1st Midterm exam

MATH 2331–Linear Algebra Spring 2014

To obtain full credit, you must **show all** work and carefully justify your assertions.
The use of a calculator is **not** permitted.

NAME: Solution

(1) (15 pts.) Use Gauss–Jordan elimination to find all solutions of the linear system

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right) \begin{array}{c} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{array} \xrightarrow{(\mathrm{II})-2\cdot(\mathrm{I})} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right) \begin{array}{c} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{array}$$

$$\xrightarrow{(\mathrm{III})-(\mathrm{II})} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} (\mathrm{I}) \\ (\mathrm{II}) \\ \end{array} \xrightarrow[\text{and } \frac{(\mathrm{II})}{2}]{(\mathrm{I})-\frac{(\mathrm{II})}{2}} \left(\begin{array}{cccc|c} 1 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = t$ and $y = s$ then,
$$\begin{cases} x + 3y = \frac{1}{2} \Rightarrow x = \frac{1}{2} - 3s \\ w + 2z = \frac{1}{2} \Rightarrow w = \frac{1}{2} - 2t \end{cases}$$

The solutions are

$$\begin{bmatrix} x \\ y \\ w \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - 3s \\ s \\ \frac{1}{2} - 2t \\ t \end{bmatrix},$$

where s, t are arbitrary real numbers.

(2) The reduced row–echelon forms of the augmented matrices of three systems are given below. How many solutions does each system have? Justify your answer.

(a) (5pts.) $\left[\begin{array}{ccc:c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

No solutions
Since the last row $[0\ 0\ 0 \vdots 1]$ representing the equation $0 = 1$.
$\therefore$ no solution.

(b) (5pts.) $\left[\begin{array}{ccc:c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Infinitely many solutions.
Since the solutions are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 2 \\ 3 \end{bmatrix}$, where t is arbitrary.

(c) (5pts.) $\left[\begin{array}{cc:c} 1 & 0 & 5 \\ 0 & 1 & 6 \end{array}\right]$

Only one solution.
The unique one solution is:
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

(3) (10 pts.) Fix two vectors $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ in $\mathbb{R}^3$. Consider the linear transformation

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \qquad \text{given by} \qquad T(\underline{x}) = (\underline{x} \cdot \underline{u})\, \underline{v}$$

where $\underline{x} \cdot \underline{u} = x_1u_1 + x_2u_2 + x_3u_3$ is the dot product of $\underline{x}$ and $\underline{u}$. Find the matrix A of T.

Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

By definition, $T(e_1) = u_1 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1v_1 \\ u_1v_2 \\ u_1v_3 \end{pmatrix}$ $T(e_2) = u_2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_1 \\ u_2v_2 \\ u_2v_3 \end{pmatrix}$

and $T(e_3) = u_3 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_3v_1 \\ u_3v_2 \\ u_3v_3 \end{pmatrix}$

Thus, the matrix A of T is

$$\begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_2v_1 & u_3v_1 \\ u_1v_2 & u_2v_2 & u_3v_2 \\ u_1v_3 & u_2v_3 & u_3v_3 \end{bmatrix}$$

(4) (10 pts.) Let L be the line in $\mathbb{R}^3$ through the vector $\underline{v} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$. Find the projection of the vector $\underline{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ on the line L.

$$\mathrm{Proj}_{L}\vec{x} = \frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} = \frac{3}{3^2+4^2}\cdot\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \frac{3}{25}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 9/25 \\ 12/25 \\ 0 \end{pmatrix}$$

OR

Find the unit vector

$$\vec{u} = \frac{1}{\sqrt{3^2+4^2}}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$

$$\mathrm{Proj}_{L}\vec{x} = (\vec{x}\cdot\vec{u})\,\vec{u} = \frac{3}{5}\cdot\frac{1}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \frac{3}{25}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$

(5) (15 pts.) Decide whether the given matrix is invertible and, if so, determine its inverse.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

$$\begin{array}{cc} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right] & \begin{array}{c} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{array} \end{array} \xrightarrow[(\mathrm{III})-(\mathrm{II})]{(\mathrm{II})-(\mathrm{I})} \begin{array}{cc} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 0 & -1 & 1 \end{array}\right] & \begin{array}{c} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{array} \end{array}$$

$$\xrightarrow{(\mathrm{III})-(\mathrm{II})} \begin{array}{cc} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] & \begin{array}{c} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{array} \end{array} \xrightarrow{(\mathrm{II})-2(\mathrm{III})} \begin{array}{cc} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] & \begin{array}{c} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{array} \end{array}$$

$$\xrightarrow{(\mathrm{I})-(\mathrm{II})-(\mathrm{III})} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

Since $\mathrm{rank}(A)=3 \Rightarrow A$ is invertible, the inverse is: $\begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

(6) Decide which of the following products of matrices are defined and, when they are, compute them.

(a) (3 pts.) $\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{B}$

Not defined, since size of A: 2×3

size of B: 2×2

column of $A = 3 \neq 2 =$ # row of B

$\therefore$ not defined.

(b) (3 pts.) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{bmatrix}$$

(c) (2 pts.) $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$$= [1\cdot 1 + 2\cdot 3 + 1\cdot 1]$$
$$= [8]$$

(d) (2 pts.) $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

(7) (15 pts.) Find a polynomial $f(t) = a + bt + ct^2$ of degree 2 whose graph goes through the points $(t = 1, f(t) = -1)$, $(t = 2, f(t) = 3)$, $(t = 3, f(t) = 13)$.

Plug in $t=1$, we have:

$$a+b+c=-1 \quad (\mathrm{I})$$

Plug in $t=2$ we have

$$a+2b+4c=3 \quad (\mathrm{II})$$

Plug in $t=3$, we have

$$a+3b+9c=13 \quad (\mathrm{III})$$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 13 \end{array}\right] \end{array} \begin{array}{c} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{array} \xrightarrow[(\mathrm{III})-(\mathrm{I})]{(\mathrm{II})-(\mathrm{I})} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 5 & 10 \end{array}\right] \begin{array}{c} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{array}$$

$$\xrightarrow{\frac{(\mathrm{III})-(\mathrm{II})}{2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{c} \\ (\mathrm{II}) \\ (\mathrm{III}) \end{array} \xrightarrow{(\mathrm{II})-3(\mathrm{III})} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{c} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{array}$$

$$\xrightarrow{(\mathrm{I})-(\mathrm{II})-(\mathrm{III})} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\therefore$ The solution is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$.

$\therefore$ The polynomial $f(t) = 1 - 5t + 3t^2$

(8) True or false? (if true, provide a justification, if false a counterexample)

(a) (2 pts.) If A, B are square matrices of the same size, then $(A+B)^2 = A^2 + 2AB + B^2$.

False. $(A+B)^2 = (A+B)(A+B)$

$= A^2 + BA + AB + B^2$

Let: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. then $AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $AB \neq BA$ Thus: $(A+B)^2 \neq A^2 + AB + AB + B^2$
$= A^2 + 2AB + B^2$.

(b) (2 pts.) If A, B are matrices such that the multiplications AB and BA are well–defined, then A, B are square matrices.

False

Let the size of A be: $m \times n$
the size of B be: $p \times q$.
AB is well-defined $\Rightarrow n = p$.
BA is well-defined $\Rightarrow q = m$.

Counterexample:
$A = [1, 2]$ $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
Then AB, BA are well-defined
But A, B are not square matrices.

(c) (2 pts.) If A is a matrix such that A^2 is defined, then A is a square matrix.

True. Let the size of A be $m \times n$

$A^2 = A \cdot A$ is defined $\Rightarrow n = m$.

$\therefore$ A has m rows, and m columns.

$\therefore$ A is a square matrix.

(d) (2 pts.) If A, B are invertibles matrices of the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

True. Since $(AB) \cdot (B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A I_n A^{-1} = AA^{-1} = I_n$

and $(B^{-1}A^{-1}) \cdot (AB) = B^{-1}(A^{-1}A)B = B^{-1} I_n B = B^{-1}B = I_n$

$\therefore$ $(AB)^{-1} = B^{-1}A^{-1}$

(e) (2 pts.) If A, B are invertibles matrices, then $(A+B)$ is invertible, and $(A+B)^{-1} = A^{-1} + B^{-1}$ (hint: think about matrices of very small size).

False. Counterexample: Let $B = -A$. then $A + B = 0$, non-invertible

OR

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ then: $A+B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

~~R~~ rank $(A+B) = 1$, $\therefore$ $A+B$ is not invertible.