Chapter 5

Section 5.1

5.1.**1** $\|\vec{v}\| = \sqrt{7^2+11^2} = \sqrt{49+121} = \sqrt{170} \approx 13.04$

5.1.**2** $\|\vec{v}\| = \sqrt{2^2+3^2+4^2} = \sqrt{4+9+16} = \sqrt{29} \approx 5.39$

5.1.**3** $\|\vec{v}\| = \sqrt{2^2+3^2+4^2+5^2} = \sqrt{4+9+16+25} = \sqrt{54} \approx 7.35$

5.1.**4** $\theta = \arccos \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \arccos \frac{7+11}{\sqrt{2}\sqrt{170}} = \arccos \frac{18}{\sqrt{340}} \approx 0.219$ (radians)

5.1.**5** $\theta = \arccos \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \arccos \frac{2+6+12}{\sqrt{14}\sqrt{29}} \approx 0.122$ (radians)

5.1.**6** $\theta = \arccos \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \arccos \frac{2-3+8-10}{\sqrt{10}\sqrt{54}} \approx 1.700$ (radians)

5.1.**7** Use the fact that $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$, so that the angle is acute if $\vec{u}\cdot\vec{v} > 0$, and obtuse if $\vec{u}\cdot\vec{v} < 0$. Since $\vec{u}\cdot\vec{v} = 10-12 = -2$, the angle is obtuse.

5.1.**8** Since $\vec{u}\cdot\vec{v} = 4-24+20 = 0$, the two vectors enclose a right angle.

5.1.**9** Since $\vec{u}\cdot\vec{v} = 3-4+5-3 = 1$, the angle is acute (see Exercise 7).

5.1.**10** $\vec{u}\cdot\vec{v} = 2+3k+4 = 6+3k$. The two vectors enclose a right angle if $\vec{u}\cdot\vec{v} = 6+3k = 0$, that is, if $k=-2$.

5.1.**11** a $\theta_n = \arccos \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \arccos \frac{1}{\sqrt{n}}$

$\theta_2 = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4} (= 45^\circ)$

$\theta_3 = \arccos \frac{1}{\sqrt{3}} \approx 0.955$ (radians)

$\theta_4 = \arccos \frac{1}{2} = \frac{\pi}{3} (= 60^\circ)$

b Since $y = \arccos(x)$ is a continuous function,

$\lim_{n\to\infty} \theta_n = \arccos\left(\lim_{n\to\infty} \frac{1}{\sqrt{n}}\right) = \arccos(0) = \frac{\pi}{2} (= 90^\circ)$

5.1.**12** $\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})$ (by hint)

$= \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2(\vec{v}\cdot\vec{w})$ (by definition of length)

$\leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\|$ (by Cauchy-Schwarz)

$= (\|\vec{v}\| + \|\vec{w}\|)^2$, so that

$\|\vec{v}+\vec{w}\|^2 \leq (\|\vec{v}\| + \|\vec{w}\|)^2$

Taking square roots of both sides, we find that $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$, as claimed.

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5.1.**13** Figure 5.1 shows that $\|\vec{F}_2+\vec{F}_3\| = 2\cos\left(\frac{\theta}{2}\right)\|\vec{F}_2\| = 20\cos\left(\frac{\theta}{2}\right)$.

It is required that $\|\vec{F}_2+\vec{F}_3\| = 16$, so that $20\cos\left(\frac{\theta}{2}\right) = 16$, or $\theta = 2\arccos(0.8) \approx 74^\circ$.

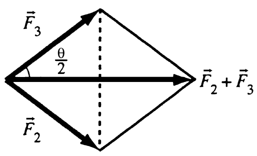

Figure 5.1: for Problem 5.1.13.

5.1.**14** The horizontal components of $\vec{F}_1$ and $\vec{F}_2$ are $-\|\vec{F}_1\|\sin\beta$ and $\|\vec{F}_2\|\sin\alpha$, respectively (the horizontal component of $\vec{F}_3$ is zero).

Since the system is at rest, the horizontal components must add up to 0, so that $-\|\vec{F}_1\|\sin\beta + \|\vec{F}_2\|\sin\alpha = 0$ or $\|\vec{F}_1\|\sin\beta = \|\vec{F}_2\|\sin\alpha$ or $\frac{\|\vec{F}_1\|}{\|\vec{F}_2\|} = \frac{\sin\alpha}{\sin\beta}$.

To find $\frac{\overline{EA}}{\overline{EB}}$, note that $\overline{EA} = \overline{ED}\tan\alpha$ and $\overline{EB} = \overline{ED}\tan\beta$ so that $\frac{\overline{EA}}{\overline{EB}} = \frac{\tan\alpha}{\tan\beta} = \frac{\sin\alpha}{\sin\beta}\cdot\frac{\cos\beta}{\cos\alpha} = \frac{\|\vec{F}_1\|}{\|\vec{F}_2\|}\frac{\cos\beta}{\cos\alpha}$. Since α and β are two distinct acute angles, it follows that $\frac{\overline{EA}}{\overline{EB}} \neq \frac{\|\vec{F}_1\|}{\|\vec{F}_2\|}$, so that Leonardo was mistaken.

5.1.**15** The subspace consists of all vectors $\vec{x}$ in $\mathbb{R}^4$ such that

$$\vec{x}\cdot\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = x_1+2x_2+3x_3+4x_4 = 0.$$

These are vectors of the form $\begin{bmatrix} -2r & -3s & -4t \\ r & & \\ & s & \\ & & t \end{bmatrix} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

The three vectors to the right form a basis.

5.1.**16** You may be able to find the solutions by educated guessing. Here is the systematic approach: we first find all vectors $\vec{x}$ that are orthogonal to $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, then we identify the unit vectors among them.

Finding the vectors $\vec{x}$ with $\vec{x}\cdot\vec{v}_1 = \vec{x}\cdot\vec{v}_2 = \vec{x}\cdot\vec{v}_3 = 0$ amounts to solving the system

$$\begin{bmatrix} x_1+x_2+x_3+x_4=0 \\ x_1+x_2-x_3-x_4=0 \\ x_1-x_2+x_3-x_4=0 \end{bmatrix}$$

(we can omit all the coefficients $\frac{1}{2}$).

The solutions are of the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ -t \\ t \end{bmatrix}$.

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Since $\|\vec{x}\| = 2|t|$, we have a unit vector if $t = \frac{1}{2}$ or $t = -\frac{1}{2}$. Thus there are two possible choices for $\vec{v}_4$:

$$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

5.1.**17** The orthogonal complement $W^\perp$ of W consists of the vectors $\vec{x}$ in $\mathbb{R}^4$ such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 0 \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 0.$$

Finding these vectors amounts to solving the system $\begin{bmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 = 0 \end{bmatrix}$.

The solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s + 2t \\ -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

The two vectors to the right form a basis of $W^\perp$.

5.1.**18** a $\|\vec{x}\|^2 = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$ (use the formula for a geometric series, with $a = \frac{1}{4}$), so that $\|\vec{x}\| = \frac{2}{\sqrt{3}} \approx 1.155$.

b If we let $\vec{u} = (1, 0, 0, \ldots)$ and $\vec{v} = \left(1, \frac{1}{2}, \frac{1}{4}, \cdots\right)$, then

$\theta = \arccos \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \arccos \frac{1}{\frac{2}{\sqrt{3}}} = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6} (= 30°)$.

c $\vec{x} = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \cdots, \frac{1}{\sqrt{n}}, \cdots\right)$ does the job, since the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges (a fact discussed in introductory calculus classes).

d If we let $\vec{v} = (1, 0, 0, \ldots), \vec{x} = \left(1, \frac{1}{2}, \frac{1}{4}, \cdots\right)$ and $\vec{u} = \frac{\vec{x}}{\|\vec{x}\|} = \frac{\sqrt{3}}{2}\left(1, \frac{1}{2}, \frac{1}{4}, \cdots\right)$ then

$\text{proj}_L \vec{v} = (\vec{u} \cdot \vec{v})\vec{u} = \frac{3}{4}\left(1, \frac{1}{2}, \frac{1}{4}, \cdots\right)$.

5.1.**19** See Figure 5.2.

5.1.**20** On the line L spanned by $\vec{x}$ we want to find the vector $m\vec{x}$ closest to $\vec{y}$ (that is, we want $\|m\vec{x} - \vec{y}\|$ to be minimal). We want $m\vec{x} - \vec{y}$ to be perpendicular to L (that is, to $\vec{x}$), which means that $\vec{x} \cdot (m\vec{x} - \vec{y}) = 0$ or $m(\vec{x} \cdot \vec{x}) - \vec{x} \cdot \vec{y} = 0$ or $m = \frac{\vec{x}\cdot\vec{y}}{\vec{x}\cdot\vec{x}} \approx \frac{4182.9}{198.53^2} \approx 0.106$.

Recall that the correlation coefficient r is $r = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\|\vec{y}\|}$, so that $m = \frac{\|\vec{y}\|}{\|\vec{x}\|} r$. See Figure 5.3.

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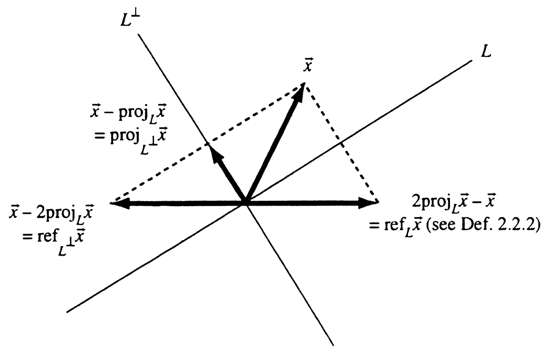
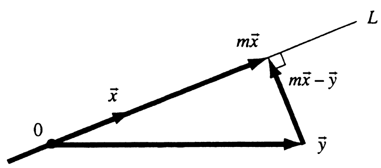


Figure 5.2: for Problem 5.1.19.



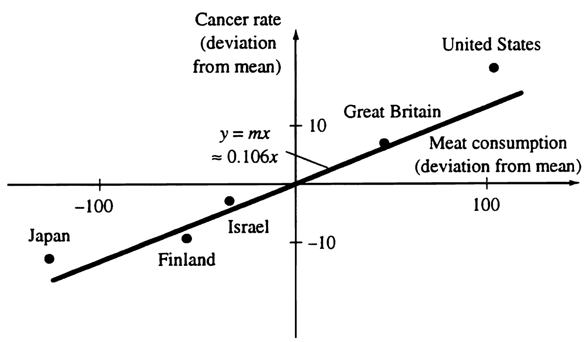


Figure 5.3: for Problem 5.1.20.

5.1.**21** Call the three given vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$. Since $\vec{v}_2$ is required to be a unit vector, we must have $b = g = 0$. Now $\vec{v}_1 \cdot \vec{v}_2 = d$ must be zero, so that $d = 0$.

Likewise, $\vec{v}_2 \cdot \vec{v}_3 = e$ must be zero, so that $e = 0$.

Since $\vec{v}_3$ must be a unit vector, we have $\|\vec{v}_3\|^2 = c^2 + \frac{1}{4} = 1$, so that $c = \pm\frac{\sqrt{3}}{2}$.

Since we are asked to find just one solution, let us pick $c = \frac{\sqrt{3}}{2}$.

The condition $\vec{v}_1 \cdot \vec{v}_3 = 0$ now implies that $\frac{\sqrt{3}}{2}a + \frac{1}{2}f = 0$, or $f = -\sqrt{3}a$.

Finally, it is required that $\|\vec{v}_1\|^2 = a^2 + f^2 = a^2 + 3a^2 = 4a^2 = 1$, so that $a = \pm\frac{1}{2}$.

Let us pick $a = \frac{1}{2}$, so that $f = -\frac{\sqrt{3}}{2}$.

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Summary:

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

There are other solutions; some components will have different signs.

5.1.**22** Let $W = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{x} \cdot \vec{v}_i = 0 \text{ for all } i = 1, \ldots, m\}$. We are asked to show that $V^\perp = W$, that is, any $\vec{x}$ in $V^\perp$ is in W, and vice versa.

If $\vec{x}$ is in $V^\perp$, then $\vec{x} \cdot \vec{v} = 0$ for all $\vec{v}$ in V; in particular, $x \cdot \vec{v}_i = 0$ for all i (since the $\vec{v}_i$ are in V), so that $\vec{x}$ is in W.

Conversely, consider a vector $\vec{x}$ in W. To show that $\vec{x}$ is in $V^\perp$, we have to verify that $\vec{x} \cdot \vec{v} = 0$ for all $\vec{v}$ in V. Pick a particular $\vec{v}$ in V. Since the $\vec{v}_i$ span V, we can write $\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$, for some scalars c_i. Then $\vec{x} \cdot \vec{v} = c_1(\vec{x} \cdot \vec{v}_1) + \cdots + c_m(\vec{x} \cdot \vec{v}_m) = 0$, as claimed.

5.1.**23** We will follow the hint. Let $\vec{v}$ be a vector in V. Then $\vec{v} \cdot \vec{x} = 0$ for all $\vec{x}$ in $V^\perp$. Since $(V^\perp)^\perp$ contains all vectors $\vec{y}$ such that $\vec{y} \cdot \vec{x} = 0$, $\vec{v}$ is in $(V^\perp)^\perp$. So V is a subspace of $(V^\perp)^\perp$.

Then, by Theorem 5.1.8c, $\dim(V) + \dim(V^\perp) = n$ and $\dim(V^\perp) + \dim((V^\perp)^\perp) = n$, so $\dim(V) + \dim(V^\perp) = \dim(V^\perp) + \dim((V^\perp)^\perp)$ and $\dim(V) = \dim((V^\perp)^\perp)$. Since V is a subspace of $(V^\perp)^\perp$, it follows that $V = (V^\perp)^\perp$, by Exercise 3.3.61.

5.1.**24** Write $T(\vec{x}) = \text{proj}_V(\vec{x})$ for simplicity.

To prove the linearity of T we will use the definition of a projection: $T(\vec{x})$ is in V, and $\vec{x} - T(\vec{x})$ is in $V^\perp$.

To show that $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$, note that $T(\vec{x}) + T(\vec{y})$ is in V (since V is a subspace), and $\vec{x} + \vec{y} - (T(\vec{x}) + T(\vec{y})) = (\vec{x} - T(\vec{x})) + (\vec{y} - T(\vec{y}))$ is in $V^\perp$ (since $V^\perp$ is a subspace, by Theorem 5.1.8a).

To show that $T(k\vec{x}) = kT(\vec{x})$, note that $kT(\vec{x})$ is in V (since V is a subspace), and $k\vec{x} - kT(\vec{x}) = k(\vec{x} - T(\vec{x}))$ is in $V^\perp$ (since $V^\perp$ is a subspace).

5.1.**25** a $\|k\vec{v}\|^2 = (k\vec{v}) \cdot (k\vec{v}) = k^2(\vec{v} \cdot \vec{v}) = k^2\|\vec{v}\|^2$

Now take square roots of both sides; note that $\sqrt{k^2} = |k|$, the absolute value of k (think about the case when k is negative). $\|k\vec{v}\| = |k|\|\vec{v}\|$, as claimed.

b $\|\vec{u}\| = \left\| \frac{1}{\|\vec{v}\|}\vec{v} \right\| \underset{\substack{\uparrow \\ \text{by part a}}}{=} \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1$, as claimed.

5.1.**26** The two given vectors spanning the subspace are orthogonal, but they are not unit vectors: both have length 7. To obtain an orthonormal basis $\vec{u}_1, \vec{u}_2$ of the subspace, we divide by 7:

$$\vec{u}_1 = \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \vec{u}_2 = \frac{1}{7}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

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Now we can use Theorem 5.1.5, with $\vec{x}=\begin{bmatrix}49\\49\\49\end{bmatrix}$:

$$\text{proj}_V\vec{x}=(\vec{u}_1\cdot\vec{x})\vec{u}_1+(\vec{u}_2\cdot\vec{x})\vec{u}_2=11\begin{bmatrix}2\\3\\6\end{bmatrix}-\begin{bmatrix}3\\-6\\2\end{bmatrix}=\begin{bmatrix}19\\39\\64\end{bmatrix}.$$

5.1.**27** Since the two given vectors in the subspace are orthogonal, we have the orthonormal basis

$$\vec{u}_1=\tfrac{1}{3}\begin{bmatrix}2\\2\\1\\0\end{bmatrix},\vec{u}_2=\tfrac{1}{3}\begin{bmatrix}-2\\2\\0\\1\end{bmatrix}.$$

Now we can use Theorem 5.1.5, with $\vec{x}=9\vec{e}_1$: $\text{proj}_V\vec{x}=(\vec{u}_1\cdot\vec{x})\vec{u}_1+(\vec{u}_2\cdot\vec{x})\vec{u}_2$

$$=2\begin{bmatrix}2\\2\\1\\0\end{bmatrix}-2\begin{bmatrix}-2\\2\\0\\1\end{bmatrix}=\begin{bmatrix}8\\0\\2\\-2\end{bmatrix}.$$

5.1.**28** Since the three given vectors in the subspace are orthogonal, we have the orthonormal basis

$$\vec{u}_1=\tfrac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\vec{u}_2=\tfrac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix},\vec{u}_3=\tfrac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}.$$

Now we can use Theorem 5.1.5, with $\vec{x}=\vec{e}_1$: $\text{proj}_V\vec{x}=(\vec{u}_1\cdot\vec{x})\vec{u}_1+(\vec{u}_2\cdot\vec{x})\vec{u}_2+(\vec{u}_3\cdot\vec{x})\vec{u}_3=\frac{1}{4}\begin{bmatrix}3\\1\\-1\\1\end{bmatrix}$.

5.1.**29** By the Pythagorean theorem (Theorem 5.1.9),

$$\begin{aligned}\|\vec{x}\|^2 &= \|7\vec{u}_1-3\vec{u}_2+2\vec{u}_3+\vec{u}_4-\vec{u}_5\|^2\\ &=\|7\vec{u}_1\|^2+\|3\vec{u}_2\|^2+\|2\vec{u}_3\|^2+\|\vec{u}_4\|^2+\|\vec{u}_5\|^2\\ &=49+9+4+1+1\\ &=64,\ \text{so that } \|\vec{x}\|=8.\end{aligned}$$

5.1.**30** Since $\vec{y}=\text{proj}_V\vec{x}$, the vector $\vec{x}-\vec{y}$ is orthogonal to $\vec{y}$, by definition of an orthogonal projection (see Theorem 5.1.4): $(\vec{x}-\vec{y})\cdot\vec{y}=0$ or $\vec{x}\cdot\vec{y}-\|\vec{y}\|^2=0$ or $\vec{x}\cdot\vec{y}=\|\vec{y}\|^2$. See Figure 5.4.

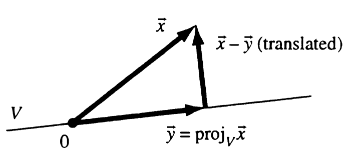

Figure 5.4: for Problem 5.1.30.

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5.1.**31** If $V = \text{span}(\vec{u}_1, \ldots, \vec{u}_m)$, then $\text{proj}_V\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$, by Theorem 5.1.5, and $\|\text{proj}_V\vec{x}\|^2 = (\vec{u}_1 \cdot \vec{x})^2 + \cdots + (\vec{u}_m \cdot \vec{x})^2 = p$, by the Pythagorean theorem (Theorem 5.1.9). Therefore $p \leq \|\vec{x}\|^2$, by Theorem 5.1.10. The two quantities are equal if (and only if) $\vec{x}$ is in V.

5.1.**32** By Theorem 2.4.9a, the matrix G is invertible if (and only if) $(\vec{v}_1 \cdot \vec{v}_1)(\vec{v}_2 \cdot \vec{v}_2) - (\vec{v}_1 \cdot \vec{v}_2)^2 = \|\vec{v}_1\|^2\|\vec{v}_2\|^2 - (\vec{v}_1 \cdot \vec{v}_2)^2 \neq 0$. The Cauchy-Schwarz inequality (Theorem 5.1.11) tells us that $\|\vec{v}_1\|^2\|\vec{v}_2\|^2 - (\vec{v}_1 \cdot \vec{v}_2)^2 \geq 0$; equality holds if (and only if) $\vec{v}_1$ and $\vec{v}_2$ are parallel (that is, linearly dependent).

5.1.**33** Let $\vec{x} = \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$ whose components add up to 1, that is, $x_1 + \cdots + x_n = 1$. Let $\vec{y} = \begin{bmatrix} 1 \\ \cdots \\ 1 \end{bmatrix}$ (all n components are 1). The Cauchy-Schwarz inequality (Theorem 5.1.11) tells us that $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$, or, $|x_1 + \cdots + x_n| \leq \|\vec{x}\|\sqrt{n}$, or $\|\vec{x}\| \geq \frac{1}{\sqrt{n}}$. By Theorem 5.1.11, the equation $\|\vec{x}\| = \frac{1}{\sqrt{n}}$ holds if (and only if) the vectors $\vec{x}$ and $\vec{y}$ are parallel, that is, $x_1 = x_2 = \cdots = x_n = \frac{1}{n}$. Thus the vector of minimal length is $\vec{x} = \begin{bmatrix} \frac{1}{n} \\ \cdots \\ \frac{1}{n} \end{bmatrix}$ $\left(\text{all components are } \frac{1}{n}\right)$.

Figure 5.5 illustrates the case $n = 2$.

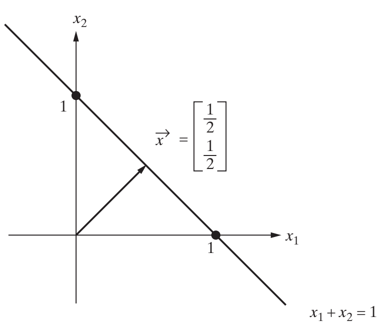

Figure 5.5: for Problem 5.1.33.

5.1.**34** Let $\vec{x}$ be a unit vector in $\mathbb{R}^n$, that is, $\|\vec{x}\| = 1$. Let $\vec{y} = \begin{bmatrix} 1 \\ \ldots \\ 1 \end{bmatrix}$ (all n components are 1). The Cauchy-Schwarz inequality (Theorem 5.1.11) tells us that $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$, or, $|x_1 + \ldots + x_n| \leq \|\vec{x}\|\sqrt{n} = \sqrt{n}$. By Theorem 5.1.11, the equation $x_1 + \ldots + x_n = \sqrt{n}$ holds if $\vec{x} = k\vec{y}$ for positive k. Thus $\vec{x}$ must be a unit vector of the form $\vec{x} = \begin{bmatrix} k \\ \ldots \\ k \end{bmatrix}$ for some positive k. It is required that $nk^2 = 1$, or, $k = \frac{1}{\sqrt{n}}$. Thus $\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \cdots \\ \frac{1}{\sqrt{n}} \end{bmatrix}$ $\left(\text{all components are } \frac{1}{\sqrt{n}}\right)$.

Figure 5.6 illustrates the case $n = 2$.

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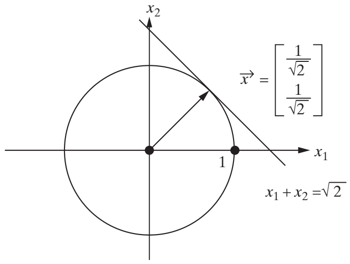

Figure 5.6: for Problem 5.1.34.

5.1.**35** Applying the Cauchy-Schwarz inequality to $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ gives $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$, or $|x+2y+3z| \leq \sqrt{14}$. The minimal value $x + 2y + 3z = -\sqrt{14}$ is attained when $\vec{u} = k\vec{v}$ for negative k. Thus $\vec{u}$ must be a unit vector of the form $\vec{u} = \begin{bmatrix} k \\ 2k \\ 3k \end{bmatrix}$, for negative k. It is required that $14k^2 = 1$, or, $k = -\frac{1}{\sqrt{14}}$. Thus $\vec{u} = \begin{bmatrix} -\frac{1}{\sqrt{14}} \\ -\frac{2}{\sqrt{14}} \\ -\frac{3}{\sqrt{14}} \end{bmatrix}$.

5.1.**36** Let $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$. It is required that $\vec{x} \cdot \vec{y} = 0.2a + 0.3b + 0.5c = 76$. Our goal is to minimize quantity $\vec{x} \cdot \vec{x} = a^2 + b^2 + c^2$. The Cauchy-Schwarz inequality (squared) tells us that $(\vec{x} \cdot \vec{y})^2 \leq \|\vec{x}\|^2\|\vec{y}\|^2$, or $76^2 \leq (a^2 + b^2 + c^2)(0.2^2 + 0.3^2 + 0.5^2)$ or $a^2 + b^2 + c^2 \geq \frac{76^2}{0.38}$. The quantity $a^2 + b^2 + c^2$ is minimal when $a^2 + b^2 + c^2 = \frac{76^2}{0.38}$. This is the case when $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.2k \\ 0.3k \\ 0.5k \end{bmatrix}$ for some positive constant k. It is required that $0.2a + 0.3b + 0.5c = (0.2)^2k + (0.3)^2k + (0.5)^2k = 0.38k = 76$, so that $k = 200$. Thus $a = 40, b = 60, c = 100$: The student must study 40 hours for the first exam, 60 hours for the second, and 100 hours for the third.

5.1.**37** Using Definition 2.2.2 as a guide, we find that $\text{ref}_V\vec{x} = 2(\text{proj}_{\text{V}}\vec{\text{x}}) - \vec{\text{x}} = 2(\vec{\text{u}}_1 \cdot \vec{\text{x}})\vec{\text{u}}_1 + 2(\vec{\text{u}}_2 \cdot \vec{\text{x}})\vec{\text{u}}_2 - \vec{\text{x}}$.

5.1.**38** Since $\vec{v}_1$ and $\vec{v}_2$ are unit vectors, the condition $\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\|\|\vec{v}_2\|\cos(\alpha) = \cos(\alpha) = \frac{1}{2}$ implies that $\vec{v}_1$ and $\vec{v}_2$ enclose an angle of $60^\circ \left(= \frac{\pi}{3}\right)$. The vectors $\vec{v}_1$ and $\vec{v}_3$ enclose an angle of 60° as well.
In the case $n = 2$ there are two possible scenarios: either $\vec{v}_2 = \vec{v}_3$, or $\vec{v}_2$ and $\vec{v}_3$ enclose an angle of 120°. Therefore, either $\vec{v}_2 \cdot \vec{v}_3 = 1$ or $\vec{v}_2 \cdot \vec{v}_3 = \cos(120^\circ) = -\frac{1}{2}$.
In the case $n = 3$, the vectors $\vec{v}_2$ and $\vec{v}_3$ could enclose any angle between 0° (if $\vec{v}_2 = \vec{v}_3$) and 120°, as illustrated in Figure 5.7. We have $-\frac{1}{2} \leq \vec{v}_2 \cdot \vec{v}_3 \leq 1$.

For example, consider $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} \left(\frac{\sqrt{3}}{2}\right)\cos\theta \\ \left(\frac{\sqrt{3}}{2}\right)\sin\theta \\ \frac{1}{2} \end{bmatrix}$

Note that $\vec{v}_2 \cdot \vec{v}_3 = \left(\frac{3}{4}\right)\sin\theta + \frac{1}{4}$ could be anything between $-\frac{1}{2}$ (when $\sin\theta = -1$) and 1 (when $\sin\theta = 1$), as claimed.

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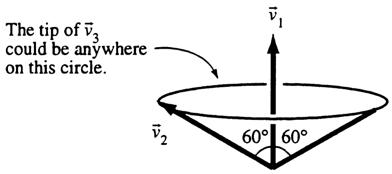

Figure 5.7: for Problem 5.1.38.

If n exceeds three, we can consider the orthogonal projection $\vec{w}$ of $\vec{v}_3$ onto the plane E spanned by $\vec{v}_1$ and $\vec{v}_2$.

Since $\text{proj}_{\vec{v}_1}\vec{w} = (\vec{v}_1 \cdot \vec{w})\vec{v}_1 = \frac{1}{2}\vec{v}_1$, and since $\|\vec{w}\| \leq \|\vec{v}_3\| = 1$, (by Theorem 5.1.10), the tip of $\vec{w}$ will be on the line segment in Figure 5.8. Note that the angle ϕ enclosed by the vectors $\vec{v}_2$ and $\vec{w}$ is between 0° and 120°, so that $\cos\phi$ is between $-\frac{1}{2}$ and 1.

Therefore, $\vec{v}_2 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{w} = \|\vec{w}\| \cos\phi$ is between $-\frac{1}{2}$ and 1.

This implies that $\angle(\vec{v}_2, \vec{v}_3)$ is between 0° and 120° as well. To see that all these values are attained, add $(n-3)$ zeros to the three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$ given above.

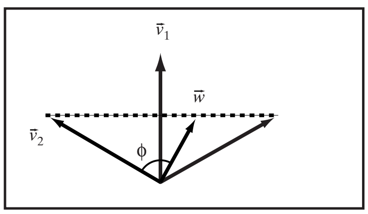

Figure 5.8: for Problem 5.1.38.

5.1.**39** No! By definition of a projection, the vector $\vec{x} - \text{proj}_L\vec{x}$ is perpendicular to $\text{proj}_L\vec{x}$, so that $(\vec{x} - \text{proj}_L\vec{x}) \cdot (\text{proj}_L\vec{x}) = \vec{x} \cdot \text{proj}_L\vec{x} - \|\text{proj}_L\vec{x}\|^2 = 0$ and $\vec{x} \cdot \text{proj}_L\vec{x} = \|\text{proj}_L\vec{x}\|^2 \geq 0$. (See Figure 5.9.)

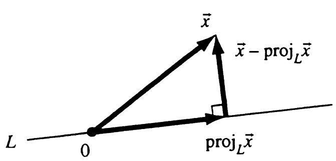

Figure 5.9: for Problem 5.1.39.

5.1.**40** $||\vec{v}_2|| = \sqrt{\vec{v}_2 \cdot \vec{v}_2} = \sqrt{a_{22}} = 3$.

5.1.**41** $\theta = \arccos(\frac{\vec{v}_2 \cdot \vec{v}_3}{||\vec{v}_2||||\vec{v}_3||}) = \arccos(\frac{a_{23}}{\sqrt{a_{22}}\sqrt{a_{33}}}) = \arccos(\frac{20}{21}) \approx 0.31$ radians.

5.1.**42** $||\vec{v}_1 + \vec{v}_2|| = \sqrt{(\vec{v}_1 + \vec{v}_2) \cdot (\vec{v}_1 + \vec{v}_2)} = \sqrt{a_{11} + 2a_{12} + a_{22}} = \sqrt{22}$.

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5.1.**43** Let $\vec{u} = \frac{\vec{v}_2}{||\vec{v}_2||} = \frac{\vec{v}_2}{3}$. Then, $\vec{u}$ is an orthonormal basis for span$(\vec{v}_2)$. Using Theorem 5.1.5, $\text{proj}_{\vec{v}_2}(\vec{v}_1) = (\vec{u}\cdot\vec{v}_1)\vec{u} = (\frac{\vec{v}_2}{3}\cdot\vec{v}_1)\frac{\vec{v}_2}{3} = \frac{1}{3}(\vec{v}_2\cdot\vec{v}_1)\frac{\vec{v}_2}{3} = \frac{1}{3}(a_{12})\frac{\vec{v}_2}{3} = \frac{5}{9}\vec{v}_2$.

5.1.**44** One method to solve this is to take $\vec{v} = \vec{v}_2 - \text{proj}_{\vec{v}_3}\vec{v}_2 = \vec{v}_2 - \frac{20}{49}\vec{v}_3$.

5.1.**45** Write the projection as a linear combination of $\vec{v}_2$ and $\vec{v}_3$, $c_2\vec{v}_2 + c_3\vec{v}_3$. Now you want $\vec{v}_1 - c_2\vec{v}_2 - c_3\vec{v}_3$ to be perpendicular to V, that is, perpendicular to both $\vec{v}_2$ and $\vec{v}_3$. Using dot products, this boils down to two linear equation in two unknowns, $9c_2 + 20c_3 = 5$, and $20c_2 + 49c_3 = 11$, with the solution $c_2 = \frac{25}{41}$ and $c_3 = -\frac{1}{41}$. Thus the answer is $\frac{25}{41}\vec{v}_2 - \frac{1}{41}\vec{v}_3$.

5.1.**46** Write the projection as a linear combination of $\vec{v}_1$ and $\vec{v}_2 : c_1\vec{v}_1 + c_2\vec{v}_2$. Now we want $\vec{v}_3 - c_1\vec{v}_1 + c_2\vec{v}_2$ to be perpendicular to V, that is, perpendicular to both $\vec{v}_1$ and $\vec{v}_2$. Using dot products, this boils down to two linear equations in two unknowns, $11 = 3c_1 + 5c_2$ and $20 = 5c_1 + 9c_2$, with the solution $c_1 = -\frac{1}{2}, c_2 = \frac{5}{2}$. Thus, the answer is $-\frac{1}{2}\vec{v}_1 + \frac{5}{2}\vec{v}_2$.

Section 5.2

In Exercises 1–14, we will refer to the given vectors as $\vec{v}_1, \ldots, \vec{v}_m$, where $m = 1, 2$, or 3.

5.2.**1** $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

5.2.**2** $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{7}\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2 - (\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{7}\begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$$

Note that $\vec{u}_1\cdot\vec{v}_2 = 0$.

5.2.**3** $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2 - (\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

5.2.**4** $\vec{u}_1 = \frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ and $\vec{u}_2 = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$ as in Exercise 3.

Since $\vec{v}_3$ is orthogonal to $\vec{u}_1$ and $\vec{u}_2, \vec{u}_3 = \frac{1}{\|\vec{v}_3\|}\vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

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5.2.5 $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{\sqrt{18}}\begin{bmatrix}-1\\-1\\4\end{bmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix}-1\\-1\\4\end{bmatrix}$$

5.2.6 $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} = \vec{e}_1$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \begin{bmatrix}0\\1\\0\end{bmatrix} = \vec{e}_2$$

$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \frac{\vec{v}_3-(\vec{u}_1\cdot\vec{v}_3)\vec{u}_1-(\vec{u}_2\cdot\vec{v}_3)\vec{u}_2}{\|\vec{v}_3-(\vec{u}_1\cdot\vec{v}_3)\vec{u}_1-(\vec{u}_2\cdot\vec{v}_3)\vec{u}_2\|} = \begin{bmatrix}0\\0\\1\end{bmatrix} = \vec{e}_3$$

5.2.7 Note that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal, so that $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}$ and $\vec{u}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{3}\begin{bmatrix}-2\\1\\2\end{bmatrix}$. Then

$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \frac{\vec{v}_3-(\vec{u}_1\cdot\vec{v}_3)\vec{u}_1-(\vec{u}_2\cdot\vec{v}_3)\vec{u}_2}{\|\vec{v}_3-(\vec{u}_1\cdot\vec{v}_3)\vec{u}_1-(\vec{u}_2\cdot\vec{v}_3)\vec{u}_2\|} = \frac{1}{\sqrt{36}}\begin{bmatrix}2\\-4\\4\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\-2\\2\end{bmatrix}.$$

5.2.8 $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{7}\begin{bmatrix}5\\4\\2\\2\end{bmatrix}$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{7}\begin{bmatrix}-2\\2\\5\\-4\end{bmatrix}$$

5.2.9 $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{10}\begin{bmatrix}-1\\7\\-7\\1\end{bmatrix}$$

5.2.10 $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

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$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

5.2.**11** $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{5}\begin{bmatrix}4\\0\\0\\3\end{bmatrix}$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{\sqrt{225}}\begin{bmatrix}-3\\2\\14\\4\end{bmatrix} = \frac{1}{15}\begin{bmatrix}-3\\2\\14\\4\end{bmatrix}$$

5.2.**12** $\vec{u}_1 = \frac{1}{7}\begin{bmatrix}2\\3\\0\\6\end{bmatrix}$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{3}\begin{bmatrix}0\\-2\\2\\1\end{bmatrix}$$

5.2.**13** $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \frac{\vec{v}_3-(\vec{u}_1\cdot\vec{v}_3)\vec{u}_1-(\vec{u}_2\cdot\vec{v}_3)\vec{u}_2}{\|\vec{v}_3-(\vec{u}_1\cdot\vec{v}_3)\vec{u}_1-(\vec{u}_2\cdot\vec{v}_3)\vec{u}_2\|} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}$$

5.2.**14** $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{10}\begin{bmatrix}1\\7\\1\\7\end{bmatrix}$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

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$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \frac{\vec{v}_3-(\vec{u}_1\cdot\vec{v}_3)\vec{u}_1-(\vec{u}_2\cdot\vec{v}_3)\vec{u}_2}{\|\vec{v}_3-(\vec{u}_1\cdot\vec{v}_3)\vec{u}_1-(\vec{u}_2\cdot\vec{v}_3)\vec{u}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

In Exercises 15–28, we will use the results of Exercises 1–14 (note that Exercise k, where $k = 1, \ldots, 14$, gives the QR factorization of the matrix in Exercise $(k+14)$). We can set $Q = [\vec{u}_1 \ldots \vec{u}_m]$; the entries of R are

$$\begin{array}{llll} r_{11} & = \|\vec{v}_1\| & & \\ r_{22} & = \|\vec{v}_2^{\perp}\| & & = \|\vec{v}_2 - (\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\| \\ r_{33} & = \|\vec{v}_3^{\perp}\| & = \|\vec{v}_3 - (\vec{u}_1\cdot\vec{v}_3)\vec{u}_1 - (\vec{u}_2\cdot\vec{v}_3)\vec{u}_2\| & \\ r_{ij} & & & = \vec{u}_i\cdot\vec{v}_j, \text{ where } i < j. \end{array}$$

5.2.**15** $Q = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, R = [3]$

5.2.**16** $Q = \frac{1}{7}\begin{bmatrix} 6 & 2 \\ 5 & -6 \\ 2 & 3 \end{bmatrix}, R = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

5.2.**17** $Q = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ 0 & 0 \\ 5 & -4 \end{bmatrix}, R = \begin{bmatrix} 5 & 5 \\ 0 & 35 \end{bmatrix}$

5.2.**18** $Q = \frac{1}{5}\begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & -5 \\ 5 & -4 & 0 \end{bmatrix}, R = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

5.2.**19** $Q = \frac{1}{3}\begin{bmatrix} 2 & -\frac{1}{\sqrt{2}} \\ 2 & -\frac{1}{\sqrt{2}} \\ 1 & \frac{4}{\sqrt{2}} \end{bmatrix}, R = 3\begin{bmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{bmatrix}$

5.2.**20** $Q = I_3$, $R = [\,\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3\,] = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$

5.2.**21** $Q = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}, R = \begin{bmatrix} 3 & 0 & 12 \\ 0 & 3 & -12 \\ 0 & 0 & 6 \end{bmatrix}$

5.2.**22** $Q = \frac{1}{7}\begin{bmatrix} 5 & -2 \\ 4 & 2 \\ 2 & 5 \\ 2 & -4 \end{bmatrix}, R = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$

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5.2.23 $Q = \begin{bmatrix} 0.5 & -0.1 \\ 0.5 & 0.7 \\ 0.5 & -0.7 \\ 0.5 & 0.1 \end{bmatrix}$, $R = \begin{bmatrix} 2 & 4 \\ 0 & 10 \end{bmatrix}$

5.2.24 $Q = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$, $R = \begin{bmatrix} 2 & 10 \\ 0 & 2 \end{bmatrix}$

5.2.25 $Q = \frac{1}{15}\begin{bmatrix} 12 & -3 \\ 0 & 2 \\ 0 & 14 \\ 9 & 4 \end{bmatrix}$, $R = \begin{bmatrix} 5 & 10 \\ 0 & 15 \end{bmatrix}$

5.2.26 $Q = \begin{bmatrix} \frac{2}{7} & 0 \\ \frac{3}{7} & -\frac{2}{3} \\ 0 & \frac{2}{3} \\ \frac{6}{7} & \frac{1}{3} \end{bmatrix}$, $R = \begin{bmatrix} 7 & 14 \\ 0 & 3 \end{bmatrix}$

5.2.27 $Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, $R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

5.2.28 $Q = \begin{bmatrix} \frac{1}{10} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{7}{10} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{10} & \frac{1}{\sqrt{2}} & 0 \\ \frac{7}{10} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$, $R = \begin{bmatrix} 10 & 10 & 10 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$

5.2.29 $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{5}\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ $\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. (See Figure 5.10.)

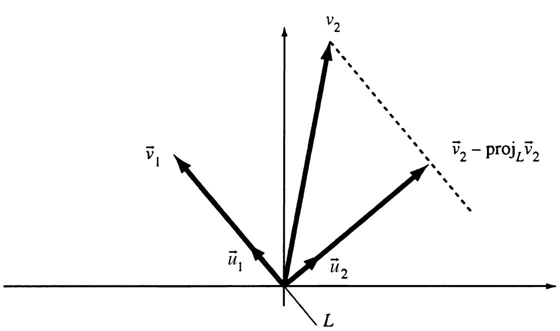

Figure 5.10: for Problem 5.2.29.

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5.2.**30** See Figure 5.11.

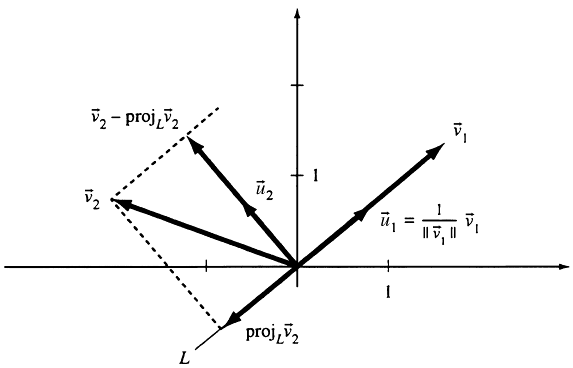

Figure 5.11: for Problem 5.2.30.

5.2.**31** $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{e}_1$

$\vec{v}_2^{\perp} = \vec{v}_2 - \text{proj}_{V_1}\vec{v}_2 = \begin{bmatrix} b \\ c \\ 0 \end{bmatrix} - \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix}$, so that $\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{e}_2$

Here $V_1 = \text{span}(\vec{e}_1) = x$ axis.

$\vec{v}_3^{\perp} = \vec{v}_3 - \text{proj}_{V_2}\vec{v}_3 = \begin{bmatrix} d \\ e \\ f \end{bmatrix} - \begin{bmatrix} d \\ e \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}$, so that $\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{e}_3$.

Here $V_2 = \text{span}(\vec{e}_1, \vec{e}_2) = x\text{-}y$ plane. (See Figure 5.12.)

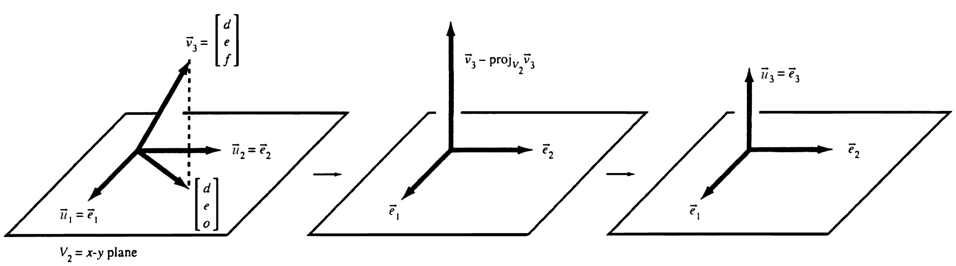

Figure 5.12: for Problem 5.2.31.

5.2.**32** A basis of the plane is $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

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Now apply the Gram-Schmidt process.

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

Your solution may be different if you start with a different basis $\vec{v}_1, \vec{v}_2$ of the plane.

5.2.**33** $\text{rref}(A) = \begin{bmatrix}1&0&0&1\\0&1&1&0\end{bmatrix}$

A basis of $\ker(A)$ is $\vec{v}_1 = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}$.

Since $\vec{v}_1$ and $\vec{v}_2$ are orthogonal already, we obtain $\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$, $\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}$.

5.2.**34** $\text{rref}(A) = \begin{bmatrix}1&0&-1&-2\\0&1&2&3\end{bmatrix}$

A basis of $\ker(A)$ is $\vec{v}_1 = \begin{bmatrix}1\\-2\\1\\0\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}2\\-3\\0\\1\end{bmatrix}$.

We apply the Gram-Schmidt process and obtain

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{\sqrt{30}}\begin{bmatrix}2\\-1\\-4\\3\end{bmatrix}$$

5.2.**35** $\text{rref}(A) = \begin{bmatrix}1&0&\frac{1}{3}\\0&1&\frac{1}{3}\\0&0&0\end{bmatrix}$

The non-redundant columns of A give us a basis of $\text{im}(A)$:

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$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Since $\vec{v}_1$ and $\vec{v}_2$ are orthogonal already, we obtain $\vec{u}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

5.2.**36** Write $M = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}_{\uparrow \; Q_0} \underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix}}_{\uparrow \; R_0}$

This is almost the QR factorization of M: the matrix Q_0 has orthonormal columns and R_0 is upper triangular; the only problem is the entry -4 on the diagonal of R_0. Keeping in mind how matrices are multiplied, we can change all the signs in the second column of Q_0 and in the second row of R_0 to fix this problem:

$$M = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}}_{\uparrow \; Q} \underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}}_{\uparrow \; R}$$

5.2.**37** Write $M = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}}_{\uparrow \; Q_0} \underbrace{\begin{bmatrix} 3 & 4 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\uparrow \; R_0}$

Note that the last two columns of Q_0 and the last two rows of R_0 have no effect on the product Q_0R_0; if we drop them, we have the QR factorization of M:

$$M = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}}_{\uparrow \; Q} \underbrace{\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}}_{\uparrow \; R}$$

5.2.**38** Since $\vec{v}_1 = 2\vec{e}_3$, $\vec{v}_2 = -3\vec{e}_1$ and $\vec{v}_3 = 4\vec{e}_4$ are orthogonal, we have

$$Q = \begin{bmatrix} \frac{\vec{v}_1}{\|\vec{v}_1\|} & \frac{\vec{v}_2}{\|\vec{v}_2\|} & \frac{\vec{v}_3}{\|\vec{v}_3\|} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } R = \begin{bmatrix} \|\vec{v}_1\| & 0 & 0 \\ 0 & \|\vec{v}_2\| & 0 \\ 0 & 0 & \|\vec{v}_3\| \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

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5.2.**39** $\vec{u}_1 = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{u}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\vec{u}_3 = \vec{u}_1 \times \vec{u}_2 = \frac{1}{\sqrt{42}}\begin{bmatrix}-5\\4\\-1\end{bmatrix}$

5.2.**40** If $\vec{v}_1, \ldots, \vec{v}_n$ are the columns of A, then $Q = \begin{bmatrix}\frac{\vec{v}_1}{\|\vec{v}_1\|} & \cdots & \frac{\vec{v}_n}{\|\vec{v}_n\|}\end{bmatrix}$ and $R = \begin{bmatrix}\|\vec{v}_1\| & & 0\\ & \ddots & \\ 0 & & \|\vec{v}_n\|\end{bmatrix}$.

(See Exercise 38 as an example.)

5.2.**41** If all diagonal entries of A are positive, then we have $Q = I_n$ and $R = A$. A small modification is necessary if A has negative entries on the diagonal: if $a_{ii} < 0$ we let $r_{ij} = -a_{ij}$ for all j, and we let $q_{ii} = -1$; if $a_{ii} > 0$ we let $r_{ij} = a_{ij}$ and $q_{ii} = 1$. Furthermore, $q_{ij} = 0$ if $i \neq j$ (that is, Q is diagonal).

For example, $$\underset{\substack{\uparrow\\ A}}{\begin{bmatrix}-1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & -6\end{bmatrix}} = \underset{\substack{\uparrow\\ Q}}{\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}} \underset{\substack{\uparrow\\ R}}{\begin{bmatrix}1 & -2 & -3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{bmatrix}}$$

5.2.**42** We have $r_{11} = \|\vec{v}_1\|$ and $r_{22} = \|\vec{v}_2^{\perp}\| = \|\vec{v}_2 - \text{proj}_L\vec{v}_2\|$, so that $r_{11}r_{22}$ is the area of the parallelogram defined by $\vec{v}_1$ and $\vec{v}_2$. See Figure 5.13.

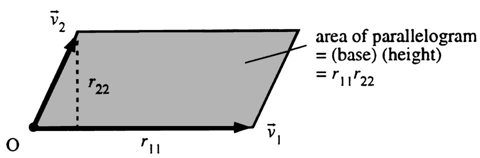

Figure 5.13: for Problem 5.2.42.

5.2.**43** Partition the matrices Q and R in the QR factorization of A as follows:

$$[A_1 \quad A_2] = A = QR = [Q_1 \quad Q_2]\begin{bmatrix}R_1 & R_2\\ 0 & R_3\end{bmatrix} = [Q_1R_1 \quad Q_1R_2 + Q_2R_3],$$

where Q_1 is $n \times m_1, Q_2$ is $n \times m_2, R_1$ is $m_1 \times m_1$, and R_3 is $m_2 \times m_2$.

Then, $A_1 = Q_1R_1$ is the QR factorization of A_1: note that the columns of A_1 are orthonormal, and R_1 is upper triangular with positive diagonal entries.

5.2.**44** No! If m exceeds n, then there is no $n \times m$ matrix Q with orthonormal columns (if the columns of a matrix are orthonormal, then they are linearly independent).

5.2.**45** Yes. Let $A = [\vec{v}_1 \quad \cdots \quad \vec{v}_m]$. The idea is to perform the Gram-Schmidt process in reversed order, starting with $\vec{u}_m = \frac{1}{\|\vec{v}_m\|}\vec{v}_m$.

Then we can express $\vec{v}_j$ as a linear combination of $\vec{u}_j, \ldots, \vec{u}_m$, so that $[\vec{v}_1 \quad \cdots \quad \vec{v}_j \quad \cdots \quad \vec{v}_m] = [\vec{u}_1 \quad \cdots \quad \vec{u}_j \quad \cdots \quad \vec{u}_m]\,L$ for some *lower* triangular matrix L, with

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