Chapter 7

Section 7.1

7.1.1 If $\vec{v}$ is an eigenvector of A, then $A\vec{v} = \lambda\vec{v}$.

Hence $A^3\vec{v} = A^2(A\vec{v}) = A^2(\lambda\vec{v}) = A(A\lambda\vec{v}) = A(\lambda A\vec{v}) = A(\lambda^2\vec{v}) = \lambda^2 A\vec{v} = \lambda^3\vec{v}$, so $\vec{v}$ is an eigenvector of A^3 with eigenvalue λ^3.

7.1.2 We know $A\vec{v} = \lambda\vec{v}$ so $\vec{v} = A^{-1}A\vec{v} = A^{-1}\lambda\vec{v} = \lambda A^{-1}\vec{v}$, so $\vec{v} = \lambda A^{-1}\vec{v}$ or $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$.

Hence $\vec{v}$ is an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

7.1.3 We know $A\vec{v} = \lambda\vec{v}$, so $(A + 2I_n)\vec{v} = A\vec{v} + 2I_n\vec{v} = \lambda\vec{v} + 2\vec{v} = (\lambda + 2)\vec{v}$, hence $\vec{v}$ is an eigenvector of $(A + 2I_n)$ with eigenvalue $\lambda + 2$.

7.1.4 We know $A\vec{v} = \lambda\vec{v}$, so $7A\vec{v} = 7\lambda\vec{v}$, hence $\vec{v}$ is an eigenvector of $7A$ with eigenvalue 7λ.

7.1.5 Assume $A\vec{v} = \lambda\vec{v}$ and $B\vec{v} = \beta\vec{v}$ for some eigenvalues λ, β. Then $(A + B)\vec{v} = A\vec{v} + B\vec{v} = \lambda\vec{v} + \beta\vec{v} = (\lambda + \beta)\vec{v}$ so $\vec{v}$ is an eigenvector of $A + B$ with eigenvalue $\lambda + \beta$.

7.1.6 Yes. If $A\vec{v} = \lambda\vec{v}$ and $B\vec{v} = \mu\vec{v}$, then $AB\vec{v} = A(\mu\vec{v}) = \mu(A\vec{v}) = \mu\lambda\vec{v}$

7.1.7 We know $A\vec{v} = \lambda\vec{v}$ so $(A - \lambda I_n)\vec{v} = A\vec{v} - \lambda I_n\vec{v} = \lambda\vec{v} - \lambda\vec{v} = \vec{0}$ so a nonzero vector $\vec{v}$ is in the kernel of $(A - \lambda I_n)$ so $\ker(A - \lambda I_n) \neq \{\vec{0}\}$ and $A - \lambda I_n$ is not invertible.

7.1.8 We want all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ hence $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, i.e. the desired matrices must have the form $\begin{bmatrix} 5 & b \\ 0 & d \end{bmatrix}$.

7.1.9 We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for any λ. Hence $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}$, i.e., the desired matrices must have the form $\begin{bmatrix} \lambda & b \\ 0 & d \end{bmatrix}$, they must be upper triangular.

7.1.10 We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, i.e. the desired matrices must have the form $\begin{bmatrix} 5 - 2b & b \\ 10 - 2d & d \end{bmatrix}$.

7.1.11 We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$. So, $2a + 3b = -2$ and $2c + 3d = -3$. Thus, $b = \frac{-2-2a}{3}$, and $d = \frac{-3-2c}{3}$. So all matrices of the form $\begin{bmatrix} a & \frac{-2-2a}{3} \\ c & \frac{-3-2c}{3} \end{bmatrix}$ will fit.

7.1.12 Solving $\begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ we get $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} t \\ -\frac{3}{2}t \end{bmatrix}$ (with $t \neq 0$) and

solving $\begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ we get $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ (with $t \neq 0$).

An eigenbasis for A is $\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We can diagonalize A with $S = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

7.1.**13** Solving $\begin{bmatrix} -6 & 6 \\ -15 & 13 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, we get $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5}t \\ t \end{bmatrix}$ (with $t \neq 0$).

7.1.**14** We want to find all 4×4 matrices A such that $A\vec{e}_2 = \lambda\vec{e}_2$, i.e. the second column of A must be of the form $\begin{bmatrix} 0 \\ \lambda \\ 0 \\ 0 \end{bmatrix}$, so $A = \begin{bmatrix} a & 0 & c & d \\ e & \lambda & f & g \\ h & 0 & i & j \\ k & 0 & l & m \end{bmatrix}$.

7.1.**15** Any vector on L is unaffected by the reflection, so that a nonzero vector on L is an eigenvector with eigenvalue 1. Any vector on $L^\perp$ is flipped about L, so that a nonzero vector on $L^\perp$ is an eigenvector with eigenvalue -1. Picking a nonzero vector from L and one from $L^\perp$, we obtain an eigenbasis. This transformation is diagonalizable.

7.1.**16** Rotation by 180° is a flip about the origin so every nonzero vector is an eigenvector with the eigenvalue -1. Any basis for $\mathbb{R}^2$ is an eigenbasis. This transformation is diagonalizable.

7.1.**17** No (real) eigenvalues

7.1.**18** Any nonzero vector in the plane is unchanged, hence is an eigenvector with the eigenvalue 1. Since any nonzero vector in $V^\perp$ is flipped about the origin, it is an eigenvector with eigenvalue -1. Pick any two noncollinear vectors from V and a nonzero vector from $V^\perp$ to form an eigenbasis. This transformation is diagonalizable.

7.1.**19** Any nonzero vector in L is an eigenvector with eigenvalue 1, and any nonzero vector in the plane $L^\perp$ is an eigenvector with eigenvalue 0. Form an eigenbasis by picking any nonzero vector in L and any two nonparallel vectors in $L^\perp$. This transformation is diagonalizable.

7.1.**20** Any nonzero vector along the $\vec{e}_3$-axis is unchanged, hence is an eigenvector with eigenvalue 1. No other (real) eigenvalues can be found. There is no eigenbasis. This transformation fails to be diagonalizable.

7.1.**21** Any nonzero vector in $\mathbb{R}^3$ is an eigenvector with eigenvalue 5. Any basis for $\mathbb{R}^3$ is an eigenbasis. This transformation is diagonalizable.

7.1.**22** Any nonzero scalar multiple of $\vec{v}$ is an eigenvector with eigenvalue 1. There is no eigenbasis. This transformation fails to be diagonalizable.

7.1.**23** Denote the columns of S with $\vec{v}_1, ..., \vec{v}_n$, and let the diagonal entries of B be $\lambda_1, ..., \lambda_n$. The given equation $S^{-1}AS = B$ implies that $AS = SB$. Now the i^{th} column of AS is $AS\vec{e}_i = A\vec{v}_i$, and the i^{th} column of SB is $SB\vec{e}_i = S(\lambda_i\vec{e}_i) = \lambda_i(S\vec{e}_i) = \lambda_i\vec{v}_i$, so that $A\vec{v}_i = \lambda_i\vec{v}_i$ as claimed. (Since S is invertible, the vectors $\vec{v}_1, ..., \vec{v}_n$ will form a basis of $\mathbb{R}^n$.)

7.1.**24** See Figure 7.1.

7.1.**25** See Figure 7.2.

7.1.**26** See Figure 7.3.

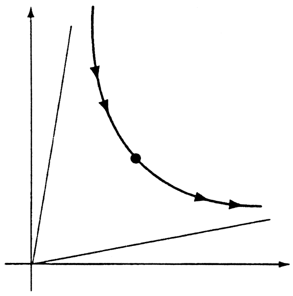


Figure 7.1: for Problem 7.1.24.

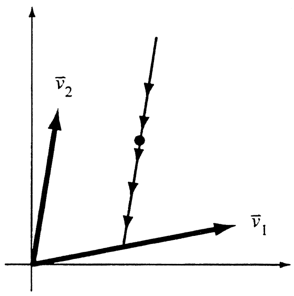


Figure 7.2: for Problem 7.1.25.

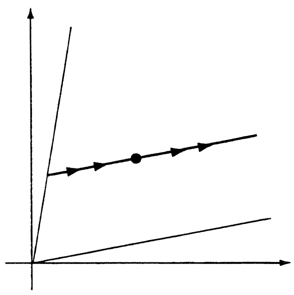


Figure 7.3: for Problem 7.1.26.

7.1.**27** See Figure 7.4.

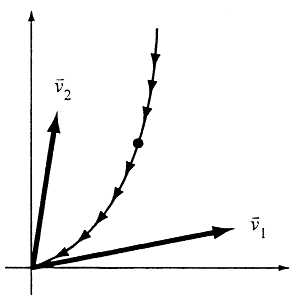


Figure 7.4: for Problem 7.1.27.

7.1.**28** See Figure 7.5.

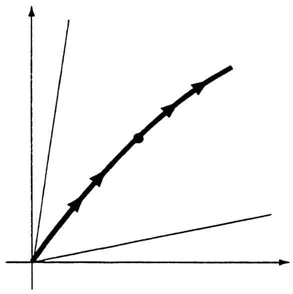


Figure 7.5: for Problem 7.1.28.

7.1.**29** See Figure 7.6.

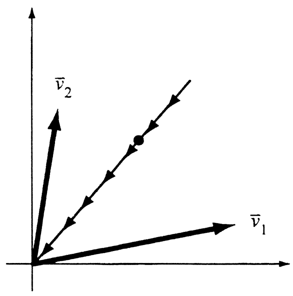


Figure 7.6: for Problem 7.1.29.

7.1.**30** Since the matrix is diagonal, $\vec{e}_1$ and $\vec{e}_2$ are eigenvectors. See Figure 7.7.

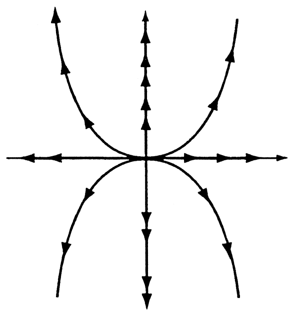


Figure 7.7: for Problem 7.1.30.

7.1.**31** See Figure 7.8.

7.1.**32** Since the matrix is diagonal, $\vec{e}_1$ and $\vec{e}_2$ are eigenvectors. See Figure 7.9.

7.1.**33** We are given that $\vec{x}(t) = 2^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 6^t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, hence we know that the eigenvalues are 2 and 6 with corresponding

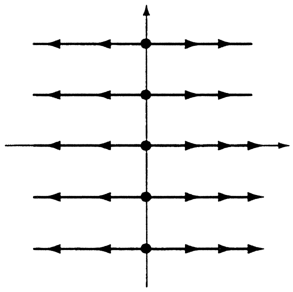


Figure 7.8: for Problem 7.1.31.

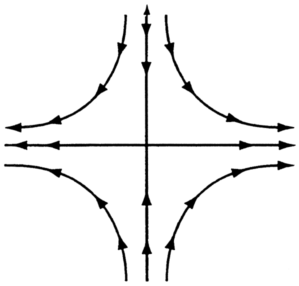


Figure 7.9: for Problem 7.1.32.

eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively (see Theorem 7.1.6), so we want a matrix A such that $A\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ 2 & 6 \end{bmatrix}$. Multiplying on the right by $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$, we get $A = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$.

7.1.**34** $(A^2 + 2A + 3I_n)\vec{v} = A^2\vec{v} + 2A\vec{v} + 3I_n\vec{v} = 4^2\vec{v} + 2 \cdot 4\vec{v} + 3\vec{v} = (16 + 8 + 3)\vec{v} = 27\vec{v}$ so $\vec{v}$ is an eigenvector of $A^2 + 2A + 3I_n$ with eigenvalue 27.

7.1.**35** Let λ be an eigenvalue of $S^{-1}AS$. Then for some nonzero vector $\vec{v}$, $S^{-1}AS\vec{v} = \lambda\vec{v}$, i.e., $AS\vec{v} = S\lambda\vec{v} = \lambda S\vec{v}$ so λ is an eigenvalue of A with eigenvector $S\vec{v}$.

Conversely, if α is an eigenvalue of A with eigenvector $\vec{w}$, then $A\vec{w} = \alpha\vec{w}$, for some nonzero $\vec{w}$.

Therefore, $S^{-1}AS(S^{-1}\vec{w}) = S^{-1}A\vec{w} = S^{-1}\alpha\vec{w} = \alpha S^{-1}\vec{w}$, so $S^{-1}\vec{w}$ is an eigenvector of $S^{-1}AS$ with eigenvalue α.

7.1.**36** We want A such that $A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, i.e. $A\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ 5 & 20 \end{bmatrix}$, so

$$A = \begin{bmatrix} 15 & 10 \\ 5 & 20 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 3 \\ -2 & 11 \end{bmatrix}.$$

7.1.**37** a $A = 5\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$ is a scalar multiple of an orthogonal matrix. By Theorem 7.1.2, the possible eigenvalues of the orthogonal matrix are ± 1, so that the possible eigenvalues of A are ± 5. In part b we see that both are indeed eigenvalues.

b Solve $A\vec{v} = \pm 5\vec{v}$ to get the eigenbasis $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

c We can diagonalize A with $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$.

7.1.**38** $\begin{bmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. The associated eigenvalue is 2.

7.1.**39** We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$. So $b = 0$, and $d = \lambda$ (for any λ). Thus, we need matrices of the form $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

So, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is a basis of V, and $\dim(V) = 3$.

7.1.**40** We need all matrices A such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} \lambda \\ -3\lambda \end{bmatrix}$.

Thus, $a - 3b = \lambda$ and $c - 3d = -3\lambda$. Thus, $c - 3d = -3(a - 3b) = -3a + 9b$, or $c = -3a + 9b + 3d$.

So A must be of the form $\begin{bmatrix} a & b \\ -3a + 9b + 3d & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ -3 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}$. Thus, a basis of V is $\begin{bmatrix} 1 & 0 \\ -3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}$, and the dimension of V is 3.

7.1.**41** We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So, $a + b = \lambda_1 = c + d$ and $a + 2b = \lambda_2$ and $2\lambda_2 = c + 2d$.

So $(a+2b)-(a+b) = \lambda_2 - \lambda_1 = b, a = \lambda_1 - b = 2\lambda_1 - \lambda_2$. Also, $(c+2d)-(c+d) = 2\lambda_2 - \lambda_1 = d, c = \lambda_1 - d = 2\lambda_1 - 2\lambda_2$. So A must be of the form: $\begin{bmatrix} 2\lambda_1 - \lambda_2 & \lambda_2 - \lambda_1 \\ 2\lambda_1 - 2\lambda_2 & 2\lambda_2 - \lambda_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$.

So a basis of V is $\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$, and $\dim(V) = 2$.

7.1.**42** We will do this in a slightly simpler manner than Exercise 40. Since $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is simply the first column of A, the first column must be a multiple of $\vec{e}_1$. Similarly, the third column must be a multiple of $\vec{e}_3$. There are no other restrictions on the form of A, meaning it can be any matrix of the form $\begin{bmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & d & e \end{bmatrix} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Thus, a basis of V is $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$ and the dimension of V is 5.

7.1.**43** $A = AI_n = A[\vec{e}_1 \quad \dots \quad \vec{e}_n] = [\lambda_1\vec{e}_1 \quad \dots \quad \lambda_n\vec{e}_n]$, where the eigenvalues $\lambda_1, \dots, \lambda_n$ are arbitrary. Thus A can be any diagonal matrix, and $\dim(V) = n$.

7.1.**44** We see that each of the columns 1 through m of A will have to be a multiple of its respective vector $\vec{e}_i$. Thus, there will be m free variables in the first m columns. The remaining $n-m$ columns will each have n free variables. Thus, in total, the dimension of V is $m + (n-m)n = m + n^2 - nm$.

7.1.**45** Consider a vector $\vec{w}$ that is not parallel to $\vec{v}$. We want $A[\vec{v}\ \vec{w}] = [\lambda\vec{v}\ a\vec{v} + b\vec{w}]$, where λ, a and b are arbitrary constants. Thus the matrices A in V are of the form $A = [\lambda\vec{v}\ a\vec{v} + b\vec{w}][\vec{v}\ \vec{w}]^{-1}$. Using Summary 4.1.6, we see that $[\vec{v}\ \vec{0}][\vec{v}\ \vec{w}]^{-1}, [\vec{0}\ \vec{v}][\vec{v}\ \vec{w}]^{-1}, [\vec{0}\ \vec{w}][\vec{v}\ \vec{w}]^{-1}$ is a basis of V, so that $\dim(V) = 3$.

7.1.**46** Since $A\vec{v} = 3\vec{v}$, we have $\vec{v} = \frac{1}{3}A\vec{v} = A\left(\frac{1}{3}\vec{v}\right)$, so that $\vec{v}$ is in the image of A, as claimed.

7.1.**47** We know that $A\vec{v} = \lambda\vec{v}$ for some λ. If $\lambda = 0$, then $A\vec{v} = \lambda\vec{v} = \vec{0}$, so that $\vec{v}$ is in the kernel of A. If $\lambda \neq 0$, then $\vec{v} = \frac{1}{\lambda}A\vec{v} = A\left(\frac{1}{\lambda}\vec{v}\right)$, and $\vec{v}$ is in the image of A.

7.1.**48** If $\vec{v}$ is any nonzero vector in the image of A, then we can write $\mathrm{Im}A = \mathrm{span}(\vec{v})$ since $1 = \mathrm{rank}\ A = \dim(\mathrm{Im}A)$. Now $A\vec{v}$ is in $\mathrm{Im}A = \mathrm{span}(\vec{v})$ as well, so that $A\vec{v} = \lambda\vec{v}$ for some λ, as claimed.

7.1.**49** For example, the matrix $A = \begin{bmatrix}0&1\\0&0\end{bmatrix}$, with rank $A = 1$, fails to be diagonalizable. Since eigenvectors of A must be in the image or in the kernel of A (by Exercise 47), the only eigenvectors of A are of the form $\begin{bmatrix}a\\0\end{bmatrix}$, where $a \neq 0$. Thus there is no eigenbasis for A.

7.1.**50** We observe that rank $A = 1$, and we will use the ideas presented in Exercises 47 and 48. Now $\ker A =$ span $\begin{bmatrix}-3\\1\end{bmatrix}$ and $ImA =$ span $\begin{bmatrix}1\\2\end{bmatrix}$, with $A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}7\\14\end{bmatrix} = 7\begin{bmatrix}1\\2\end{bmatrix}$. Thus A is diagonalizable, with $S = \begin{bmatrix}-3&1\\1&2\end{bmatrix}$ and $B = \begin{bmatrix}0&0\\0&7\end{bmatrix}$.

7.1.**51** We observe that rank $A = 1$, and we will use the ideas presented in Exercises 47 and 48. Now $\ker A =$ span $\begin{bmatrix}-1\\1\end{bmatrix}$ and $\mathrm{Im}A =$ span $\begin{bmatrix}1\\1\end{bmatrix}$, with $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix}$. Thus A is diagonalizable, with $S = \begin{bmatrix}-1&1\\1&1\end{bmatrix}$ and $B = \begin{bmatrix}0&0\\0&2\end{bmatrix}$.

7.1.**52** We observe that rank $A = 1$, and we will use the ideas presented in Exercises 47 and 48. Now $\ker A =$ span $\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right)$ and $\mathrm{Im}A =$ span $\begin{bmatrix}1\\1\\1\end{bmatrix}$, with $A\begin{bmatrix}1\\1\\1\end{bmatrix} = 3\begin{bmatrix}1\\1\\1\end{bmatrix}$. Thus A is diagonalizable, with $S = \begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}$ and $B = \begin{bmatrix}0&0&0\\0&0&0\\0&0&3\end{bmatrix}$. (Verify that the column vectors of S are linearly independent.)

7.1.53 We observe that rank $A = 1$, and we will use the ideas presented in Exercises 47 and 48. Now $\ker A = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right)$ and $\operatorname{Im}A = \operatorname{span}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, with $A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 14\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Thus A is diagonalizable, with $S = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 14 \end{bmatrix}$. (Verify that the column vectors of S are linearly independent.)

7.1.54 We observe that rank $A = 1$, and we will use the ideas presented in Exercises 47 and 48. Now $\ker A = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$ and $\operatorname{Im}A = \operatorname{span}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, with $A\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Thus A is diagonalizable, with $S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. (Verify that the column vectors of S are linearly independent.)

7.1.55 As in Example 3, we can pick a nonzero vector on L, for example, $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, and a nonzero vector perpendicular to L, for example, $\vec{w} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$. Now $\vec{v}, \vec{w}$ is an eigenbasis, with $A\vec{v} = \vec{v} = 1\vec{v}$ and $A\vec{w} = \vec{0} = 0\vec{w}$. Thus A is diagonalizable, with $S = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

7.1.56 Adapting the ideas developed in Example 3 and Exercise 55, we can pick a nonzero vector on L, for example, $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and a nonzero vector perpendicular to L, for example, $\vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Now $\vec{v}, \vec{w}$ is an eigenbasis, with $A\vec{v} = \vec{v} = 1\vec{v}$ and $A\vec{w} = -\vec{w} = (-1)\vec{w}$. Thus A is diagonalizable, with $S = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

7.1.57 Notice that the matrix A is of the form $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, with $a^2 + b^2 = 1$, so that it represents the reflection about a line L. Solving the equation $A\vec{x} = \vec{x}$, we find that $L = \operatorname{span}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. To construct an eigenbasis $\vec{v}, \vec{w}$, we can pick a nonzero vector on L, for example, $\vec{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, and a nonzero vector perpendicular to L, for example, $\vec{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Now$A\vec{v} = \vec{v} = 1\vec{v}$ and $A\vec{w} = -\vec{w} = (-1)\vec{w}$. Thus A is diagonalizable, with $S = \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

7.1.58 The matrix A represents the reflection about the line $L = \operatorname{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Using the approach outlined in Exercise 57, we find that A is diagonalizable, with $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

7.1.59 Adapting the ideas developed in Example 3, we can pick two linearly independent vectors on the given plane E, for example, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. (Use the given equation to find those by inspection.) A vector perpendicular to E is $\vec{u} = \vec{v} \times \vec{w} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$. Now $\vec{u}, \vec{v}, \vec{w}$ is an eigenbasis, with $A\vec{u} = 0\vec{u}$, $A\vec{v} = 1\vec{v}$ and $A\vec{w} = 1\vec{w}$. Thus A is diagonalizable, with $S = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & 1 \\ -2 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

7.1.**60** Adapting the ideas developed in Example 3 and Exercise 59, we find the eigenbasis $\vec{u} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, with $A\vec{u} = (-1)\vec{u}$, $A\vec{v} = 1\vec{v}$ and $A\vec{w} = 1\vec{w}$. Thus A is diagonalizable, with $S = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & 1 \\ -2 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

7.1.**61** Solving the linear system $A\vec{x} = \vec{x}$, we see that the plane E is given by the equation $x+2y+3z = 0$. Proceeding as in Exercise 59, we construct the eigenbasis $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. Now $A\vec{u} = (-1)\vec{u}$, $A\vec{v} = 1\vec{v}$ and $A\vec{w} = 1\vec{w}$. Thus A is diagonalizable, with $S = \begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

7.1.**62** A basis of $E = \text{Im}A$ is $\vec{v} = \begin{bmatrix} 13 \\ -2 \\ -3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -2 \\ 10 \\ -6 \end{bmatrix}$, corresponding to the first two columns of matrix A. A vector perpendicular to E is $\vec{v} \times \vec{w} = \begin{bmatrix} 42 \\ 84 \\ 126 \end{bmatrix}$; we scale this vector down and let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Now $\vec{u}, \vec{v}, \vec{w}$ is an eigenbasis, with $A\vec{u} = 0\vec{u}$, $A\vec{v} = 1\vec{v}$ and $A\vec{w} = 1\vec{w}$. Thus A is diagonalizable, with $S = \begin{bmatrix} 1 & 13 & -2 \\ 2 & -2 & 10 \\ 3 & -3 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

7.1.**63** As in Exercise 53, we find the eigenbasis $\vec{u} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. We have $A\vec{u} = 0\vec{u}$, $A\vec{v} = 0\vec{v}$, and $A\vec{w} = 1\vec{w}$. Thus A is diagonalizable, with $S = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The matrix A represents the orthogonal projection onto the line $L = \text{span} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

7.1.**64** a We need all matrices A such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} k \\ 2k \end{bmatrix}$.

Thus, $a + 2b = k$ and $c + 2d = 2k$. So, $c + 2d = 2a + 4b$, or $c = -2d + 2a + 4b$ and A must be of the form $\begin{bmatrix} a & b \\ -2d+2a+4b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$. So a basis of V is $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$, and the dimension of V is 3.

b Clearly $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a basis of the image of T by definition of V, so that the rank of T is 1. The kernel of T consists of all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{0}$, or $a + 2b = 0, c + 2d = 0$. These are the matrices of the form $\begin{bmatrix} -2b & b \\ -2d & d \end{bmatrix} = b\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$. Thus a basis of the kernel of T is $\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$.

c Let's find the kernel of L first. In part (a) we saw that the matrices in V are of the form $A = \begin{bmatrix} a & b \\ -2d+2a+4b & d \end{bmatrix}$. A matrix A in V is in the kernel of L if $\begin{bmatrix} a & b \\ -2d+2a+4b & d \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \vec{0}$, or $a + 3b = 0, 2a + 4b + d = 0$. This system simplifies to $a = -3b$ and $d = 2b$, so that the matrices in the kernel of L are of the form $\begin{bmatrix} -3b & b \\ -6b & 2b \end{bmatrix} = b\begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix}$. The matrix $\begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix}$ forms a basis of the kernel of L. By the rank-nullity theorem, the rank of L is $\dim(V) - \dim(\ker L) = 3 - 1 = 2$, and the image of L is all of $\mathbb{R}^2$.

7.1.**65** Suppose V is a one-dimensional A-invariant subspace of $\mathbb{R}^n$, and $\vec{v}$ is a non-zero vector in V. Then $A\vec{v}$ will be in V, so that $A\vec{v} = \lambda\vec{v}$ for some λ, and $\vec{v}$ is an eigenvector of A. Conversely, if $\vec{v}$ is any eigenvector of A, then $V = \text{span}(\vec{v})$ will be a one-dimensional A-invariant subspace. Thus the one-dimensional A-invariant subspaces V are of the form $V = \text{span}(\vec{v})$, where $\vec{v}$ is an eigenvector of A.

7.1.**66** a Since $\text{span}(\vec{e}_1)$ is an A-invariant subspace of $\mathbb{R}^3$, it must be that $\vec{e}_1$ is an eigenvector of A, as revealed in Exercise 65. Thus, the first column of A must be of the form $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$. Since $\text{span}(\vec{e}_1, \vec{e}_2)$ is also an A-invariant subspace, it must be that $A\vec{e}_2$ is in $\text{span}(\vec{e}_1, \vec{e}_2)$. Thus, the second column of A must have the form $\begin{bmatrix} b \\ c \\ 0 \end{bmatrix}$. The third column may be any vector in $\mathbb{R}^3$. Thus, we can choose $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ to maximize the number of non-zero entries.

b We see, from our construction above, that upper-triangular matrices fit this description. This space, V consists of all matrices of the form $\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}$ and has a dimension of 6.

7.1.**67** The eigenvalues of the system are $\lambda_1 = 1.1$, and $\lambda_2 = 0.9$ and corresponding eigenvectors are $\vec{v}_1 = \begin{bmatrix} 100 \\ 300 \end{bmatrix}$

and $\vec{v}_2 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$, respectively. So if $\vec{x}_0 = \begin{bmatrix} 100 \\ 800 \end{bmatrix}$, we can see that $\vec{x}_0 = 3\vec{v}_1 - \vec{v}_2$. Therefore, by Theorem 7.1.6, we have $\vec{x}(t) = 3(1.1)^t \begin{bmatrix} 100 \\ 300 \end{bmatrix} - (0.9)^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}$, i.e. $c(t) = 300(1.1)^t - 200(0.9)^t$ and $r(t) = 900(1.1)^t - 100(0.9)^t$.

7.1.**68** a $\vec{v}(0) = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, and we see that $A\vec{v}(0) = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 200 \\ 200 \end{bmatrix} = 2 \begin{bmatrix} 100 \\ 100 \end{bmatrix}$. So, $\vec{v}(t) = A^t \vec{v}(0) = A^t \begin{bmatrix} 100 \\ 100 \end{bmatrix} = 2^t \begin{bmatrix} 100 \\ 100 \end{bmatrix}$.

So $c(t) = r(t) = 100(2)^t$.

b $\vec{v}(0) = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$, and we see that $A\vec{v}(0) = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 600 \\ 300 \end{bmatrix} = 3 \begin{bmatrix} 200 \\ 100 \end{bmatrix}$. So, $\vec{v}(t) = A^t \vec{v}(0) = A^t \begin{bmatrix} 200 \\ 100 \end{bmatrix} = 3^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}$.

So $c(t) = 200(3)^t$ and $r(t) = 100(3)^t$.

c $\vec{v}(0) = \begin{bmatrix} 600 \\ 500 \end{bmatrix}$. We can write this in terms of the previous eigenvectors as $\vec{v}(0) = 4 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 200 \\ 100 \end{bmatrix}$. So, $\vec{v}(t) = A^t \vec{v}(0) = A^t 4 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + A^t \begin{bmatrix} 200 \\ 100 \end{bmatrix} = 4(2)^t \begin{bmatrix} 100 \\ 100 \end{bmatrix} + (3)^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}$.

So $c(t) = 400(2)^t + 200(3)^t$ and $r(t) = 400(2)^t + 100(3)^t$.

7.1.**69** a $\vec{v}(0) = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$, and we see that $A\vec{v}(0) = \begin{bmatrix} 0 & .75 \\ -1.5 & 2.25 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 150 \\ 300 \end{bmatrix} = 1.5 \begin{bmatrix} 100 \\ 200 \end{bmatrix}$. So, $\vec{v}(t) = A^t \vec{v}(0) = A^t \begin{bmatrix} 100 \\ 200 \end{bmatrix} = (1.5)^t \begin{bmatrix} 100 \\ 200 \end{bmatrix}$.

So $c(t) = 100(1.5)^t$ and $r(t) = 200(1.5)^t$.

b $\vec{v}(0) = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, and we see that $A\vec{v}(0) = \begin{bmatrix} 0 & .75 \\ -1.5 & 2.25 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 75 \\ 75 \end{bmatrix} = .75 \begin{bmatrix} 100 \\ 100 \end{bmatrix}$. So, $\vec{v}(t) = A^t \vec{v}(0) = A^t \begin{bmatrix} 100 \\ 100 \end{bmatrix} = (.75)^t \begin{bmatrix} 100 \\ 100 \end{bmatrix}$.

So $c(t) = 100(.75)^t$ and $r(t) = 100(.75)^t$.

c $\vec{v}(0) = \begin{bmatrix} 500 \\ 700 \end{bmatrix}$. We can write this in terms of the previous eigenvectors as $\vec{v}(0) = 3 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + 2 \begin{bmatrix} 100 \\ 200 \end{bmatrix}$. So, $\vec{v}(t) = A^t \vec{v}(0) = A^t 3 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + A^t 2 \begin{bmatrix} 100 \\ 200 \end{bmatrix} = 3(.75)^t \begin{bmatrix} 100 \\ 100 \end{bmatrix} + 2(1.5)^t \begin{bmatrix} 100 \\ 200 \end{bmatrix}$.

So $c(t) = 300(.75)^t + 200(1.5)^t$ and $r(t) = 300(.75)^t + 400(1.5)^t$.

7.1.**70** a $\begin{bmatrix} 0.978 & -0.006 \\ 0.004 & 0.992 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.99 \\ 1.98 \end{bmatrix} = 0.99 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 0.978 & -0.006 \\ 0.004 & 0.992 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2.94 \\ -0.98 \end{bmatrix} = 0.98 \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. The eigenvalues are $\lambda_1 = 0.99$ and $\lambda_2 = 0.98$.

b $\vec{x}_0 = \begin{bmatrix} g_0 \\ l_0 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} = 20 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 40 \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ so $\vec{x}(t) = 20(0.99)^t \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 40(0.98)^t \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, hence

$g(t) = -20(0.99)^t + 120(0.98)^t$ and $h(t) = 40(0.99)^t - 40(0.98)^t$.

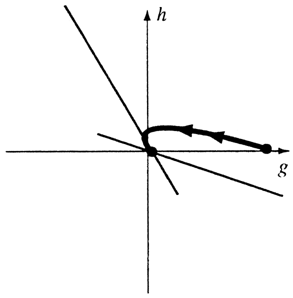


Figure 7.10: for Problem 7.1.70b.

$h(t)$ first rises, then falls back to zero. $g(t)$ falls a little below zero, then goes back up to zero. See Figure 7.10.

c We set $g(t) = -20(0.99)^t + 120(0.98)^t = 0$.

Solving for t we get that $g(t) = 0$ for $t \approx 176$ minutes. (After $t = 176$, $g(t) < 0$).

7.1.**71** a $\vec{v}(0) = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

So, $\vec{v}(t) = A^t \vec{v}(0) = A^t \left(3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$

$= 3A^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2A^t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + A^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 3\lambda_1^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\lambda_2^t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \lambda_3^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$= 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2(-\frac{1}{2})^t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (-\frac{1}{2})^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

So $a(t) = 3 + 3(-\frac{1}{2})^t$, $b(t) = 3 - 2(-\frac{1}{2})^t$ and $c(t) = 3 - (-\frac{1}{2})^t$.

b $a(365) = 3 + 3(-\frac{1}{2})^{365} = 3 - \frac{3}{2^{365}}$, $b(365) = 3 - 2(-\frac{1}{2})^{365} = 3 + \frac{1}{2^{364}}$ and

$c(365) = 3 - (-\frac{1}{2})^{365} = 3 + \frac{1}{2^{365}}$. So, Benjamin will have the most gold.

7.1.**72** a We are given that

$n(t + 1) = 2a(t)$

$a(t+1) = n(t) + a(t)$,

so that the matrix is $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$.

b $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, hence 2 and -1 are the eigenvalues associated with $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ respectively.

c We are given $\vec{x}_0 = \begin{bmatrix} n_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $\vec{x}_0 = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and $\vec{x}(t) = \frac{1}{3}2^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}(-1)^t\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (by Theorem 7.1.6), hence $n(t) = \frac{1}{3}2^t + \frac{2}{3}(-1)^t$ and $a(t) = \frac{1}{3}2^t - \frac{1}{3}(-1)^t$.

Section 7.2

7.2.1 $\lambda_1 = 1$, $\lambda_2 = 3$ by Theorem 7.2.2.

7.2.2 $\lambda_1 = 2$ (Algebraic multiplicity 2)

$\lambda_2 = 1$ (Algebraic multiplicity 2), by Theorem 7.2.2.

7.2.3 $\det(A - \lambda I_2) = \det\begin{bmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3) = 0$ so $\lambda_1 = 1, \lambda_2 = 3$.

7.2.4 $\det(A - \lambda I_2) = \det\begin{bmatrix} -\lambda & 4 \\ -1 & 4-\lambda \end{bmatrix} = -\lambda(4-\lambda) + 4 = (\lambda-2)^2 = 0$ so $\lambda = 2$ with algebraic multiplicity 2.

7.2.5 $\det(A - \lambda I_2) = \det\begin{bmatrix} 11-\lambda & -15 \\ 6 & -7-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 13$ so $\det(A - \lambda I_2) = 0$ for no real λ.

7.2.6 $\det(A - \lambda I_2) = \det\begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} = \lambda^2 - 5\lambda - 2 = 0$ so $\lambda_{1,2} = \frac{5\pm\sqrt{33}}{2}$.

7.2.7 $\lambda = 1$ with algebraic multiplicity 3, by Theorem 7.2.2.

7.2.8 $f_A(\lambda) = -\lambda^2(\lambda+3)$ so

$\lambda_1 = 0$ (Algebraic multiplicity 2)

$\lambda_2 = -3$.

7.2.9 $f_A(\lambda) = -(\lambda-2)^2(\lambda-1)$ so

$\lambda_1 = 2$ (Algebraic multiplicity 2)

$\lambda_2 = 1$.

7.2.10 $f_A(\lambda) = (1+\lambda)^2(1-\lambda)$ so $\lambda_1 = -1$ (Algebraic multiplicity 2), $\lambda_2 = 1$.

7.2.11 $f_A(\lambda) = -\lambda^3 - \lambda^2 - \lambda - 1 = -(\lambda+1)(\lambda^2+1) = 0$

$\lambda = -1$ (Algebraic multiplicity 1).

7.2.12 $f_A(\lambda) = \lambda(\lambda+1)(\lambda-1)^2$ so $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = 1$ (Algebraic multiplicity 2).

7.2.13 $f_A(\lambda) = -\lambda^3 + 1 = -(\lambda-1)(\lambda^2+\lambda+1)$ so $\lambda = 1$ (Algebraic multiplicity 1).

7.2.14 $f_A(\lambda) = \det(B - \lambda I_2)\det(D - \lambda I_2)$ (see Theorem 6.1.5).

The eigenvalues of A are the eigenvalues of B and D. The eigenvalues of C are irrelevant.

7.2.15 $f_A(\lambda) = \lambda^2 - 2\lambda + (1-k) = 0$ if $\lambda_{1,2} = \frac{2\pm\sqrt{4-4(1-k)}}{2} = 1 \pm \sqrt{k}$

The matrix A has 2 distinct real eigenvalues when $k > 0$, no real eigenvalues when $k < 0$.

7.2.16 $f_A(\lambda) = \lambda^2 - (a+c)\lambda + (ac - b^2)$

The discriminant of this quadratic equation is $(a+c)^2 - 4(ac - b^2) = a^2 + 2ac + c^2 - 4ac + 4b^2 = (a-c)^2 + 4b^2$; this quantity is always positive (since $b \neq 0$). There will always be two distinct real eigenvalues.

7.2.17 $f_A(\lambda) = \lambda^2 - a^2 - b^2 = 0$ so $\lambda_{1,2} = \pm\sqrt{a^2+b^2}$.

The matrix A represents a reflection about a line followed by a scaling by $\sqrt{a^2+b^2}$, hence the eigenvalues.

7.2.18 $f_A(\lambda) = \lambda^2 - 2a\lambda + a^2 - b^2$ so $\lambda_{1,2} = \frac{+2a\pm\sqrt{4a^2-4(a^2-b^2)}}{2} = a \pm b$.

Hence the eigenvalues are $a \pm b$.

7.2.19 True, since $f_A(\lambda) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$ and the discriminant $[\operatorname{tr}(A)]^2 - 4\det(A)$ is positive if $\det(A)$ is negative.

7.2.20 If $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors with eigenvalues λ_1 and λ_2, respectively, then $\vec{v}_1, \vec{v}_2$ will be an eigenbasis, so that matrix A is diagonalizable as claimed.

7.2.21 If A has n eigenvalues, then $f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$. Then $f_A(\lambda) = (-\lambda)^n + (\lambda_1 + \lambda_2 + \cdots + \lambda_n)(-\lambda)^{n-1} + \cdots + (\lambda_1\lambda_2\cdots\lambda_n)$. But, by Theorem 7.2.5, the coefficient of $(-\lambda)^{n-1}$ is $\operatorname{tr}(A)$. So, $\operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n$.

7.2.22 By Theorem 6.2.1, $f_A(\lambda) = \det(A - \lambda I_n) = \det(A - \lambda I_n)^T = \det(A^T - \lambda I_n) = f_{A^T}(\lambda)$. Since the characteristic polynomials of A and A^T are identical, the two matrices have the same eigenvalues, with the same algebraic multiplicities.

7.2.23 $f_B(\lambda) = \det(B - \lambda I_n) = \det(S^{-1}AS - \lambda I_n)$

$= \det(S^{-1}AS - \lambda S^{-1}I_nS)$

$= \det(S^{-1}(A - \lambda I_n)S) = \det(S^{-1})\det(A - \lambda I_n)\det(S)$

$= (\det S)^{-1}\det(A-\lambda I_n)\det(S) = \det(A-\lambda I_n) = f_A(\lambda)$

Hence, since $f_A(\lambda) = f_B(\lambda)$, A and B have the same eigenvalues.

7.2.**24** $\lambda_1 = 0.25$, $\lambda_2 = 1$

7.2.**25** $A\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} ab+cb \\ cb+cd \end{bmatrix} = \begin{bmatrix} (a+c)b \\ (b+d)c \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$ since $a+c = b+d = 1$; therefore, $\begin{bmatrix} b \\ c \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 1$.

Also, $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a-b \\ c-d \end{bmatrix} = (a-b)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ since $a-b = -(c-d)$; therefore, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = a-b$. Note that $|a-b| < 1$; a possible phase portrait is shown in Figure 7.11.

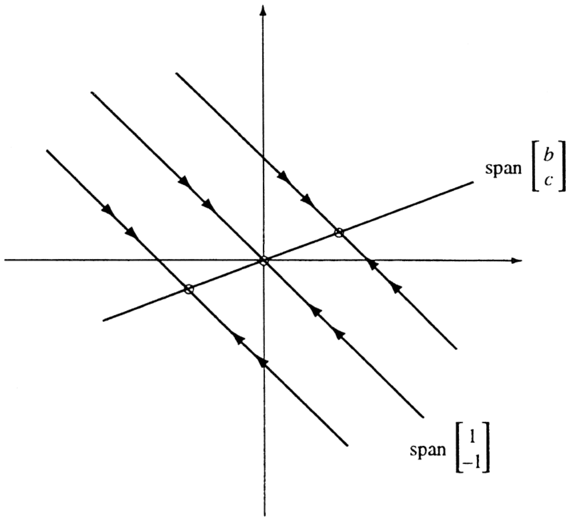


Figure 7.11: for Problem 7.2.25.

7.2.**26** Here $\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}$ with $\lambda_1 = 1$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with $\lambda_2 = a-b = 0.25$. See Figure 7.12.

7.2.**27** a We know $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\lambda_1 = 1$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\lambda_2 = \frac{1}{4}$. If $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then $\vec{x}_0 = \frac{1}{3}\vec{v}_1 + \frac{2}{3}\vec{v}_2$, so by Theorem 7.1.6,

$x_1(t) = \frac{1}{3} + \frac{2}{3}\left(\frac{1}{4}\right)^t$

$x_2(t) = \frac{2}{3} - \frac{2}{3}\left(\frac{1}{4}\right)^t$.

If $\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $\vec{x}_0 = \frac{1}{3}\vec{v}_1 - \frac{1}{3}\vec{v}_2$, so by Theorem 7.1.6,

$x_1(t) = \frac{1}{3} - \frac{1}{3}\left(\frac{1}{4}\right)^t$

$x_2(t) = \frac{2}{3} + \frac{1}{3}\left(\frac{1}{4}\right)^t$. See Figure 7.13.

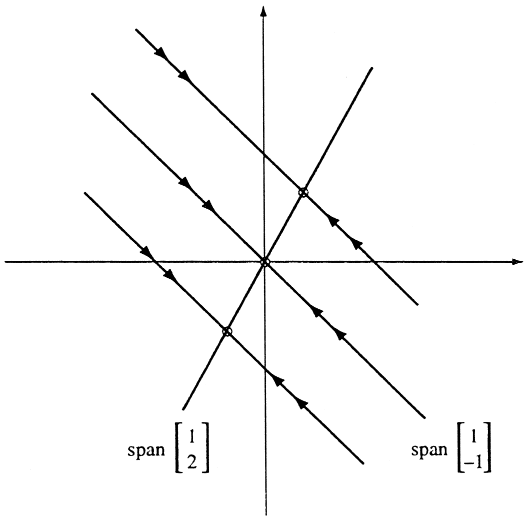


Figure 7.12: for Problem 7.2.26.

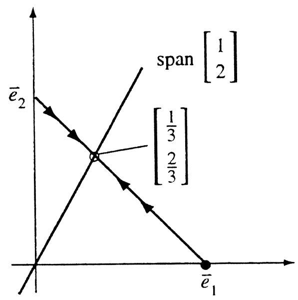


Figure 7.13: for Problem 7.2.27a.

b A^t appears to approach $\frac{1}{3}\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}$ as $t \to \infty$. To justify this result, we can diagonalize A with $S = \begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}$ and $B = \begin{bmatrix}1 & 0\\0 & 1/4\end{bmatrix}$; see Exercise 25. Now $S^{-1}AS = B$, $A = SBS^{-1}$, and $A^t = SB^tS^{-1}$. We find that $S = \begin{bmatrix}1 & 4\\0 & 1\end{bmatrix}$ and $\lim\limits_{t\to\infty} A^t = \lim\limits_{t\to\infty}(SB^tS^{-1}) = S(\lim\limits_{t\to\infty} B^t)S^{-1} = \begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\left(\frac{1}{3}\begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}\right)$
$= \frac{1}{3}\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}$, as we conjectured.

c Let us think about the first column of A^t, which is $A^t\vec{e}_1$. We can use Theorem 7.1.6 to compute $A^t\vec{e}_1$.

Start by writing $\vec{e}_1 = c_1\begin{bmatrix}b\\c\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix}$; a straightforward computation shows that $c_1 = \frac{1}{b+c}$ and $c_2 = \frac{c}{b+c}$.

Now $A^t\vec{e}_1 = \frac{1}{b+c}\begin{bmatrix}b\\c\end{bmatrix} + \frac{c}{b+c}(\lambda_2)^t\begin{bmatrix}1\\-1\end{bmatrix}$, where $\lambda_2 = a - b$.

Since $|\lambda_2| < 1$, the second summand goes to zero, so that $\lim\limits_{t\to\infty}(A^t\vec{e}_1) = \frac{1}{b+c}\begin{bmatrix}b\\c\end{bmatrix}$.

Likewise, $\lim_{t\to\infty}(A^t\vec{e}_2) = \frac{1}{b+c}\begin{bmatrix} b \\ c \end{bmatrix}$, so that $\lim_{t\to\infty} A^t = \frac{1}{b+c}\begin{bmatrix} b & b \\ c & c \end{bmatrix}$.

7.2.**28** a $w(t+1) = 0.8w(t) + 0.1m(t)$

$m(t+1) = 0.2w(t) + 0.9m(t)$

so $A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$ which is a regular transition matrix since its columns sum to 1 and its entries are positive.

b The eigenvectors of A are $\begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with $\lambda_1 = 1$, and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with $\lambda_2 = 0.7$.

$\vec{x}_0 = \begin{bmatrix} 1200 \\ 0 \end{bmatrix} = 400\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 800\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so $\vec{x}(t) = 400\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 800(0.7)^t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or

$w(t) = 400 + 800(0.7)^t$

$m(t) = 800 - 800(0.7)^t$.

c As $t \to \infty$, $w(t) \to 400$ so Wipfs won't have to close the store.

7.2.**29** The ith entry of $A\vec{e}$ is $[a_{i1}a_{i2}\cdots a_{in}]\vec{e} = \sum_{j=1}^{n} a_{ij} = 1$, so $A\vec{e} = \vec{e}$ and $\lambda = 1$ is an eigenvalue of A, corresponding to the eigenvector $\vec{e}$.

7.2.**30** a Suppose $A\vec{v} = \lambda\vec{v}$. Let v_i be the largest component of $\vec{v}$, meaning that $v_i \geq v_j$ for all $j = 1, ..., n$. Then the i^{th} component of $A\vec{v}$ is

$$\lambda v_i = \sum_{j=1}^{n} a_{ij}v_j \underset{step2}{\leq} \sum_{j=1}^{n} a_{ij}v_i = \left(\sum_{j=1}^{n} a_{ij}\right) v_i \underset{step4}{=} v_i.$$

We can conclude that $\lambda v_i \leq v_i$ and therefore $\lambda \leq 1$, as claimed.

In step 2 we use the fact that $v_j \leq v_i$ for all j, and in step 4 we are using the given property that the sum of the entries in each row of A is 1.

b. Let v_i be the component of $\vec{v}$ with the largest absolute value, meaning that $|v_i| \geq |v_j|$ for all $j = 1, ..., n$. Then the absolute value of the i^{th} component of $A\vec{v}$ is

$$|\lambda|\,|v_i| = \left|\sum_{j=1}^{n} a_{ij}v_j\right| \leq \sum_{j=1}^{n} a_{ij}\,|v_j| \leq \sum_{j=1}^{n} a_{ij}\,|v_i| = \left(\sum_{j=1}^{n} a_{ij}\right)|v_i| = |v_i|\,.$$

It follows that $|\lambda| \leq 1$, as claimed.

c. In part b, we see that $|\lambda| = 1$ if (and only if) both of the equations $\left|\sum_{j=1}^{n} a_{ij} v_j\right| = \sum_{j=1}^{n} a_{ij} |v_j|$ and $\sum_{j=1}^{n} a_{ij} |v_j| = \sum_{j=1}^{n} a_{ij} |v_i|$ hold. The first equation holds if (and only if) all the v_j are nonnegative or they are all nonpositive. The second equation holds if (and only if) $|v_j|$ is the same for all $j = 1, ..., n$. Combining these two observations, we see that $|\lambda| = 1$ if (and only if) all the components v_j of $\vec{v}$ are the same (and $v_j \neq 0$, since $\vec{v}$ is an eigenvector), meaning that

$$\vec{v} = \begin{bmatrix} c \\ c \\ ... \\ c \end{bmatrix} \text{ for some nonzero } c.$$

Note that $A\vec{v} = \vec{v} = 1\vec{v}$ since $\sum_{j=1}^{n} a_{ij} = 1$ for all i. It follows that -1 fails to be an eigenvalue of A.

7.2.**31** Since A and A^T have the same eigenvalues (by Exercise 22), Exercise 29 states that $\lambda = 1$ is an eigenvalue of A, and Exercise 30 says that $|\lambda| \leq 1$ for all eigenvalues λ. Vector $\vec{e}$ need not be an eigenvector of A; consider $A = \begin{bmatrix} 0.9 & 0.9 \\ 0.1 & 0.1 \end{bmatrix}$.

7.2.**32** $f_A(\lambda) = -\lambda^3 + 3\lambda + k$. The eigenvalues of A are the solutions of the equation $-\lambda^3 + 3\lambda + k = 0$, or, $\lambda^3 - 3\lambda = k$. Following the hint, we graph the function $g(\lambda) = \lambda^3 - 3\lambda$ as shown in Figure 7.14. We use the derivative $f'(\lambda) = 3\lambda^2 - 3$ to see that $g(\lambda)$ has a global minimum at $(1, -2)$ and a global maximum at $(-1, 2)$.

To count the eigenvalues of A, we need to find out how many times the horizontal line $y = k$ intersects the graph of $g(\lambda)$. In Figure 7.14, we see that there are three solutions if k satisfies the inequality $2 > k > -2$, two solutions if $k = 2$ or $k = -2$, and one solution if $|k| > 2$.

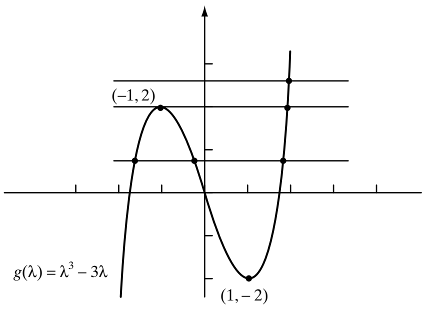


Figure 7.14: for Problem 7.2.32.

7.2.**33** a $f_A(\lambda) = \det(A - \lambda I_3) = -\lambda^3 + c\lambda^2 + b\lambda + a$

b By part a, we have $c = 17$, $b = -5$ and $a = \pi$, so $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \pi & -5 & 17 \end{bmatrix}$.

7.2.**34** Consider the possible graphs of $f_A(\lambda)$ assuming that it has 2 distinct real roots.

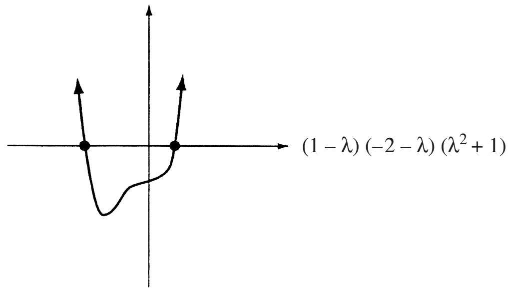


Figure 7.15: for Problem 7.2.34.

Algebraic multiplicity of each eigenvalue is 1. Example: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. See Figure 7.15.

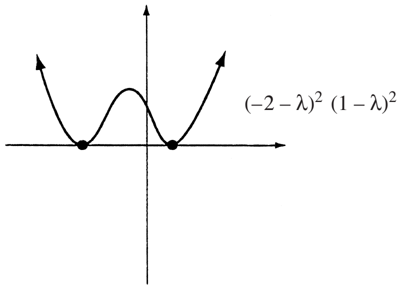


Figure 7.16: for Problem 7.2.34.

Algebraic multiplicity of each eigenvalue is 2. Example: $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. See Figure 7.16.

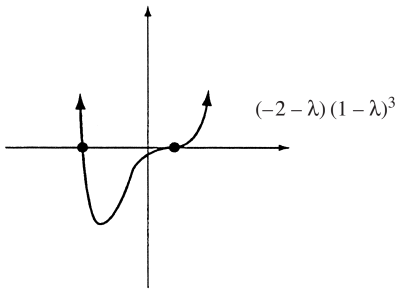


Figure 7.17: for Problem 7.2.34.

Algebraic multiplicity of λ_1 is 1, and of λ_2 is 3.

Example: $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. See Figure 7.17.

7.2.**35** $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, with $f_A(\lambda) = (\lambda^2+1)^2$

7.2.**36** Let $A = \begin{bmatrix} B & & & 0 \\ & B & & \\ & & \ddots & \\ 0 & & & B \end{bmatrix}$ where $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $f_A(\lambda) = (\lambda^2+1)^n$.

7.2.**37** We can write $f_A(\lambda) = (\lambda - \lambda_0)^2 g(\lambda)$, for some polynomial g. The product rule for derivatives tells us that $f'_A(\lambda) = 2(\lambda-\lambda_0)g(\lambda) + (\lambda-\lambda_0)^2 g'(\lambda)$, so that $f'_A(\lambda_0) = 0$, as claimed.

7.2.**38** By Theorem 7.2.4, the characteristic polynomial of A is $f_A(\lambda) = \lambda^2 - 5\lambda - 14 = (\lambda-7)(\lambda+2)$, so that the eigenvalues are 7 and -2.

7.2.**39** $\text{tr}(AB) = \text{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = \text{tr}\left(\begin{bmatrix} ae+bg & --- \\ --- & cf+dh \end{bmatrix}\right) = ae+bg+cf+dh.$

$\text{tr}(BA) = \text{tr}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \text{tr}\left(\begin{bmatrix} ea+fc & --- \\ --- & gb+hd \end{bmatrix}\right) = ea+fc+gb+hd.$ So they are equal.

7.2.**40** Let the entries of A be a_{ij} and the entries of B be b_{ij}.

Now, $\text{tr}(AB) = (a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) + (a_{21}b_{12} + \cdots + a_{2n}b_{n2}) + \cdots + (a_{n1}b_{1n} + \cdots$

$+ a_{nn}b_{nn})$. This is the sum of all products of the form $a_{ij}b_{ji}$.

We see that $\text{tr}(BA) = (b_{11}a_{11} + \cdots + b_{1n}a_{n1}) + \cdots + (b_{n1}a_{1n} + \cdots + b_{nn}a_{nn})$, which also is the sum of all products of the form $b_{ji}a_{ij} = a_{ij}b_{ji}$. Thus, $\text{tr}(AB) = \text{tr}(BA)$.

7.2.**41** So there exists an invertible S such that $B = S^{-1}AS$, and $\text{tr}(B) = \text{tr}(S^{-1}AS)$

$= \text{tr}((S^{-1}A)S)$. By Exercise 40, this equals $\text{tr}(S(S^{-1}A)) = \text{tr}(A)$.

7.2.**42** $\text{tr}\big((A+B)^2\big) = \text{tr}(A^2 + AB + BA + B^2) = \text{tr}(A^2) + \text{tr}(AB) + \text{tr}(BA) + \text{tr}(B^2)$.. By Exercise 40, $\text{tr}(AB) = \text{tr}(BA)$. Thus, $\text{tr}\big((A+B)^2\big) = \text{tr}(A^2) + 2\text{tr}(BA) + \text{tr}(B^2) = \text{tr}(A^2) + \text{tr}(B^2)$, since $BA = 0$.

7.2.**43** $\text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = \text{tr}(AB) - \text{tr}(AB) = 0$, but $\text{tr}(I_n) = n$, so no such A, B exist. We have used Exercise 40.

7.2.**44** No, there are no such matrices A and B. We will argue indirectly, assuming that invertible matrices A and B with $AB - BA = A$ do exist. Then $AB = BA + A = (B + I_n)A$, and $ABA^{-1} = B + I_n$. Using Exercise 41, we see that $\text{tr}(B) = \text{tr}(ABA^{-1}) = \text{tr}(B + I_n) = \text{tr}(B) + n$, a contradiction.

7.2.**45** $f_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + (-3-4k)$. We want $f_A(5) = 25 - 10 - 3 - 4k = 0$, or, $12 - 4k = 0$, or $k = 3$.

7.2.**46** a $\lambda_1^2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 = (\text{tr}A)^2 - 2\det(A) = (a+d)^2 - 2(ad - bc) = a^2 + d^2 + 2bc$.

b Based on part (a), we need to show that $a^2 + d^2 + 2bc \leq a^2 + b^2 + c^2 + d^2$, or $2bc \leq b^2 + c^2$, or $0 \leq (b-c)^2$. But the last inequality is obvious.

c By parts (a) and (b), the equality $\lambda_1^2 + \lambda_2^2 = a^2 + b^2 + c^2 + d^2$ holds if (and only if) $0 = (b-c)^2$, or $b = c$. Thus equality holds for symmetric matrices A.

7.2.**47** Let $M = [\,\vec{v}_1 \quad \vec{v}_2\,]$. We want $A[\,\vec{v}_1 \quad \vec{v}_2\,] = [\,\vec{v}_1 \quad \vec{v}_2\,]\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, or, $[\,A\vec{v}_1 \quad A\vec{v}_2\,] = [\,2\vec{v}_1 \quad 3\vec{v}_2\,]$. Since $\vec{v}_1$ or $\vec{v}_2$ must be nonzero, 2 or 3 must be an eigenvalue of A.

7.2.**48** Let $S = [\vec{v}_1\ \vec{v}_2]$. Then $AS = [A\vec{v}_1\ A\vec{v}_2]$ and $SD = [2\vec{v}_1\ 3\vec{v}_2]$, so that $\vec{v}_1$ must be an eigenvector with eigenvalue 2, and $\vec{v}_2$ must be an eigenvector with eigenvalue 3. Thus, both 2 and 3 must be eigenvalues of A.

7.2.**49** As in problem 47, such an M will exist if A has an eigenvalue 2, 3 or 4.

7.2.**50** a If $f(x) = x^3 + 6x - 20$ then $f'(x) = 3x^2 + 6$ so $f'(x) > 0$ for all x, i.e. f is always increasing, hence has only one real root.

b If $v^3 - u^3 = 20$ and $vu = 2$ then

$(v-u)^3 + 6(v-u) = v^3 - 3v^2u + 3vu^2 - u^3 + 6(v-u) = v^3 - u^3 - 3vu(v-u) + 6(v-u)$

$= 20 - 6(v-u) + 6(v-u) = 20$

Hence $x = v - u$ satisfies the equation $x^3 + 6x = 20$.

c The second equation tells us that $u = \frac{2}{v}$ or $u^3 = \frac{8}{v^3}$. Substituting into the first equation we find that

$v^3 - \frac{8}{v^3} = 20$, or, $(v^3)^2 - 8 = 20v^3$ or $(v^3)^2 - 20v^3 - 8 = 0$, with solutions

$v^3 = \frac{20 \pm \sqrt{400+32}}{2} = 10 \pm \sqrt{108} = 10 \pm 6\sqrt{3}$ and $v = \sqrt[3]{10 \pm \sqrt{108}}$.

Now $u^3 = v^3 - 20 = -10 \pm \sqrt{108}$ and $u = \sqrt[3]{-10 \pm \sqrt{108}}$.

d Let $v = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$ and $u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$.

Then $v^3 - u^3 = q$ and $vu = \sqrt[3]{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2} = \frac{p}{3}$.

Since $x = v - u$ we have

$x^3 + px = v^3 - 3v^2u + 3vu^2 - u^3 + p(v-u) = v^3 - u^3 - 3vu(v-u) + p(v-u)$

$= q - p(v-u) + p(v-u) = q$, as claimed.

If p is negative, the expression $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$ may be negative. Also, the equation $x^3 + px = q$ may have more than one solution in this case.

e Setting $x = t - \left(\frac{a}{3}\right)$ we get $\left(t - \frac{a}{3}\right)^3 + a\left(t - \frac{a}{3}\right)^2 + b\left(t - \frac{a}{3}\right) + c = 0$ or

$t^3 - at^2 + at^2 +$ (linear and constant terms) $= 0$ or $t^3 +$ (linear and constant terms) $= 0$, as claimed (bring the constant terms to the right-hand side).

Section 7.3

7.3.1 $\lambda_1 = 7$, $\lambda_2 = 9$, $E_7 = \ker\begin{bmatrix} 0 & 8 \\ 0 & 2 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, E_9 = \ker\begin{bmatrix} -2 & 8 \\ 0 & 0 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Eigenbasis: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

We can diagonalize A with $S = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix}$.

7.3.2 $\lambda_1 = 2, \lambda_2 = 0, E_2 = \operatorname{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, E_0 = \operatorname{span}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigenbasis: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

We can diagonalize A with $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

7.3.3 $\lambda_1 = 4$, $\lambda_2 = 9$, $E_4 = \operatorname{span}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $E_9 = \operatorname{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenbasis: $\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We can diagonalize A with $S = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$.

7.3.4 $\lambda_1 = \lambda_2 = 1, E_1 = \operatorname{span}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

No eigenbasis. This matrix fails to be diagonalizable.

7.3.5 No real eigenvalues as $f_A(\lambda) = \lambda^2 - 2\lambda + 2$. This matrix fails to be diagonalizable.

7.3.6 $\lambda_{1,2} = \frac{7 \pm \sqrt{57}}{2}$

Eigenbasis: $\begin{bmatrix} 3 \\ \lambda_1 - 2 \end{bmatrix} \approx \begin{bmatrix} 3 \\ 5.27 \end{bmatrix}, \begin{bmatrix} 3 \\ \lambda_2 - 2 \end{bmatrix} \approx \begin{bmatrix} 3 \\ -2.27 \end{bmatrix}$

We can diagonalize A with $S = \begin{bmatrix} 3 & 3 \\ \lambda_1 - 2 & \lambda_2 - 2 \end{bmatrix}$ and $B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

7.3.7 $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, eigenbasis: $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$

We can diagonalize the diagonal matrix A with $S = I_3$ and $B = A$.

7.3.8 $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, eigenbasis: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}$

We can diagonalize A with $S = \begin{bmatrix}1&1&1\\0&1&2\\0&0&1\end{bmatrix}$ and $B = \begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$.

7.3.9 $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 0$, eigenbasis: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$

We can diagonalize A with $S = \begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}$ and $B = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$.

7.3.10 $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 0$, $E_1 = \text{span}\begin{bmatrix}1\\0\\0\end{bmatrix}$, $E_0 = \text{span}\begin{bmatrix}0\\0\\1\end{bmatrix}$

No eigenbasis. This matrix fails to be diagonalizable.

7.3.11 $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 3$, eigenbasis: $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$

We can diagonalize A with $S = \begin{bmatrix}1&1&1\\-1&0&1\\0&-1&1\end{bmatrix}$ and $B = \begin{bmatrix}0&0&0\\0&0&0\\0&0&3\end{bmatrix}$.

7.3.12 $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $E_1 = \text{span}\begin{bmatrix}1\\0\\0\end{bmatrix}$, no eigenbasis

7.3.13 $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$, eigenbasis: $\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-3\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\2\end{bmatrix}$

We can diagonalize A with $S = \begin{bmatrix}0&1&1\\1&-3&-1\\0&1&2\end{bmatrix}$ and $B = \begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}$.

7.3.14 $\lambda_1 = 0$, $\lambda_1 = \lambda_3 = 1$, eigenbasis: $\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-5\\0\end{bmatrix}, \begin{bmatrix}0\\2\\1\end{bmatrix}$

We can diagonalize A with $S = \begin{bmatrix}0&1&0\\1&-5&2\\0&0&1\end{bmatrix}$ and $B = \begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$.

7.3.15 $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$, $E_0 = \text{span}\begin{bmatrix}0\\1\\0\end{bmatrix}$. We can use Kyle Numbers to see that

$$E_1 = \ker \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ -3 & -1 & 1 \\ -4 & 0 & 2 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

There is no eigenbasis since the eigenvalue 1 has algebraic multiplicity 2, but the geometric multiplicity is only 1. This matrix fails to be diagonalizable.

7.3.**16** $\lambda_1 = 0$ (no other real eigenvalues), with eigenvector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

No real eigenbasis. This matrix fails to be diagonalizable.

7.3.**17** $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = 1$

with eigenbasis $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

We can diagonalize A with $S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

7.3.**18** $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = 1$, $E_0 = \operatorname{span}(\vec{e}_1, \vec{e}_3)$, $E_1 = \operatorname{span}(\vec{e}_2)$

No eigenbasis. This matrix fails to be diagonalizable.

7.3.**19** Fails to be diagonalizable. The eigenvalues are 1,0,1, and the eigenspace $E_1 = \ker(A - I_3)$

$= \operatorname{span}(\vec{e}_1)$ is only one-dimensional.

7.3.**20** Diagonalizable. The eigenvalues are 1,2,0, with associated eigenvectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. If we let $S = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, then $S^{-1}AS = B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

7.3.**21** We want A such that $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, i.e. $A\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$ so $A = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}$.

The answer is unique.

7.3.**22** We want A such that $A\vec{e}_1 = 7\vec{e}_1$ and $A\vec{e}_2 = 7\vec{e}_2$ hence $A = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$.

7.3.23 $\lambda_1 = \lambda_2 = 1$ and $E_1 = \text{span}(\vec{e}_1)$, hence there is no eigenbasis. The matrix represents a shear parallel to the x-axis.

7.3.24 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. First we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, or $2a + b = 2, 2c + d = 1$. This condition is satisfied by all matrices of the form $A = \begin{bmatrix} a & 2-2a \\ c & 1-2c \end{bmatrix}$. Next, we want there to be no other eigenvalue, besides 1, so that 1 must have an algebraic multiplicity of 2.

We want the characteristic polynomial to be $(\lambda - 1)^2 = \lambda^2 - 2\lambda + 1$, so that the trace must be 2, and $a + (1-2c) = 2$, or, $a = 1 + 2c$. Thus we want a matrix of the form $A = \begin{bmatrix} 1+2c & -4c \\ c & 1-2c \end{bmatrix}$.

Finally, we have to make sure the $E_1 = \text{span}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ instead of $E_1 = \mathbb{R}^2$. This means that we must exclude the case $A = I_2$. In order to ensure this, we state simply that $A = \begin{bmatrix} 1+2c & -4c \\ c & 1-2c \end{bmatrix}$, where c is any nonzero constant.

7.3.25 If λ is an eigenvalue of A, then $E_\lambda = \ker(A - \lambda I_3) = \ker \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ a & b & c-\lambda \end{bmatrix}$.

The second and third columns of the above matrix aren't parallel, hence E_λ is always 1-dimensional, i.e., the geometric multiplicity of λ is 1.

7.3.26 Note that $f_A(0) = \det(A - 0I_6) = \det(A)$ is negative. Since $\lim_{\lambda\to\infty} f_A(\lambda) = \infty$, there must be a positive root, by the Intermediate Value Theorem (see Exercise 2.2.47). Therefore, the matrix A has a positive eigenvalue. See Figure 7.18.

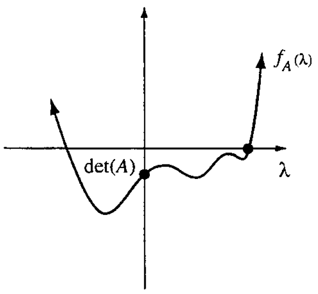


Figure 7.18: for Problem 7.3.26.

7.3.27 By Theorem 7.2.4, we have $f_A(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2)$ so $\lambda_1 = 2$, $\lambda_2 = 3$.

7.3.28 Since $J_n(k)$ is triangular, its eigenvalues are its diagonal entries, hence its only eigenvalue is k. Moreover,

$$E_k = \ker(J_n(k) - kI_n) = \ker \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & 0 & & \vdots \\ \vdots & \vdots & \vdots & & 1 \\ 0 & 0 & 0 & & 0 \end{bmatrix} = \text{span }(\vec{e}_1).$$

The geometric multiplicity of k is 1 while its algebraic multiplicity is n.

7.3.**29** Note that r is the number of nonzero diagonal entries of A, since the nonzero columns of A form a basis of $\text{im}(A)$. Therefore, there are $n-r$ zeros on the diagonal, so that the algebraic multiplicity of the eigenvalue 0 is $n-r$. It is true for any $n \times n$ matrix A that the geometric multiplicity of the eigenvalue 0 is $\dim(\ker(A)) = n - \text{rank}(A) = n - r$.

7.3.**30** Since A is triangular, $f_A(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{mm} - \lambda)(0 - \lambda)^{n-m}$.

Hence the algebraic multiplicity of $\lambda = 0$ is $(n - m)$.

Also note that the rank of A is at least m, since the first m columns of A are linearly independent. Therefore, the geometric multiplicity of the eigenvalue 0 is $\dim(\ker(A)) = n - \text{rank}(A) \leq n - m$.

7.3.**31** They must be the same. For if they are not, by Theorem 7.3.6, the geometric multiplicities would not add up to n.

7.3.**32** Recall that a matrix and its transpose have the same rank (Theorem 5.3.9c). The geometric multiplicity of λ as an eigenvalue of A is $\dim(\ker(A - \lambda I_n)) = n - \text{rank}(A - \lambda I_n)$.

The geometric multiplicity of λ as an eigenvalue of A^T is $\dim(\ker(A^T - \lambda I_n))$

$= \dim(\ker(A - \lambda I_n)^T) = n - \text{rank}(A - \lambda I_n)^T = n - \text{rank}(A - \lambda I_n)$.

We can see that the two multiplicities are the same.

7.3.**33** If $S^{-1}AS = B$, then

$S^{-1}(A - \lambda I_n)S = S^{-1}(AS - \lambda S) = S^{-1}AS - \lambda S^{-1}S = B - \lambda I_n$.

7.3.**34** a If $\vec{x}$ is in the kernel of B, then $AS\vec{x} = SB\vec{x} = S\vec{0} = \vec{0}$, so that $S\vec{x}$ is in $\ker(A)$.

b T is clearly linear, and the transformation $R(\vec{x}) = S^{-1}\vec{x}$ is the inverse of T (if $\vec{x}$ is in the kernel of B, then $S^{-1}\vec{x}$ is in the kernel of A, by part (a)).

c The equation $\text{nullity}(A) = \text{nullity}(B)$ follows from part (b); the equation $\text{rank}(A) = \text{rank}(B)$ then follows from the rank-nullity theorem (Theorem 3.3.7).

7.3.**35** No, since the two matrices have different eigenvalues (see Theorem 7.3.5c).

7.3.**36** No, since the two matrices have different traces (see Theorem 7.3.5d)

7.3.**37** a $A\vec{v} \cdot \vec{w} = (A\vec{v})^T\vec{w} = (\vec{v}^TA^T)\vec{w} \underset{A \text{ symmetric}}{\underset{\uparrow}{=}} (\vec{v}^TA)\vec{w} = v^T(A\vec{w}) = \vec{v} \cdot A\vec{w}$

b Assume $A\vec{v} = \lambda\vec{v}$ and $A\vec{w} = \alpha\vec{w}$ for $\lambda \neq \alpha$, then $(A\vec{v}) \cdot \vec{w} = (\lambda\vec{v}) \cdot \vec{w} = \lambda(\vec{v} \cdot \vec{w})$, and $\vec{v} \cdot A\vec{w} = \vec{v} \cdot \alpha\vec{w} = \alpha(\vec{v} \cdot \vec{w})$.

By part a, $\lambda(\vec{v}\cdot\vec{w}) = \alpha(\vec{v}\cdot\vec{w})$ i.e., $(\lambda-\alpha)(\vec{v}\cdot\vec{w}) = 0$.

Since $\lambda \neq \alpha$, it must be that $\vec{v}\cdot\vec{w} = 0$, i.e., $\vec{v}$ and $\vec{w}$ are perpendicular.

7.3.**38** Note that $f_A(0) = \det(A - 0I_3) = \det(A) = 1$.

Since $\lim_{\lambda\to\infty} f_A(\lambda) = -\infty$, the polynomial $f_A(\lambda)$ must have a positive root λ_0, by the Intermediate Value Theorem in single variable calculus. In other words, the matrix A will have a positive eigenvalue λ_0. Since A is orthogonal, this eigenvalue λ_0 will be 1, by Theorem 7.1.4. This means that there is a nonzero vector $\vec{v}$ in $\mathbb{R}^3$ such that $A\vec{v} = 1\vec{v} = \vec{v}$, as claimed. See Figure 7.19.

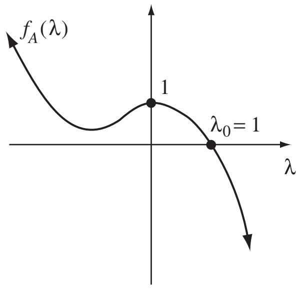


Figure 7.19: for Problem 7.3.38.

7.3.**39** a There are two eigenvalues, $\lambda_1 = 1$ (with $E_1 = V$) and $\lambda_2 = 0$ (with $E_0 = V^\perp$).

Now geometric multiplicity(1) $= \dim(E_1) = \dim(V) = m$, and

geometric multiplicity(0) $= \dim(E_0) = \dim(V^\perp) = n - \dim(V) = n - m$.

Since geometric multiplicity(λ) $\leq$ algebraic multiplicity(λ), by Theorem 7.3.6, and the algebraic multiplicities cannot add up to more than n, the geometric and algebraic multiplicities of the eigenvalues are the same here.

b Analogous to part a: $E_1 = V$ and $E_{-1} = V^\perp$.

geometric multiplicity(1) = algebraic multiplicity(1) $= \dim(V) = m$, and

geometric multiplicity(-1) = algebraic multiplicity(-1) $= \dim(V^\perp) = n - m$.

7.3.**40** The sole eigenvalue is 1, with algebraic multiplicity 2. This matrix is diagonalizable only if $a = 0$.

7.3.**41** Diagonalizable for all values of a, since there are always two distinct eigenvalues, 1 and 2. See Theorem 7.1.3.

7.3.**42** Diagonalizable except if $b = 1$ and $a \neq 0$. (In that case we have only one eigenvalue, 1, with a one-dimensional eigenspace.).

7.3.**43** Diagonalizable for positive a. The characteristic polynomial is $(\lambda-1)^2 - a$, so that the eigenvalues are $\lambda = 1 \pm \sqrt{a}$. If a is positive, then we have two distinct real eigenvalues, so that the matrix is diagonalizable. If a is negative, then there are no real eigenvalues. If a is 0, then 1 is the only eigenvalue, with a one-dimensional eigenspace.

7.3.44 Diagonalizable for all values of a, b, and c. The characteristic polynomial is $\lambda^2-(a+c)\lambda+ac-b^2$, so that the eigenvalues are $\lambda=\frac{a+c\pm\sqrt{(a+c)^2-4(ac-b^2)}}{2}=\frac{a+c\pm\sqrt{(a-c)^2+4b^2}}{2}$. Note that the expression whose square root we take (the "discriminant") is always positive or 0, since it is the sum of two squares. If the discriminant is positive, then we have two distinct real eigenvalues, and everything is fine. The discriminant is 0 only if $a=c$ and $b=0$. In that case the matrix is diagonal already, and certainly diagonalizable as well.

7.3.45 Diagonalizable for all values of a, b, and c, since we have three distinct eigenvalues, 1, 2, and 3.

7.3.46 The eigenvalues are 1, 2, 1, and the matrix is diagonalizable if (and only if) the eigenspace E_1 is two-dimensional. Now $E_1=\ker(A-I_3)=\ker\begin{bmatrix}0&a&b\\0&1&c\\0&0&0\end{bmatrix}=\ker\begin{bmatrix}0&1&c\\0&0&b-ac\\0&0&0\end{bmatrix}$ is two-dimensional if (and only if) $b-ac=0$. Thus the matrix is diagonalizable if and only if $b=ac$.

7.3.47 Diagonalizable only if $a=b=c=0$. Since 1 is the only eigenvalue, it is required that $E_1=\mathbb{R}^3$, that is, the matrix must be the identity matrix.

7.3.48 Diagonalizable for positive values of a. The characteristic polynomial is $-\lambda^3+a\lambda=-\lambda(\lambda^2-a)$. If a is positive, then we have three distinct real eigenvalues, $0,\pm\sqrt{a}$, so that the matrix will be diagonalizable. If a is negative or 0, then 0 is the only real eigenvalue, and the matrix fails to be diagonalizable.

7.3.49 Not diagonalizable for any a. The characteristic polynomial is $-\lambda^3+a$, so that there is only one real eigenvalue, $\sqrt[3]{a}$, for all a. Since the corresponding eigenspace isn't all of $\mathbb{R}^3$, the matrix fails to be diagonalizable.

7.3.50 First we observe that all the eigenspaces of $A=\begin{bmatrix}0&0&a\\1&0&3\\0&1&0\end{bmatrix}$ are one-dimensional, regardless of the value of a, since $\text{rref}(A-\lambda I_3)$ is of the form $\begin{bmatrix}1&0&*\\0&1&*\\0&0&*\end{bmatrix}$ for all λ. Thus A is diagonalizable if and only if there are three distinct real eigenvalues. The characteristic polynomial of A is $-\lambda^3+3\lambda+a$. Thus the eigenvalues of A are the solutions of the equation $\lambda^3-3\lambda=a$. See Figure 7.24 with the function $f(\lambda)=\lambda^3-3\lambda$; using calculus, we find the local maximum $f(-1)=2$ and the local minimum $f(1)=-2$. To count the distinct eigenvalues of A, we have to examine how many times the horizontal line $y=a$ intersects the graph of $f(\lambda)$. The answer is three if $|a|<2$, two if $a=\pm2$, and one if $|a|>2$. Thus A is diagonalizable if and only if $|a|<2$, that is, $-2<a<2$.

7.3.51 $f_A(\lambda)=\det(A-\lambda I_3)=\det\begin{bmatrix}-\lambda&0&a\\1&-\lambda&b\\0&1&c-\lambda\end{bmatrix}=-\lambda^3+c\lambda^2+b\lambda+a.$

7.3.52 The characteristic polynomial is $f_A(\lambda)=(-1)^n\left(\lambda^n-a_{n-1}\lambda^{n-1}-\ldots-a_1\lambda-a_0\right)$. We can prove this by induction, using Laplace expansion along the first row:
$f_A(\lambda)=\det(A-\lambda I_n)=-\lambda\det(A-\lambda I_n)_{11}+(-1)^{n+1}a_0\det(A-\lambda I_n)_{1n}$
$=-\lambda(-1)^{n-1}\left(\lambda^{n-1}-a_{n-1}\lambda^{n-2}-\ldots-a_2\lambda-a_1\right)+(-1)^{n+1}a_0$
$=(-1)^n\left(\lambda^n-a_{n-1}\lambda^{n-1}-\ldots-a_1\lambda-a_0\right)$. Note that $(A-\lambda I_n)_{1n}$ is upper triangular with all 1's on the diagonal, so that $\det(A-\lambda I_n)_{1n}=1$.

7.3.53 a $B = \begin{bmatrix} 0 & 0 & a & * & * \\ 1 & 0 & b & * & * \\ 0 & 1 & c & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}$.

b Let $B_{11} = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{bmatrix}$ and B_{22} be the diagonal blocks of the matrix B we found in part (a). Since A is similar to B, we have $f_A(\lambda) = f_B(\lambda) = f_{B_{22}}(\lambda) f_{B_{11}}(\lambda) = f_{B_{22}}(\lambda)\left(-\lambda^3 + c\lambda^2 + b\lambda + a\right)$, by Exercise 51. Now we observe that $h(\lambda) = f_{B_{22}}(\lambda)$is a quadratic polynomial, completing our proof.

c From part (b) we know that $f_A(A) = h(A)\left(-A^3 + cA^2 + bA + aI_5\right)$. The given equation $A^3\vec{v} = a\vec{v} + bA\vec{v} + cA^2\vec{v}$ can be written as $\left(-A^3 + cA^2 + bA + aI_5\right)\vec{v} = \vec{0}$, implying that $f_A(A)\vec{v} = h(A)\left(-A^3 + cA^2 + bA + aI_5\right)\vec{v} = \vec{0}$, as claimed.

7.3.54 a. $B_{11} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & 0 & \dots & 0 & a_1 \\ 0 & 1 & 0 & \dots & 0 & a_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & a_{m-2} \\ 0 & 0 & 0 & \dots & 1 & a_{m-1} \end{bmatrix}$, $B_{21} = 0$.

b. We find $f_A(\lambda) = f_B(\lambda) = f_{B_{22}}(\lambda) f_{B_{11}}(\lambda)$ as in part (b) of Exercise 53. Now matrix B_{11} has the form discussed in Exercise 52, where we show that
$\det(B_{11}) = (-1)^m\left(\lambda^m - a_{m-1}\lambda^{m-1} - \dots - a_1\lambda - a_0\right)$. It follows that
$f_A(\lambda) = (-1)^m f_{B_{22}}(\lambda)\left(\lambda^m - a_{m-1}\lambda^{m-1} - \dots - a_1\lambda - a_0\right)$, as claimed.

c. Using the same approach as in Exercise 53, part (c), we can write the given equation for $A^m\vec{v}$ as
$\left(A^m - a_{m-1}A^{m-1} - \dots - a_1A - a_0I_n\right)\vec{v} = \vec{0}$. Thus
$f_A(A)\vec{v} = (-1)^m f_{B_{22}}(A)\left(A^m - a_{m-1}A^{m-1} - \dots - a_1A - a_0I_n\right)\vec{v} = \vec{0}$, as claimed.

d. Our work in parts (a) through (c) shows that $f_A(A)\vec{v} = \vec{0}$ for all $\vec{v}$ in $\mathbb{R}^n$, meaning that $f_A(A) = 0$, the zero matrix.

7.3.55 The non-invertible matrix $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ has 0 as one of its eigenvalues, so that $A = B + 7I_3 = \begin{bmatrix} 8 & 1 & 1 \\ 1 & 8 & 1 \\ 1 & 1 & 8 \end{bmatrix}$ has 7 as one of its eigenvalues.

7.3.56 There are many ways to construct such a matrix $A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$. One idea is to make sure that the matrices $A - I_3, A - 2I_3$, and $A - 3I_3$ all fail to be invertible. We can construct A so that the first two columns of $A - I_3$ are identical. Likewise, we can make the first and the last columns of $A - 2I_3$ identical. It is easy to see that these two conditions are both satisfied if A is of the form $A = \begin{bmatrix} a & a-1 & a-2 \\ b & b+1 & b \\ c & c & c+2 \end{bmatrix}$. To make sure that 3 is an eigenvalue as well, we want the trace of A to be $\text{tr}A = 1+2+3 = 6 = a+(b+1)+(c+2) = a+b+c+3$, meaning that $a+b+c = 3$. For example, we can make $a = 3,\ b = 1$ and $c = -1$, giving

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 1 \end{bmatrix}.$$

Section 7.4

7.4.**1** The eigenvalues of A are 1 and 3, the diagonal entries of A, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Matrix A is diagonalizable, with $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. By Theorem 7.4.2, we have

$$A^t = SB^tS^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3^t - 1 \\ 0 & 3^t \end{bmatrix}.$$

7.4.**2** The eigenvalues of A are 2 and 3, the diagonal entries of A, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Matrix A is diagonalizable, with $S = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. By Theorem 7.4.2, we have

$$A^t = SB^tS^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2^t & 0 \\ 0 & 3^t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2^t & 0 \\ 3^t - 2^t & 3^t \end{bmatrix}.$$

7.4.**3** The eigenvalues of A are -1 and 5, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Matrix A is diagonalizable, with $S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$. By Theorem 7.4.2, we have

$$A^t = SB^tS^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} (-1)^t & 0 \\ 0 & 5^t \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5^t + 2(-1)^t & 5^t - (-1)^t \\ 2 \cdot 5^t - 2(-1)^t & 2 \cdot 5^t + (-1)^t \end{bmatrix}.$$

7.4.**4** The eigenvalues of A are 3 and 2, and corresponding eigenvectors are $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Matrix A is diagonalizable, with $S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. By Theorem 7.4.2, we have

$$A^t = SB^tS^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^t & 0 \\ 0 & 2^t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3^t - 2^t & -2 \cdot 3^t + 2 \cdot 2^t \\ 3^t - 2^t & -3^t + 2 \cdot 2^t \end{bmatrix}.$$

7.4.5 The eigenvalues of A are 0 and 7, and corresponding eigenvectors are $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Matrix A is diagonalizable, with $S = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$. By Theorem 7.4.2, we have

$$A^t = SB^tS^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 7^t \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7^t & 2\cdot 7^t \\ 3\cdot 7^t & 6\cdot 7^t \end{bmatrix}.$$

7.4.6 The eigenvalues of A are 0 and 3, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Matrix A is diagonalizable, with $S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$. By Theorem 7.4.2, we have

$$A^t = SB^tS^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3^t \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3^t & 3^t \\ 2\cdot 3^t & 2\cdot 3^t \end{bmatrix}.$$

7.4.7 The eigenvalues of A are 0.25 and 1, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Matrix A is diagonalizable, with $S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0.25 & 0 \\ 0 & 1 \end{bmatrix}$. By Theorem 7.4.2, we have

$$A^t = SB^tS^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (0.25)^t & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2(0.25)^t + 1 & -(0.25)^t + 1 \\ -2(0.25)^t + 2 & (0.25)^t + 2 \end{bmatrix}.$$

7.4.8 The eigenvalues of A are 1 and 0.2, and corresponding eigenvectors are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Matrix A is diagonalizable, with $S = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}$. By Theorem 7.4.2, we have

$$A^t = SB^tS^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (0.2)^t \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 + (0.2)^t & 3 - 3(0.2)^t \\ 1 - (0.2)^t & 1 + 3(0.2)^t \end{bmatrix}.$$

7.4.9 The eigenvalues of A are 0, -1, and 1, the diagonal entries of the triangular matrix A, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Matrix A is diagonalizable, with $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. By Theorem 7.4.2, we have

$$A^t = SB^tS^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & (-1)^t & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ -2(-1)^t & 2(-1)^t & 0 \\ 1 + (-1)^t & 1 - (-1)^t & 2 \end{bmatrix}.$$

7.4.10 The eigenvalues of A are 1, 0, and 2, the diagonal entries of the triangular matrix A, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Matrix A is diagonalizable, with $S = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. By Theorem 7.4.2, we have

$$A^t = SB^tS^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2^t \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3\cdot 2^{t-1}-2 \\ 0 & 0 & 2^{t-1} \\ 0 & 0 & 2^t \end{bmatrix}.$$

7.4.11 We use technology to find that the eigenvalues of A are 1, 0, and 2, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Matrix A is diagonalizable, with $S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -4 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. By Theorem 7.4.2, we have

$$A^t = SB^tS^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -4 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2^t \end{bmatrix}\begin{bmatrix} 4 & 2 & 4 \\ -1 & -1 & -1 \\ -1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 2-2^{t-1} & 1-2^{t-1} & 2-3\cdot 2^{t-1} \\ -2 & -1 & -2 \\ 2^{t-1} & 2^{t-1} & 3\cdot 2^{t-1} \end{bmatrix}.$$

7.4.12 We use technology to find that the eigenvalues of A are 1, 0.2, and 0, and corresponding eigenvectors are $\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Matrix A is diagonalizable, with $S = \begin{bmatrix} 2 & 1 & 1 \\ 5 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. By Theorem 7.4.2, we have $A^t = SB^tS^{-1}$

$$= \frac{1}{10}\begin{bmatrix} 2 & 1 & 1 \\ 5 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & (0.2)^t & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 5 & -5 & 5 \\ 3 & 3 & -7 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 2+5(0.2)^t & 2-5(0.2)^t & 2+5(0.2)^t \\ 5-5(0.2)^t & 5+5(0.2)^t & 5-5(0.2)^t \\ 3 & 3 & 3 \end{bmatrix}$$

7.4.13 The eigenvalues of A are 1 and 3, the diagonal entries of A, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Now $\vec{x}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $A^t\vec{x}_0 = A^t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2A^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\cdot 3^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2\cdot 3^t \\ 2\cdot 3^t \end{bmatrix}$.

7.4.14 The eigenvalues of A are 3 and 2, and corresponding eigenvectors are $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Now $\vec{x}_0 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $A^t\vec{x}_0 = A^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2A^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\cdot 2^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 3^t + 2^{t+1} \\ 3^t + 2^{t+1} \end{bmatrix}$.

7.4.15 The eigenvalues of A are 0.25 and 1, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Now $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $A^t\vec{x}_0 = \frac{2}{3}A^t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{3}A^t\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{2}{3}(0.25)^t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2(0.25)^t + 1 \\ -2(0.25)^t + 2 \end{bmatrix}$.

7.4.**16** The eigenvalues of A are 1, 2, and 3, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Now $\vec{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, so $A^t\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\cdot 2^t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3^t\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-2^{t+1}+3^t \\ -2^{t+1}+2\cdot 3^t \\ 2\cdot 3^t \end{bmatrix}$.

7.4.**17** The eigenvalues of A are 1, 0, and 2, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Now $\vec{x}_0 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, so $A^t\vec{x}_0 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 2^t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4-2^t \\ -4 \\ 2^t \end{bmatrix}$.

7.4.**18** The eigenvalues of A are 3, 1, and 0, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Now $\vec{x}_0 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, so $A^t\vec{x}_0 = 3^t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3^t+1 \\ 0 \\ 3^t-1 \end{bmatrix}$.

7.4.**19** The eigenvalues of A are 6, 2, and 1, and corresponding eigenvectors are $\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Now $\vec{x}_0 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, so $A^t\vec{x}_0 = 6^t\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} + 2^t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3\cdot 6^t+2^t+1 \\ 5\cdot 6^t-2^t \\ 2\cdot 6^t-1 \end{bmatrix}$.

7.4.**20** Matrix A is a regular transition matrix with $E_1 = \text{span}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{9}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$. By Theorem 7.4.1c, we have $\lim_{t\to\infty} A^t = \frac{1}{9}\begin{bmatrix} 5 & 5 \\ 4 & 4 \end{bmatrix}$.

7.4.**21** Matrix A is a regular transition matrix with $E_1 = \text{span}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By Theorem 7.4.1c, we have $\lim_{t\to\infty} A^t = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

7.4.**22** Matrix A is a regular transition matrix with $E_1 = \text{span}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By Theorem 7.4.1c, we have $\lim_{t\to\infty} A^t = \frac{1}{4}\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$.

7.4.**23** Matrix A is a regular transition matrix with $E_1 = \text{span}\begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{22}\begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$. By Theorem 7.4.1c, we have $\lim_{t\to\infty} A^t = \frac{1}{22}\begin{bmatrix} 7 & 7 & 7 \\ 10 & 10 & 10 \\ 5 & 5 & 5 \end{bmatrix}$.

7.4.**24** Matrix A is a regular transition matrix with $E_1 = \text{span}\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{10}\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$. By Theorem 7.4.1c, we have $\lim_{t\to\infty} A^t = \frac{1}{10}\begin{bmatrix} 2 & 2 & 2 \\ 5 & 5 & 5 \\ 3 & 3 & 3 \end{bmatrix}$.

7.4.**25** Note that A is a regular transition matrix and $\vec{x}_0$ is a distribution vector. Now $E_1 = \text{span}\begin{bmatrix} 10 \\ 7 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{17}\begin{bmatrix} 10 \\ 7 \end{bmatrix}$. By Theorem 7.4.1b. we have $\lim_{t\to\infty}(A^t\vec{x}_0) = \vec{x}_{equ} = \frac{1}{17}\begin{bmatrix} 10 \\ 7 \end{bmatrix}$.

7.4.**26** Note that A is a regular transition matrix and $\vec{x}_0$ is a distribution vector. Now $E_1 = \text{span}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{11}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$. By Theorem 7.4.1b. we have $\lim_{t\to\infty}(A^t\vec{x}_0) = \vec{x}_{equ} = \frac{1}{11}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

7.4.**27** Note that A is a regular transition matrix and $\vec{x}_0$ is a distribution vector. Now $E_1 = \text{span}\begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{22}\begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$. By Theorem 7.4.1b. we have $\lim_{t\to\infty}(A^t\vec{x}_0) = \vec{x}_{equ} = \frac{1}{22}\begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$.

7.4.**28** Note that A is a regular transition matrix and $\vec{x}_0$ is a distribution vector. Now $E_1 = \text{span}\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{10}\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$. By Theorem 7.4.1b. we have $\lim_{t\to\infty}(A^t\vec{x}_0) = \vec{x}_{equ} = \frac{1}{10}\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$.

7.4.**29** Note that A is a regular transition matrix and $\vec{x}_0$ is a distribution vector. Now $E_1 = \text{span}\begin{bmatrix} 24 \\ 29 \\ 31 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{84}\begin{bmatrix} 24 \\ 29 \\ 31 \end{bmatrix}$. By Theorem 7.4.1b. we have $\lim_{t\to\infty}(A^t\vec{x}_0) = \vec{x}_{equ} = \frac{1}{84}\begin{bmatrix} 24 \\ 29 \\ 31 \end{bmatrix}$.

7.4.**30** $f_A(\lambda) = (2-\lambda)^2 - 3$ so $\lambda_{1,2} = 2 \pm \sqrt{3}$ (or approximately 3.73 and 0.27) with eigenvectors $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$. See Figure 7.20.

b The trajectory starting at $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is above the line E_{λ_1}, so that $A^t\begin{bmatrix} 0 \\ 1 \end{bmatrix} =$

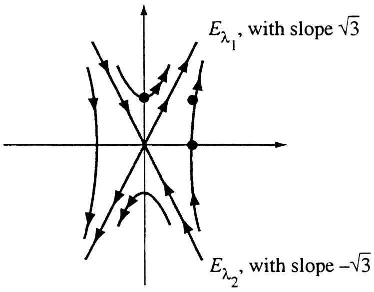


Figure 7.20: for Problem 7.4.30a.

(second column of A^t) has a slope of more than $\sqrt{3}$, for all t. Applying this to $t = 6$ gives the estimate $\sqrt{3} < \frac{1351}{780}$.

Likewise, the trajectory starting at $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is below E_{λ_1}, so that $A^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$

(sum of the two columns of
A^t) has a slope of less than $\sqrt{3}$. Applying this to $t = 4$ gives $\frac{265}{153} < \sqrt{3}$.

c $\det(A^6) = (\det A)^6 = 1$ and $\det(A^6) = 1351^2 - 780 \cdot 2340$, so that $1351^2 - 780 \cdot 2340 = 1$.

Dividing both sides by $1351 \cdot 780$ we obtain $\frac{1351}{780} - \frac{2340}{1351} = \frac{1}{780 \cdot 1351} < 10^{-6}$.

Now note that $\frac{2340}{1351}$ is the slope of $A^6 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which is less than $\sqrt{3}$.

Therefore $\frac{1351}{780} - \sqrt{3} < \frac{1351}{780} - \frac{2340}{1351} < 10^{-6}$, as claimed.

d The slope of $A^6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2131 \\ 3691 \end{bmatrix}$ is less than $\sqrt{3}$, i.e. $\frac{3691}{2131} < \sqrt{3}$.

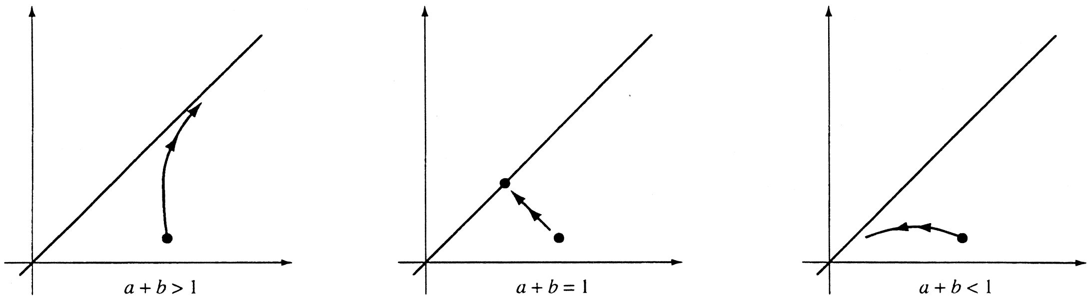
7.4.**31** The matrix of the dynamical system is $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ so $f_A(\lambda) = (a - \lambda)^2 - b^2$.

Hence, $\lambda_{1,2} = a \pm b$, and the respective eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Since $\vec{x}(0) = \begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix} = \frac{7}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{5}{4} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, by Theorem 7.1.6, $\vec{x}(t) = \frac{7}{4}(a + b)^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{5}{4}(a - b)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Note that $a - b$ is between 0 and 1, so that the second summand in the formula above goes to $\vec{0}$ as t goes to infinity. Qualitatively different outcomes occur depending on whether $a + b$ exceeds 1, equals 1, or is less than 1. See Figure 7.21.

7.4.**32** $C(t + 1) = 0.8C(t) + 10$ so if $A \begin{bmatrix} C(t) \\ 1 \end{bmatrix} = \begin{bmatrix} C(t + 1) \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 0.8 & 10 \\ 0 & 1 \end{bmatrix}$. A has eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 50 \\ 1 \end{bmatrix}$ corresponding to $\lambda_1 = 0.8$ and $\lambda_2 = 1$. Since $\begin{bmatrix} C(0) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -50 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 50 \\ 1 \end{bmatrix}$, we have $C(t) = -50(0.9)^t + 50$, hence in the long run, there will be 50 spectators. The graph of $C(t)$ looks similar to the graph in Figure 7.22.



The spending of both nations grows: an arms race

The spending of both nations stabilizes at an equal level of \$b 1.75.

The spending of both nations decreases to zero: disarmament

Figure 7.21: for Problem 7.4.31.

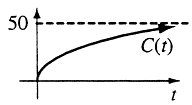


Figure 7.22: for Problem 7.4.32.

7.4.**33** a $A = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

b After 10 rounds, we have $A^{10}\begin{bmatrix} 7 \\ 11 \\ 5 \end{bmatrix} \approx \begin{bmatrix} 7.6660156 \\ 7.6699219 \\ 7.6640625 \end{bmatrix}$.

After 50 rounds, we have $A^{50}\begin{bmatrix} 7 \\ 11 \\ 5 \end{bmatrix} \approx \begin{bmatrix} 7.66666666667 \\ 7.66666666667 \\ 7.66666666667 \end{bmatrix}$.

c The eigenvalues of A are 1 and $-\frac{1}{2}$ with

$E_1 = \text{span}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $E_{-\frac{1}{2}} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}\right)$

so $\vec{x}(t) = \left(1 + \frac{c_0}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \left(-\frac{1}{2}\right)^t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \left(-\frac{1}{2}\right)^t \frac{c_0}{3}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

After 1001 rounds, Alberich will be ahead of Brunnhilde $\left(\text{by } \left(\frac{1}{2}\right)^{1001}\right)$, so that Carl needs to beat Alberich to win the game. A straightforward computation shows that $c(1001) - a(1001) = \left(\frac{1}{2}\right)^{1001}(1 - c_0)$; Carl wins if this quantity is positive, which is the case if c_0 is less than 1.
Alternatively, observe that the ranking of the players is reversed in each round: whoever is first will be last after the next round. Since the total number of rounds is odd (1001), Carl wants to be last initially to win the game; he wants to choose a smaller number than both Alberich and Brunnhilde.

7.4.**34** a $a_{11} = 0.7$ means that only 70% of the pollutant present in Lake Sils at a given time is still there a week later; some is carried down to Lake Silvaplana by the river Inn, and some is absorbed or evaporates. The other diagonal entries can be interpreted analogously: $a_{21} = 0.1$ means that 10% of the pollutant present in Lake Sils at any given time can be found in Lake Silvaplana a week later, carried down by the river Inn. The significance of the coefficient $a_{32} = 0.2$ is analogous; $a_{31} = 0$ means that no pollutant is carried down from Lake Sils to Lake St. Moritz in just one week. The matrix is lower triangular since no pollutant is carried from Lake Silvaplana to Lake Sils. The river Inn would have to flow the other way.

b The eigenvalues of A are 0.8, 0.6, 0.7, with corresponding eigenvectors

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

$$\vec{x}(0) = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = 100 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 100 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 100 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

so $\vec{x}(t) = 100(0.8)^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 100(0.6)^t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 100(0.7)^t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ or

$x_1(t) = 100(0.7)^t$

$x_2(t) = 100(0.7)^t - 100(0.6)^t$

$x_3(t) = 100(0.8)^t + 100(0.6)^t - 200(0.7)^t$. See Figure 7.23.

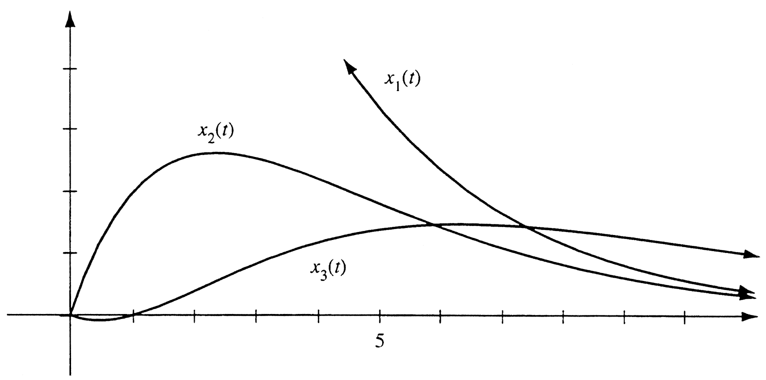


Figure 7.23: for Problem 7.4.34b.

Using calculus, we find that the function $x_2(t) = 100(0.7)^t - 100(0.6)^t$ reaches its maximum at $t \approx 2.33$. Keep in mind, however, that our model holds for integer t only.

7.4.**35** a $A = \begin{bmatrix} 0.1 & 0.2 \\ 0.4 & 0.3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b $B = \begin{bmatrix} A & \vec{b} \\ 0 & 1 \end{bmatrix}$

c The eigenvalues of A are 0.5 and -0.1 with associated eigenvectors $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$.

The eigenvalues of B are 0.5, -0.1, and 1. If $A\vec{v}=\lambda\vec{v}$ then $B\begin{bmatrix}\vec{v}\\0\end{bmatrix}=\lambda\begin{bmatrix}\vec{v}\\0\end{bmatrix}$ so $\begin{bmatrix}\vec{v}\\0\end{bmatrix}$ is an eigenvector of B.

Furthermore, $\begin{bmatrix}2\\4\\1\end{bmatrix}$ is an eigenvector of B corresponding to the eigenvalue 1. Note that this vector is $\begin{bmatrix}-(A-I_2)^{-1}\vec{b}\\1\end{bmatrix}$.

d Write $\vec{y}(0)=\begin{bmatrix}x_1(0)\\x_2(0)\\1\end{bmatrix}=c_1\begin{bmatrix}1\\2\\0\end{bmatrix}+c_2\begin{bmatrix}1\\-1\\0\end{bmatrix}+c_3\begin{bmatrix}2\\4\\1\end{bmatrix}$.

Note that $c_3=1$.

Now $\vec{y}(t)=c_1(0.5)^t\begin{bmatrix}1\\2\\0\end{bmatrix}+c_2(-0.1)^t\begin{bmatrix}1\\-1\\0\end{bmatrix}+\begin{bmatrix}2\\4\\1\end{bmatrix}\overset{t\to\infty}{\longrightarrow}\begin{bmatrix}2\\4\\1\end{bmatrix}$ so that $\vec{x}(t)\overset{t\to\infty}{\longrightarrow}\begin{bmatrix}2\\4\end{bmatrix}$.

7.4.**36** a $T_1(t+1)=0.6T_1(t)+0.1T_2(t)+20$

$T_2(t+1)=0.1T_1(t)+0.6T_2(t)+0.1T_3(t)+20$

$T_3(t+1)=0.1T_2(t)+0.6T_3(t)+40$

so $A=\begin{bmatrix}0.6&0.1&0\\0.1&0.6&0.1\\0&0.1&0.6\end{bmatrix}$, $\vec{b}=\begin{bmatrix}20\\20\\40\end{bmatrix}$

b $B=\begin{bmatrix}A&\vec{b}\\0&1\end{bmatrix}$

c $\vec{y}(10)=B^{10}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\approx\begin{bmatrix}70.86\\93.95\\120.56\\1\end{bmatrix}$

$\vec{y}(30)=B^{30}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\approx\begin{bmatrix}74.989\\99.985\\124.989\\1\end{bmatrix}$

$\vec{y}(t)$ seems to approach $\begin{bmatrix}75\\100\\125\\1\end{bmatrix}$ as $t\to\infty$

d The eigenvalues of A are $\lambda_1\approx0.45858$, $\lambda_2=0.6$, $\lambda_3\approx0.74142$ so the eigenvalues of B are $\lambda_1\approx0.45858$, $\lambda_2=0.6$, $\lambda_3\approx0.74142$, $\lambda_4=1$.

If $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are eigenvectors of A (with $A\vec{v}_i = \lambda_i \vec{v}_i$), then $\begin{bmatrix} \vec{v}_1 \\ 0 \end{bmatrix}, \begin{bmatrix} \vec{v}_2 \\ 0 \end{bmatrix}, \begin{bmatrix} \vec{v}_3 \\ 0 \end{bmatrix}$ are corresponding eigenvectors of B. Furthermore, $\begin{bmatrix} 75 \\ 100 \\ 125 \\ 1 \end{bmatrix}$ is an eigenvector if B with eigenvalue 1. Since λ_1, λ_2, λ_3 are all less than 1, $\lim_{t\to\infty} \vec{x}(t) = \begin{bmatrix} 75 \\ 100 \\ 125 \end{bmatrix}$, as in Exercise 35.

7.4.**37** a If $\vec{x}(t) = \begin{bmatrix} r(t) \\ p(t) \\ w(t) \end{bmatrix}$, then $\vec{x}(t+1) = A\vec{x}(t)$ with $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$.

The eigenvalues of A are $0, \frac{1}{2}, 1$ with eigenvectors $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Since $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{x}(t) = \frac{1}{2}\left(\frac{1}{2}\right)^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ for $t > 0$.

b As $t \to \infty$ the ratio is $1:2:1$ (since the first term of $\vec{x}(t)$ drops out).

7.4.**38** a We are told that

$a(t+1) = a(t) + j(t)$

$j(t+1) = a(t)$, so that $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

b $f_A(\lambda) = \lambda(\lambda-1) - 1 = \lambda^2 - \lambda - 1$ so $\lambda_{1,2} = \frac{1\pm\sqrt{5}}{2}$ with eigenvectors $\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$.

Since $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ we have $\vec{x}(t) = \frac{1}{\sqrt{5}}(\lambda_1)^t \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}}(\lambda_2)^t \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$, i.e.

$a(t) = \frac{1}{\sqrt{5}}((\lambda_1)^{t+1} - (\lambda_2)^{t+1})$

$j(t) = \frac{1}{\sqrt{5}}((\lambda_1)^t - (\lambda_2)^t)$.

c As $t \to \infty$, $\frac{a(t)}{j(t)} \to \lambda_1 = \frac{1+\sqrt{5}}{2}$, since $|\lambda_2| < 1$.

7.4.**39** The eigenfunctions with eigenvalue λ are the nonzero functions $f(x)$ such that $T(f(x)) = f'(x) - f(x) = \lambda f(x)$, or $f'(x) = (\lambda+1)f(x)$. From calculus we recall that those are the exponential functions of the form $f(x) = Ce^{(\lambda+1)x}$, where C is a nonzero constant. Thus all real numbers are eigenvalues of T, and the eigenspace E_λ is one-dimensional, spanned by $e^{(\lambda+1)x}$.

7.4.**40** The eigenfunctions with eigenvalue λ are the nonzero functions $f(x)$ such that $T(f(x)) = 5f'(x) - 3f(x) = \lambda f(x)$, or $f'(x) = \frac{\lambda+3}{5}f(x)$. From calculus we recall that those are the exponential functions of the form

$f(x) = Ce^{(\lambda+3)x/5}$, where C is a nonzero constant. Thus all real numbers are eigenvalues of T, and the eigenspace E_λ is one-dimensional, spanned by $e^{(\lambda+3)x/5}$.

7.4.**41** The nonzero symmetric matrices are eigenmatrices with eigenvalue 2, since $L(A) = A + A^T = 2A$ in this case. The nonzero skew-symmetric matrices have eigenvalue 0, since $L(A) = A + A^T = A - A = 0$. Yes, L is diagonalizable, since we have the eigenbasis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (three symmetric matrices, and one skew-symmetric one).

7.4.**42** The nonzero symmetric matrices are eigenmatrices with eigenvalue 0, since $L(A) = A - A^T = A - A = 0$ in this case. The nonzero skew-symmetric matrices have eigenvalue 2, since $L(A) = A - A^T = A + A = 2A$. Yes, L is diagonalizable, since we have the eigenbasis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (three symmetric matrices, and one skew-symmetric one).

7.4.**43** The nonzero real numbers are "eigenvectors" with eigenvalue 1, and the nonzero imaginary numbers (of the form iy) are "eigenvectors" with eigenvalue -1. Yes, T is diagonalizable, since we have the eigenbasis $1, i$.

7.4.**44** The nonzero sequence $(x_0, x_1, x_2, \ldots)$ is an eigensequence with eigenvalue λ if

$T(x_0, x_1, x_2, \ldots) = (x_2, x_3, x_4, \ldots) = \lambda(x_0, x_1, x_2, \ldots) = (\lambda x_0, \lambda x_1, \lambda x_2, \ldots)$. This means that $x_2 = \lambda x_0, x_3 = \lambda x_1, \ldots, x_{n+2} = \lambda x_n, \ldots$. These are the sequences of the form $(a, b, \lambda a, \lambda b, \lambda^2 a, \lambda^2 b, \ldots)$, where at least one of the first two terms, a and b, is nonzero.

Thus all real numbers λ are eigenvalues of T, and the eigenspace E_λ is two-dimensional, with basis $(1, 0, \lambda, 0, \lambda^2, 0, \ldots), (0, 1, 0, \lambda, 0, \lambda^2, \ldots)$.

7.4.**45** The nonzero sequence $(x_0, x_1, x_2, \ldots)$ is an eigensequence with eigenvalue λ if $T(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots) = \lambda(x_0, x_1, x_2, \ldots) = (\lambda x_0, \lambda x_1, \lambda x_2, \ldots)$. This means that $0 = \lambda x_0, x_0 = \lambda x_1,\ x_1 = \lambda x_2, \ldots, x_n = \lambda x_{n+1}, \ldots$. If λ is nonzero, then these equations imply that $x_0 = \frac{1}{\lambda}0 = 0, x_1 = \frac{1}{\lambda}x_0 = 0, x_2 = \frac{1}{\lambda}x_1 = 0, \ldots$, so that there are no eigensequences in this case. If $\lambda = 0$, then we have $x_0 = \lambda x_1 = 0, x_1 = \lambda x_2 = 0, x_2 = \lambda x_3 = 0, \ldots$, so that there aren't any eigensequences either. In summary: There are no eigenvalues and eigensequences for T.

7.4.**46** The nonzero sequence $(x_0, x_1, x_2, \ldots)$ is an eigensequence with eigenvalue λ if

$T(x_0, x_1, x_2, \ldots) = (x_0, x_2, x_4, \ldots) = \lambda(x_0, x_1, x_2, \ldots) = (\lambda x_0, \lambda x_1, \lambda x_2, \ldots)$. This means that $x_0 = \lambda x_0, x_2 = \lambda x_1, x_4 = \lambda x_2, \ldots, x_{2n} = \lambda x_n, \ldots$. For each λ, there are lots of eigensequences: we can choose the terms x_k for odd k freely and then fix the x_k for even k according to the formula $x_{2n} = \lambda x_n$. For example, eigenspace E_3 consists of the sequences of the form $(x_0 = 0, x_1, x_2 = 3x_1, x_3, x_4 = 9x_1, x_5, x_6 = 3x_3, x_7, x_8 = 27x_1, x_9, \ldots)$, where $x_1, x_3, x_5, x_7, x_9, \ldots$ are arbitrary. Note that all the eigenspaces are infinite-dimensional. The condition $x_0 = \lambda x_0$ implies that $x_0 = 0$, except for $\lambda = 1$, in which case x_0 is arbitrary.

7.4.**47** The nonzero even functions, of the form $f(x) = a + cx^2$, are eigenfunctions with eigenvalue 1, and the nonzero odd functions, of the form $f(x) = bx$, have eigenvalue -1. Yes, T is diagonalizable, since the standard basis, $1, x, x^2$, is an eigenbasis for T.

7.4.**48** Apply T to the standard basis: $T(1) = 1, T(x) = 2x$, and $T(x^2) = (2x)^2 = 4x^2$. This gives the eigenvalues 1, 2, and 4, with corresponding eigenfunctions $1, x, x^2$. Yes, T is diagonalizable, since the standard basis is an eigenbasis for T.

7.4.**49** The matrix of T with respect to the standard basis $1, x, x^2$ is $B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -6 \\ 0 & 0 & 9 \end{bmatrix}$. The eigenvalues of B are 1, 3, 9, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$. The eigenvalues of T are 1,3,9, with corresponding eigenfunctions $1, 2x-1, 4x^2-4x+1 = (2x-1)^2$. Yes, T is diagonalizable, since the functions $1, 2x-1, (2x-1)^2$ from an eigenbasis.

7.4.**50** The matrix of T with respect to the standard basis $1, x, x^2$ is $B = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$. The only eigenvalue of B is 1, with corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The only eigenvalue of T is 1 as well, with corresponding eigenfunction $f(x) = 1$. T fails to be diagonalizable, since there is only one eigenvalue, with a one-dimensional eigenspace.

7.4.**51** The nonzero constant functions $f(x) = b$ are the eigenfunctions with eigenvalue 0. If $f(x)$ is a polynomial of degree ≥ 1, then the degree of $f(x)$ exceeds the degree of $f'(x)$ by 1 (by the power rule of calculus), so that $f'(x)$ cannot be a scalar multiple of $f(x)$. Thus 0 is the only eigenvalue of T, and the eigenspace E_0 consists of the constant functions.

7.4.**52** Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, with $a_n \neq 0$, be an eigenfunction of T with eigenvalue λ. Then $T(f(x)) = x(a_1 + 2a_2x + \cdots + na_nx^{n-1}) = a_1x + 2a_2x^2 + \cdots + na_nx^n = \lambda(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = \lambda a_0 + \lambda a_1x + \lambda a_2x^2 + \cdots + \lambda a_nx^n$. This means that $\lambda a_0 = 0, \lambda a_1 = a_1, \lambda a_2 = 2a_2, \ldots, \lambda a_n = na_n$. Since we assumed that $a_n \neq 0$, we can conclude that $\lambda = n$. Now it follows that $a_0 = a_1 = \cdots = a_{n-1} = 0$, so that the eigenfunctions with eigenvalue n are the nonzero scalar multiples of x^n, of the form $f(x) = a_nx^n$. This makes good sense, since $T(x^n) = x(nx^{n-1}) = nx^n$. In summary: The eigenvalues are the integers $n = 0, 1, 2, \ldots$, and the eigenspace E_n is span (x^n)

7.4.**53** In Exercises 7.2.30, 7.2.31, and 7.3.32, we prove the following facts concerning the eigenvalues of a positive transition matrix B:

1. $\lambda = 1$ is an eigenvalue of B with gemu(1) = 1.
2. If λ is any eigenvalue of B, then $-1 < \lambda \leq 1$.

Here we will show that the same properties hold for the eigenvalues of a *regular* transition matrix A. If A is a regular transition matrix, then there exists an odd positive integer m such that $B = A^m$ is positive, by Exercise 2.3.75.

a. Exercises 7.2.22 and 7.2.29 imply that 1 is an eigenvalue of A. Now the eigenspace $E_{1,A} = \ker(A - I_n)$ is a subspace of the eigenspace $E_{1,B} = \ker(B - I_n)$, where $B = A^m$. Since $\dim(E_{1,B}) = 1$, we have $\dim(E_{1,A}) = 1$ as well, meaning that the eigenvalue $\lambda = 1$ of A has geometric multiplicity 1.

b. If λ is an eigenvalue of A, then λ^m will be an eigenvalue of $B = A^m$. Since B is positive, we know that $-1 < \lambda^m \leq 1$. Taking the m^{th}root, we see that $-1 < \lambda \leq 1$, as claimed.

7.4.**54** Note that $A^2 = 0$, but $B^2 \neq 0$. Since A^2 fails to be similar to B^2, matrix A isn't similar to B (see Example 7 of Section 3.4).

7.4.55 Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, for example.

7.4.56 The hint shows that matrix $M = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ is similar to $N = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$; thus matrices M and N have the same characteristic polynomial, by Theorem 7.3.5a. Now $f_M(\lambda) = \det \begin{bmatrix} AB - \lambda I_n & 0 \\ B & -\lambda I_n \end{bmatrix} = (-\lambda)^n \det(AB - \lambda I_n) = (-\lambda)^n f_{AB}(\lambda)$. To understnd the second equality, consider Theorem 6.1.5. Likewise, $f_N(\lambda) = (-\lambda)^n f_{BA}(\lambda)$. It follows that $(-\lambda)^n f_{AB}(\lambda) = (-\lambda)^n f_{BA}(\lambda)$ and therefore $f_{AB}(\lambda) = f_{BA}(\lambda)$, as claimed.

7.4.57 Modifying the hint in Exercise 56 slightly, we can write $\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$. Thus matrix $M = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ is similar to $N = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$. By Theorem 7.3.5a, matrices M and N have the same characteristic polynomial.

Now $f_M(\lambda) = \det \begin{bmatrix} AB - \lambda I_m & 0 \\ B & -\lambda I_n \end{bmatrix} = (-\lambda)^n \det(AB - \lambda I_m) = (-\lambda)^n f_{AB}(\lambda)$. To understand the second equality, consider Theorem 6.1.5. Likewise, $f_N(\lambda)$

$= \det \begin{bmatrix} -\lambda I_m & 0 \\ B & BA - \lambda I_n \end{bmatrix} = (-\lambda)^m f_{BA}(\lambda)$.

It follows that $(-\lambda)^n f_{AB}(\lambda) = (-\lambda)^m f_{BA}(\lambda)$. Thus matrices AB and BA have the same *nonzero* eigenvalues, with the same algebraic multiplicities.

If $\text{mult}(AB)$ and $\text{mult}(BA)$ are the algebraic multiplicities of 0 as an eigenvalue of AB and BA, respectively, then the equation $(-\lambda)^n f_{AB}(\lambda) = (-\lambda)^m f_{BA}(\lambda)$ implies that

$n + \text{mult}(AB) = m + \text{mult}(BA)$.

7.4.58 a. If $\vec{v}$ is in the image of A, then $\vec{v} = A\vec{w}$ for some vector $\vec{w}$. Now $A\vec{v} = A^2\vec{w} = \vec{0}$, showing that $\vec{v}$ is in the kernel of A.

b. From part (a) we know that $\dim \text{im} A \le \dim \ker A$. Also, $\dim \text{im} A > 0$ since A is nonzero, and $\dim \ker A + \dim \text{im} A = 3$ by the rank-nullity theorem. This leaves us with only one possibility, namely, $\dim \text{im} A = 1$ and $\dim \ker A = 2$.

c. Consider a relation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Multiplying both sides with A from the left, keeping in mind that $\vec{v}_1$ and $\vec{v}_3$ are in the kernel, we find $c_2A\vec{v}_2 = c_2\vec{v}_1 = \vec{0}$, so $c_2 = 0$. Now $c_1 = c_3 = 0$ since $\vec{v}_1$ and $\vec{v}_3$ are independent by construction.

d. We have $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ since $A\vec{v}_1 = \vec{0}$, $A\vec{v}_2 = \vec{v}_1$ and $A\vec{v}_3 = \vec{0}$.

7.4.59 Yes, A is similar to B, since both A and B are similar to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, by Exercise 58.

7.4.**60** We will use the method outlined in Exercise 58. We start with the vector $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the image of A. Since $\vec{v}_1 =$ (first column of A) $= A\vec{e}_1$, we can let $\vec{v}_2 = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Finally, we let $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, another vector in the kernel (use Kyle numbers!). Thus $S = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$ will do the job, but there are other answers.

7.4.**61** First we need to verify that the vectors $\vec{v}_1, ..., \vec{v}_r, \vec{w}_1, ..., \vec{w}_r, \vec{u}_1, ..., \vec{u}_m$ are linearly independent. Consider a relation $a_1\vec{v}_1 + ... + a_r\vec{v}_r + b_1\vec{w}_1 + ... + b_r\vec{w}_r + c_1\vec{u}_1 + ... + c_m\vec{u}_m = \vec{0}$. Multiplying both sides with A, we find $b_1\vec{v}_1 + ... + b_r\vec{v}_r = \vec{0}$, so that $b_1 = ... = b_r = 0$ since the $\vec{v}_i$ are independent by construction. Now the a_i and the c_i must be 0 as well, since the vectors $\vec{v}_1, ..., \vec{v}_r, \vec{u}_1, ..., \vec{u}_m$ are independent by construction.

Secondly, we need to show that $2r + m = n$, meaning that we have the right number of vectors to form a basis of $\mathbb{R}^n$. Indeed. $n = \dim\ker A + \dim \operatorname{im} A = (r + m) + r = 2r + m$. Each of the r pairs $\vec{v}_i, \vec{w}_i$ gives us a block $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ along the diagonal of B, since $A\vec{v}_i = \vec{0}$ and $A\vec{w}_i = \vec{v}_i$, while all other entries of matrix B are zero.

7.4.**62** A nonzero function f is an eigenfunction of T with eigenvalue λ if $T(f) = f'' + af' + bf = \lambda f$, or, $f'' + af' + (b - \lambda)f = 0$. By Theorem 4.1.7, this differential equation has a two-dimensional solution space. Thus all real numbers are eigenvalues of T, and all the eigenspaces are two-dimensional.

7.4.**63** a We need to solve the differential equation $f''(x) = f(x)$. As in Example 18 of Section 4.1, we will look for *exponential* solutions. The function $f(x) = e^{kx}$ is a solution if $k^2 = 1$, or $k = \pm 1$. Thus the eigenspace E_1 is the span of functions e^x and e^{-x}.

b We need to solve the differential equation $f''(x) = 0$. Integration gives $f'(x) = C$, a constant. If we integrate again, we find $f(x) = Cx + c$, where c is another arbitrary constant. Thus $E_0 = \operatorname{span}(1, x)$.

c The solutions of the differential equation $f''(x) = -f(x)$ are the functions $f(x) = a\cos(x) + b\sin(x)$, so that $E_{-1} = \operatorname{span}(\cos x, \sin x)$. See the introductory example of Section 4.1 and Exercise 4.1.58.

d Modifying part c, we see that the solutions of the differential equation $f''(x) = -4f(x)$ are the functions $f(x) = a\cos(2x) + b\sin(2x)$, so that $E_{-4} = \operatorname{span}(\cos(2x), \sin(2x))$.

7.4.**64** The eigenvalues of A are 1 and 3, with associated eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Arguing as in Exercise 65, we find the basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ for V, so that $\dim(V) = 2$.

7.4.**65** Let's write S in terms of its columns, as $S = [\,\vec{v} \quad \vec{w}\,]$.

We want $A[\,\vec{v} \quad \vec{w}\,] = [\,\vec{v} \quad \vec{w}\,]\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$, or, $[\,A\vec{v} \quad A\vec{w}\,] = [\,5\vec{v} \quad -\vec{w}\,]$, that is, we want $\vec{v}$ to be in the eigenspace E_5, and $\vec{w}$ in E_{-1}. We find that $E_5 = \operatorname{span}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $E_{-1} = \operatorname{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so that S must be of the form $\left[a\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad b\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right] = a\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. Thus, a basis of the space V is $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$, and $\dim(V) = 2$.

7.4.**66** For A we find the eigenspaces $E_1 = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right)$ and $E_2 = \text{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)$. If we write $S = [\vec{u}\ \vec{v}\ \vec{w}]$, then we want $A[\vec{u}\ \vec{v}\ \vec{w}] = [\vec{u}\ \vec{v}\ \vec{w}]\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$, or $[A\vec{u}\ A\vec{v}\ A\vec{w}] = [\vec{u}\ \vec{v}\ 2\vec{w}]$, that is, $\vec{u}$ and $\vec{v}$ must be in E_1, and $\vec{w}$ must be in E_2. The matrices S we seek are of the form $S = \begin{bmatrix} a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\-1\end{bmatrix} & c\begin{bmatrix}1\\0\\0\end{bmatrix}+d\begin{bmatrix}0\\1\\-1\end{bmatrix} & e\begin{bmatrix}1\\1\\0\end{bmatrix}\end{bmatrix}$, and a basis of V is $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\-1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&-1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&1\\0&0&0\end{bmatrix}$. The dimension of V is five.

7.4.**67** Let $E_{\lambda_1} = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ and $E_{\lambda_2} = \text{span}(\vec{w}_1, \vec{w}_2)$. As in Exercise 65, we can see that S must be of the form $[\,\vec{x}_1\quad \vec{x}_2\quad \vec{x}_3\quad \vec{x}_4\quad \vec{x}_5\,]$ where $\vec{x}_1, \vec{x}_2$ and $\vec{x}_3$ are in E_{λ_1} and $\vec{x}_4$ and $\vec{x}_5$ are in E_{λ_2}. Thus, we can write $\vec{x}_1 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, for example, or $\vec{x}_5 = d_1\vec{w}_1 + d_2\vec{w}_2$.

Using Summary 4.1.6, we find a basis: $[\,\vec{v}_1\quad \vec{0}\quad \vec{0}\quad \vec{0}\quad \vec{0}\,], [\,\vec{v}_2\quad \vec{0}\quad \vec{0}\quad \vec{0}\quad \vec{0}\,]$,
$[\,\vec{v}_3\quad \vec{0}\quad \vec{0}\quad \vec{0}\quad \vec{0}\,], [\,\vec{0}\quad \vec{v}_1\quad \vec{0}\quad \vec{0}\quad \vec{0}\,], [\,\vec{0}\quad \vec{v}_2\quad \vec{0}\quad \vec{0}\quad \vec{0}\,], [\,\vec{0}\quad \vec{v}_3\quad \vec{0}\quad \vec{0}\quad \vec{0}\,]$,
$[\,\vec{0}\quad \vec{0}\quad \vec{v}_1\quad \vec{0}\quad \vec{0}\,], [\,\vec{0}\quad \vec{0}\quad \vec{v}_2\quad \vec{0}\quad \vec{0}\,], [\,\vec{0}\quad \vec{0}\quad \vec{v}_3\quad \vec{0}\quad \vec{0}\,], [\,\vec{0}\quad \vec{0}\quad \vec{0}\quad \vec{w}_1\quad \vec{0}\,]$,
$[\,\vec{0}\quad \vec{0}\quad \vec{0}\quad \vec{w}_2\quad \vec{0}\,], [\,\vec{0}\quad \vec{0}\quad \vec{0}\quad \vec{0}\quad \vec{w}_1\,], [\,\vec{0}\quad \vec{0}\quad \vec{0}\quad \vec{0}\quad \vec{w}_2\,]$.

Thus, the dimension of the space of matrices S is $3+3+3+2+2 = 13$.

7.4.**68** Let $\vec{v}_1, \ldots, \vec{v}_n$ be an eigenbasis for A, with $A\vec{v}_i = \lambda_i\vec{v}_i$. Arguing as in Exercises 64 through 67, we see that the i^{th} column of S must be in E_{λ_i}, so that it must be of the form $c_i\vec{v}_i$ for some scalar c_i. The matrices S we seek are of the form $S = [c_1\vec{v}_1\ \ldots\ c_n\vec{v}_n]$, involving the n arbitrary constants $c_1, \ldots, c_n$, so that the dimension of V is n.

7.4.**69** a B is diagonalizable since it has three distinct eigenvalues, so that $S^{-1}BS$ is diagonal for some invertible S. But $S^{-1}AS = S^{-1}I_3S = I_3$ is diagonal as well. Thus A and B are indeed simultaneously diagonalizable.

b There is an invertible S such that $S^{-1}AS = D_1$ and $S^{-1}BS = D_2$ are both diagonal. Then $A = SD_1S^{-1}$ and $B = SD_2S^{-1}$, so that $AB = (SD_1S^{-1})(SD_2S^{-1}) = SD_1D_2S^{-1}$ and $BA = (SD_2S^{-1})(SD_1S^{-1}) = SD_2D_1S^{-1}$. These two results agree, since $D_1D_2 = D_2D_1$ for the diagonal matrices D_1 and D_2.

c Let A be I_n and B a nondiagonalizable $n \times n$ matrix, for example, $A = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $B = \begin{bmatrix}1&1\\0&1\end{bmatrix}$.

d Suppose $BD = DB$ for a diagonal D with distinct diagonal entries. The ij-th entry of the matrix $BD = DB$ is $b_{ij}d_{jj} = d_{ii}b_{ij}$. For $i \neq j$ this implies that $b_{ij} = 0$. Thus B must be diagonal.

e Since A has n distinct eigenvalues, A is diagonalizable, that is, there is an invertible S such that $S^{-1}AS = D$ is a diagonal matrix with n distinct diagonal entries. We claim that $S^{-1}BS$ is diagonal as well; by part d it suffices to show that $S^{-1}BS$ commutes with $D = S^{-1}AS$. This is easy to verify:

$$(S^{-1}BS)D = (S^{-1}BS)(S^{-1}AS) = S^{-1}BAS = S^{-1}ABS = (S^{-1}AS)(S^{-1}BS) = D(S^{-1}BS)\,.$$

7.4.**70** Consider an $n \times n$ matrix A with m distinct eigenvalues $\lambda_1, ..., \lambda_m$.

If $\vec{v}$ is an eigenvector of A with eigenvalue λ_m, then $(A - \lambda_m I_n)\,\vec{v} = \vec{0}$, so that $(A - \lambda_1 I_n)\,(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n)\,\vec{v} = \vec{0}$. Since the factors in the product $(A - \lambda_1 I_n)\,(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n)$ commute, we can conclude that $(A - \lambda_1 I_n)\,(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n)\,\vec{v} = \vec{0}$ for all eigenvectors $\vec{v}$ of A.

Now assume that A is diagonalizable. Then there exists a basis consisting of eigenvectors, so that $(A - \lambda_1 I_n)\,(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n)\,\vec{v} = \vec{0}$ for all vectors in $\mathbb{R}^n$. We can conclude that $(A - \lambda_1 I_n)\,(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n) = 0$.

Conversely, assume that $(A - \lambda_1 I_n)\,(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n) = 0$. Then $n = \dim\ker\left((A - \lambda_1 I_n)\,(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n)\right)$ $\leq \dim\ker(A - \lambda_1 I_n) + ... + \dim\ker(A - \lambda_m I_n)$ by Exercise 4.2.83. This means that the geometric multiplicities of the eigenvalues add up to n (we know that this sum cannot exceed n), so that A is diagonalizable, by Theorem 7.3.3b.

7.4.**71** The eigenvalues are 1 and 2, and $(A - I_3)\,(A - 2I_3) = 0$. Thus A is diagonalizable.

7.4.**72** The eigenvalues of this upper triangular matrix are 0 and 1. Now $A\,(A - I_3) = \begin{bmatrix} 0 & 0 & b+ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Matrix A is diagonalizable if (and only if) $b = -ac$.

7.4.**73** If an $n \times n$ matrix A has m distinct eigenvalues $\lambda_1, ..., \lambda_m$, then we can write its characteristic polynomial as $f_A(\lambda) = (\lambda - \lambda_1)\,(\lambda - \lambda_2) \cdots (\lambda - \lambda_m)\,g(\lambda)$ for some polynomial $g(\lambda)$ of degree $n - m$. Now $f_A(A) = (A - \lambda_1 I_n)\,(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n)\,g(A) = 0$ for a diagonalizable matrix A since $(A - \lambda_1 I_n)\,(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n) = 0$, as we have seen in Exercise 70. (In Exercise 7.3.54 we prove the Cayley-Hamilton theorem for all $n \times n$ matrices A, but that proof is a bit longer.)

7.4.**74** a For a diagonalizable $n \times n$ matrix A with only two distinct eigenvalues, λ_1 and λ_2, we have $(A - \lambda_1 I_n)(A - \lambda_2 I_n) = 0$, by Exercise 70. Thus the column vectors of $A - \lambda_2 I_n$ are in the kernel of $A - \lambda_1 I_n$, that is, they are eigenvectors of A with eigenvalue λ_1 (or else they are $\vec{0}$). Conversely, the column vectors of $A - \lambda_1 I_n$ are eigenvectors of A with eigenvalue λ_2 (or else they are $\vec{0}$).

b If A is a 2×2 matrix with distinct eigenvalues λ_1 and λ_2, then the nonzero columns of $A - \lambda_1 I_2$ are eigenvectors of A with eigenvalue λ_2, as we observed in part (a). Since the matrices $A - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $A - \lambda_1 I_2$ have the same first column, the first column of $A - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ will be an eigenvector of A with eigenvalue λ_2 as well (or it is zero). Likewise, the second column of $A - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ will be an eigenvector of A with eigenvalue λ_1 (or it is zero).

Section 7.5

7.5.**1** $z = 3 - 3i$ so $|z| = \sqrt{3^2 + (-3)^2} = \sqrt{18}$ and $\arg(z) = -\frac{\pi}{4}$,

so $z = \sqrt{18}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$.

7.5.2 If $z = r(\cos\theta + i\sin\theta)$ then $z^4 = r^4(\cos 4\theta + i\sin 4\theta)$.

$z^4 = 1$ if $r = 1, \cos 4\theta = 1$ and $\sin 4\theta = 0$ so $4\theta = 2k\pi$ for an integer k, and $\theta = \frac{k\pi}{2}$,

i.e. $z = \cos\left(\frac{k\pi}{2}\right) + i\sin\left(\frac{k\pi}{2}\right)$, $k = 0, 1, 2, 3$. Thus $z = 1, i, -1, -i$. See Figure 7.24.

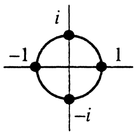


Figure 7.24: for Problem 7.5.2.

7.5.3 If $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

$z^n = 1$ if $r = 1, \cos(n\theta) = 1, \sin(n\theta) = 0$ so $n\theta = 2k\pi$ for an integer k, and $\theta = \frac{2k\pi}{n}$,

i.e. $z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$, $k = 0, 1, 2, \ldots, n-1$. See Figure 7.25.

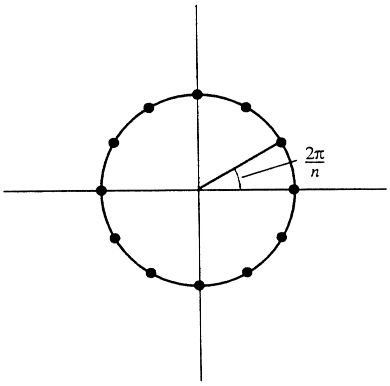


Figure 7.25: for Problem 7.5.3.

7.5.4 Let $z = r(\cos\theta + i\sin\theta)$ then $w = \sqrt{r}\left(\cos\left(\frac{\theta+2\pi k}{2}\right) + i\sin\left(\frac{\theta+2\pi k}{2}\right)\right)$, $k = 0,\ 1$.

7.5.5 Let $z = r(\cos\theta + i\sin\theta)$ then $w = \sqrt[n]{r}\left(\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right)$, $k = 0, 1, 2, \ldots, n-1$.

7.5.6 If we have $z = r(\cos\theta + i\sin\theta)$ then $\frac{1}{z}$ must have the property that $z \cdot \frac{1}{z} = 1 = \cos 0 + i\sin 0$

i.e. $|z| \cdot \left[\frac{1}{z}\right] = 1$ and $\arg\left(z \cdot \frac{1}{z}\right) = \arg(z) + \arg\left(\frac{1}{z}\right) = 0$ so $\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) =$ $\frac{1}{r}(\cos\theta - i\sin\theta)$ (since cosine is even, sine odd). Hence $\frac{1}{z}$ is a real scalar multiple of $\bar{z}$. See Figure 7.26.

7.5.7 $|T(z)| = |z|\sqrt{2}$ and $\arg(T(z)) = \arg(1-i) + \arg(z) = -\frac{\pi}{4} + \arg(z)$ so T is a clockwise rotation by $\frac{\pi}{4}$ followed by a scaling of $\sqrt{2}$.

7.5.8 By Theorem 7.5.1, $(\cos 3\theta + i\sin 3\theta) = (\cos\theta + i\sin\theta)^3$, i.e.

$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\phi$

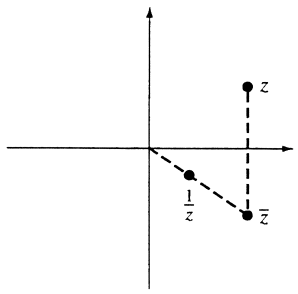


Figure 7.26: for Problem 7.5.6.

$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$.

Equating real and imaginary parts, we get

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

7.5.**9** $|z| = \sqrt{0.8^2 + 0.7^2} = \sqrt{1.15}$, $\arg(z) = \arctan\left(-\frac{0.7}{0.8}\right) \approx -0.72$. See Figure 7.27.

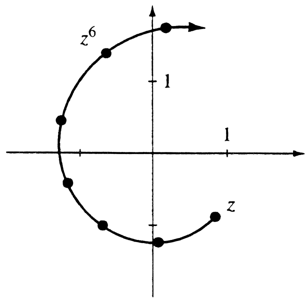


Figure 7.27: for Problem 7.5.9.

The trajectory spirals outward, in the clockwise direction.

7.5.**10** Let $p(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. Since p must have a real root, say λ_1, we can write $p(x) = a(x - \lambda_1)g(x)$ where $g(x)$ is of the form $g(x) = x^2 + px + q$. On page 367, we see that $g(x) = (x - \lambda_2)(x - \lambda_3)$, so that $p(x) = a(x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$, as claimed.

7.5.**11** Notice that $f(1) = 0$ so $\lambda = 1$ is a root of $f(\lambda)$. Hence $f(\lambda) = (\lambda - 1)g(\lambda)$, where $g(\lambda) = \frac{f(\lambda)}{\lambda - 1} = \lambda^2 - 2\lambda + 5$. Setting $g(\lambda) = 0$ we get $\lambda = 1 \pm 2i$ so that $f(\lambda) = (\lambda - 1)(\lambda - 1 - 2i)(\lambda - 1 + 2i)$.

7.5.**12** We will use the facts:

i) $\overline{z + w} = \bar{z} + \bar{w}$ and

ii) $(\overline{z^n}) = \bar{z}^n$
which are easy to check. Assume λ_0 is a complex root of $f(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$, where the coefficients a_i are real. Since λ_0 is a root of f, we have $a_n\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots + a_1\lambda_0 + a_0 = 0$.

Taking the conjugate of both sides we get $\overline{a_n\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots + a_1\lambda_0 + a_0} = \bar{0}$ so by fact i), and factoring the real constants we get $a_n\overline{\lambda_0^n} + a_{n-1}\overline{\lambda_0^{n-1}} + \cdots + a_1\overline{\lambda_0} + a_0 = 0$.

Now, by fact ii), $a_n(\overline{\lambda}_0)^n + a_{n-1}(\overline{\lambda}_0)^{n-1} + \cdots + a_1\overline{\lambda}_0 + a_0 = 0$, i.e. $\overline{\lambda}_0$ is also a root of f, as claimed.

7.5.**13** $\begin{bmatrix} 2i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{v}} + i\underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\vec{w}}$ is an eigenvector with eigenvalue $2i$, so that we can let $S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, with $S^{-1}AS = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

7.5.**14** $\begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v}} + i\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{w}}$ is an eigenvector with eigenvalue i, so that we can let $S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, with $S^{-1}AS = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

7.5.**15** $\begin{bmatrix} 1 \\ 2+i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{v}} + i\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{w}}$ is an eigenvector with eigenvalue $2+i$, so that we can let $S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, with $S^{-1}AS = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

7.5.**16** $\begin{bmatrix} 1 \\ 1+i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v}} + i\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{w}}$ is an eigenvector with eigenvalue $4+i$, so that we can let $S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, with $S^{-1}AS = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$.

7.5.**17** $\begin{bmatrix} 2 \\ -1+2i \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\vec{v}} + i\underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{\vec{w}}$ is an eigenvector with eigenvalue $3+4i$, so that we can let $S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}$, with $S^{-1}AS = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.

7.5.**18** Let $\vec{v}_1, \vec{v}_2$ be two eigenvectors of A. They define a parallelogram of area $S = |\det[\vec{v}_1\ \vec{v}_2]|$. Now $A\vec{v}_1 = \lambda_1\vec{v}_1$ and $A\vec{v}_2 = \lambda_2\vec{v}_2$ define a parallelogram of area $S_1 = |\det[\lambda_1\vec{v}_1\ \lambda_2\vec{v}_2]| = |\lambda_1\lambda_2 \det[\vec{v}_1\ \vec{v}_2]|$ so $\frac{S_1}{S} = |\lambda_1\lambda_2| = |\det(A)|$. Hence $|\det(A)| = |\lambda_1\lambda_2|$, as claimed. In $\mathbb{R}^3$, a similar argument holds if we replace areas by volumes. See Figure 7.28.

7.5.**19** a Since A has eigenvalues 1 and 0 associated with V and $V^\perp$ respectively and since V is the eigenspace of $\lambda = 1$, by Theorem 7.5.5, $\operatorname{tr}(A) = m$, $\det(A) = 0$.

b Since B has eigenvalues 1 and -1 associated with V and $V^\perp$ respectively and since V is the eigenspace associated with $\lambda = 1$, $\operatorname{tr}(A) = m - (n-m) = 2m - n$, $\det B = (-1)^{n-m}$.

7.5.**20** $f_A(\lambda) = (3-\lambda)(-3-\lambda) + 10 = \lambda^2 + 1$ so $\lambda_{1,2} = \pm i$.

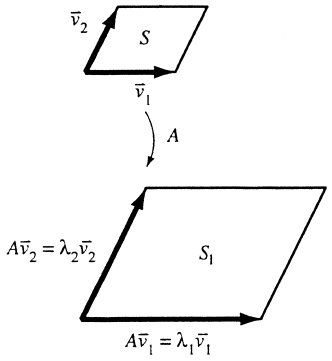


Figure 7.28: for Problem 7.5.18.

7.5.**21** $f_A(\lambda) = (11-\lambda)(-7-\lambda)+90 = \lambda^2-4\lambda+13$ so $\lambda_{1,2} = 2\pm 3i$.

7.5.**22** $f_A(\lambda) = (1-\lambda)(10-\lambda)+12 = \lambda^2-11\lambda+22$ so $\lambda_{1,2} = \frac{11\pm\sqrt{33}}{2}$.

7.5.**23** $f_A(\lambda) = -\lambda^3+1 = -(\lambda-1)(\lambda^2+\lambda+1)$ so $\lambda_1 = 1, \lambda_{2,3} = \frac{-1\pm\sqrt{3}i}{2}$.

7.5.**24** $f_A(\lambda) = -\lambda^3+3\lambda^2-7\lambda+5$ so $\lambda_1 = 1, \lambda_{2,3} = 1\pm 2i$. (See Exercise 11.)

7.5.**25** $f_A(\lambda) = \lambda^4-1 = (\lambda^2-1)(\lambda^2+1) = (\lambda-1)(\lambda+1)(\lambda-i)(\lambda+i)$ so $\lambda_{1,2} = \pm 1$ and $\lambda_{3,4} = \pm i$

7.5.**26** $f_A(\lambda) = (\lambda^2-2\lambda+2)(\lambda^2-2\lambda) = (\lambda^2-2\lambda+2)(\lambda-2)\lambda = 0$, so $\lambda_{1,2} = 1\pm i, \lambda_3 = 2, \lambda_4 = 0$.

7.5.**27** By Theorem 7.5.5, $\mathrm{tr}(A) = \lambda_1+\lambda_2+\lambda_3$, $\det(A) = \lambda_1\lambda_2\lambda_3$ but $\lambda_1 = \lambda_2 \neq \lambda_3$ by assumption, so $\mathrm{tr}(A) = 1 = 2\lambda_2+\lambda_3$ and $\det(A) = 3 = \lambda_2^2\lambda_3$.

Solving for λ_2, λ_3 we get $-1, 3$ hence $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 3$. (Note that the eigenvalues must be real; why?)

7.5.**28** Suppose the complex eigenvalues are $z = a+ib$ and $\bar{z} = a-ib$. By Theorem 7.5.5, we have $\mathrm{tr}(A) = 2+z+\bar{z} = 2+2a = 8$, so that $a = 3$. Furthermore , $\det(A) = 2z\bar{z} = 2(a^2+b^2) = 2(9+b^2) = 50$, so that $b = 4$. Hence the complex eigenvalues are $3\pm 4i$.

7.5.**29** $\mathrm{tr}(A) = 0$ so $\lambda_1+\lambda_2+\lambda_3 = 0$.

Also, we can compute $\det(A) = bcd > 0$ since $b, c, d > 0$. Therefore, $\lambda_1\lambda_2\lambda_3 > 0$.

Hence two of the eigenvalues must be negative, and the largest one (in absolute value) must be positive.

7.5.**30** a. The eigenvalues of a 2×2 matrix are the roots of a quadratic equation. Since one of the eigenvalues of A is $2i$, the other must be its complex conjugate, $-2i$. Now A is similar to $B = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}$, meaning that $A = SBS^{-1}$ for some invertible S with complex entries. We find

$$A^2 = SB^2S^{-1} = S\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}S^{-1} = SS^{-1}\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}.$$

b. The characteristic polynomial of A must be $f_A(\lambda) = (\lambda - 2i)(\lambda + 2i) = \lambda^2 + 4$, meaning that $\det A = 4$ and $\text{tr}A = 0$. An example is $A = \begin{bmatrix} 1 & -1 \\ 5 & -1 \end{bmatrix}$, with $A^2 = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$, as expected.

7.5.**31** a. Computing A^{20}, we conjecture that

$$\lim_{t\to\infty} A^t = \frac{1}{5}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

b. The complex eigenvalues of A are

$$\lambda_1 = 1, \quad \lambda_{2,3} \approx -0.200 \pm 0.136i, \quad \lambda_{4,5} \approx 0.134 \pm 0.132i.$$

Since A has five distinct complex eigenvalues, A will be diagonalizable over the complex numbers.

c. By Exercise 7.2.30c, we have the eigenspace $E_1 = \text{span}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{5}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

d. We can adapt the proof of Theorem 7.4.1; see the second paragraph on Page 350 in particular. Since A is diagonalizable over the complex numbers, there exists a complex eigenbasis $\vec{v}_1, \vec{v}_2, ..., \vec{v}_5$ for A, where $A\vec{v}_i = \lambda_i\vec{v}_i$, with the eigenvalues λ_i we found in part b. Note that $\lambda_1 = 1$ and $|\lambda_i| < 1$ for $i = 2, 3, 4, 5$. If $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_5\vec{v}_5$, then $A^t\vec{x}_0 = c_1\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + ... + c_5\lambda_5^t\vec{v}_5$ and $\lim_{t\to\infty} A^t\vec{x}_0 = c_1\vec{v}_1$, since $\lim_{t\to\infty} \lambda_i^t = 0$ for $i = 2, 3, 4, 5$. Now, $c_1\vec{v}_1$ is an eigenvector of A with eigenvalue 1. Since $\vec{x}_0$ is a distribution vector, so is $A^t\vec{x}_0$ (for all t), and so is $c_1\vec{v}_1 = \lim_{t\to\infty} A^t\vec{x}_0$. Since the eigenspace E_1 is one dimensional, $c_1\vec{v}_1$is the unique equilibrium distribution vector, $\vec{x}_{equ}$, so that $\lim_{t\to\infty} A^t\vec{x}_0 = \vec{x}_{equ}$, as claimed.

e. $\lim_{t\to\infty} A^t = \lim_{t\to\infty}\begin{bmatrix} A^t\vec{e}_1 & ... & A^t\vec{e}_5 \end{bmatrix} = \begin{bmatrix} \lim_{t\to\infty} A^t\vec{e}_1 & ... & \lim_{t\to\infty} A^t\vec{e}_5 \end{bmatrix} = \begin{bmatrix} \vec{x}_{equ} & ... & \vec{x}_{equ} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

7.5.**32** a. Computing A^{20}, we conjecture that

$$\lim_{t\to\infty} A^t = \begin{bmatrix} 0.36 & 0.36 & 0.36 \\ 0.26 & 0.26 & 0.26 \\ 0.38 & 0.38 & 0.38 \end{bmatrix}$$

b. The complex eigenvalues of A are

$$\lambda_1 = 1, \quad \lambda_{2,3} = 0.05 \pm \sqrt{39/400}\, i.$$

Since A has three distinct complex eigenvalues, A will be diagonalizable over the complex numbers.

c. We find the eigenspace $E_1 = \text{span}\begin{bmatrix} 18 \\ 13 \\ 19 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{50}\begin{bmatrix} 18 \\ 13 \\ 19 \end{bmatrix}$.

d. This proof is analogous to Exercise 31d.

e. This proof is analogous to Exercise 31e.

7.5.**33** a C is obtained from B by dividing each column of B by its first component. Thus, the first row of C will consist of 1's.

b We observe that the columns of C are almost identical, so that the columns of B are "almost parallel" (that is, almost scalar multiples of each other).

c Let $\lambda_1, \lambda_2, \ldots, \lambda_5$ be the eigenvalues. Assume λ_1 real and positive and $\lambda_1 > |\lambda_j|$ for $2 \leq j \leq 5$.

Let $\vec{v}_1, \ldots, \vec{v}_5$ be corresponding eigenvectors. For a fixed i, write $\vec{e}_i = \sum_{j=1}^{5} c_j \vec{v}_j$; then

(ith column of A^t) $= A^t \vec{e}_i = c_1 \lambda_1^t \vec{v}_1 + \cdots + c_5 \lambda_5^t \vec{v}_5$.

But in the last expression, for large t, the first term is dominant, so the ith column of A^t is almost parallel to $\vec{v}_1$, the eigenvector corresponding to the dominant eigenvalue.

d By part c, the columns of B and C are almost eigenvectors of A associated with the largest eigenvalue, λ_1. Since the first row of C consists of 1's, the entries in the first row of AC will be close to λ_1.

7.5.**34** a The eigenvalues of $A - \lambda I_n$ are $\lambda_1 - \lambda, \lambda_2 - \lambda, \ldots, \lambda_n - \lambda$, and we were told that $|\lambda_1 - \lambda| < |\lambda_i - \lambda|$ for $i = 2, \ldots, n$. We may assume that $\lambda_1 \neq \lambda$ (otherwise we are done).

The eigenvalues of $(A - \lambda I_n)^{-1}$ are $(\lambda_1 - \lambda)^{-1}, (\lambda_2 - \lambda)^{-1}, \ldots, (\lambda_n - \lambda)^{-1}$, and $(\lambda_1 - \lambda)^{-1}$ has the largest modulus. The matrices $A, A - \lambda I_n$, and $(A - \lambda I_n)^{-1}$ have the same eigenvectors.

For large t, the columns of the tth power of $(A - \lambda I_n)^{-1}$ will be almost eigenvectors of A. If $\vec{v}$ is such a column, compare $\vec{v}$ and $A\vec{v}$ to find an approximation of λ_1.

b See Figure 7.29.

Let $\lambda = -1$.

$$A - \lambda I_3 = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 6 & 6 \\ 7 & 8 & 11 \end{bmatrix}, N = (A - \lambda I_3)^{-1} = \begin{bmatrix} 9 & 1 & -3 \\ -1 & 0.5 & 0 \\ -5 & -1 & 2 \end{bmatrix}, \text{ and } B = N^{20}.$$

Obtain C from B as in Exercise 33:

$$C \approx \begin{bmatrix} 1 & 1 & 1 \\ -0.098922005729 & -0.098922005729 & -0.098922005729 \\ -0.569298722688 & -0.569298722688 & -0.569298722688 \end{bmatrix}$$

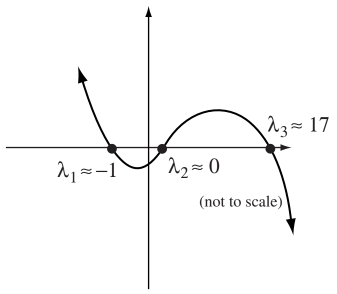


Figure 7.29: for Problem 7.5.34b.

$$AC \approx \begin{bmatrix} -0.905740179522 & -0.905740179522 & -0.905740179522 \\ * & * & * \\ * & * & * \end{bmatrix}$$

The entries in the first row of AC give us a good approximation for λ_1, and the columns of C give us a good approximation for a corresponding eigenvector.

7.5.**35** We have $f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$

$= (-\lambda)^n + (\lambda_1 + \lambda_2 + \cdots + \lambda_n)(-\lambda)^{n-1} + \cdots + (\lambda_1\lambda_2\cdots\lambda_n)$. But, by Theorem 7.2.5, the coefficient of $(-\lambda)^{n-1}$ is $\text{tr}(A)$. So, $\text{tr}(A) = \lambda_1 + \cdots + \lambda_n$.

7.5.**36** a The entries in the first row are age-specific birth rates and the entries just below the diagonal are age-specific survival rates. For example, the entry 1.6 in the first row tells us that during the next 15 years the people who are 15–30 years old today will *on average* have 1.6 children (3.2 per couple) who will survive to the next census. The entry 0.53 tells us that 53% of those in the age group 45–60 today will still be alive in 15 years (they will then be in the age group 60–75).

b Using technology, we find the largest eigenvalue $\lambda_1 = 1.908$ with associated eigenvector

$$\vec{v}_1 \approx \begin{bmatrix} 0.574 \\ 0.247 \\ 0.115 \\ 0.047 \\ 0.014 \\ 0.002 \end{bmatrix}.$$

The components of $\vec{v}_1$ give the distribution of the population among the age groups in the long run, assuming that current trends continue. λ_1 gives the factor by which the population will grow in the long run in a period of 15 years; this translates to an annual growth factor of $\sqrt[15]{1.908} \approx 1.044$, or an annual growth of about 4.4%.

7.5.**37** a Use that $\overline{w} + \overline{z} = \overline{w+z}$ and $\overline{w}\overline{z} = \overline{wz}$.

$$\begin{bmatrix} w_1 & -\overline{z}_1 \\ z_1 & \overline{w}_1 \end{bmatrix} + \begin{bmatrix} w_2 & -\overline{z}_2 \\ z_2 & \overline{w}_2 \end{bmatrix} = \begin{bmatrix} w_1 + w_2 & -(\overline{z_1 + z_2}) \\ z_1 + z_2 & \overline{w_1 + w_2} \end{bmatrix} \text{ is in } \mathbb{H}.$$

$$\begin{bmatrix} w_1 & -\overline{z}_1 \\ z_1 & \overline{w}_1 \end{bmatrix} \begin{bmatrix} w_2 & -\overline{z}_2 \\ z_2 & \overline{w}_2 \end{bmatrix} = \begin{bmatrix} w_1w_2 - \overline{z}_1z_2 & -(\overline{z_1w_2 + \overline{w}_1z_2}) \\ z_1w_2 + \overline{w}_1z_2 & \overline{w_1w_2 - \overline{z}_1z_2} \end{bmatrix} \text{ is in } \mathbb{H}.$$

b If A in $\mathbb{H}$ is nonzero, then $\det(A) = w\overline{w} + z\overline{z} = |w|^2 + |z|^2 > 0$, so that A is invertible.

c Yes; if $A = \begin{bmatrix} w & -\overline{z} \\ z & \overline{w} \end{bmatrix}$, then $A^{-1} = \frac{1}{|w|^2+|z|^2} \begin{bmatrix} \overline{w} & \overline{z} \\ -z & w \end{bmatrix}$ is in $\mathbb{H}$.

d For example, if $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$.

7.5.**38** a $C_4^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, C_4^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, C_4^4 = I_4$, then $C_4^{4+k} = C_4^k$.

Figure 7.30 illustrates how C_4 acts on the basis vectors $\vec{e}_i$.

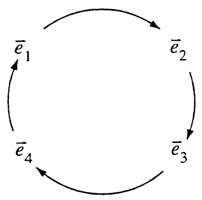


Figure 7.30: for Problem 7.5.38a.

b The eigenvalues are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i$, and $\lambda_4 = -i$, and for each eigenvalue $\lambda_k, \vec{v}_k = \begin{bmatrix} \lambda_k^3 \\ \lambda_k^2 \\ \lambda_k \\ 1 \end{bmatrix}$ is an associated eigenvector.

c $M = aI_4 + bC_4 + cC_4^2 + dC_4^3$

If $\vec{v}$ is an eigenvector of C_4 with eigenvalue λ, then $M\vec{v} = a\vec{v} + b\lambda\vec{v} + c\lambda^2\vec{v} + d\lambda^3\vec{v} = (a + b\lambda + c\lambda^2 + d\lambda^3)\vec{v}$, so that $\vec{v}$ is an eigenvector of M as well, with eigenvalue $a + b\lambda + c\lambda^2 + d\lambda^3$.

The eigenbasis for C_4 we found in part b is an eigenbasis for all circulant 4×4 matrices.

7.5.**39** Figure 7.31 illustrates how C_n acts on the standard basis vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ of $\mathbb{R}^n$.

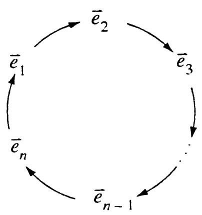


Figure 7.31: for Problem 7.5.39.

7.5.**39** a Based on Figure 7.31, we see that C_n^k takes $\vec{e}_i$ to $\vec{e}_{i+k}$ "modulo n," that is, if $i+k$ exceeds n then C_n^k takes $\vec{e}_i$ to $\vec{e}_{i+k-n}$ (for $k = 1, \ldots, n-1$).

To put it differently: C_n^k is the matrix whose ith column is $\vec{e}_{i+k}$ if $i+k \leq n$ and $\vec{e}_{i+k-n}$ if $i+k > n$ (for $k = 1, \ldots, n-1$).

b The characteristic polynomial is $1-\lambda^n$, so that the eigenvalues are the n distinct solutions of the equation $\lambda^n = 1$ (the so-called nth roots of unity), equally spaced points along the unit circle, $\lambda_k = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$, for $k = 0, 1, \ldots, n-1$ (compare with Exercise 5 and Figure 7.7.). For each eigenvalue λ_k,

$$\vec{v}_k = \begin{bmatrix} \lambda_k^{n-1} \\ \vdots \\ \lambda_k^2 \\ \lambda_k \\ 1 \end{bmatrix} \text{ is an associated eigenvector.}$$

c The eigenbasis $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{n-1}$ for C_n we found in part b is in fact an eigenbasis for all circulant $n \times n$ matrices.

7.5.**40** In Exercise 7.2.50 we derived the formula $x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$

for the solution of the equation $x^3 + px = q$. Here $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$ is negative, and we can write

$$x = \sqrt[3]{\frac{q}{2} + i\sqrt{-\left(\frac{q}{2}\right)^2 + \left(\frac{-p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - i\sqrt{-\left(\frac{q}{2}\right)^2 + \left(\frac{-p}{3}\right)^3}}.$$

Let us write this solution in polar coordinates:

$$x = \sqrt[3]{\left(-\frac{p}{3}\right)^{3/2}(\cos\alpha + i\sin\alpha)} + \sqrt[3]{\left(-\frac{p}{3}\right)^{3/2}(\cos\alpha - i\sin\alpha)}$$

$$= \sqrt{-\frac{p}{3}}\left(\cos\frac{\alpha+2\pi k}{3} + i\sin\frac{\alpha+2\pi k}{3}\right) + \sqrt{\frac{-p}{3}}\left(\cos\frac{\alpha+2\pi k}{3} - i\sin\frac{\alpha+2\pi k}{3}\right)$$

$$= 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\alpha+2\pi k}{3}\right), k = 0, 1, 2. \text{ See Figure 7.32.}$$

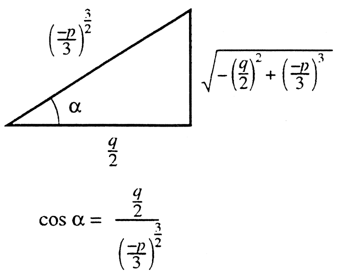


Figure 7.32: for Problem 7.5.40.

Answer:

$x_{1,2,3} = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{\alpha+2\pi k}{3}\right), k = 0, 1, 2,$ where $\alpha = \arccos\left(\frac{\frac{q}{2}}{\left(-\frac{p}{3}\right)^{3/2}}\right)$.

Note that x is on the interval $\left(\sqrt{\frac{-p}{3}}, 2\sqrt{\frac{-p}{3}}\right)$ when $k=0$, on $\left(-2\sqrt{\frac{-p}{3}}, -\sqrt{\frac{-p}{3}}\right)$ when $k=1$ and on $\left(-\sqrt{\frac{-p}{3}}, \sqrt{\frac{-p}{3}}\right)$ when $k=-1$ (Think about it!).

7.5.**41** Substitute $\rho = \frac{1}{x}$ into $14\rho^2 + 12\rho^3 - 1 = 0$;

$\frac{14}{x^2} + \frac{12}{x^3} - 1 = 0$

$14x + 12 - x^3 = 0$

$x^3 - 14x = 12$

Now use the formula derived in Exercise 40 to find x, with $p = -14$ and $q = 12$. There is only one positive solution, $x \approx 4.114$, so that $\rho = \frac{1}{x} \approx 0.243$.

7.5.**42** a We will use the fact that for any two complex numbers z and w, $\overline{z+w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$.

The ijth entry of $\overline{AB}$ is $\overline{\sum_{k=1}^{p} a_{ik}b_{kj}} = \sum_{k=1}^{p} \overline{a_{ik}b_{kj}} = \sum_{k=1}^{p} \overline{a_{ik}}\,\overline{b_{kj}}$, which is the ijth entry of $\bar{A}\bar{B}$, as claimed.

b Use part a, where B is the $n \times 1$ matrix $\vec{v} + i\vec{w}$. We are told that $AB = \lambda B$, where $\lambda = p + iq$. Then $A\bar{B} = \bar{A}\,\bar{B} = \overline{AB} = \overline{\lambda B} = \bar{\lambda}\,\bar{B}$, or $A(\vec{v} - i\vec{w}) = (p - iq)(\vec{v} - i\vec{w})$.

7.5.**43** Note that $f(z)$ is not the zero polynomial, since $f(i) = \det(S_1 + iS_2) = \det(S) \neq 0$, as S is invertible. A nonzero polynomial has only finitely many zeros, so that there is a real number x such that $f(x) = \det(S_1 + xS_2) \neq 0$, that is, $S_1 + xS_2$ is invertible. Now $SB = AS$ or $(S_1 + iS_2)B = A(S_1 + iS_2)$. Considering the real and the imaginary part, we can conclude that $S_1B = AS_1$ and $S_2B = AS_2$ and therefore $(S_1 + xS_2)B = A(S_1 + xS_2)$. Since $S_1 + xS_2$ is invertible, we have $B = (S_1 + xS_2)^{-1}A(S_1 + xS_2)$, as claimed.

7.5.**44** Let A be a complex 2×2 matrix. Let λ be a complex eigenvalue of A, and consider an associated eigenvector $\vec{v}$, so that $A\vec{v} = \lambda\vec{v}$. Now let P be an invertible 2×2 matrix of the form $P = [\vec{v}\,\vec{w}]$ (the first column of P is our eigenvector $\vec{v}$). Then $P^{-1}AP$ will be of the form $\begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix}$, so that we have found an upper triangular matrix similar to A (compare with the proof of Theorem 7.1.3).

Yes, any complex square matrix is similar to an upper triangular matrix, although the proof is challenging at this stage of the course. Following the hint, we will assume that the claim holds for $n \times n$ matrices, and we will prove it for an $(n+1) \times (n+1)$ matrix A. As in the case of a 2×2 matrix discussed above, we can find an invertible P such that $P^{-1}AP$ is of the form $\begin{bmatrix} \lambda & \vec{w} \\ 0 & B \end{bmatrix}$ for some scalar λ, a row vector $\vec{w}$ with n components, and an $n \times n$ matrix B (just make the first column of P an eigenvector of A). By the induction hypothesis, B is similar to some upper triangular matrix T, that is, $R^{-1}BR = T$ for some invertible R.

Now let $S = P\begin{bmatrix} 1 & \vec{0} \\ 0 & R \end{bmatrix}$, an invertible $(n+1) \times (n+1)$ matrix. Then

$S^{-1}AS = \begin{bmatrix} 1 & \vec{0} \\ 0 & R^{-1} \end{bmatrix} P^{-1}AP \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} 1 & \vec{0} \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} \lambda & \vec{w} \\ 0 & B \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} \lambda & \vec{w}R \\ 0 & T \end{bmatrix}$, an upper triangular matrix, showing that A is indeed similar to an upper triangular matrix. You will see an analogous proof in Section 8.1 (proof of Theorem 8.1.1, Page 389).

7.5.**45** If $a \neq 0$, then there are two distinct eigenvalues, $1 \pm \sqrt{a}$, so that the matrix is diagonalizable. If $a = 0$, then $\begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ fails to be diagonalizable.

7.5.**46** If $a \neq 0$, then there are two distinct eigenvalues, $\pm ia$, so that the matrix is diagonalizable. If $a = 0$, then $\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable as well. Thus the matrix is diagonalizable for all a.

7.5.**47** If $a \neq 0$, then there are three distinct eigenvalues, $0, \pm\sqrt{a}$, so that the matrix is diagonalizable. If $a = 0$, then $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & a \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ fails to be diagonalizable.

7.5.**48** The characteristic polynomial is $f(\lambda) = -\lambda^3 + 3\lambda + a$. We need to find the values a such that this polynomial has multiple roots. Now λ is a multiple root if (and only if) $f(\lambda) = f'(\lambda) = 0$ (see Exercise 7.2.37). Since $f'(\lambda) = -3\lambda^2 + 3 = -3(\lambda - 1)(\lambda + 1)$, the only possible multiple roots are 1 and -1. Now 1 is a multiple root if $f(1) = 2 + a = 0$, or, $a = -2$, and -1 is a multiple root if $a = 2$. Thus, if a is neither 2 nor -2, then the matrix is diagonalizable. Conversely, if $a = 2$ or $a = -2$, then the matrix fails to be diagonalizable, since all the eigenspaces will be one-dimensional (verify this!).

7.5.**49** The eigenvalues are $0, 1, a - 1$. If a is neither 1 nor 2, then there are three distinct eigenvalues, so that the matrix is diagonalizable. Conversely, if $a = 1$ or $a = 2$, then the matrix fails to be diagonalizable, since all the eigenspaces will be one-dimensional (verify this!).

7.5.**50** The eigenvalues are 0, 0, 1. Since the kernel is always two-dimensional, with basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, the matrix is diagonalizable for all values of constant a.

7.5.**51** Yes, $\mathbb{Q}$ is a field. Check the axioms on page 368.

7.5.**52** No, $\mathbb{Z}$ is not a field since multiplicative inverses do not exist, i.e. division within $\mathbb{Z}$ is not possible. (Axiom 8 does not hold).

7.5.**53** Yes, check the axioms on page 368. (additive identity $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, multiplicative identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$).

7.5.**54** Yes, check the axioms on page 368. (additive identity $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, multiplicative identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$). Also notice that rotation-scaling matrices commute when multiplied.

7.5.**55** No, since multiplication is not commutative; Axiom 5 does not hold.

Section 7.6

7.6.**1** $\lambda_1 = 0.9, \lambda_2 = 0.8$, so, by Theorem 7.6.2, $\vec{0}$ is a stable equilibrium.

7.6.2 $\lambda_1 = -1.1, \lambda_2 = 0.9$, so by Theorem 7.6.2, $\vec{0}$ is not a stable equilibrium. ($|\lambda_1| > 1$)

7.6.3 $\lambda_{1,2} = 0.8 \pm (0.7)i$ so $|\lambda_1| = |\lambda_2| = \sqrt{0.64 + 0.49} > 1$ so $\vec{0}$ is not a stable equilibrium.

7.6.4 $\lambda_{1,2} = -0.9 \pm (0.4)i$ so $|\lambda_1| = |\lambda_2| = \sqrt{0.81 + 0.16} < 1$ so $\vec{0}$ is a stable equilibrium.

7.6.5 $\lambda_1 = 0.8, \lambda_2 = 1.1$ so $\vec{0}$ is not a stable equilibrium.

7.6.6 $\lambda_{1,2} = 0.8 \pm (0.6)i$ so $|\lambda_1| = |\lambda_2| = \sqrt{0.64 + 0.36} = 1$ and $\vec{0}$ is not a stable equilibrium.

7.6.7 $\lambda_{1,2} = 0.9 \pm (0.5)i$ so $|\lambda_1| = |\lambda_2| = \sqrt{0.81 + 0.25} > 1$ and $\vec{0}$ is not a stable equilibrium.

7.6.8 $\lambda_1 = 0.9, \lambda_2 = 0.8$ so $\vec{0}$ is a stable equilibrium.

7.6.9 $\lambda_{1,2} = 0.8 \pm (0.6)i, \lambda_3 = 0.7$, so $|\lambda_1| = |\lambda_2| = 1$ and $\vec{0}$ is not a stable equilibrium.

7.6.10 $\lambda_{1,2} = 0, \lambda_3 = 0.9$ so $\vec{0}$ is a stable equilibrium.

7.6.11 $\lambda_1 = k, \lambda_2 = 0.9$ so $\vec{0}$ is a stable equilibrium if $|k| < 1$.

7.6.12 $\lambda_{1,2} = 0.6 \pm ik$ so $\vec{0}$ is a stable equilibrium if $|\lambda_1| = |\lambda_2| = \sqrt{0.36 + k^2} < 1$ i.e. if $k^2 < 0.64$ or $|k| < 0.8$.

7.6.13 Since $\lambda_1 = 0.7, \lambda_2 = -0.9, \vec{0}$ is a stable equilibrium regardless of the value of k.

7.6.14 $\lambda_1 = 0, \lambda_2 = 2k$ so $\vec{0}$ is a stable equilibrium if $|2k| < 1$ or $|k| < \frac{1}{2}$.

7.6.15 $\lambda_{1,2} = 1 \pm \frac{1}{10}\sqrt{k}$

If $k \geq 0$ then $\lambda_1 = 1 + \frac{1}{10}\sqrt{k} \geq 1$. If $k < 0$ then $|\lambda_1| = |\lambda_2| > 1$. Thus, the zero state isn't a stable equilibrium for any real k.

7.6.16 $\lambda_{1,2} = \frac{2 \pm \sqrt{1+30k}}{10}$ so $|2 \pm \sqrt{1 + 30k}|$ must be less than 10. $\lambda_{1,2}$ are *real* if $k \geq -\frac{1}{30}$. In this case it is required that $2 + \sqrt{1 + 30k} < 10$ and $-10 < 2 - \sqrt{1 + 30k}$, which means that $\sqrt{1 + 30k} < 8$ or $k < \frac{21}{10}$.

$\lambda_{1,2}$ are *complex* if $k < -\frac{1}{30}$. Here it is required that $4 + (-1 - 30k) < 100$ or $k > -\frac{97}{30}$. Overall, $\vec{0}$ is a stable equilibrium if $-\frac{97}{30} < k < \frac{21}{10}$.

7.6.17 $\lambda_{1,2} = 0.6 \pm (0.8)i = 1(\cos\theta \pm i \sin\theta)$, where

$\theta = \arctan\left(\frac{0.8}{0.6}\right) = \arctan\left(\frac{0.8}{0.6}\right) = \arctan\left(\frac{4}{3}\right) \approx 0.927.$

$E_{\lambda_1} = \ker \begin{bmatrix} -0.8i & -0.8 \\ 0.8 & -0.8i \end{bmatrix} = \text{span} \begin{bmatrix} -1 \\ i \end{bmatrix}$ so $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

$\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\vec{w} + 0\vec{v}$, so $a = 1$ and $b = 0$. Now we use Theorem 7.6.3:

$\vec{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta t \\ \sin\theta t \end{bmatrix} = \begin{bmatrix} -\sin\theta t \\ \cos\theta t \end{bmatrix}$, where

$\theta = \arctan\left(\frac{4}{3}\right) \approx 0.927.$

The trajectory is the circle shown in Figure 7.33.

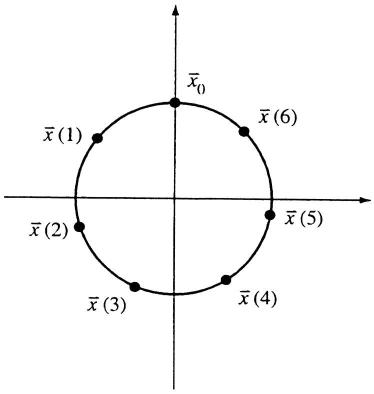


Figure 7.33: for Problem 7.6.17.

7.6.18 $\lambda_{1,2} = \frac{-4\pm 2\sqrt{3}i}{5} = r(\cos\theta \pm i\sin\theta)$, where $r \approx 1.058$ and $\theta = \pi - \arctan\left(\frac{\frac{2\sqrt{3}}{5}}{\frac{4}{5}}\right) \approx 2.428$ (second quadrant).

$E_{\lambda_1} = \operatorname{span}\begin{bmatrix}\frac{\sqrt{3}}{2}\\ i\end{bmatrix}$, so $\vec{w} = \begin{bmatrix}0\\1\end{bmatrix}, \vec{v} = \begin{bmatrix}\frac{\sqrt{3}}{2}\\0\end{bmatrix}$.

$\vec{x}_0 = 1\vec{w} + 0\vec{v}$, so $a = 1, b = 0$.

$$\vec{x}(t) = r^t\begin{bmatrix}\vec{w} & \vec{v}\end{bmatrix}\begin{bmatrix}\cos(\theta t) & -\sin(\theta t)\\ \sin(\theta t) & \cos(\theta t)\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \approx (1.058)^t\begin{bmatrix}0 & 0.866\\1 & 0\end{bmatrix}\begin{bmatrix}\cos(\theta t)\\ \sin(\theta t)\end{bmatrix}$$

$$\approx (1.058)^t\begin{bmatrix}0.866\cdot\sin(2.428t)\\ \cos(2.428t)\end{bmatrix}.$$ See Figure 7.34.

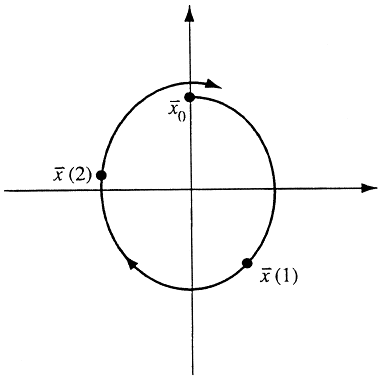


Figure 7.34: for Problem 7.6.18.

Spirals slowly outwards (plot the first few points).

7.6.19 $\lambda_{1,2} = 2 \pm 3i$, $r = \sqrt{13}$, and $\theta = \arctan\left(\frac{3}{2}\right) \approx 0.98$, so

$\lambda_1 \approx \sqrt{13}(\cos(0.98) + i\sin(0.98)), [\vec{w}\ \vec{v}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{x}(t)$

$\approx \sqrt{13}^t \begin{bmatrix} -\sin(0.98t) \\ \cos(0.98t) \end{bmatrix}$.

The trajectory spirals outwards; see Figure 7.35.

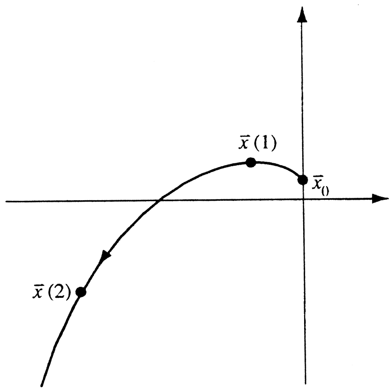


Figure 7.35: for Problem 7.6.19.

7.6.**20** $\lambda_{1,2} = 4 \pm 3i, r = 5, \theta = \arctan\left(\frac{3}{4}\right) \approx 0.64$, so $\lambda_1 \approx 5(\cos(0.64) + i\sin(0.64)), [\vec{w}\ \vec{v}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and $\vec{x}(t) \approx 5^t \begin{bmatrix} \sin(0.64t) \\ \cos(0.64t) \end{bmatrix}$. See Figure 7.36.

Spirals outwards (rotation-dilation).

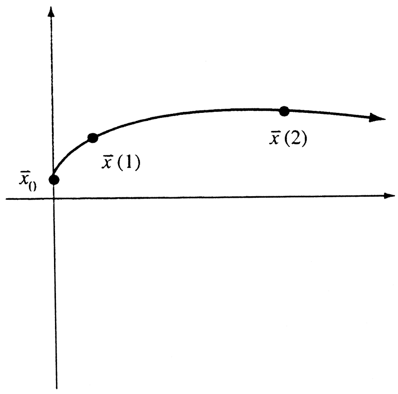


Figure 7.36: for Problem 7.6.20.

7.6.**21** $\lambda_{1,2} = 4 \pm i, r = \sqrt{17}, \theta = \arctan\left(\frac{1}{4}\right) \approx 0.245$ so

$\lambda_1 \approx \sqrt{17}(\cos(0.245) + i\sin(0.245)), [\vec{w}\ \vec{v}] = \begin{bmatrix} 0 & 5 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and $\vec{x}(t) \approx \sqrt{17}^t \begin{bmatrix} 5\sin(0.245t) \\ \cos(0.245t) + 3\sin(0.245t) \end{bmatrix}$

The trajectory spirals outwards; see Figure 7.37.

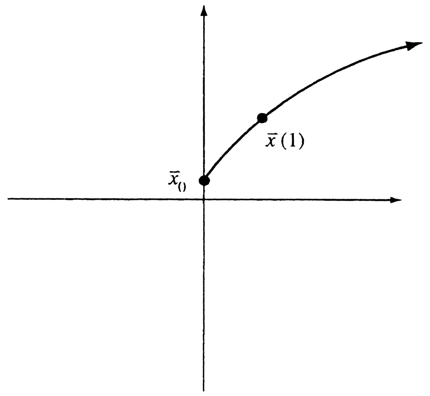


Figure 7.37: for Problem 7.6.21.

7.6.**22** $\lambda_{1,2} = -2 \pm 3i, r = \sqrt{13}, \theta \approx 2.16$ (in second quadrant)

$[\vec{w}\ \vec{v}] = \begin{bmatrix} 0 & -5 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $\vec{x}(t) = \sqrt{13}^t \begin{bmatrix} -5\sin(\theta t) \\ \cos(\theta t) - 3\sin(\theta t) \end{bmatrix}$, where $\theta \approx 2.16$.

Spirals outwards, as in Figure 7.38.

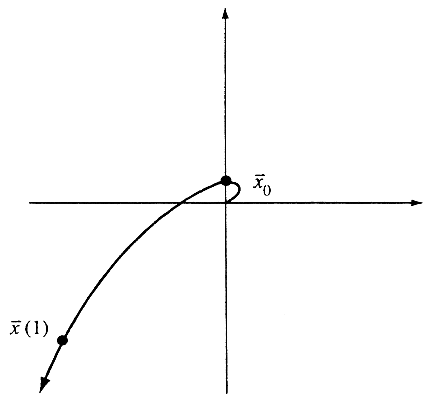


Figure 7.38: for Problem 7.6.22.

7.6.**23** $\lambda_{1,2} = 0.4 \pm 0.3i, r = \frac{1}{2}, \theta = \arctan\left(\frac{0.3}{0.4}\right) \approx 0.643$

$[\vec{w}\ \vec{v}] = \begin{bmatrix} 0 & 5 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $\vec{x}(t) = \left(\frac{1}{2}\right)^t \begin{bmatrix} 5\sin(\theta t) \\ \cos(\theta t) + 3\sin\theta(t) \end{bmatrix}$.

The trajectory spirals inwards as shown in Figure 7.39.

7.6.**24** $\lambda_{1,2} = -0.8 \pm 0.6i, r = 1, \theta = \pi - \arctan\left(\frac{.6}{.8}\right) \approx 2.5$ (second quadrant)

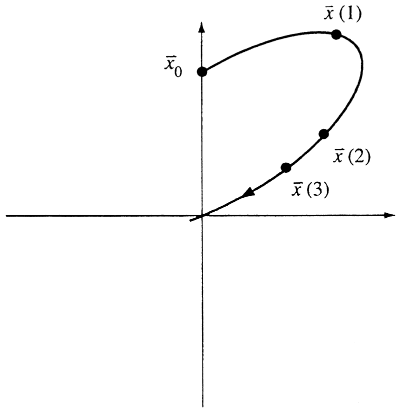


Figure 7.39: for Problem 7.6.23.

$[\vec{w}\ \vec{v}] = \begin{bmatrix} 0 & -5 \\ 1 & -3 \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $\vec{x}(t) = \begin{bmatrix} -5\sin(\theta t) \\ \cos(\theta t) - 3\sin(\theta t) \end{bmatrix}$, an ellipse, as shown in Figure 7.40.

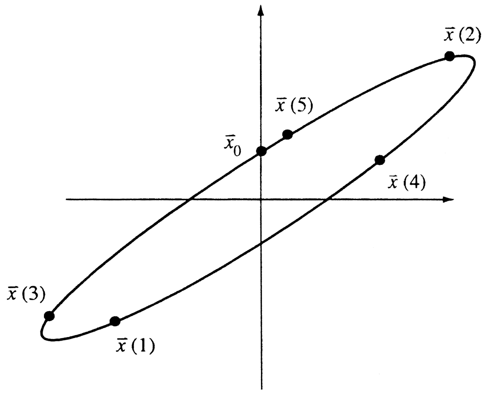


Figure 7.40: for Problem 7.6.24.

7.6.**25** Not stable since if λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and $\left[\frac{1}{\lambda}\right] = \frac{1}{|\lambda|} > 1$.

7.6.**26** Stable since A and A^T have the same eigenvalues.

7.6.**27** Stable since if λ is an eigenvalue of $-A$, then $-\lambda$ is an eigenvalue of $-A$ and $|-\lambda| = |\lambda|$.

7.6.**28** Not stable, since if λ is an eigenvalue of A, then $(\lambda - 2)$ is an eigenvalue of $(A - 2I_n)$ and $|\lambda - 2| > 1$.

7.6.**29** Cannot tell; for example, if $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, then $A + I_2$ is $\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$ and the zero state is not stable, but if $A = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ then $A + I_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and the zero state is stable.

7.6.**30** Consider the dynamical systems $\vec{x}(t+1) = A^2\vec{x}(t)$ and $\vec{y}(t+1) = A\vec{y}(t)$ with equal initial values, $\vec{x}(0) = \vec{y}(0)$.

Then $\vec{x}(t) = \vec{y}(2t)$ for all positive integers t. We know that $\lim_{t\to\infty} \vec{y}(t) = \vec{0}$; thus $\lim_{t\to\infty} \vec{x}(t) = \vec{0}$, proving that the zero state is a stable equilibrium of the system $\vec{x}(t+1) = A^2\vec{x}(t)$.

7.6.**31** We need to determine for which values of $\det(A)$ and $\mathrm{tr}(A)$ the modulus of both eigenvalues is less than 1. We will first think about the border line case and examine when one of the moduli is exactly 1: If one of the eigenvalues is 1 and the other is λ, then $\mathrm{tr}(A) = \lambda + 1$ and $\det(A) = \lambda$, so that $\det(A) = \mathrm{tr}(A) - 1$. If one of the eigenvalues is -1 and the other is λ, then $\mathrm{tr}(A) = \lambda - 1$ and $\det(A) = -\lambda$, so that $\det(A) = -\mathrm{tr}(A) - 1$. If the eigenvalues are complex conjugates with modulus 1, then $\det(A) = 1$ and $|\mathrm{tr}(A)| < 2$ (think about it!). It is convenient to represent these conditions in the tr-det plane, where each 2×2 matrix A is represented by the point $(\mathrm{tr}A, \det A)$, as shown in Figure 7.41.

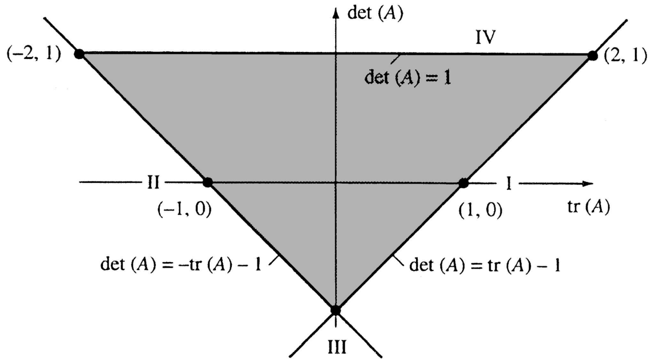


Figure 7.41: for Problem 7.6.31.

If $\mathrm{tr}(A) = \det(A) = 0$, then both eigenvalues of A are zero. We can conclude that throughout the shaded triangle in Figure 7.41 the modulus of both eigenvalues will be less than 1, since the modulus of the eigenvalues changes continuously with $\mathrm{tr}(A)$ and $\det(A)$ (consider the quadratic formula!). Conversely, we can choose sample points to show that in all the other four regions in Figure 7.41 the modulus of at least one of the eigenvalues exceeds 1; consider

$$\text{the matrices} \quad \underset{\text{in (I)}}{\underset{\uparrow}{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}}, \quad \underset{\text{in (II)}}{\underset{\uparrow}{\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}}}, \quad \underset{\text{in (III)}}{\underset{\uparrow}{\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}}}, \quad \text{and} \quad \underset{\text{in (IV)}}{\underset{\uparrow}{\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}}}.$$

It follows that throughout these four regions, (I), (II), (III), and (IV), at least one of the eigenvalues will have a modulus exceeding one.

The point $(\mathrm{tr}A, \det A)$ is in the shaded triangle if $\det(A) < 1$, $\det(A) > \mathrm{tr}(A) - 1$ and $\det(A) > -\mathrm{tr}(A) - 1$. This means that $|\mathrm{tr}A| - 1 < \det(A) < 1$, as claimed.

7.6.**32** Take conjugates of both sides of the equation $\vec{x}_0 = c_1(\vec{v} + i\vec{w}) + c_2(\vec{v} - i\vec{w})$:

$\vec{x}_0 = \overline{\vec{x}_0} = \overline{c_1(\vec{v} + i\vec{w}) + c_2(\vec{v} - i\vec{w})} = \bar{c}_1(\vec{v} - i\vec{w}) + \bar{c}_2(\vec{v} + i\vec{w}) = \bar{c}_2(\vec{v} + i\vec{w}) + \bar{c}_1(\vec{v} - i\vec{w})$.

The claim that $c_2 = \bar{c}_1$ now follows from the fact that the representation of $\vec{x}_0$ as a linear combination of the linearly independent vectors $\vec{v} + i\vec{w}$ and $\vec{v} - i\vec{w}$ is unique.

7.6.**33** Take conjugates of both sides of the equation $\vec{x}_0 = c_1(\vec{v} + i\vec{w}) + c_2(\vec{v} - i\vec{w})$:

$$\vec{x}_0 = \overline{\vec{x}}_0 = \overline{c_1(\vec{v}+i\vec{w})+c_2(\vec{v}-i\vec{w})} = \bar{c}_1(\vec{v}-i\vec{w})+\bar{c}_2(\vec{v}+i\vec{w}) = \bar{c}_2(\vec{v}+i\vec{w})+\bar{c}_1(\vec{v}-i\vec{w}).$$

The claim that $c_2 = \bar{c}_1$ now follows from the fact that the representation of $\vec{x}_0$ as a linear combination of the linearly independent vectors $\vec{v}+i\vec{w}$ and $\vec{v}-i\vec{w}$ is unique.

7.6.**34** a If $|\det A| = |\lambda_1\lambda_2\cdots\lambda_n| = |\lambda_1\lambda_2|\cdots|\lambda_n| > 1$ then at least one eigenvalue is greater than one in modulus and the zero state fails to be stable.

b If $|\det A| = |\lambda_1||\lambda_2|\cdots|\lambda_n| < 1$ we cannot conclude anything about the stability of $\vec{0}$.

$|2||0.1| < 1$ and $|0.2||0.1| < 1$ but in the first case we would not have stability, in the second case we would.

7.6.**35** a Let $\vec{v}_1,\ldots,\vec{v}_n$ be an eigenbasis for A. Then $\vec{x}(t) = \sum_{i=1}^{n} c_i\lambda_i^t\vec{v}_i$ and

$$\|\vec{x}(t)\| = |\sum_{i=1}^{n} c_i\lambda_i^t\vec{v}_i| \le \sum_{i=1}^{n}\|c_i\lambda_i^t\vec{v}_i\| = \sum_{i=1}^{n}\underset{\substack{\uparrow \\ \le 1}}{|\lambda_i|^t}\|c_i\vec{v}_i\| \le \sum_{i=1}^{n}\|c_i\vec{v}_i\|.$$

The last quantity, $\sum_{i=1}^{n}\|c_i\vec{v}_i\|$, gives the desired bound M.

b $A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ represents a shear parallel to the x-axis, with $A\begin{bmatrix}k\\1\end{bmatrix} = \begin{bmatrix}k+1\\1\end{bmatrix}$, so that $\vec{x}(t) = A^t\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}t\\1\end{bmatrix}$ is not bounded. This does not contradict part a, since there is no eigenbasis for A.

7.6.**36** If the zero state is stable, then $\lim_{t\to\infty}(i\text{th column of } A^t) = \lim_{t\to\infty}(A^t\vec{e}_i) = \vec{0}$, so that all columns and therefore all entries of A^t approach 0.

Conversely, if $\lim_{t\to\infty} A^t = 0$, then $\lim_{t\to\infty}(A^t\vec{x}_0) = \left(\lim_{t\to\infty}A^t\right)\vec{x}_0 = \vec{0}$ for all $\vec{x}_0$ (check the details).

7.6.**37** a Write $Y(t+1) = Y(t) = Y, C(t+1) = C(t) = C, I(t+1) = I(t) = I$.

$$\begin{bmatrix} Y = C + I + G_0 \\ C = \gamma Y \\ I = 0 \end{bmatrix} \to \begin{matrix} Y = \gamma Y + G_0 \\ Y = \frac{G_0}{1-\gamma} \end{matrix}$$

$Y = \frac{G_0}{1-\gamma}, C = \frac{\gamma G_0}{1-\gamma}, I = 0$

b $y(t) = Y(t) - \frac{G_0}{1-\gamma}, c(t) = C(t) - \frac{\gamma G_0}{1-\gamma}, i(t) = I(t)$

Substitute to verify the equations.

$$\begin{bmatrix}C(t+1)\\ i(t+1)\end{bmatrix} = \begin{bmatrix}\gamma & \gamma\\ \alpha\gamma-\alpha & \alpha\gamma\end{bmatrix}\begin{bmatrix}c(t)\\ i(t)\end{bmatrix}$$

c $A = \begin{bmatrix} 0.2 & 0.2 \\ -4 & 1 \end{bmatrix}$ eigenvalues $0.6 \pm 0.8i$

not stable

d $A = \begin{bmatrix} \gamma & \gamma \\ \gamma - 1 & \gamma \end{bmatrix}$, tr$A = 2\gamma$, det$A = \gamma$, stable (use Exercise 31)

e $A = \begin{bmatrix} \gamma & \gamma \\ \alpha\gamma - \alpha & \alpha\gamma \end{bmatrix}$ tr$A = \gamma(1+\alpha) > 0$, det$A = \alpha\gamma$

Use Exercise 31; stable if $\det(A) = \alpha\gamma < 1$ and tr$A - 1 = \alpha\gamma + \gamma - 1 < \alpha\gamma$.

The second condition is satisfied since $\gamma < 1$.

Stable if $\gamma < \frac{1}{\alpha}$

$\left(\text{eigenvalues are real if } \gamma \geq \frac{4\alpha}{(1+\alpha)^2}\right)$

7.6.**38** a $T(\vec{v}) = A\vec{v} + \vec{b} = \vec{v}$ if $\vec{v} - A\vec{v} = \vec{b}$ or $(I_n - A)\vec{v} = \vec{b}$.

$I_n - A$ is invertible since 1 is not an eigenvalue of A. Therefore, $\vec{v} = (I_n - A)^{-1}\,\vec{b}$ is the only solution.

b Let $\vec{y}(t) = \vec{x}(t) - \vec{v}$ be the deviation of $\vec{x}(t)$ from the equilibrium $\vec{v}$.

Then $\vec{y}(t+1) = \vec{x}(t+1) - \vec{v} = A\vec{x}(t) + \vec{b} - \vec{v} = A(\vec{y}(t) + \vec{v}) + \vec{b} - \vec{v} = A\vec{y}(t) + A\vec{v} + \vec{b} - \vec{v} = A\vec{y}(t)$, so that $\vec{y}(t) = A^t\vec{y}(0)$, or $\vec{x}(t) = \vec{v} + A^t(\vec{x}_0 - \vec{v})$.

$\lim\limits_{t\to\infty} \vec{x}(t) = \vec{v}$ for all $\vec{x}_0$ if $\lim\limits_{t\to\infty} A^t(\vec{x}_0 - \vec{v}) = \vec{0}$. This is the case if the modulus of all the eigenvalues of A is less than 1.

7.6.**39** Use Exercise 38: $\vec{v} = (I_2 - A)^{-1}\vec{b} = \begin{bmatrix} 0.9 & -0.2 \\ -0.4 & 0.7 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is a stable equilibrium since the eigenvalues of A are 0.5 and -0.1.

7.6.**40** a $A^TA = \begin{bmatrix} B^T & C^T \\ -C & B \end{bmatrix} \begin{bmatrix} B & -C^T \\ C & B^T \end{bmatrix} = (p^2 + q^2 + r^2 + s^2)I_4$

b By part a, $A^{-1} = \frac{1}{p^2+q^2+r^2+s^2}A^T$ if $A \neq 0$.

c $(\det A)^2 = (p^2 + q^2 + r^2 + s^2)^4$, by part a, so that $\det A = \pm(p^2 + q^2 + r^2 + s^2)^2$.

Laplace Expansion along the first row produces the term $+p^4$,

so that $\det(A) = (p^2 + q^2 + r^2 + s^2)^2$.

d Consider $\det(A - \lambda I_4)$. Note that the matrix $A - \lambda I_4$ has the same "format" as A, with p replaced by $p - \lambda$ and q, r, s remaining unchanged. By part c, $\det(A - \lambda I_4) = ((p-\lambda)^2 + q^2 + r^2 + s^2)^2 = 0$ when

$(p-\lambda)^2 = -q^2 - r^2 - s^2$

$p - \lambda = \pm i\sqrt{q^2+r^2+s^2}$

$\lambda = p \pm i\sqrt{q^2+r^2+s^2}$

Each of these eigenvalues has algebraic multiplicity 2 (if $q = r = s = 0$ then $\lambda = p$ has algebraic multiplicity 4).

e By part a we can write $A = \sqrt{p^2+q^2+r^2+s^2}\underbrace{\left(\frac{1}{\sqrt{p^2+q^2+r^2+s^2}}A\right)}_{S}$, where S is orthogonal.

Therefore, $\|A\vec{x}\| = \|\sqrt{p^2+q^2+r^2+s^2}(S\vec{x})\| = \sqrt{p^2+q^2+r^2+s^2}\|\vec{x}\|$.

f Let $A = \begin{bmatrix} 3 & -3 & -4 & -5 \\ 3 & 3 & 5 & -4 \\ 4 & -5 & 3 & 3 \\ 5 & 4 & -3 & 3 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}$; then $A\vec{x} = \begin{bmatrix} -39 \\ 13 \\ 18 \\ 13 \end{bmatrix}$.

By part e, $\|A\vec{x}\|^2 = (3^2+3^2+4^2+5^2)\|\vec{x}\|^2$, or

$39^2 + 13^2 + 18^2 + 13^2 = (3^2+3^2+4^2+5^2)(1^2+2^2+4^2+4^2)$, as desired.

g Any positive integer m can be written as $m = p_1p_2\dots p_n$. Using part f repeatedly we see that the numbers $p_1, p_1p_2, p_1p_2p_3, \dots, p_1p_2p_3\cdots p_{n-1}$, and finally $m = p_1\cdots p_n$ can be expressed as the sums of four squares.

7.6.**41** Find the 2×2 matrix A that transforms $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$ into $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ into $\begin{bmatrix} -8 \\ -6 \end{bmatrix}$:

$A\begin{bmatrix} 8 & -3 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -8 \\ 4 & -6 \end{bmatrix}$ and $A = \begin{bmatrix} -3 & -8 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} 8 & -3 \\ 6 & 4 \end{bmatrix}^{-1} = \frac{1}{50}\begin{bmatrix} 36 & -73 \\ 52 & -36 \end{bmatrix}$.

There are many other correct answers.

7.6.**42** a $x(t+1) = x(t) - ky(t)$

$y(t+1) = kx(t) + y(t) = kx(t) + (1-k^2)y(t)$ so $\begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = \begin{bmatrix} 1 & -k \\ k & 1-k^2 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

b $f_A(\lambda) = \lambda^2 - (2-k^2)\lambda + 1 = 0$

The discriminant is $(2-k^2)^2 - 4 = -4k^2 + k^4 = k^2(k^2-4)$, which is negative if k is a small positive number ($k < 2$). Therefore, the eigenvalues are complex. By Theorem 7.6.4 the trajectory will be an ellipse, since $\det(A) = 1$.

True or False

Ch 7.TF.**1** T, by Summary 7.1.5.

Ch 7.TF.**2** T, by Theorem 7.2.4

Ch 7.TF.**3** T, by Theorem 7.2.2.

Ch 7.TF.**4** T, by Definition 7.2.3.

Ch 7.TF.**5** F; If $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then eigenvalue 1 has geometric multiplicity 1 and algebraic multiplicity. 2.

Ch 7.TF.**6** T, by Theorem 7.1.3.

Ch 7.TF.**7** T; $A = AI_n = A[\vec{e}_1 \ldots \vec{e}_n] = [\lambda_1\vec{e}_1 \ldots \lambda_n\vec{e}_n]$ is diagonal.

Ch 7.TF.**8** T; If $A\vec{v} = \lambda\vec{v}$, then $A^3\vec{v} = \lambda^3\vec{v}$.

Ch 7.TF.**9** T; Consider a diagonal 5×5 matrix with only two distinct diagonal entries.

Ch 7.TF.**10** F, by Theorem 7.2.7.

Ch 7.TF.**11** T, by Theorem 7.5.5.

Ch 7.TF.**12** F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, for example.

Ch 7.TF.**13** T, by Example 6 of Section 7.5.

Ch 7.TF.**14** T; The geometric multiplicity of eigenvalue 0 is $\dim(\ker A) = n - \text{rank}(A)$.

Ch 7.TF.**15** T; If $S^{-1}AS = B$, then $S^TA^T(S^T)^{-1} = B$.

Ch 7.TF.**16** F; Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, for example.

Ch 7.TF.**17** F; Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Ch 7.TF.**18** F; Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \alpha = 2, B = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \beta = 5$, for example. Then $\alpha\beta = 10$ isn't an eigenvalue of $AB = \begin{bmatrix} 8 & 0 \\ 0 & 15 \end{bmatrix}$.

Ch 7.TF.**19** T; If $A\vec{v} = 3\vec{v}$, then $A^2\vec{v} = 9\vec{v}$.

Ch 7.TF.**20** T; Construct an eigenbasis by concatenating a basis of V with a basis of $V^\perp$.

Ch 7.TF.**21** F; Consider the zero matrix.

Ch 7.TF.**22** T; If $A\vec{v} = \alpha\vec{v}$ and $B\vec{v} = \beta\vec{v}$, then $(A+B)\vec{v} = A\vec{v} + B\vec{v} = \alpha\vec{v} + \beta\vec{v} = (\alpha+\beta)\vec{v}$.

Ch 7.TF.**23** F; Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, with $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Ch 7.TF.**24** T, by Theorem 7.5.5

Ch 7.TF.**25** F; Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, for example.

Ch 7.TF.**26** F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, with $AB = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, for example.

Ch 7.TF.**27** T; If $S^{-1}AS = B$, then $S^{-1}A^{-1}S = B^{-1}$ is diagonal.

Ch 7.TF.**28** F; the equation $\det(A) = \det(A^T)$ holds for all square matrices, by Theorem 6.2.1.

Ch 7.TF.**29** T; The sole eigenvalue, 7, must have geometric multiplicity 3.

Ch 7.TF.**30** F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, with $A+B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, for example.

Ch 7.TF.**31** T, Consider the proof of Theorem 7.3.3a.

Ch 7.TF.**32** T; An eigenbasis for A is an eigenbasis for $A + 4I_4$ as well.

Ch 7.TF.**33** F; Consider the identity matrix.

Ch 7.TF.**34** T; Both A and B are similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, by Theorem 7.1.3.

Ch 7.TF.**35** F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for example.

Ch 7.TF.**36** F; Consider $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Ch 7.TF.**37** F; Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for example.

Ch 7.TF.**38** T; A nonzero vector on L and a nonzero vector on $L^\perp$ form an eigenbasis.

Ch 7.TF.**39** T; The eigenvalues are 3 and -2.

Ch 7.TF.**40** T, We will use Theorem 7.3.6 throughout: The geometric multiplicity of an eigenvalue is $\leq$ its algebraic multiplicity.

Now let's show the contrapositive of the given statement: If the geometric multiplicity of some eigenvalue is less than its algebraic multiplicity, then the matrix A fails to be diagonalizable. Indeed, in this case the sum of the geometric multiplicities of all the eigenvalues is less than the sum of their algebraic multiplicities, which in turn is $\leq n$ (where A is an $n \times n$ matrix). Thus the geometric multiplicities do not add up to n, so that A fails to be diagonalizable, by Theorem 7.3.3b.

Ch 7.TF.**41** F; Consider a rotation through $\pi/2$.

Ch 7.TF.**42** T; Suppose $\begin{bmatrix} A & A \\ 0 & A \end{bmatrix}\begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix} = \begin{bmatrix} A(\vec{v}+\vec{w}) \\ A\vec{w} \end{bmatrix} = \begin{bmatrix} \lambda\vec{v} \\ \lambda\vec{w} \end{bmatrix}$ for a nonzero vector $\begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix}$. If $\vec{w}$ is nonzero, then it is an eigenvector of A with eigenvalue λ; otherwise $\vec{v}$ is such an eigenvector.

Ch 7.TF.**43** F; Consider $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Ch 7.TF.**44** T; Note that $S^{-1}AS = B$, so that $B^4 = S^{-1}A^4S = S^{-1}0S = 0$, and therefore $B = 0$ (since B is diagonal) and $A = SBS^{-1} = 0$.

Ch 7.TF.**45** T; There is an eigenbasis $\vec{v}_1, \dots, \vec{v}_n$, and we can write $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. The vectors $c_i\vec{v}_i$ are either eigenvectors or zero.

Ch 7.TF.**46** T; If $A\vec{v} = \alpha\vec{v}$ and $B\vec{v} = \beta\vec{v}$, then $AB\vec{v} = \alpha\beta\vec{v}$.

Ch 7.TF.**47** T, by Theorem 7.3.5a.

Ch 7.TF.**48** F; Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, for example.

Ch 7.TF.**49** T; Recall that the rank is the dimension of the image. If $\vec{v}$ is in the image of A, then $A\vec{v}$ is in the image of A as well, so that $A\vec{v}$ is parallel to $\vec{v}$.

Ch 7.TF.**50** F; Consider $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Ch 7.TF.**51** T; If $A\vec{v} = \lambda\vec{v}$ for a nonzero $\vec{v}$, then $A^4\vec{v} = \lambda^4\vec{v} = \vec{0}$, so that $\lambda^4 = 0$ and $\lambda = 0$.

Ch 7.TF.**52** F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, for example.

Ch 7.TF.**53** T; If the eigenvalue associated with $\vec{v}$ is $\lambda = 0$, then $A\vec{v} = \vec{0}$, so that $\vec{v}$ is in the kernel of A; otherwise $\vec{v} = A\left(\frac{1}{\lambda}\vec{v}\right)$, so that $\vec{v}$ is in the image of A.

Ch 7.TF.**54** T; either there are two distinct real eigenvalues, or the matrix is of the form kI_2.

Ch 7.TF.**55** T; Either $A\vec{u} = 3\vec{u}$ or $A\vec{u} = 4\vec{u}$.

Ch 7.TF.**56** T; Note that $(\vec{u}\vec{u}^T)\vec{u} = \|\vec{u}\|^2\vec{u}$.

Ch 7.TF.**57** T; Suppose $A\vec{v}_i = \alpha_i\vec{v}_i$ and $B\vec{v}_i = \beta_i\vec{v}_i$, and let $S = [\vec{v}_1 \ldots \vec{v}_n]$.
Then $ABS = BAS = [\alpha_1\beta_1\vec{v}_1 \ldots \alpha_n\beta_n\vec{v}_n]$, so that $AB = BA$.

Ch 7.TF.**58** T; Note that a nonzero vector $\vec{v} = \begin{bmatrix} p \\ q \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if (and only if) $A\vec{v} = \begin{bmatrix} ap+bq \\ cp+dq \end{bmatrix}$ is parallel to $\vec{v} = \begin{bmatrix} p \\ q \end{bmatrix}$, that is, if $\det \begin{bmatrix} p & ap+bq \\ q & cp+dq \end{bmatrix} = 0$. Check that this is the case if (and only if) $\vec{v}$ is an eigenvector of $\operatorname{adj}(A)$ (use the same criterion).