Chapter 2

Section 2.1

2.1.1 Not a linear transformation, since $y_2 = x_2 + 2$ is not linear in our sense.

2.1.2 Linear, with matrix $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}$

2.1.3 Not linear, since $y_2 = x_1x_3$ is nonlinear.

2.1.4 $A = \begin{bmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{bmatrix}$

2.1.5 By Theorem 2.1.2, the three columns of the 2×3 matrix A are $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$, so that

$$A = \begin{bmatrix} 7 & 6 & -13 \\ 11 & 9 & 17 \end{bmatrix}.$$

2.1.6 Note that $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so that T is indeed linear, with matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

2.1.7 Note that $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = [\vec{v}_1 \dots \vec{v}_m] \begin{bmatrix} x_1 \\ \cdots \\ x_m \end{bmatrix}$, so that T is indeed linear, with matrix $[\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_m]$.

2.1.8 Reducing the system $\begin{bmatrix} x_1 + 7x_2 & = y_1 \\ 3x_1 + 20x_2 & = y_2 \end{bmatrix}$, we obtain $\begin{bmatrix} x_1 & & = & -20y_1 & + & 7y_2 \\ & x_2 & = & 3y_1 & - & y_2 \end{bmatrix}$.

2.1.9 We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2. Reducing the system $\begin{bmatrix} 2x_1 & + & 3x_2 & = & y_1 & \\ 6x_1 & + & 9x_2 & = & & y_2 \end{bmatrix}$ we obtain $\begin{bmatrix} x_1 + 1.5x_2 & = 0.5y_1 \\ 0 & = -3y_1 + y_2 \end{bmatrix}$.

No unique solution (x_1, x_2) can be found for a given (y_1, y_2); the matrix is noninvertible.

2.1.10 We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2. Reducing the system $\begin{bmatrix} x_1 & + & 2x_2 & = & y_1 & \\ 4x_1 & + & 9x_2 & = & & y_2 \end{bmatrix}$ we find that $\begin{bmatrix} x_1 & & = & 9y_1 & + & 2y_2 \\ & x_2 & = & -4y_1 & + & y_2 \end{bmatrix}$ or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

The inverse matrix is $\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$.

2.1.11 We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2. Reducing the system $\begin{bmatrix} x_1 & + & 2x_2 & = & y_1 & \\ 3x_1 & + & 9x_2 & = & & y_2 \end{bmatrix}$ we find that $\begin{bmatrix} x_1 & & = & 3y_1 - \frac{2}{3}y_2 \\ & x_2 & = & -y_1 + \frac{1}{3}y_2 \end{bmatrix}$. The inverse matrix is $\begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}$.

2.1.12 Reducing the system $\begin{bmatrix} x_1 + kx_2 & = y_1 \\ x_2 & = y_2 \end{bmatrix}$ we find that $\begin{bmatrix} x_1 & & = & y_1 & - & ky_2 \\ & x_2 & = & & & y_2 \end{bmatrix}$. The inverse matrix is $\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$.

2.1.13 a First suppose that $a \neq 0$. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2.

$$\begin{bmatrix} ax_1 & + & bx_2 & = & y_1 & \\ cx_1 & + & dx_2 & = & & y_2 \end{bmatrix} \begin{matrix} \div a \\ \\ \end{matrix} \to \begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & \\ cx_1 & + & dx_2 & = & & y_2 \end{bmatrix} \begin{matrix} \\ -c(I) \end{matrix} \to$$

$$\begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & (d - \frac{bc}{a})x_2 & = & -\frac{c}{a}y_1 & + & y_2 \end{bmatrix} \to$$

$$\begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & (\frac{ad-bc}{a})x_2 & = & -\frac{c}{a}y_1 & + & y_2 \end{bmatrix}$$

We can solve this system for x_1 and x_2 if (and only if) $ad - bc \neq 0$, as claimed.

If $a = 0$, then we have to consider the system

$$\begin{bmatrix} & & bx_2 & = & y_1 & \\ cx_1 & + & dx_2 & = & & y_2 \end{bmatrix} \text{swap} : I \leftrightarrow II \begin{bmatrix} cx_1 & + & dx_2 & = & & y_2 \\ & & bx_2 & = & y_1 & \end{bmatrix}$$

We can solve for x_1 and x_2 provided that both b and c are nonzero, that is if $bc \neq 0$. Since $a = 0$, this means that $ad - bc \neq 0$, as claimed.

b First suppose that $ad - bc \neq 0$ and $a \neq 0$. Let $D = ad - bc$ for simplicity. We continue our work in part (a):

$$\begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & \frac{D}{a}x_2 & = & -\frac{c}{a}y_1 & + & y_2 \end{bmatrix} \begin{matrix} \\ \cdot \frac{a}{D} \end{matrix} \to$$

$$\begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & x_2 & = & -\frac{c}{D}y_1 & + & \frac{a}{D}y_2 \end{bmatrix} \begin{matrix} -\frac{b}{a}(II) \\ \\ \end{matrix} \to$$

$$\begin{bmatrix} x_1 & & = & (\frac{1}{a} + \frac{bc}{aD})y_1 & - & \frac{b}{D}y_2 \\ & x_2 & = & -\frac{c}{D}y_1 & + & \frac{a}{D}y_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & & = & \frac{d}{D}y_1 & - & \frac{b}{D}y_2 \\ & x_2 & = & -\frac{c}{D}y_1 & + & \frac{a}{D}y_2 \end{bmatrix}$$

(Note that $\frac{1}{a} + \frac{bc}{aD} = \frac{D+bc}{aD} = \frac{ad}{aD} = \frac{d}{D}$.)

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, as claimed. If $ad - bc \neq 0$ and $a = 0$, then we have to solve the system

$$\left[\begin{array}{llll} cx_1+ & dx_2 & = y_2 \\ & bx_2 & = y_1 \end{array}\right] \begin{array}{l} \div c \\ \div b \end{array}$$

$$\left[\begin{array}{llll} x_1+ & \frac{d}{c}x_2 & = \frac{1}{c}y_2 \\ & x_2 & = \frac{1}{b}y_1 \end{array}\right] -\frac{d}{c}(II)$$

$$\left[\begin{array}{lllll} x_1 & & = & -\frac{d}{bc}y_1 & +\frac{1}{c}y_2 \\ & x_2 & = & \frac{1}{b}y_1 & \end{array}\right]$$

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (recall that $a = 0$), as claimed.

2.1.**14** a By Exercise 13a, $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$ is invertible if (and only if) $2k - 15 \neq 0$, or $k \neq 7.5$.

b By Exercise 13b, $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1} = \frac{1}{2k-15}\begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix}$.

If all entries of this inverse are integers, then $\frac{3}{2k-15} \quad - \frac{2}{2k-15} = \frac{1}{2k-15}$ is a (nonzero) integer n, so that $2k-15 = \frac{1}{n}$ or $k = 7.5 + \frac{1}{2n}$. Since $\frac{k}{2k-15} = kn = 7.5n + \frac{1}{2}$ is an integer as well, n must be odd.

We have shown: If all entries of the inverse are integers, then $k = 7.5 + \frac{1}{2n}$, where n is an odd integer. The converse is true as well: If k is chosen in this way, then the entries of $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$ will be integers.

2.1.**15** By Exercise 13a, the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is invertible if (and only if) $a^2 + b^2 \neq 0$, which is the case unless $a = b = 0$. If $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is invertible, then its inverse is $\frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, by Exercise 13b.

2.1.**16** If $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then $A\vec{x} = 3\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^2$, so that A represents a scaling by a factor of 3. Its inverse is a scaling by a factor of $\frac{1}{3}$: $A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$. (See Figure 2.1.)

2.1.**17** If $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A\vec{x} = -\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^2$, so that A represents a reflection about the origin.

This transformation is its own inverse: $A^{-1} = A$. (See Figure 2.2.)

2.1.**18** Compare with Exercise 16: This matrix represents a scaling by the factor of $\frac{1}{2}$; the inverse is a scaling by 2. (See Figure 2.3.)

2.1.**19** If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$, so that A represents the orthogonal projection onto the $\vec{e}_1$ axis. (See Figure 2.1.) This transformation is not invertible, since the equation $A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has infinitely many solutions $\vec{x}$. (See Figure 2.4.)

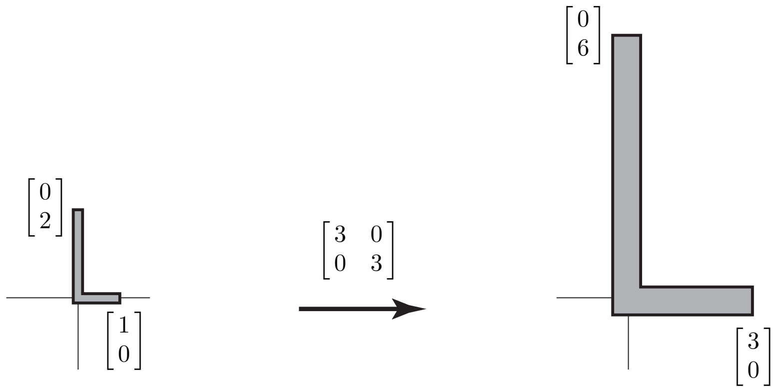

Figure 2.1: for Problem 2.1.16.

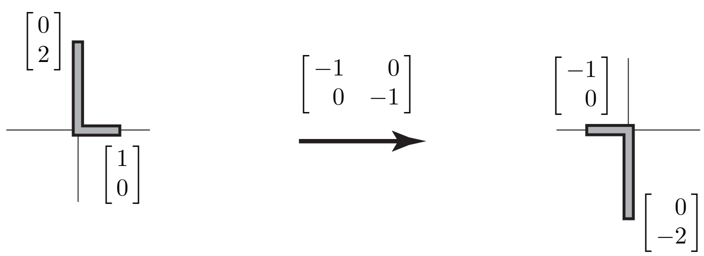

Figure 2.2: for Problem 2.1.17.

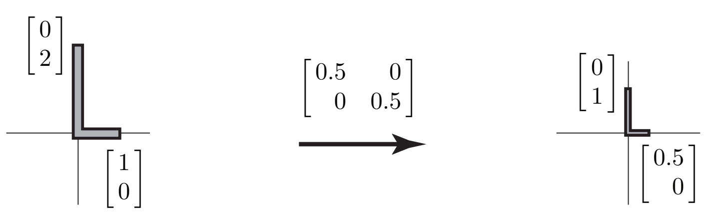

Figure 2.3: for Problem 2.1.18.

2.1.**20** If $A = \begin{bmatrix}0&1\\1&0\end{bmatrix}$, then $A\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_2\\x_1\end{bmatrix}$, so that A represents the reflection about the line $x_2 = x_1$. This transformation is its own inverse: $A^{-1} = A$. (See Figure 2.5.)

2.1.**21** Compare with Example 5.

If $A = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$, then $A\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_2\\-x_1\end{bmatrix}$. Note that the vectors $\vec{x}$ and $A\vec{x}$ are perpendicular and have the same

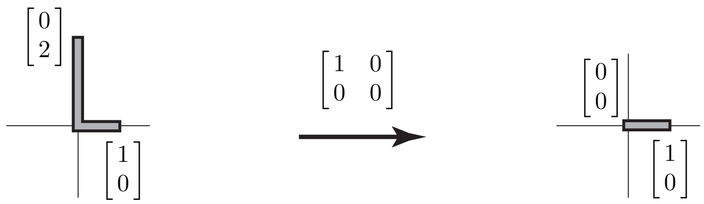

Figure 2.4: for Problem 2.1.19.

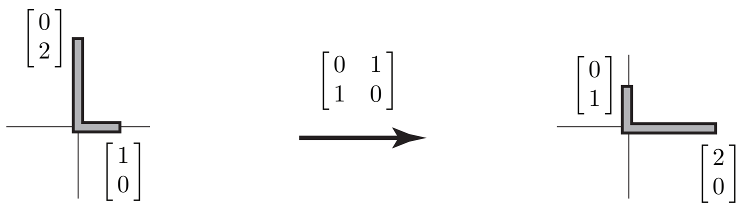

Figure 2.5: for Problem 2.1.20.

length. If $\vec{x}$ is in the first quadrant, then $A\vec{x}$ is in the fourth. Therefore, A represents the rotation through an angle of 90° in the clockwise direction. (See Figure 2.6.) The inverse $A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents the rotation through 90° in the counterclockwise direction.

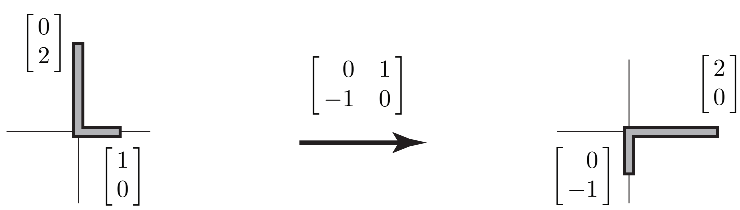

Figure 2.6: for Problem 2.1.21.

2.1.**22** If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$, so that A represents the reflection about the $\vec{e}_1$ axis. This transformation is its own inverse: $A^{-1} = A$. (See Figure 2.7.)

2.1.**23** Compare with Exercise 21.

Note that $A = 2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so that A represents a rotation through an angle of 90° in the clockwise direction, followed by a scaling by the factor of 2.

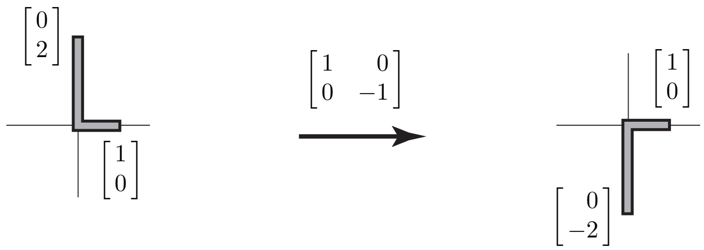

Figure 2.7: for Problem 2.1.22.

The inverse $A^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ represents a rotation through an angle of 90° in the counterclockwise direction, followed by a scaling by the factor of $\frac{1}{2}$. (See Figure 2.8.)

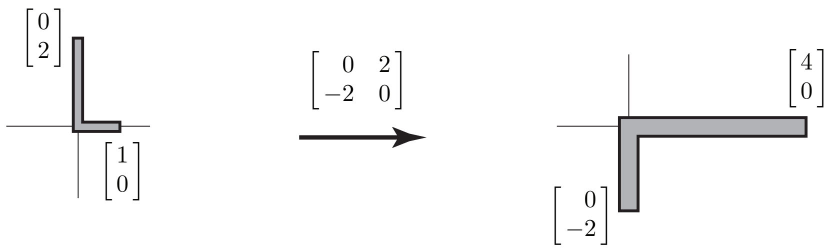

Figure 2.8: for Problem 2.1.23.

2.1.**24** Compare with Example 5. (See Figure 2.9.)

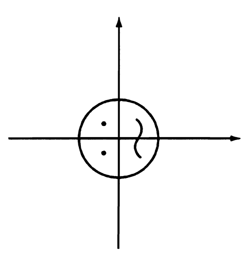

Figure 2.9: for Problem 2.1.24.

2.1.**25** The matrix represents a scaling by the factor of 2. (See Figure 2.10.)

2.1.**26** This matrix represents a reflection about the line $x_2 = x_1$. (See Figure 2.11.)

2.1.**27** This matrix represents a reflection about the $\vec{e}_1$ axis. (See Figure 2.12.)

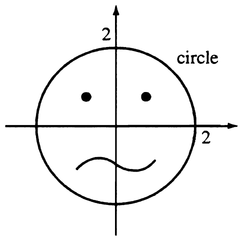

Figure 2.10: for Problem 2.1.25.

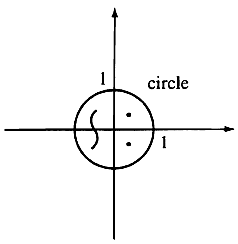

Figure 2.11: for Problem 2.1.26.

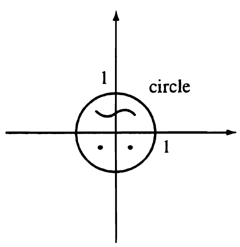

Figure 2.12: for Problem 2.1.27.

2.1.**28** If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$, so that the x_2 component is multiplied by 2, while the x_1 component remains unchanged. (See Figure 2.13.)

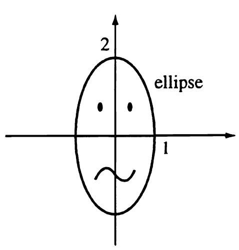

Figure 2.13: for Problem 2.1.28.

2.1.**29** This matrix represents a reflection about the origin. Compare with Exercise 17. (See Figure 2.14.)

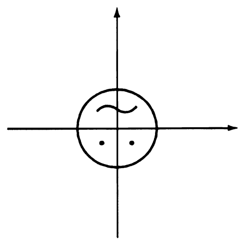

Figure 2.14: for Problem 2.1.29.

2.1.**30** If $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$, so that A represents the projection onto the $\vec{e}_2$ axis. (See Figure 2.15.)

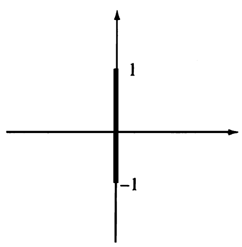

Figure 2.15: for Problem 2.1.30.

2.1.**31** The image must be reflected about the $\vec{e}_2$ axis, that is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ must be transformed into $\begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$: This can be accomplished by means of the linear transformation $T(\vec{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}$.

2.1.**32** Using Theorem 2.1.2, we find $A = \begin{bmatrix} 3 & 0 & \cdot & 0 \\ 0 & 3 & \cdot & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{bmatrix}$. This matrix has 3's on the diagonal and 0's everywhere else.

2.1.**33** By Theorem 2.1.2, $A = \begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$. (See Figure 2.16.)

Therefore, $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

2.1.**34** As in Exercise 2.1.33, we find $T(\vec{e}_1)$ and $T(\vec{e}_2)$; then by Theorem 2.1.2, $A = [T(\vec{e}_1)\ T(\vec{e}_2)]$. (See Figure 2.17.)

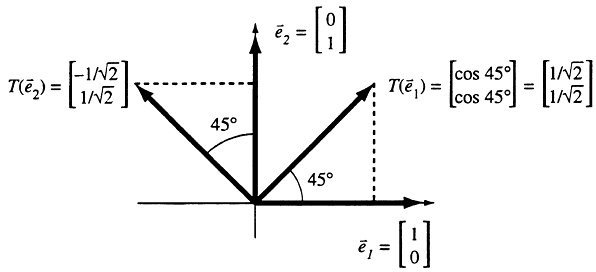

Figure 2.16: for Problem 2.1.33.

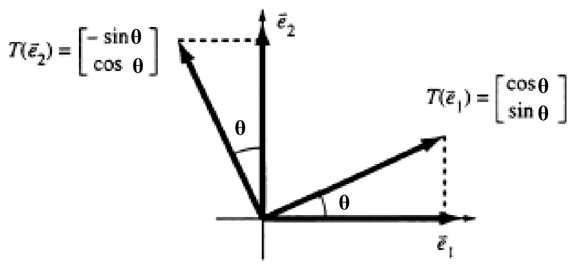

Figure 2.17: for Problem 2.1.34.

Therefore, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

2.1.**35** We want to find a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $A\begin{bmatrix} 5 \\ 42 \end{bmatrix} = \begin{bmatrix} 89 \\ 52 \end{bmatrix}$ and $A\begin{bmatrix} 6 \\ 41 \end{bmatrix} = \begin{bmatrix} 88 \\ 53 \end{bmatrix}$. This amounts to solving the system $\begin{bmatrix} 5a+42b & & =89 \\ 6a+41b & & =88 \\ & 5c+42d & =52 \\ & 6c+41d & =53 \end{bmatrix}$.

(Here we really have two systems with two unknowns each.)

The unique solution is $a = 1,\ b = 2,\ c = 2$, and $d = 1$, so that $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

2.1.**36** First we draw $\vec{w}$ in terms of $\vec{v}_1$ and $\vec{v}_2$ so that $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$ for some c_1 and c_2. Then, we scale the $\vec{v}_2$-component by 3, so our new vector equals $c_1\vec{v}_1 + 3c_2\vec{v}_2$.

2.1.**37** Since $\vec{x} = \vec{v} + k(\vec{w} - \vec{v})$, we have $T(\vec{x}) = T(\vec{v} + k(\vec{w} - \vec{v})) = T(\vec{v}) + k(T(\vec{w}) - T(\vec{v}))$, by Theorem 2.1.3

Since k is between 0 and 1, the tip of this vector $T(\vec{x})$ is on the line segment connecting the tips of $T(\vec{v})$ and $T(\vec{w})$. (See Figure 2.18.)

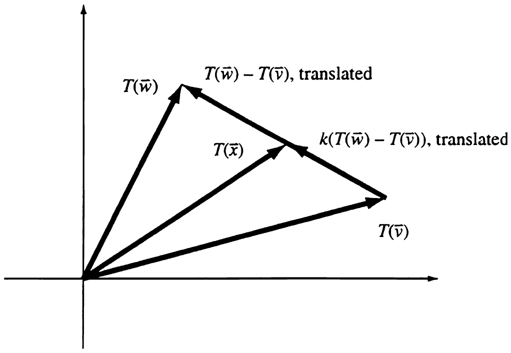

Figure 2.18: for Problem 2.1.37.

2.1.**38** $T\begin{bmatrix} 2 \\ -1 \end{bmatrix} = [\vec{v}_1 \quad \vec{v}_2]\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\vec{v}_1 - \vec{v}_2 = 2\vec{v}_1 + (-\vec{v}_2)$. (See Figure 2.19.)

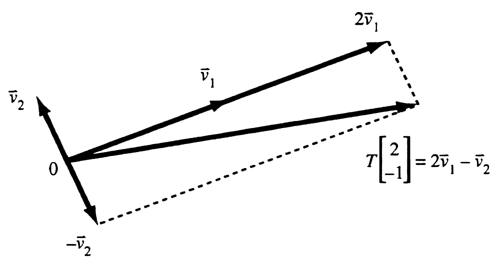

Figure 2.19: for Problem 2.1.38.

2.1.**39** By Theorem 2.1.2, we have $T\begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & \dots & T(\vec{e}_m) \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = x_1T(\vec{e}_1) + \cdots + x_mT(\vec{e}_m)$.

2.1.**40** These linear transformations are of the form $[y] = [a][x]$, or $y = ax$. The graph of such a function is a line through the origin.

2.1.**41** These linear transformations are of the form $[y] = [a\ b]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, or $y = ax_1 + bx_2$. The graph of such a function is a plane through the origin.

2.1.**42** a See Figure 2.20.

b The image of the point $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ is the origin, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

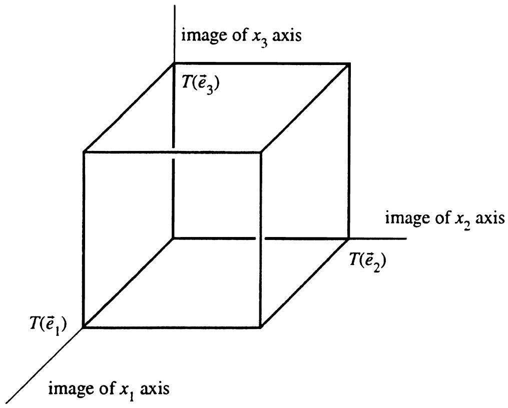

Figure 2.20: for Problem 2.1.42.

c Solve the equation $\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, or $\begin{bmatrix} -\frac{1}{2}x_1 & + & x_2 & & = 0 \\ -\frac{1}{2}x_1 & + & & x_3 & = 0 \end{bmatrix}$. (See Figure 2.16.)

The solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix}$, where t is an arbitrary real number. For example, for $t = \frac{1}{2}$, we find the point $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ considered in part b.These points are on the line through the origin and the observer's eye.

2.1.**43** a $T(\vec{x}) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 3x_2 + 4x_3 = [2\ 3\ 4] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The transformation is indeed linear, with matrix [2 3 4].

b If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then T is linear with matrix $[v_1\ v_2\ v_3]$, as in part (a).

c Let $[a\ b\ c]$ be the matrix of T. Then $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [a\ b\ c] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ax_1 + bx_2 + cx_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ does the job.

2.1.**44** $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that T is linear, with matrix $\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$.

2.1.**45** Yes, $\vec{z} = L(T(\vec{x}))$ is also linear, which we will verify using Theorem 2.1.3. Part a holds, since $L(T(\vec{v}+\vec{w})) = L(T(\vec{v}) + T(\vec{w})) = L(T(\vec{v})) + L(T(\vec{w}))$, and part b also works, because $L(T(k\vec{v})) = L(kT(\vec{v})) = kL(T(\vec{v}))$.

2.1.**46** $T\begin{bmatrix}1\\0\end{bmatrix} = B\left(A\begin{bmatrix}1\\0\end{bmatrix}\right) = B\begin{bmatrix}a\\c\end{bmatrix} = \begin{bmatrix}pa+qc\\ra+sc\end{bmatrix}$

$T\begin{bmatrix}0\\1\end{bmatrix} = B\left(A\begin{bmatrix}0\\1\end{bmatrix}\right) = B\begin{bmatrix}b\\d\end{bmatrix} = \begin{bmatrix}pb+qd\\rb+sd\end{bmatrix}$

So, $T\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\left(T\begin{bmatrix}1\\0\end{bmatrix}\right) + x_2\left(T\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}b\\d\end{bmatrix} = \begin{bmatrix}pb+qd\\rb+sd\end{bmatrix}$

2.1.**47** Write $\vec{w}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2 : \vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$. (See Figure 2.21.)

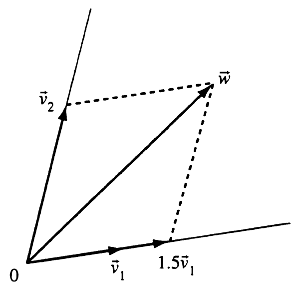

Figure 2.21: for Problem 2.1.47.

Measurements show that we have *roughly* $\vec{w} = 1.5\vec{v}_1 + \vec{v}_2$.

Therefore, by linearity, $T(\vec{w}) = T(1.5\vec{v}_1 + \vec{v}_2) = 1.5T(\vec{v}_1) + T(\vec{v}_2)$. (See Figure 2.22.)

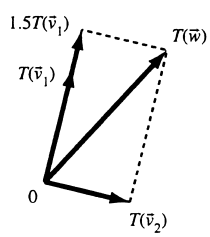

Figure 2.22: for Problem 2.1.47.

2.1.**48** Let $\vec{x}$ be some vector in $\mathbb{R}^2$. Since $\vec{v}_1$ and $\vec{v}_2$ are not parallel, we can write $\vec{x}$ in terms of components of $\vec{v}_1$ and $\vec{v}_2$. So, let c_1 and c_2 be scalars such that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. Then, by Theorem 2.1.3, $T(\vec{x}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = T(c_1\vec{v}_1) + T(c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) = c_1L(\vec{v}_1) + c_2L(\vec{v}_2) = L(c_1\vec{v}_1 + c_2\vec{v}_2) = L(\vec{x})$. So $T(\vec{x}) = L(\vec{x})$ for all $\vec{x}$ in $\mathbb{R}^2$.

2.1.**49** Denote the components of $\vec{x}$ with x_j and the entries of A with a_{ij}. We are told that $\sum_{j=1}^n x_j = 1$ and $\sum_{i=1}^n a_{ij} = 1$ for all $j = 1, ..., n$. Now the i^{th} component of $A\vec{x}$ is $\sum_{j=1}^n a_{ij}x_j$, so that the sum of all components of $A\vec{x}$ is $\sum_{i=1}^n \sum_{j=1}^n a_{ij}x_j = \sum_{j=1}^n \sum_{i=1}^n a_{ij}x_j = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}\right) x_j = \sum_{j=1}^n x_j = 1$, as claimed.

Also, the components of $A\vec{x}$ are nonnegative since all the scalars a_{ij} and x_j are nonnegative. Therefore, $A\vec{x}$ is a distribution vector.

2.1.**50** Proceeding as in Exercise 51, we find

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 1 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix} \text{ and } \vec{x}_{equ} = \tfrac{1}{11}\begin{bmatrix} 4 \\ 4 \\ 2 \\ 1 \end{bmatrix}.$$

Pages 1 and 2 have the highest naive PageRank.

2.1.**51** a. We can construct the transition matrix A column by column, as discussed in Example 9:

$$A = \begin{bmatrix} 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 1/3 & 0 \end{bmatrix}.$$

For example, the first column represents the fact that half of the surfers from page 1 take the link to page 2, while the other half go to page 3.

b. To find the equilibrium vector, we need to solve the system $A\vec{x} = \vec{x} = I_4\vec{x}$ or $(A - I_4)\vec{x} = \vec{0}$. We use technology to find

$$\text{rref}(A - I_4) = \begin{bmatrix} 1 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solutions are of the form $\vec{x} = \begin{bmatrix} t \\ 4t \\ 3t \\ 5t \end{bmatrix}$, where t is arbitrary. The distribution vector among these solutions must satisfy the condition $t + 4t + 3t + 5t = 13t = 1$, or $t = \frac{1}{13}$. Thus $\vec{x}_{equ} = \frac{1}{13}\begin{bmatrix} 1 \\ 4 \\ 3 \\ 5 \end{bmatrix}$.

c. Page 4 has the highest naive PageRank.

2.1.**52** Proceeding as in Exercise 51, we find

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \text{ and } \vec{x}_{equ} = \tfrac{1}{5}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Pages 1 and 2 have the highest naive PageRank.

2.1.**53** a. Constructing the matrix B column by column, as explained for the second column, we find

$$B = \begin{bmatrix} 0.05 & 0.45 & 0.05 & 0.05 \\ 0.45 & 0.05 & 0.05 & 0.85 \\ 0.45 & 0.45 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.85 & 0.05 \end{bmatrix}$$

b. The matrix $0.05E$ accounts for the jumpers, since 5% of the surfers from a given page jump to any other page (or stay put). The matrix $0.8A$ accounts for the 80% of the surfers who follow links.

c. To find the equilibrium vector, we need to solve the system $B\vec{x} = \vec{x} = I_4\vec{x}$ or $(B - I_4)\vec{x} = \vec{0}$. We use technology to find

$$\text{rref}(B - I_4) = \begin{bmatrix} 1 & 0 & 0 & -5/7 \\ 0 & 1 & 0 & -9/7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solutions are of the form $\vec{x} = \begin{bmatrix} 5t \\ 9t \\ 7t \\ 7t \end{bmatrix}$, where t is arbitrary. Now $\vec{x}$ is a distribution vector when $t = \frac{1}{28}$. Thus $\vec{x}_{equ} = \frac{1}{28}\begin{bmatrix} 5 \\ 9 \\ 7 \\ 7 \end{bmatrix}$. Page 2 has the highest PageRank.

2.1.**54** a. Here we consider the same mini-Web as in Exercise 50. Using the formula for B from Exercise 53b , we find

$$B = \begin{bmatrix} 0.05 & 0.85 & 0.05 & 0.05 \\ 0.45 & 0.05 & 0.45 & 0.85 \\ 0.45 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.45 & 0.05 \end{bmatrix}.$$

b. Proceeding as in Exercise 53, we find $\vec{x}_{equ} = \frac{1}{1124}\begin{bmatrix} 377 \\ 401 \\ 207 \\ 139 \end{bmatrix}$.

c. Page 2 has the highest PageRank.

2.1.**55** Here we consider the same mini-Web as in Exercise 51. Proceeding as in Exercise 53, we find

$B = \begin{bmatrix} 0.05 & 0.05 & 19/60 & 0.05 \\ 0.45 & 0.05 & 19/60 & 0.45 \\ 0.45 & 0.05 & 0.05 & 0.45 \\ 0.05 & 0.85 & 19/60 & 0.05 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{2860}\begin{bmatrix} 323 \\ 855 \\ 675 \\ 1007 \end{bmatrix}$.

Page 4 has the highest PageRank.

2.1.**56** Here we consider the same mini-Web as in Exercise 52. Proceeding as in Exercise 53, we find

$B = \frac{1}{15}\begin{bmatrix} 1 & 13 & 1 \\ 7 & 1 & 13 \\ 7 & 1 & 1 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{159}\begin{bmatrix} 61 \\ 63 \\ 35 \end{bmatrix}$.

Page 2 has the highest PageRank

2.1.**57** a Let x_1 be the number of 2 Franc coins, and x_2 be the number of 5 Franc coins. Then $\begin{bmatrix} 2x_1 & +5x_2 & = & 144 \\ x_1 & +x_2 & = & 51 \end{bmatrix}$.

From this we easily find our solution vector to be $\begin{bmatrix} 37 \\ 14 \end{bmatrix}$.

b $\begin{bmatrix} \text{total value of coins} \\ \text{total number of coins} \end{bmatrix} = \begin{bmatrix} 2x_1 & +5x_2 \\ x_1 & +x_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

So, $A = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}$.

c By Exercise 13, matrix A is invertible (since $ad - bc = -3 \neq 0$), and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix}$.

Then $-\frac{1}{3} \begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 144 \\ 51 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 144 & -5(51) \\ -144 & +2(51) \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -111 \\ -42 \end{bmatrix} = \begin{bmatrix} 37 \\ 14 \end{bmatrix}$, which was the vector we found in part a.

2.1.**58** a Let $\begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} \text{mass of the platinum alloy} \\ \text{mass of the silver alloy} \end{bmatrix}$. Using the definition density = mass/volume, or volume = mass/density, we can set up the system:

$\begin{bmatrix} p & +s & = & 5,000 \\ \frac{p}{20} & +\frac{s}{10} & = & 370 \end{bmatrix}$, with the solution $p = 2,600$ and $s = 2,400$. We see that the platinum alloy makes up only 52 percent of the crown; this gold smith is a crook!

b We seek the matrix A such that $A \begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix} = \begin{bmatrix} p+s \\ \frac{p}{20} + \frac{s}{10} \end{bmatrix}$. Thus $A = \begin{bmatrix} 1 & 1 \\ \frac{1}{20} & \frac{1}{10} \end{bmatrix}$.

c Yes. By Exercise 13, $A^{-1} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix}$. Applied to the case considered in part a, we find that $\begin{bmatrix} p \\ s \end{bmatrix} = A^{-1} \begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix} \begin{bmatrix} 5,000 \\ 370 \end{bmatrix} = \begin{bmatrix} 2,600 \\ 2,400 \end{bmatrix}$, confirming our answer in part a.

2.1.**59** a $\begin{bmatrix} C \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9}(F-32) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9}F - \frac{160}{9} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & -\frac{160}{9} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F \\ 1 \end{bmatrix}$.

So $A = \begin{bmatrix} \frac{5}{9} & -\frac{160}{9} \\ 0 & 1 \end{bmatrix}$.

b Using Exercise 13, we find $\frac{5}{9}(1) - (-\frac{160}{9})0 = \frac{5}{9} \neq 0$, so A is invertible.

$A^{-1} = \frac{9}{5} \begin{bmatrix} 1 & \frac{160}{9} \\ 0 & \frac{5}{9} \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & 32 \\ 0 & 1 \end{bmatrix}$. So, $F = \frac{9}{5}C + 32$.

2.1.**60** a $A\vec{x} = \begin{bmatrix} 300 \\ 2,400 \end{bmatrix}$, meaning that the total value of our money is C\$300, or, equivalently, $ZAR2400$.

b From Exercise 13, we test the value $ad - bc$ and find it to be zero. Thus A is not invertible. To determine when A is consistent, we begin to compute rref $\left[A \vdots \vec{b} \right]$:

$$\left[\begin{array}{cc:c} 1 & \frac{1}{8} & b_1 \\ 8 & 1 & b_2 \end{array}\right] \begin{array}{c} \\ -8I \end{array} \rightarrow \left[\begin{array}{cc:c} 1 & \frac{1}{8} & b_1 \\ 0 & 0 & b_2 - 8b_1 \end{array}\right].$$

Thus, the system is consistent only when $b_2 = 8b_1$. This makes sense, since b_2 is the total value of our money in terms of Rand, while b_1 is the value in terms of Canadian dollars. Consider the example in part a. If the system $A\vec{x} = \vec{b}$ is consistent, then there will be infinitely many solutions $\vec{x}$, representing various compositions of our portfolio in terms of Rand and Canadian dollars, all representing the same total value.

2.1.**61** All four entries along the diagonal must be 1: they represent the process of converting a currency to itself. We also know that $a_{ij} = a_{ji}^{-1}$ for all i and j because converting currency i to currency j is the inverse of converting currency j to currency i. This gives us three more entries, $A = \begin{bmatrix} 1 & 4/5 & * & 5/4 \\ 5/4 & 1 & * & * \\ * & * & 1 & 10 \\ 4/5 & * & 1/10 & 1 \end{bmatrix}$. Next let's find the entry a_{31}, giving the value of one Euro expressed in Yuan. Now $E1 = \pounds(4/5)$ and $\pounds 1 = ¥10$ so that $E1 = ¥10(4/5) = ¥8$. We have found that $a_{31} = a_{34}a_{41} = 8$. Similarly we have $a_{ij} = a_{ik}a_{kj}$ for all indices $i, j, k = 1, 2, 3, 4$. This gives $a_{24} = a_{21}a_{14} = 25/16$ and $a_{23} = a_{24}a_{43} = 5/32$. Using the fact that $a_{ij} = a_{ji}^{-1}$, we can complete the matrix:

$$A = \begin{bmatrix} 1 & 4/5 & 1/8 & 5/4 \\ 5/4 & 1 & 5/32 & 25/16 \\ 8 & 32/5 & 1 & 10 \\ 4/5 & 16/25 & 1/10 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.8 & 0.125 & 1.25 \\ 1.25 & 1 & 0.15625 & 1.5625 \\ 8 & 6.4 & 1 & 10 \\ 0.8 & 0.64 & 0.1 & 1 \end{bmatrix}$$

2.1.**62** a 1: this represents converting a currency to itself.

b a_{ij} is the reciprocal of a_{ji}, meaning that $a_{ij}a_{ji} = 1$. This represents converting on currency to another, then converting it back.

c Note that a_{ik} is the conversion factor from currency k to currency i meaning that

(1 unit of currency k) = (a_{ik} units of currency i)

Likewise,

(1 unit of currency j) = (a_{kj} units of currency k).

It follows that

(1 unit of currency j) = ($a_{kj}a_{ik}$ units of currency i) = (a_{ij} units of currency i), so that $a_{ik}a_{kj} = a_{ij}$.

d The rank of A is only 1, because every row is simply a scalar multiple of the top row. More precisely, since $a_{ij} = a_{i1}a_{1j}$, by part c, the i^{th} row is a_{i1} times the top row. When we compute the rref, every row but the top will be removed in the first step. Thus, rref(A) is a matrix with the top row of A and zeroes for all other entries.

2.1.**63** a We express the leading variables x_1, x_3, x_4 in terms of the free variables x_2, x_5:

$$\begin{aligned} x_1 &= x_2 - 4x_5 \\ x_3 &= x_5 \\ x_4 &= 2x_5 \end{aligned}$$