2nd Midterm exam

MATH 2331–Linear Algebra Summer 1 2015

To obtain full credit, you must **show all work** and carefully **justify all assertions**. The use of notes, books and a calculator is **not** permitted.

NAME:

(1) Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\to x_2 + 2x_3 + 3x_5 = 0$

$x_4 + 4x_5 = 0$

$x_6 = 0$

(a) (5 pts.) Find a basis of the Kernel of A.

free variables x_1, x_3, x_5.

$\cdot\ x_1 = 1,\ x_3 = 0,\ x_5 = 0$: $x_2 = 0$, $x_4 = 0$, $x_6 = 0$ $\to \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\cdot\ x_1 = 0,\ x_3 = 1,\ x_5 = 0$: $x_2 = -2$, $x_4 = 0$, $x_6 = 0$ $\to \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\cdot\ x_1 = 0,\ x_3 = 0,\ x_5 = 1$: $x_2 = -3$, $x_4 = -4$, $x_6 = 0$ $\to \begin{bmatrix} 0 \\ -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$

basis vectors for $\ker(A)$.

(b) (5 pts.) Find a basis of the Image of A.

column vectors with leading variables

$\underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

(2) (10 pts.) Determine whether the following vectors form a basis of $\mathbb{R}^4$

$$\underline{v}_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix}1\\2\\4\\8\end{pmatrix} \quad \underline{v}_4 = \begin{pmatrix}1\\-2\\1\\-8\end{pmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ 1 & 1 & 4 & 1 \\ 1 & -1 & 8 & -8 \end{bmatrix}$$ check if invertible :

$$A \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 0 \\ 0 & -2 & 7 & -9 \end{bmatrix} \begin{matrix} \\ \\ \\ -\mathrm{II} \end{matrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 6 & -6 \end{bmatrix} \begin{matrix} \\ \\ :3 \\ -6\,\mathrm{III} \end{matrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix} \Rightarrow$$ triangular with nonzero diagonal entries $\Rightarrow$ A is invertible $\Rightarrow$ $\underline{v}_1, \dots, v_4$ form a basis.

(3) (10 pts.) Find the orthogonal projection of $9\underline{e}_1$ onto the subspace W of $\mathbb{R}^4$ spanned by the vectors

$$\underline{v}_1 = \begin{pmatrix}2\\2\\1\\0\end{pmatrix} \quad \text{and} \quad \underline{v}_2 = \begin{pmatrix}-2\\2\\0\\1\end{pmatrix}$$

$\underline{v}_1 \circ \underline{v}_2 = -4 + 4 + 0 + 0 = 0 \to$ orthogonal.

$\|\underline{v}_1\| = \sqrt{4+4+1} = 3 = \|v_2\| \Rightarrow \underline{u}_1 = \frac{1}{3}\underline{v}_1$ and $\underline{u}_2 = \frac{1}{3}\underline{v}_2$ orthonormal

$$\Rightarrow \mathrm{proj}_W(9\underline{e}_1) = (9\underline{e}_1 \cdot \underline{u}_1)\underline{u}_1 + (9\underline{e}_1 \cdot \underline{u}_2)\underline{u}_2$$

$$= 6 \cdot \frac{1}{3}\begin{bmatrix}2\\2\\1\\0\end{bmatrix} - 6 \cdot \frac{1}{3}\begin{bmatrix}-2\\2\\0\\1\end{bmatrix} = \begin{bmatrix}8\\0\\2\\-2\end{bmatrix}$$

(4) Consider the following vectors in $\mathbb{R}^4$:

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix} \qquad \underline{v}_2 = \begin{pmatrix} 0 \\ 7 \\ 2 \\ 7 \end{pmatrix} \qquad \underline{v}_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \\ 6 \end{pmatrix}$$

(a) (6 pts.) Perform the Gram-Schmidt process on $\underline{v}_1, \underline{v}_2, \underline{v}_3$ to obtain orthonormal vectors $\underline{u}_1, \underline{u}_2, \underline{u}_3$.

$$\|\underline{v}_1\| = \sqrt{1+49+1+49} = 10, \quad \underline{u}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|} = \frac{1}{10}\begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \quad \underline{v}_2 \cdot \underline{u}_1 = 10$$

$$\underline{v}_2^{\perp} = \underline{v}_2 - (\underline{v}_2 \cdot \underline{u}_1)\,\underline{u}_1 = \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix} - 10 \cdot \frac{1}{10}\begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \|\underline{v}_2^{\perp}\| = \sqrt{2}$$

$$\boxed{\underline{u}_2 = \frac{\underline{v}_2^{\perp}}{\|\underline{v}_2^{\perp}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \qquad \underline{v}_3 \cdot \underline{u}_1 = 10, \quad \underline{v}_3 \cdot \underline{u}_2 = 0$$

$$\underline{v}_3^{\perp} = \underline{v}_3 - (\underline{v}_3 \cdot \underline{u}_1)\,\underline{u}_1 - (\underline{v}_3 \cdot \underline{u}_2)\,\underline{u}_2 = \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix} - \underline{0} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\|\underline{v}_3^{\perp}\| = \sqrt{2}$$

$$\boxed{\underline{u}_3 = \frac{\underline{v}_3^{\perp}}{\|\underline{v}_3^{\perp}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}}$$

(b) (4 pts.) Express your answer as the **QR** factorisation of a matrix V (**note**: you are simply asked to give the matrix V and matrices Q, R such that $V = QR$.)

$$\begin{array}{ccc} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{array} \qquad\qquad \begin{array}{ccc} \underline{u}_1 & \underline{u}_2 & \underline{u}_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{bmatrix} = V = QR = \begin{bmatrix} 1/10 & -1/\sqrt{2} & 0 \\ 7/10 & 0 & 1/\sqrt{2} \\ 1/10 & 1/\sqrt{2} & 0 \\ 7/10 & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 10 & 10 & 10 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

(5) Let ϕ, θ be two fixed angles and consider the matrix

$$A = \begin{pmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{pmatrix}$$

(a) (5 pts.) Show that A is an orthogonal matrix.

check: $A^T A = I_3$:

$$\begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

(b) (5 pts.) Find the inverse A^{-1} of A.

$$A^{-1} = A^T = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}$$

(checked in pt (a) !)

(6) Consider the system $A\underline{x} = \underline{b}$ where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \underline{b} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

(a) (7pts.) Find a least squares solution $\underline{x}^*$ of $A\underline{x} = \underline{b}$.

$$\underline{x}^* = (A^TA)^{-1}A^T\underline{b}.$$

$$A^T\underline{b} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad (A^TA)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow \underline{x}^* = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

(b) (3 pts.) Is this solution unique? Justify your answer.

yes: x^* is solution of $A^TAx = A^Tb$.
since $\ker(A^TA) = \ker(A) = \{\underline{0}\}$
system is consistent and has unique solution!

(7) Consider the matrix $A = \begin{pmatrix} \cos k & 1 & -\sin k \\ 0 & 2 & 0 \\ \sin k & 0 & \cos k \end{pmatrix}$.

(a) (5 pts.) Compute the determinant of A.

$$\det(A) = 2\cos^2 k + 0 + 0 - (-2\sin^2 k) - 0 - 0 = 2$$

(b) (5 pts.) Determine the values of k for which A is invertible.

$\det(A) = 2 \neq 0$ for all k, so A is invertible for all values of k.

(8) (10 pts.) Compute the determinant Laplace expansion:

$$\begin{vmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{vmatrix} = +5 \begin{vmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = +5 \cdot 4 \cdot \begin{vmatrix} 3 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} =$$

$$= 5 \cdot 4 \cdot 3 \cdot \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = 5 \cdot 4 \cdot 3 \cdot (-2) = -120$$

(9) (10 pts.) Use row operations to compute the determinant

$$\begin{vmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{vmatrix} \begin{matrix} \\ +\mathrm{I} \\ -2\mathrm{I} \\ +2\mathrm{I} \end{matrix} \longrightarrow \begin{vmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & 10 & 14 \\ 0 & 4 & 14 & 29 \end{vmatrix} \begin{matrix} \\ \\ -3\mathrm{II} \\ -4\mathrm{II} \end{matrix} \longrightarrow$$

$$\longrightarrow \begin{vmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 13 \end{vmatrix} \begin{matrix} \\ \\ \\ -2\cdot\mathrm{III} \end{matrix} \longrightarrow \begin{vmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 9 \end{vmatrix}$$

$\Rightarrow$ $\det A = 9$ (product of diagonal entries of triangular matrix)

(10) True or false? (No justification is required for your answers to the questions below. A correct answer is worth 1 point, no answer is worth 0 points, and an incorrect answer is worth -1 points).

(a) If $\underline{v}_1, \ldots, \underline{v}_n$ and $\underline{u}_1, \ldots, \underline{u}_m$ are two bases of $\mathbb{R}^{10}$, then n must be equal to m.

T — all bases of $\mathbb{R}^{10}$ must have the same cardinality $= 10$

(b) If A is a 5×6 matrix of rank 4, then $\dim \operatorname{Ker} A$ is equal to 1.

F — rank nullity thm.:

$\dim(\ker A) + \underbrace{\dim(\operatorname{im} A)}_{= \operatorname{rank}(A) = 4} = 6 \rightarrow \dim(\ker A) = 2$

(c) The image of a 3×4 matrix is a subspace of $\mathbb{R}^4$.

F — subspace of $\mathbb{R}^3$

(d) If $\underline{v}_1, \ldots, \underline{v}_n$ span $\mathbb{R}^4$, then n must be equal to 4.

F — n will be greater than or equal to 4. $n=4$ only if they are linearly independent.

(e) There exists a 5×4 matrix whose image consists of all of $\mathbb{R}^5$.

F — by rank nullity thm.,

$\dim(\operatorname{im} A) \leq 4$

(f) If the vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ are linearly independent in $\mathbb{R}^n$, then the vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$ must be linearly independent.

T — if $c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 = 0$, not all $c_i = 0$,

then $c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 + 0 \cdot \underline{v}_4 = 0$.

(g) If the vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ span a subspace W of $\mathbb{R}^n$, then the vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$ must be also span W.

F — e.g. $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}$, $\rightarrow \underline{v}_1, \underline{v}_2, \underline{v}_3$ span smaller subspace

(h) If a subspace W of $\mathbb{R}^3$ contains the standard vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$, then W must be $\mathbb{R}^3$.

T — $\dim W = \operatorname{rank}([\underline{e}_1, \underline{e}_2, \underline{e}_3]) = 3$

(i) If W is a subspace of $\mathbb{R}^n$ of dimension m, then $m \leq n$.

T — basis of W has m elements that are independent in $\mathbb{R}^n$ $\Rightarrow m \leq n$

(j) If the kernel of a matrix A consists of the zero vector only, then the columns of A must be linearly independent.

T — if $A = [\underline{v}_1, \ldots, \underline{v}_n]$ and

$c_1\underline{v}_1 + \ldots + c_n\underline{v}_n = \underline{0}$, then $\underline{c} = \begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix}$ is in $\ker(A)$.

since $\ker A = \{\underline{0}\}$, the columns must be linearly independent.