Chapter 6

Section 6.1

6.1.**1** Fails to be invertible; since $\det\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix} = 6-6=0$.

6.1.**2** Invertible; since $\det\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix} = 10-12=-2$.

6.1.**3** Invertible; since $\det\begin{bmatrix}3 & 5\\7 & 11\end{bmatrix} = 33-35=-2$.

6.1.**4** Fails to be invertible; since $\det\begin{bmatrix}1 & 4\\2 & 8\end{bmatrix} = 8-8=0$.

6.1.**5** Invertible; since $\det\begin{bmatrix}2 & 5 & 7\\0 & 11 & 7\\0 & 0 & 5\end{bmatrix} = 2\cdot 11\cdot 5+0+0-0-0-0=110$.

6.1.**6** Invertible; since $\det\begin{bmatrix}6 & 0 & 0\\5 & 4 & 0\\3 & 2 & 1\end{bmatrix} = 6\cdot 4\cdot 1+0+0-0-0-0=24$.

6.1.**7** This matrix is clearly not invertible, so the determinant must be zero.

6.1.**8** This matrix fails to be invertible, since the $\det(A)=0$.

6.1.**9** Invertible; since $\det\begin{bmatrix}0 & 1 & 2\\7 & 8 & 3\\6 & 5 & 4\end{bmatrix} = 0+3\cdot 6+2\cdot 7\cdot 5-7\cdot 4-2\cdot 8\cdot 6=-36$.

6.1.**10** Invertible; since $\det\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 3 & 6\end{bmatrix} = 1\cdot 2\cdot 6+1\cdot 3\cdot 1+1\cdot 1\cdot 3-3\cdot 3\cdot 1-2\cdot 1\cdot 1-6\cdot 1\cdot 1=1$.

6.1.**11** $\det\begin{bmatrix}k & 2\\3 & 4\end{bmatrix}\neq 0$ when $4k\neq 6$, or $k\neq \frac{3}{2}$.

6.1.**12** $\det\begin{bmatrix}1 & k\\k & 4\end{bmatrix}\neq 0$ when $k^2\neq 4$, or $k\neq 2,-2$.

6.1.**13** $\det\begin{bmatrix}k & 3 & 5\\0 & 2 & 6\\0 & 0 & 4\end{bmatrix} = 8k$, so $k\neq 0$ will ensure that this matrix is invertible.

6.1.14 $\det\begin{bmatrix}4&0&0\\3&k&0\\2&1&0\end{bmatrix}=0$, so the matrix will never be invertible, no matter which k is chosen.

6.1.15 $\det\begin{bmatrix}0&k&1\\2&3&4\\5&6&7\end{bmatrix}=6k-3$. This matrix is invertible when $k\neq\frac{1}{2}$.

6.1.16 $\det\begin{bmatrix}1&2&3\\4&k&5\\6&7&8\end{bmatrix}=60+84+8k-18k-35-64=45-10k$. So this matrix is invertible when $k\neq 4.5$.

6.1.17 $\det\begin{bmatrix}1&1&1\\1&k&-1\\1&k^2&1\end{bmatrix}=2k^2-2=2(k^2-1)=2(k-1)(k+1)$. So k cannot be 1 or -1.

6.1.18 $\det\begin{bmatrix}0&1&k\\3&2k&5\\9&7&5\end{bmatrix}=30+21k-18k^2=-3(k-2)(6k+5)$. So k cannot be 2 or $-\frac{5}{6}$.

6.1.19 $\det\begin{bmatrix}1&1&k\\1&k&k\\k&k&k\end{bmatrix}=-k^3+2k^2-k=-k(k-1)^2$. So k cannot be 0 or 1.

6.1.20 $\det\begin{bmatrix}1&k&1\\1&k+1&k+2\\1&k+2&2k+4\end{bmatrix}=(k+1)(2k+4)+k(k+2)+(k+2)-(k+1)-k(2k+4)-(k+2)(k+2)=$ $(k+1)(3k+6)-(3k^2+9k+5)=1$. Thus, A will always be invertible, no matter the value of k, meaning that k can have any value.

6.1.21 $\det\begin{bmatrix}k&1&1\\1&k&1\\1&1&k\end{bmatrix}=k^3-3k+2=(k-1)^2(k+2)$. So k cannot be -2 or 1.

6.1.22 $\det\begin{bmatrix}\cos k&1&-\sin k\\0&2&0\\\sin k&0&\cos k\end{bmatrix}=2\cos^2 k+2\sin^2 k=2$. So k can have any value.

6.1.23 $\det(A-\lambda I_2)=\det\begin{bmatrix}1-\lambda&2\\0&4-\lambda\end{bmatrix}=(1-\lambda)(4-\lambda)=0$ if λ is 1 or 4.

6.1.24 $\det(A-\lambda I_2)=\det\begin{bmatrix}2-\lambda&0\\1&0-\lambda\end{bmatrix}=(2-\lambda)(-\lambda)=0$ if λ is 2 or 0.

6.1.25 $\det(A-\lambda I_2)=\det\begin{bmatrix}4-\lambda&2\\4&6-\lambda\end{bmatrix}=(4-\lambda)(6-\lambda)-8=(\lambda-8)(\lambda-2)=0$ if λ is 2 or 8.

6.1.**26** $\det(A - \lambda I_2) = \det\begin{bmatrix} 4-\lambda & 2 \\ 2 & 7-\lambda \end{bmatrix} = (4-\lambda)(7-\lambda) - 4 = (\lambda - 8)(\lambda - 3) = 0$ if λ is 3 or 8.

6.1.**27** $A - \lambda I_3$ is a lower triangular matrix with the diagonal entries $(2-\lambda), (3-\lambda)$ and $(4-\lambda)$. Now, $\det(A - \lambda I_3) = (2-\lambda)(3-\lambda)(4-\lambda) = 0$ if λ is 2, 3 or 4.

6.1.**28** $A - \lambda I_3$ is an upper triangular matrix with the diagonal entries $(2-\lambda), (3-\lambda)$ and $(5-\lambda)$. Now, $\det(A - \lambda I_3) = (2-\lambda)(3-\lambda)(5-\lambda) = 0$ if λ is 2, 3 or 5.

6.1.**29** $\det(A - \lambda I_3) = \det\begin{bmatrix} 3-\lambda & 5 & 6 \\ 0 & 4-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{bmatrix} = (3-\lambda)(\lambda-8)(\lambda-3) = 0$ if λ is 3 or 8.

6.1.**30** $\det(A - \lambda I_3) = \det\begin{bmatrix} 4-\lambda & 2 & 0 \\ 4 & 6-\lambda & 0 \\ 5 & 2 & 3-\lambda \end{bmatrix} = (4-\lambda)(6-\lambda)(3-\lambda) - 8(3-\lambda)$

$= (3-\lambda)(8-\lambda)(2-\lambda) = 0$ if λ is 3, 8 or 2.

6.1.**31** This matrix is upper triangular, so the determinant is the product of the diagonal entries, which is 24.

6.1.**32** This matrix is upper triangular, so the determinant is the product of the diagonal entries, which is 210.

6.1.**33** The determinant of this block matrix is $\det\begin{bmatrix} 1 & 2 \\ 8 & 7 \end{bmatrix} \det\begin{bmatrix} 2 & 3 \\ 7 & 5 \end{bmatrix} = (7-16)(10-21) = 99$,by Theorem 6.1.5.

6.1.**34** The determinant of this block matrix is $\det\begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix} \det\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = (24-15)(3-8) = -45$, by Theorem 6.1.5.

6.1.**35** There are two patterns with a nonzero product, $(a_{12}, a_{23}, a_{31}, a_{44}) = (3, 2, 6, 4)$, with two inversions, and $(a_{12}, a_{23}, a_{34}, a_{41}) = (3, 2, 3, 7)$, with 3 inversions. Thus $\det A = 3 \cdot 2 \cdot 6 \cdot 4 - 3 \cdot 2 \cdot 3 \cdot 7 = 18$.

6.1.**36** There is one pattern with a nonzero product, containing all the 1's, with six inversions. Thus $\det A = 1$.

6.1.**37** The determinant of this block matrix is
$\det\begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix} \det\begin{bmatrix} 5 & 6 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = (35-24)(5 \cdot 1 \cdot 1) = 55$, by Theorem 6.1.5.

6.1.**38** The determinant of this block matrix is
$\det\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ 2 & 1 & 2 \end{bmatrix} \det\begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} = (2 \cdot 4 \cdot 2 + 3 \cdot 3 \cdot 1 - 1 \cdot 4 \cdot 1 - 2 \cdot 3 \cdot 2)(6 \cdot 6 - 5 \cdot 5) = 99$, by Theorem 6.1.5.

6.1.**39** There is only one pattern with a nonzero product, containing all the nonzero entries of the matrix, with eight inversions. Thus $\det A = 1 \cdot 2 \cdot 4 \cdot 3 \cdot 5 = 120$.

6.1.**40** There is only one pattern with a nonzero product, containing all the nonzero entries of the matrix, with seven inversions. Thus $\det A = -3 \cdot 2 \cdot 4 \cdot 1 \cdot 5 = -120$.

6.1.**41** There are two patterns with a nonzero product, $(a_{15}, a_{24}, a_{32}, a_{41}, a_{53}) = (2, 2, 3, 2, 3)$, with eight inversions, and $(a_{13}, a_{24}, a_{32}, a_{41}, a_{55}) = (1, 2, 3, 2, 4)$, with five inversions. Thus $\det A = 2 \cdot 2 \cdot 3 \cdot 2 \cdot 3 - 1 \cdot 2 \cdot 3 \cdot 2 \cdot 4 = 24$.

6.1.**42** There is only one pattern with a nonzero product, $(a_{13}, a_{24}, a_{32}, a_{45}, a_{51}) = (2, 2, 9, 5, 3)$, with six inversions. Thus $\det A = 2 \cdot 2 \cdot 9 \cdot 5 \cdot 3 = 540$.

6.1.**43** For each pattern P in A, consider the corresponding pattern P_{opp} in $-A$, with all the n entries being opposites. Then $\text{prod}\,(P_{opp}) = (-1)^n \text{prod}(P)$ and $sgn\,(P_{opp}) = sgn\,(P)$, so that $\det(-A) = (-1)^n \det A$.

6.1.**44** For each pattern P in A, consider the corresponding pattern P_m in kA, with all the n entries being multiplied by the scalar k. Then $\text{prod}\,(P_m) = k^n \text{prod}(P)$ and $sgn\,(P_m) = sgn\,(P)$, so that $\det(kA) = k^n \det A$.

6.1.**45** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A^T) = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - cb = \det(A)$. It turns out that $\det(A^T) = \det(A)$.

6.1.**46** Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$. If $a_1a_4 - a_2a_3 \neq 0$, then $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{bmatrix}$.

By Exercise 44, $\det(A^{-1}) = \left(\frac{1}{\det(A)}\right)^2 (a_1a_4 - a_2a_3) = \left(\frac{1}{\det(A)}\right)^2 \cdot \det(A)$ so $\det(A^{-1}) = \frac{1}{\det(A)}$.

6.1.**47** We have $\det(A) = (ah - cf)k + bef + cdg - aeg - bdh$. Thus matrix A is invertible for all k if (and only if) the coefficient $(ah - cf)$ of k is 0, while the sum $bef + cdg - aeg - bdh$ is nonzero. A numerical example is $a = c = d = f = h = g = 1$ and $b = e = 2$, but there are infinitely many other solutions as well.

6.1.**48** Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so $\det(A) = \det(B) = \det(C) = \det(D) = 0$ hence $\det(A)\det(D) - \det(B)\det(C) = 0$ but $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = -1$.

6.1.**49** The kernel of T consists of all vectors $\vec{x}$ such that the matrix $[\vec{x}\ \vec{v}\ \vec{w}]$ fails to be invertible. This is the case if $\vec{x}$ is a linear combination of $\vec{v}$ and $\vec{w}$ as discussed on Pages 249 and 250. Thus $\ker(T) = \text{span}(\vec{v}, \vec{w})$. The image of T isn't $\{0\}$, since $T(\vec{v} \times \vec{w}) \neq 0$, for example. Being a subspace of $\mathbb{R}$, the image must be all of $\mathbb{R}$.

6.1.**50** Theorem 6.1.1 tells us that $\det[\vec{u}\ \vec{v}\ \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w}) = \|\vec{u}\| \cos(\theta) \|\vec{v} \times \vec{w}\| = \|\vec{u}\| \cos(\theta) \|\vec{v}\| \sin(\alpha) \|\vec{w}\| = \cos(\theta)\sin(\alpha)$, where θ is the angle enclosed by vectors $\vec{u}$ and $\vec{v} \times \vec{w}$, and α is the angle between $\vec{v}$ and $\vec{w}$. Thus $\det[\vec{u}\ \vec{v}\ \vec{w}]$ can be any number on the closed interval $[-1, 1]$.

6.1.**51** Let a_{ii} be the first entry on the diagonal that fails to belong to the pattern. The pattern must contain an entry in the i^{th} row to the right of a_{ii}, above the diagonal, and also an entry in the i^{th} column below a_{ii}, below the diagonal.

6.1.**52** By Definition 6.1.1, we have $\det \begin{bmatrix} \vec{v} \times \vec{w} & \vec{v} & \vec{w} \end{bmatrix} = (\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w}) = \|\vec{v} \times \vec{w}\|^2$.

6.1.**53** There is one pattern with a nonzero product, containing all the 1's. We have n^2 inversions, since each of the 1's in the lower left block forms an inversion with each of the 1's in the upper right block. Thus $\det A = (-1)^{n^2} = (-1)^n$.

6.1.**54** The pattern containing all the 1000's has 4 inversions so it contributes $(1000)^5 = 10^{15}$ to the determinant. There are $5! - 1 = 119$ other patterns with at most 3 entries being 1000, the others being ≤ 9. Thus the product associated with each of those patterns is less than$(1000)^3\,(10)^2 = 10^{11}$. Now $\det A > 10^{15} - 119 \cdot 10^{11} > 0$.

6.1.**55** By Exercise 2.4.93, a square matrix admits an LU factorization if (and only if) all its principal submatrices are invertible. Now

$A^{(1)} = [7]$, $A^{(2)} = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$, $A^{(3)} = A = \begin{bmatrix} 7 & 4 & 2 \\ 5 & 3 & 1 \\ 3 & 1 & 4 \end{bmatrix}$,

with $\det(A^{(1)}) = 7, \det(A^{(2)}) = 1$, $\det(A^{(3)}) = 1$.

Since all principal submatrices turn out to be invertible, the matrix A does indeed admit an LU factorization.

6.1.**56** There is only one pattern with a nonzero product, containing all the 1's. The number of inversions is $(n-1) + (n-2) + ... + 2 + 1 = \sum_{k=1}^{n-1} k = \frac{(n-1)n}{2}$. This number is even if either n or $n-1$ is divisible by 4, that is, for $n = 4, 5, 8, 9, 12, 13, ...$

a. $\det M_4 = \det M_5 = 1, \quad \det M_2 = \det M_3 = \det M_6 = \det M_7 = -1$.

b. $\det M_n = (-1)^{n(n-1)/2}$

6.1.**57** In a permutation matrix P, there is only one pattern with a nonzero product, containing all the 1's. Depending on the number of inversions in that pattern, we have $\det P = 1$ or $\det P = -1$.

6.1.**58** a If a, b, c, d are distinct prime numbers, then $ad \neq bc$, since the prime factorization of a positive integer is unique. Thus $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$: No matrix of the required form exists.

b We are looking for a noninvertible matrix $A = [\vec{u}\ \vec{v}\ \vec{w}]$ whose entries are nine distinct prime numbers. The last column vector, $\vec{w}$, must be redundant; to keep things simple, we will make $\vec{w} = \vec{u} + 2\vec{v}$. Now we have to pick six distinct prime entries for the first two columns, $\vec{u}$ and $\vec{v}$, such that the entries of $\vec{w} = \vec{u} + 2\vec{v}$ are prime as well. This can be done in many different ways; one solution is $A = \begin{bmatrix} 7 & 2 & 11 \\ 17 & 3 & 23 \\ 19 & 5 & 29 \end{bmatrix}$.

6.1.**59** $F \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} = ab + cd$

a. Yes, F is linear in both columns. To prove linearity in the second column, observe that $ab + cd$ is a linear combination of the variables b and d, with the constant coefficients a and c. An analogous argument proves linearity in the first column.

b. No, since $ab + cd$ fails to be a linear combination of a and b.

c. No, $F \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix} = F \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} = \vec{v} \cdot \vec{w}$: swapping the columns leaves F unchanged.

6.1.**60** The functions in parts a, b, and d are linear in both columns; the functions in parts a, c, and d are linear in both rows; and $F(A) = -\det A$ in part d is alternating on the columns. For example, the function $F(A) = cd$ in part b is linear in the first column since cd is a linear combination of the entries a and c in the first column. However, $F(A) = cd$ fails to be linear in the second row since cd fails to be a linear combination of the entries c and d in the second row. Furthermore, $F\begin{bmatrix} a & b \\ c & d \end{bmatrix} = F\begin{bmatrix} b & a \\ d & c \end{bmatrix} = cd$, showing that F fails to be alternating on the columns.

6.1.**61** The function $F(A) = bfg$ is linear in all three columns and in all three rows since the product bfg contains exactly one factor from each row and from each column; it is the product associated with a pattern. For example, F is linear in the second row since bfg is a scalar multiple of f and thus a linear combination of the entries d, e, f in the second row. F fails to be alternating; for example, $F\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = 1$ but $F\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = 0$ after swapping the first two columns.

6.1.**62** If $A = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$, then $D(A) = D\begin{bmatrix} a & a \\ c & c \end{bmatrix} \underbrace{=}_{\substack{\textit{swap} \\ \textit{columns}}} -D\begin{bmatrix} a & a \\ c & c \end{bmatrix} = -D(A)$, so that $D(A) = 0$.

6.1.**63** $D\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \underbrace{=}_{\textit{Step 1}} D\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} + D\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \underbrace{=}_{\textit{Step 2}} ab\ D\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + ad\ D\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{=}_{\textit{Step 3}} ad$. In Step 1, we write $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \end{bmatrix}$ and use linearity in the second column. In Step 2, we write $\begin{bmatrix} a \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ etc. and use linearity in both columns. In Step 3, we use Exercise 62 for the first summand and the given property $D(I_2) = 1$ for the second summand.

6.1.**64** Writing $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$ and repeatedly using linearity in the columns, we find

$$D\begin{bmatrix} a & b \\ c & d \end{bmatrix} = D\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + D\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \underbrace{=}_{\textit{Step 2}} ad + D\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} + D\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$$

$$= ad + bc\ D\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + cd\ D\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \underbrace{=}_{\textit{Step 4}} ad - bc\ D\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = ad - bc = \det A$$

See the analogous computations in Exercise 63. In Step 2, we are using the result $D\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = ad$ from Exercise 63. In Step 4, we swap the columns of the matrix in the second summand and we apply Exercise 62 to the third summand.

6.1.**65** Freely using the linearity and alternating properties in the columns, and omitting terms with two equal columns (see Exercise 62), we find

$$D\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\ D\begin{bmatrix} 1 & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix} + d\ D\begin{bmatrix} 0 & b & c \\ 1 & e & f \\ 0 & h & i \end{bmatrix} + g\ D\begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 1 & h & i \end{bmatrix}$$

$$= ae\ D\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & i \end{bmatrix} + ah\ D\begin{bmatrix} 1 & 0 & c \\ 0 & 0 & f \\ 0 & 1 & i \end{bmatrix} + db\ D\begin{bmatrix} 0 & 1 & c \\ 1 & 0 & f \\ 0 & 0 & i \end{bmatrix} + dh\ D\begin{bmatrix} 0 & 0 & c \\ 1 & 0 & f \\ 0 & 1 & i \end{bmatrix}$$