \\ -1 \end{bmatrix}$, so by Theorem 6.3.8,

$x = \frac{\det\begin{bmatrix} 8 & 3 & 0 \\ 3 & 4 & 5 \\ -1 & 0 & 7 \end{bmatrix}}{146} = 1$, $y = \frac{\det\begin{bmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 6 & -1 & 7 \end{bmatrix}}{146} = 2$, $z = \frac{\det\begin{bmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 6 & 0 & -1 \end{bmatrix}}{146} = -1$.

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6.3.25 By Theorem 6.3.9, the ij^{th} entry of adj(A) is given by $(-1)^{i+j}\det(A_{ji})$, so since

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ for $i = 1, j = 1$, we get $(-1)^2 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$, and for $i = 1, j = 2$ we get $(-1)^3 \det \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0$, and so forth.

Completing this process gives $\text{adj}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, hence by Theorem 6.3.9,

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{-1}\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

6.3.26 By Theorem 6.3.9, $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$, so if $\det(A) = 1, A^{-1} = \text{adj}(A)$. If A has integer entries then $(-1)^{i+j}\det(A_{ji})$ will be an integer for all $1 \leq i, j \leq n$, hence adj(A) will have integer entries. Therefore, A^{-1} will also have integer entries.

6.3.27 By Theorem 6.3.8, using $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \det(A) = a^2 + b^2, \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we get

$$x = \det \begin{bmatrix} 1 & -b \\ 0 & a \end{bmatrix}\left(\frac{1}{a^2+b^2}\right) = \frac{a}{a^2+b^2}, y = \det \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}\left(\frac{1}{a^2+b^2}\right) = \frac{-b}{a^2+b^2},$$

so x is positive, y is negative (since $a, b > 0$), and x decreases as b increases.

6.3.28 Here $A = \begin{bmatrix} s & a \\ m & -h \end{bmatrix}, \det(A) = -sh - ma, \vec{b} = \begin{bmatrix} I^\circ + G \\ M_s + M^\circ \end{bmatrix}$ so, by Theorem 6.3.8

$$Y = \frac{\det \begin{bmatrix} I^\circ + G & a \\ M_s - M^\circ & -h \end{bmatrix}}{-sh-ma} = \frac{-h(I^\circ+G)-a(M_s-M^\circ)}{-sh-ma} = \frac{h(I^\circ+G)+a(M_s-M^\circ)}{sh+ma},$$

$$r = \frac{\det \begin{bmatrix} s & I^\circ + G \\ m & M_s - M^\circ \end{bmatrix}}{-sh-ma} = \frac{s(M_s-M^\circ)-m(I^\circ+G)}{-sh-ma} = \frac{m(I^\circ+G)-s(M_s-M^\circ)}{sh+ma}.$$

6.3.29 By Theorem 6.3.8,

$$dx_1 = \frac{\det \begin{bmatrix} 0 & R_1 & -(1-\alpha) \\ 0 & 1-\alpha & -(1-\alpha)^2 \\ -R_2 de_2 & -R_2 & -\frac{(1-\alpha)^2}{\alpha} \end{bmatrix}}{D} = \frac{R_1R_2(1-\alpha)^2 de_2 - R_2(1-\alpha)^2 de_2}{D}$$

$$dy_1 = \frac{\det \begin{bmatrix} -R_1 & 0 & -(1-\alpha) \\ \alpha & 0 & -(1-\alpha)^2 \\ R_2 & -R_2 de_2 & -\frac{(1-\alpha)^2}{\alpha} \end{bmatrix}}{D} = \frac{R_2 de_2(R_1(1-\alpha)^2 + \alpha(1-\alpha))}{D} > 0$$

$$dp = \frac{\det \begin{bmatrix} -R_1 & R_1 & 0 \\ \alpha & 1-\alpha & 0 \\ R_2 & -R_2 & -R_2 de_2 \end{bmatrix}}{D} = \frac{R_1 R_2 de_2}{D} > 0.$$

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6.3.**30** Using the procedure outlined in Exercise 25, we find $\text{adj}(A) = \begin{bmatrix} 18 & 0 & 0 \\ -12 & 6 & 0 \\ -2 & -5 & 3 \end{bmatrix}$.

6.3.**31** Using the procedure outlined in Exercise 25, we find $\text{adj}(A) = \begin{bmatrix} -6 & 0 & 1 \\ -3 & 5 & -2 \\ 4 & -5 & 1 \end{bmatrix}$.

6.3.**32** Using the procedure outlined in Exercise 25, we find that $\text{adj}(A) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$.

6.3.**33** Using the procedure outlined in Exercise 25, we find that $\text{adj}(A) = \begin{bmatrix} 24 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$. Note that the matrix adj(A) is diagonal, and the i^{th} diagonal entry of adj(A) is the product of all a_{jj} where $j \neq i$.

6.3.**34** For an invertible $n \times n$ matrix A, $A\text{adj}(A) = A(\det(A)A^{-1}) = \det(A)AA^{-1} = \det(A)I_n$. The same is true for adj$(A)A$.

6.3.**35** $\det(\text{adj}(A)) = \det(\det(A)A^{-1})$. Taking the product $\det(A)A^{-1}$ amounts to multiplying each row of A^{-1} by $\det(A)$, so that $\det(\text{adj}(A)) = (\det A)^n \det(A^{-1}) = (\det A)^n \frac{1}{\det(A)} = (\det A)^{n-1}$.

6.3.**36** $\text{adj}(\text{adj}A) = \text{adj}(\det(A)A^{-1})$

$= \det(\det(A)A^{-1})(\det(A)A^{-1})^{-1} = (\det A)^n \det(A^{-1})(\det(A)A^{-1})^{-1}$

$= (\det A)^{n-1}(\det(A)A^{-1})^{-1} = (\det A)^{n-1}\frac{1}{\det(A)}(A^{-1})^{-1}$

$= (\det A)^{n-2}A$.

6.3.**37** $\text{adj}(A^{-1}) = \det(A^{-1})(A^{-1})^{-1} = (\det A)^{-1}(A^{-1})^{-1} = (\text{adj}A)^{-1}$.

6.3.**38** $\text{adj}(AB) = \det(AB)(AB)^{-1}$

$= \det(A)(\det(B)B^{-1})A^{-1}$

$= \det(B)B^{-1}(\det(A)A^{-1})$

$= \text{adj}(B)\text{adj}(A)$.

6.3.**39** Yes, let S be an invertible matrix such that $AS = SB$, or $SB^{-1} = A^{-1}S$. Multiplying both sides by $\det(A) = \det(B)$, we find that $S(\det(B)B^{-1}) = (\det(A)A^{-1})S$, or, $S(\text{adj}B) = (\text{adj}A)S$, as claimed.

6.3.**40** The ij^{th} entry of the matrix B of T is (i^{th} component of $T(\vec{e}_j)) = \det(A_{\vec{e}_j,i})$. Expand down the $i'th$ column to see that this is the ij^{th} entry of adj(A). Thus $B = \text{adj}(A)$. See Theorem 6.3.9.

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6.3.**41** If A has a nonzero minor $\det(A_{ij})$, then the $n-1$ columns of the invertible matrix A_{ij} will be independent, so that the $n-1$ columns of A, minus the j^{th}, will be independent as well. Thus, the rank of A (the dimension of the image) is at least $n-1$.

Conversely, if $\text{rank}(A) \geq n-1$, then we can find $n-1$ independent columns of A. The $n\times(n-1)$ matrix consisting of those $n-1$ columns will have rank $n-1$, so that there will be exactly one redundant row (compare with Exercises 3.3.71 through 3.3.73). Omitting this redundant row produces an invertible $(n-1)\times(n-1)$ submatrix of A, giving us a nonzero minor of A.

6.3.**42** By Theorem 6.3.9, $\text{adj}(A) = 0$ if (and only if) all the minors A_{ji} of A are zero. By Exercise 41, this is the case if (and only if) $\text{rank}(A) \leq n-2$.

6.3.**43** A direct computation shows that $A(\text{adj}A) = (\text{adj}A)A = (\det A)(I_n)$ for all square matrices. Thus we have $A(\text{adj}A) = (\text{adj}A)A = 0$ for noninvertible matrices, as claimed.

Let's write $B = \text{adj}(A)$, and let's verify the equation $AB = (\det A)(I_n)$ for the diagonal entries; the verification for the off-diagonal entries is analogous. The i^{th} diagonal entry of AB is

$$[i^{th}\text{ row of }A]\begin{bmatrix} i^{th} \\ \text{column} \\ \text{of } B\end{bmatrix} = a_{i1}b_{1i} + \cdots + a_{in}b_{ni} = \sum_{j=1}^{n} a_{ij}b_{ji}.$$

Since B is the adjunct of A, $b_{ji} = (-1)^{j+i}\det(A_{ij})$.

So, our summation equals

$$\sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij})$$

which is our formula for Laplace expansion across the i^{th} row, and equals $\det(A)$, proving our claim for the diagonal entries.

6.3.**44** The equation $A(\text{adj}A) = 0$ from Exercise 43 means that $\text{im}(\text{adj}A)$ is a subspace of $\ker(A)$. Thus $\text{rank}(\text{adj}A) = \dim(\text{im}(\text{adj}A)) \leq \dim(\ker A) = n - \text{rank}(A) = n-(n-1) = 1$, implying that $\text{rank}(\text{adj}A) \leq 1$. Since $\text{adj}(A) \neq 0$, by Exercise 42, we can conclude that $\text{rank}(\text{adj}A) = 1$.

6.3.**45** Let $A = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$. We want $A^T = \text{adj}(A)$, or $\begin{bmatrix} a & c \\ b & d\end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$. So, $a = d$ and $b = -c$. Thus, the equation $A^T = \text{adj}(A)$ holds for all matrices of the form $\begin{bmatrix} a & b \\ -b & a\end{bmatrix}$.

6.3.**46** In the simple case when $f(x,y) = 1$ we have $\int_{\Omega_2} f(x,y)dA = \int_{\Omega_2} dA = \text{area of }\Omega_2 = |\det M|$ and

$\int_{\Omega_1} g(u,v)dA = \int_{\Omega_1} dA = \text{area of }\Omega_1 = 1$, so that $\int_{\Omega_2} f(x,y)dA = |\det M|\cdot\int_{\Omega_1} g(u,v)dA$.

This formula holds, in fact, for any continuous function $f(x,y)$; see an introductory text in multivariable calculus for a justification.

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6.3.**47** Note that $\frac{1}{2}\det\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$ is the area of the triangle OP_1P_2, where O denotes the origin. This is likewise true for one-half the second matrix. See Theorem 2.4.10. However, because of the reversal in orientation, $\frac{1}{2}\det\begin{bmatrix} x_3 & x_4 \\ y_3 & y_4 \end{bmatrix}$ is negative the area of the triangle OP_3P_4; likewise for the last matrix. Finally, note that the area of the quadrilateral $P_1P_2P_3P_4$ is equal to:

$$\text{the area of triangle } OP_1P_2 + \text{ area of triangle } OP_2P_3$$
$$- \text{ area of triangle } OP_3P_4 - \text{ area of triangle } OP_4P_1.$$

6.3.**48** In what follows, we will freely use the fact that an invertible linear transformation L from $\mathbb{R}^2$ to $\mathbb{R}^2$ maps an ellipse into an ellipse (see Exercise 2.2.52).

Now consider a linear transformation L that transforms our 3-4-5 right triangle R into an equilateral triangle T. If we place the vertices of the right triangle R at the points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$, and the vertices of the equilateral triangle T at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$, then the transformation L has the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$, with $\det(A) = \frac{1}{2\sqrt{3}}$.

According to the hint, L will map the largest ellipse E inscribed into R into the circle C inscribed into T. The Figure 6.9 illustrates that the radius of C is $\tan(\pi/6) = 1/\sqrt{3}$, so that the area of C is $\pi/3$. Using the interpretation of the determinant as an expansion fact, we find that (area of C) = (det A)(area of E), or (area of E) = $\frac{\text{area of } C}{\det(A)} = \frac{2\pi}{\sqrt{3}} \approx 3.6$

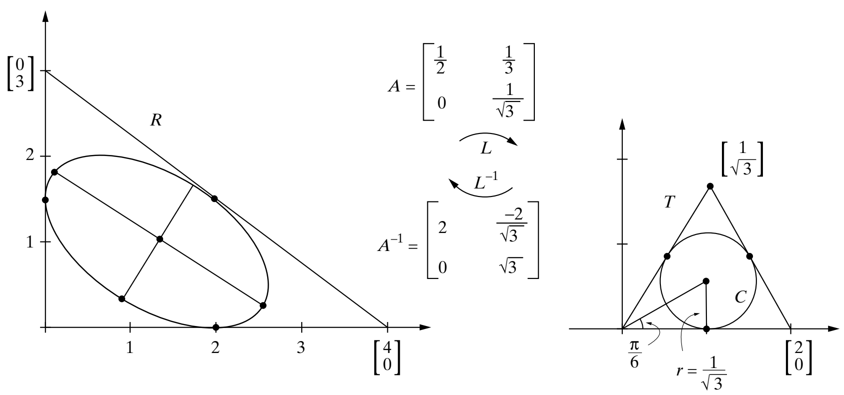

Figure 6.9: for Problem 6.3.48 and Problem 6.3.49.

6.3.**49** We will use the terminology introduced in the solution of Exercise 48 throughout. Note that the transformation L^{-1}, with matrix $A^{-1} = \begin{bmatrix} 2 & -2/\sqrt{3} \\ 0 & \sqrt{3} \end{bmatrix}$, maps the circle C (with radius $1/\sqrt{3}$) into the ellipse E. Now consider a radial vector $\vec{v} = \frac{1}{\sqrt{3}}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ of C, and find the maximal value M and the minimal value m of $\|A^{-1}\vec{v}\|^2 = \frac{4}{3} + \frac{1}{9}\sin^2\theta - \frac{8}{3\sqrt{3}}(\sin\theta)(\cos\theta) = \frac{25}{18} - \frac{1}{18}(\cos 2\theta) - \frac{4}{3\sqrt{3}}(\sin 2\theta)$ (we are taking the square to facilitate the computations). Then $\sqrt{M}$ and $\sqrt{m}$ will be the lengths of the semi-axis of E. The function above is sinusoidal

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with average value $\frac{25}{18}$ and amplitude $\sqrt{\frac{1}{18^2}+\frac{16}{27}} = \frac{\sqrt{193}}{18}$. Thus $M = \frac{25+\sqrt{193}}{18}$ and $m = \frac{25-\sqrt{193}}{18}$, so that the length of the semi-major axis of E is

$\sqrt{M} = \sqrt{\frac{25+\sqrt{193}}{18}} \approx 1.47$, and for the semi-minor axis we get

$\sqrt{m} = \sqrt{\frac{25-\sqrt{193}}{18}} \approx 0.79$.

True or False

Ch 6.TF.**1** T, by Theorem 6.2.3a, applied to the columns.

Ch 6.TF.**2** T, by Theorem 6.2.6.

Ch 6.TF.**3** T, By theorem 6.1.4, a diagonal matrix is triangular as well.

Ch 6.TF.**4** T, by Theorem 6.2.3b.

Ch 6.TF.**5** T, by Definition 6.1.1

Ch 6.TF.**6** F; We have $\det(4A) = 4^4 \det(A)$, by Theorem 6.2.3a.

Ch 6.TF.**7** F; Let $A = B = I_5$, for example

Ch 6.TF.**8** T; We have $\det(-A) = (-1)^6 \det(A) = \det(A)$, by Theorem 6.2.3a.

Ch 6.TF.**9** F; In fact, $\det(A) = 0$, since A fails to be invertible

Ch 6.TF.**10** F; The matrix A fails to be invertible if $\det(A) = 0$ by Theorem 6.2.4.

Ch 6.TF.**11** T; The determinant is 0 for $k = -1$ or $k = -2$, so that the matrix is invertible for all *positive* k.

Ch 6.TF.**12** F. There is only one pattern with a nonzero product, containing all the 1's. Since there are three inversions in this pattern, $\det A = -1$.

Ch 6.TF.**13** T. Without computing its exact value, we will show that the determinant is positive. The pattern that contains all the entries 100 has a product of $100^4 = 10^8$, with two inversions. Each of the other $4! - 1 = 23$ patterns contains at most two entries 100, with the other entries being less than 10, so that the product of each of these patterns is less than $100^2 \cdot 10^2 = 10^6$. Thus the determinant in more than $10^8 - 23 \cdot 10^6 > 0$, so that the matrix in invertible.

Ch 6.TF.**14** F; The correct formula is $\det(A^{-1}) = \frac{1}{\det(A^T)}$, by Theorems 6.2.1 and 6.2.8.

Ch 6.TF.**15** T; The matrix A is invertible.

Ch 6.TF.**16** T; Any nonzero noninvertible matrix A will do.

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Ch 6.TF.**17** T, by Theorem 6.2.7.

Ch 6.TF.**18** F, by Theorem 6.3.1. The determinant can be -1.

Ch 6.TF.**19** T, by Theorem 6.2.6.

Ch 6.TF.**20** F; The second and the fourth column are linearly dependent.

Ch 6.TF.**21** F; Note that $\det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = 2$.

Ch 6.TF.**22** T, by Theorem 6.3.9.

Ch 6.TF.**23** T, by Theorem 6.3.3, since $\|\vec{v}_i^{\perp}\| \leq \|\vec{v}_i\| = 1$ for all column vectors $\vec{v}_i$.

Ch 6.TF.**24** T; We have $\det(A) = \det(\text{rref } A) = 0$.

Ch 6.TF.**25** F; Let $A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$, for example. See Theorem 6.2.10.

Ch 6.TF.**26** F; Let $A = 2I_2$, for example

Ch 6.TF.**27** T; Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$. The column vectors of A are orthogonal and they all have length 2.

Ch 6.TF.**28** F; Let $A = \begin{bmatrix} 8 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for example.

Ch 6.TF.**29** F; In fact, $\det(A) = \det[\vec{u}\ \vec{v}\ \vec{w}] = -\det[\vec{v}\ \vec{u}\ \vec{w}] = -\vec{v} \cdot (\vec{u} \times \vec{w})$. We have used Theorem 6.2.3b and Definition 6.1.1.

Ch 6.TF.**30** T; Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, for example.

Ch 6.TF.**31** F; Note that $\det(S^{-1}AS) = \det(A)$ but $\det(2A) = 2^3(\det A) = 8(\det A)$.

Ch 6.TF.**32** F; Note that $\det(S^TAS) = (\det S)^2(\det A)$ and $\det(-A) = -(\det A)$ have opposite signs.

Ch 6.TF.**33** F; Let $A = 2I_2$, for example.

Ch 6.TF.**34** F; Let $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, for example.

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Ch 6.TF.**35** F; Let $A = I_2$ and $B = -I_2$, for example.

Ch 6.TF.**36** T; Note that $\det(B) = -\det(A) < \det(A)$, so that $\det(A) > 0$.

Ch 6.TF.**37** T; Let's do Laplace expansion along the first row, for example (see Theorem 6.2.10).

Then $\det(A) = \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det(A_{1j}) \neq 0$. Thus $\det(A_{1j}) \neq 0$ for at least one j, so that A_{1j} is invertible.

Ch 6.TF.**38** T; Note that $\det(A)$ and $\det(A^{-1})$ are both integers, and $(\det A)(\det A^{-1}) = 1$. This leaves only the possibilities $\det(A) = \det(A^{-1}) = 1$ and $\det(A) = \det(A^{-1}) = -1$.

Ch 6.TF.**39** T, since $\operatorname{adj}(A) = (\det A)(A^{-1})$, by Theorem 6.3.9.

Ch 6.TF.**40** F; Note that $\det(A^2) = (\det A)^2$ cannot be negative, but $\det(-I_3) = -1$.

Ch 6.TF.**41** T; The product associated with the diagonal pattern is odd, while the products associated with all other patterns are even. Thus the determinant of A is odd, so that A is invertible, as claimed.

Ch 6.TF.**42** F; Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 5 & 2 \end{bmatrix}$, for example

Ch 6.TF.**43** T; Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $a \neq 0$, let $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; if $b \neq 0$, let $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; if $c \neq 0$, let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and if $d \neq 0$, let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Ch 6.TF.**44** T; Use Gaussian elimination for the first column only to transform A into a matrix of the form

$$B = \begin{bmatrix} 1 & \pm1 & \pm1 & \pm1 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

Note that $\det(B) = \det(A)$ or $\det(B) = -(\det A)$. The stars in matrix B all represent numbers $(\pm1) \pm (\pm1)$, so that they are 2, 0, or -2. Thus the determinant of the 3×3 matrix M containing the stars is divisible by 8, since each of the 6 terms in Sarrus' rule is 8, 0 or -8. Now perform Laplace expansion down the first column of B to see that $\det(M) = \det(B) = +/-\det(A)$.

Ch 6.TF.**45** T; $A(\operatorname{adj}A) = A(\det(A)A^{-1}) = \det(A)I_n = \det(A)A^{-1}A = \operatorname{adj}(A)A$.

Ch 6.TF.**46** T; Laplace expansion along the second row gives $\det(A) = -k\det\begin{bmatrix} 1 & 2 & 4 \\ 8 & 9 & 7 \\ 0 & 0 & 5 \end{bmatrix} + C = 35k + C$, for some constant C (we need not compute that $C = -259$). Thus A is invertible except for $k = \frac{-C}{35}$ (which turns out to be $\frac{259}{35} = \frac{37}{5} = 7.4$).

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Ch 6.TF.**47** F; $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are both orthogonal and $\det(A) = \det(B) = 1$. However, $AB \neq BA$.

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