Chapter 3

Section 3.1

3.1.1 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc:c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right], \text{ so that } x_1 = x_2 = 0.$$

$\ker(A) = \{\vec{0}\}$.

3.1.2 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$, or $x_1 + 2x_2 + 3x_2 = 0$.

The solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t-3r \\ t \\ r \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, so that

$\ker(A) = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right)$.

3.1.3 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$; note that all $\vec{x}$ in $\mathbb{R}^2$ satisfy the equation, so that $\ker(A) = \mathbb{R}^2 = \operatorname{span}(\vec{e}_1, \vec{e}_2)$.

3.1.4 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc:c} 2 & 3 & 0 \\ 6 & 9 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ so that } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3t}{2} \\ t \end{bmatrix}$$

Setting $t = 2$ we find $\ker(A) = \operatorname{span}\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

3.1.5 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]; \begin{array}{ll} x_1 & = x_3 \\ x_2 & = -2x_3 \end{array}; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$$

$\ker(A) = \operatorname{span}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

3.1.6 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$. Solving this system yields $\ker(A) = \operatorname{span}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

3.1.7 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$. Since $\operatorname{rref}(A) = I_3$ we have $\ker(A) = \{\vec{0}\}$.

3.1.8 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]; x_1 + x_2 + x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -r-t \\ r \\ t \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\ker(A) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

3.1.**9** Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$. Solving this system yields $\ker(A) = \{\vec{0}\}$.

3.1.**10** Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right)$.

3.1.**11** Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span} \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$.

3.1.**12** Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

3.1.**13** Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

3.1.**14** By Theorem 3.1.3, the image of A is the span of the column vectors of A:

$$\text{im}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right).$$

Since these three vectors are parallel, we need only one of them to span the image:

$$\text{im}(A) = \text{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

3.1.**15** By Theorem 3.1.3, the image of A is the span of the columns of A:

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}1\\4\end{bmatrix}\right).$$

Since any two of these vectors span all of $\mathbb{R}^2$ already, we can write

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right).$$

3.1.**16** By Theorem 3.1.3, the image of A is the span of the column vectors of A:

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right).$$

3.1.**17** By Theorem 3.1.3, $\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right) = \mathbb{R}^2$ (the whole plane).

3.1.**18** By Theorem 3.1.3, $\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}4\\12\end{bmatrix}\right) = \operatorname{span}\begin{bmatrix}1\\3\end{bmatrix}$ (a line in $\mathbb{R}^2$).

3.1.**19** Since the four column vectors of A are parallel, we have $\operatorname{im}(A) = \operatorname{span}\begin{bmatrix}1\\-2\end{bmatrix}$, a line in $\mathbb{R}^2$.

3.1.**20** Compare with the solution to Exercise 21.

$$\left[\begin{array}{cc|c}2 & 1 & 3\\3 & 4 & 2\\6 & 5 & 7\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 0 & 2\\0 & 1 & -1\\0 & 0 & 0\end{array}\right]$$

This computation shows that the third column vector of A, $\vec{v}_3$, is a linear combination of the first two, Thus, only the first two vectors are independent, and the image is a plane in $\mathbb{R}^3$.

3.1.**21** By Theorem 3.1.3, $\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}4\\1\\5\end{bmatrix}, \begin{bmatrix}7\\9\\6\end{bmatrix}, \begin{bmatrix}3\\2\\8\end{bmatrix}\right)$.

We must simply find out how many of the column vectors are not redundant to determine a basis of the image. We can detemine this by taking the rref of the matrix:

$\left[\begin{array}{cc|c}4 & 7 & 3\\1 & 9 & 2\\5 & 6 & 8\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right]$, which shows us that all three column vectors are independent: the span is all of $\mathbb{R}^3$.

3.1.**22** Since the three column vectors of A are parallel, we have $\operatorname{im}(A) = \operatorname{span}\begin{bmatrix}1\\1\\1\end{bmatrix}$, a line in $\mathbb{R}^3$.

3.1.**23** $\text{im}(T) = \mathbb{R}^2$ and $\ker(T) = \{\vec{0}\}$, since T is invertible (see Summary 3.1.8).

3.1.**24** $\text{im}(T)$ is the plane $x + 2y + 3z = 0$, and $\ker(T)$ is the line perpendicular to this plane, spanned by the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (compare with Examples 5 and 9).

3.1.**25** $\text{im}(T) = \mathbb{R}^2$ and $\ker(T) = \{\vec{0}\}$, since T is invertible (see Summary 3.1.8).

3.1.**26** Since $\lim_{t\to\infty} f(t) = \infty$ and $\lim_{t\to-\infty} f(t) = -\infty$, we have $\text{im}(f) = \mathbb{R}$.

A careful proof involves the intermediate value theorem (see Exercise 2.2.47),

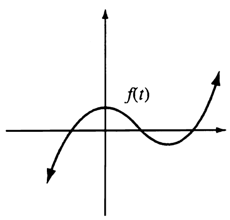

Figure 3.1: for Problem 3.1.26.

Any horizontal line intersects this graph at least once (compare with Example 3 and see Figure 3.1).

3.1.**27** Let $f(x) = x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$.

Then $\text{im}(f) = \mathbb{R}$, since

$$\lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to-\infty} f(x) = -\infty$$

but the function fails to be invertible since the equation $f(x) = 0$ has three solutions, $x = 0$, 1, and -1.

3.1.**28** This ellipse can be obtained from the unit circle by means of the linear transformation with matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, as shown in Figure 3.2 (compare with Exercise 2.2.53).

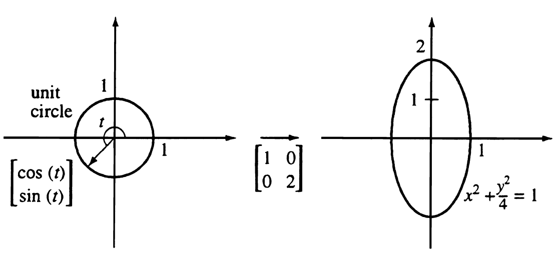

Figure 3.2: for Problem 3.1.28.

We obtain the parametrization $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix}$ for the ellipse.

We can check that $x^2 + \frac{y^2}{4} = \cos^2(t) + \frac{4\sin^2(t)}{4} = 1$.

3.1.**29** Use spherical coordinates (see any good text on multivariable calculus): $f\begin{bmatrix}\phi\\ \theta\end{bmatrix} = \begin{bmatrix}\sin(\phi)\cos(\theta)\\ \sin(\phi)\sin(\theta)\\ \cos(\phi)\end{bmatrix}$

3.1.**30** By Theorem 3.1.3, $A = \begin{bmatrix}1\\5\end{bmatrix}$ does the job. There are many other possible answers: any nonzero $2\times n$ matrix A whose column vectors are scalar multiples of vector $\begin{bmatrix}1\\5\end{bmatrix}$.

3.1.**31** The plane $x + 3y + 2z = 0$ is spanned by the two vectors $\begin{bmatrix}-2\\0\\1\end{bmatrix}$ and $\begin{bmatrix}-3\\1\\0\end{bmatrix}$, for example. Therefore, $A = \begin{bmatrix}-2 & -3\\ 0 & 1\\ 1 & 0\end{bmatrix}$ does the job. There are many other correct answers.

3.1.**32** By Theorem 3.1.3, $A = \begin{bmatrix}7\\6\\5\end{bmatrix}$ does the job. There are many other correct answers: any nonzero $3\times n$ matrix A whose column vectors are scalar multiples of $\begin{bmatrix}7\\6\\5\end{bmatrix}$.

3.1.**33** The plane is the kernel of the linear transformation $T\begin{bmatrix}x\\y\\z\end{bmatrix} = x + 2y + 3z$ from $\mathbb{R}^3$ to $\mathbb{R}$.

3.1.**34** To describe a subset of $\mathbb{R}^3$ as a kernel means to describe it as an intersection of planes (think about it). By inspection, the given line is the intersection of the planes

$$\begin{aligned} x + y &= 0 \quad \text{and} \\ 2x + z &= 0. \end{aligned}$$

This means that the line is the kernel of the linear transformation $T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x+y\\2x+z\end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

3.1.**35** $\ker(T) = \{\vec{x} : T(\vec{x}) = \vec{v}\cdot\vec{x} = 0\}$ = the plane with normal vector $\vec{v}$.

$\operatorname{im}(T) = \mathbb{R}$, since for every real number k there is a vector $\vec{x}$ such that $T(\vec{x}) = k$, for example, $\vec{x} = \frac{k}{\vec{v}\cdot\vec{v}}\vec{v}$.

3.1.**36** $\ker(T) = \{\vec{x} : T(\vec{x}) = \vec{v}\times\vec{x} = \vec{0}\}$ = the line spanned by $\vec{v}$

(see Theorem A.10d in the Appendix)

$\operatorname{im}(T)$ = the plane with normal vector $\vec{v}$

By Definition A.9, $T(\vec{x}) = \vec{v} \times \vec{x}$ is in this plane, for all $\vec{x}$ in $\mathbb{R}^3$. Conversely, for every vector $\vec{w}$ in this plane there is an $\vec{x}$ in $\mathbb{R}^3$ such that $T(\vec{x}) = \vec{w}$, namely $\vec{x} = -\frac{1}{\vec{v}\cdot\vec{v}}T(\vec{w})$ (verify this!).

3.1.**37** $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so that

$\ker(A) = \text{span}(\vec{e}_1)$, $\ker(A^2) = \text{span}(\vec{e}_1, \vec{e}_2)$, $\ker(A^3) = \mathbb{R}^3$, and

$\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2)$, $\text{im}(A^2) = \text{span}(\vec{e}_1)$, $\text{im}(A^3) = \{\vec{0}\}$.

3.1.**38** a If a vector $\vec{x}$ is in $\ker(A^k)$, that is, $A^k\vec{x} = \vec{0}$, then $\vec{x}$ is also in $\ker(A^{k+1})$, since $A^{k+1}\vec{x} = AA^k\vec{x} = A\vec{0} = \vec{0}$.

Therefore, $\ker(A) \subseteq \ker(A^2) \subseteq \ker(A^3) \subseteq \ldots$

Exercise 37 shows that these kernels need not be equal.

b If a vector $\vec{y}$ is in $\text{im}(A^{k+1})$, that is, $\vec{y} = A^{k+1}\vec{x}$ for some $\vec{x}$, then $\vec{y}$ is also in $\text{im}(A^k)$, since we can write $\vec{y} = A^k(A\vec{x})$. Therefore, $\text{im}(A) \supseteq \text{im}(A^2) \supseteq \text{im}(A^3) \supseteq \ldots$.

Exercise 37 shows that these images need not be equal.

3.1.**39** a If a vector $\vec{x}$ is in $\ker(B)$, that is, $B\vec{x} = \vec{0}$, then $\vec{x}$ is also in $\ker(AB)$, since $AB(\vec{x}) = A(B\vec{x}) = A\vec{0} = \vec{0}$:

$\ker(B) \subseteq \ker(AB)$.

Exercise 37 (with $A = B$) illustrates that these kernels need not be equal.

b If a vector $\vec{y}$ is in $\text{im}(AB)$, that is, $\vec{y} = AB\vec{x}$ for some $\vec{x}$, then $\vec{y}$ is also in $\text{im}(A)$, since we can write

$\vec{y} = A(B\vec{x})$:

$\text{im}(AB) \subseteq \text{im}(A)$.

Exercise 37 (with $A = B$) illustrates that these images need not be equal.

3.1.**40** For any $\vec{x}$ in $\mathbb{R}^m$, the vector $B\vec{x}$ is in $\text{im}(B) = \ker(A)$, so that $AB\vec{x} = \vec{0}$. If we apply this fact to $\vec{x} = \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m$, we find that all the columns of the matrix AB are zero, so that $AB = 0$.

3.1.**41** a $\text{rref}(A) = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 0 \end{bmatrix}$, so that $\ker(A) = \text{span}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

$\text{im}(A) = \text{span}\begin{bmatrix} 0.36 \\ 0.48 \end{bmatrix} = \text{span}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Note that $\text{im}(A)$ and $\ker(A)$ are perpendicular lines.

b $A^2 = A$

If $\vec{v}$ is in $\text{im}(A)$, with $\vec{v} = A\vec{x}$, then $A\vec{v} = A^2\vec{x} = A\vec{x} = \vec{v}$.

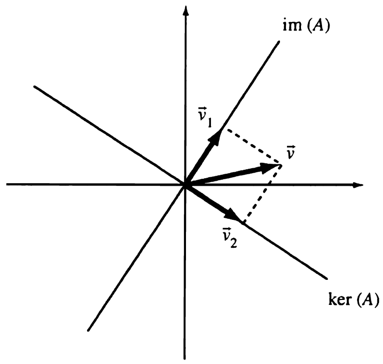

Figure 3.3: for Problem 3.1.41c.

c Any vector $\vec{v}$ in $\mathbb{R}^2$ can be written uniquely as $\vec{v} = \vec{v}_1 + \vec{v}_2$, where $\vec{v}_1$ is in $\text{im}(A)$ and $\vec{v}_2$ is in $\ker(A)$. (See Figure 3.3.) Then $A\vec{v} = A\vec{v}_1 + A\vec{v}_2 = \vec{v}_1(A\vec{v}_1 = \vec{v}_1$ by part b, $A\vec{v}_2 = \vec{0}$ since $\vec{v}_2$ is in $\ker(A))$, so that A represents the *orthogonal projection* onto $\text{im}(A) = \text{span}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

3.1.**42** Using the hint, we see that the vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ is in the image of A if

$$\begin{array}{ccccl} y_1 & & -3y_3 & +2y_4 & = 0 \quad \text{and} \\ & y_2 & -2y_3 & +y_4 & = 0. \end{array}$$

This means that $\text{im}(A)$ is the kernel of the matrix $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$.

3.1.**43** Using our work in Exercise 42 as a guide, we come up with the following procedure to express the image of an $n \times m$ matrix A as the kernel of a matrix B:

If $\text{rank}(A) = n$, let B be the $n \times n$ zero matrix.

If $r = \text{rank}(A) < n$, let B be the $(n-r) \times n$ matrix obtained by omitting the first r rows and the first m columns of $\text{rref}[A \vdots I_n]$.

3.1.**44** a Yes; by construction of the echelon form, the systems $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solutions (it is the whole point of Gaussian elimination not to change the solutions of a system).

b No; as a counterexample, consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, with $\text{im}(A) = \text{span}(\vec{e}_2)$, but $B = \text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, with $\text{im}(B) = \text{span}(\vec{e}_1)$.

3.1.**45** As we solve the system $A\vec{x} = \vec{0}$, we obtain r leading variables and $m - r$ free variables. The "general vector" in $\ker(A)$ can be written as a linear combination of $m - r$ vectors, with the free variables as coefficients. (See Example 11, where $m - r = 5 - 3 = 2$.)

3.1.**46** If rank$(A) = r$, then im$(A) = \text{span}(\vec{e}_1, \ldots, \vec{e}_r)$. See Figure 3.4.

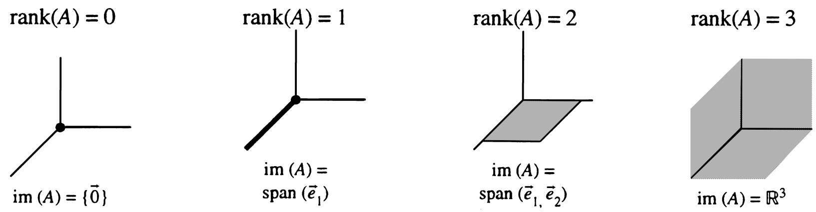

Figure 3.4: for Problem 3.1.46.

3.1.**47** $\text{im}(T) = L_2$ and $\ker(T) = L_1$.

3.1.**48** a $\vec{w} = A\vec{x}$, for some $\vec{x}$, so that $A\vec{w} = A^2\vec{x} = A\vec{x} = \vec{w}$.

b If rank$(A) = 2$, then A is invertible, and the equation $A^2 = A$ implies that $A = I_2$ (multiply by A^{-1}).

If rank$(A) = 0$ then $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

c First note that im(A) and ker(A) are lines (there is one nonleading variable).

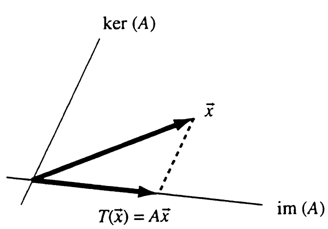

Figure 3.5: for Problem 3.1.48c.

By definition of a projection, we need to verify that $\vec{x} - A\vec{x}$ is in ker(A). This is indeed the case, since

$A(\vec{x} - A\vec{x}) = A\vec{x} - A^2\vec{x} = A\vec{x} - A\vec{x} = \vec{0}$ (we are told that $A^2 = A$). See Figure 3.5.

3.1.**49** If $\vec{v}$ and $\vec{w}$ are in ker(T), then $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) = \vec{0} + \vec{0} = \vec{0}$, so that $\vec{v} + \vec{w}$ is in ker(T) as well.

If $\vec{v}$ is in ker(T) and k is an arbitrary scalar, then $T(k\vec{v}) = kT(\vec{v}) = k\vec{0} = \vec{0}$, so that $k\vec{v}$ is in ker(T) as well.

3.1.**50** From Exercise 38 we know that $\ker(A^3) \subseteq \ker(A^4)$. Conversely, if $\vec{x}$ is in $\ker(A^4)$, then $A^4\vec{x} = A^3(A\vec{x}) = \vec{0}$, so that $A\vec{x}$ is in $\ker(A^3) = \ker(A^2)$, which implies that $A^2(A\vec{x}) = A^3\vec{x} = \vec{0}$, that is, $\vec{x}$ is in $\ker(A^3)$. We have shown that $\ker(A^3) = \ker(A^4)$.

3.1.**51** We need to find all $\vec{x}$ such that $AB\vec{x} = \vec{0}$. If $AB\vec{x} = \vec{0}$, then $B\vec{x}$ is in $\ker(A) = \{\vec{0}\}$, so that $B\vec{x} = \vec{0}$.

Since $\ker(B) = \{\vec{0}\}$, we can conclude that $\vec{x} = \vec{0}$. It follows that $\ker(AB) = \{\vec{0}\}$.

3.1.**52** Since $C\vec{x} = \begin{bmatrix} A \\ B \end{bmatrix} \vec{x} = \begin{bmatrix} A\vec{x} \\ B\vec{x} \end{bmatrix}$, we can conclude that $C\vec{x} = \vec{0}$ if (and only if) both $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$. It follows that $\ker(C)$ is the intersection of $\ker(A)$ and $\ker(B)$: $\ker(C) = \ker(A) \cap \ker(B)$.

3.1.**53** a Using the equation $1 + 1 = 0$ (or $-1 = 1$), we can write the general vector $\vec{x}$ in $\ker(H)$ as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} p+r+s \\ p+q+s \\ p+q+r \\ p \\ q \\ r \\ s \end{bmatrix}$$

$$= p \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \quad & \uparrow & \quad & \uparrow & \quad & \uparrow \\ \vec{v}_1 & & \vec{v}_2 & & \vec{v}_3 & & \vec{v}_4 \end{matrix}$$

b $\ker(H) = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ by part (a), and $\operatorname{im}(M) = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ by Theorem 3.1.3, so that $\operatorname{im}(M) = \ker(H)$. $M\vec{x}$ is in $\operatorname{im}(M) = \ker(H)$, so that $H(M\vec{x}) = \vec{0}$.

3.1.**54** a If no error occurred, then $\vec{w} = \vec{v} = M\vec{u}$, and $H\vec{w} = H(M\vec{u}) = \vec{0}$, by Exercise 53b.

If an error occurred in the ith component, then $\vec{w} = \vec{v} + \vec{e}_i = M\vec{u} + \vec{e}_i$, so that

$H\vec{w} = H(M\vec{u}) + H\vec{e}_i = i$th column of H.

Since the columns of H are all different, this method allows us to find out where an error occurred.

b $H\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ = seventh column of H: an error occurred in the seventh component of $\vec{v}$.

Therefore $\vec{v} = \vec{w} + \vec{e}_7 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Section 3.2

3.2.**1** Not a subspace, since W does not contain the zero vector.