Chapter 8

Section 8.1

8.1.**1** $\vec{e}_1$, $\vec{e}_2$ is an orthonormal eigenbasis.

8.1.**2** $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ is an orthonormal eigenbasis.

8.1.**3** $\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}$, $\frac{1}{\sqrt{5}}\begin{bmatrix}-1\\2\end{bmatrix}$ is an orthonormal eigenbasis.

8.1.**4** $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$, $\frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\2\end{bmatrix}$ is an orthonormal eigenbasis.

8.1.**5** Eigenvalues -1, -1, 2

Choose $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}$ in E_{-1} and $\vec{v}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ in E_2 and let $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$.

8.1.**6** $\frac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}$, $\frac{1}{3}\begin{bmatrix}2\\-1\\-2\end{bmatrix}$, $\frac{1}{3}\begin{bmatrix}1\\-2\\2\end{bmatrix}$ is an orthonormal eigenbasis.

8.1.**7** $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ is an orthonormal eigenbasis, so $S = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$ and $D = \begin{bmatrix}5 & 0\\0 & 1\end{bmatrix}$.

8.1.**8** $\frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix}$, $\frac{1}{\sqrt{10}}\begin{bmatrix}-1\\3\end{bmatrix}$ is an orthonormal eigenbasis, with $\lambda_1 = 4$ and $\lambda_2 = -6$, so $S = \frac{1}{\sqrt{10}}\begin{bmatrix}3 & -1\\1 & 3\end{bmatrix}$ and $D = \begin{bmatrix}4 & 0\\0 & -6\end{bmatrix}$.

8.1.**9** $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$ is an orthonormal eigenbasis, with $\lambda_1 = 3$, $\lambda_2 = -3$, and $\lambda_3 = 2$, so

$S = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1 & 0\\0 & 0 & \sqrt{2}\\1 & 1 & 0\end{bmatrix}$ and $D = \begin{bmatrix}3 & 0 & 0\\0 & -3 & 0\\0 & 0 & 2\end{bmatrix}$.

8.1.**10** $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 9$.

$\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\\0\end{bmatrix}$ is in E_0 and $\vec{v}_2 = \frac{1}{3}\begin{bmatrix}1\\-2\\2\end{bmatrix}$ is in E_9.

Let $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 \\ -4 \\ -5 \end{bmatrix}$; then $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ is an orthonormal eigenbasis.

$$S = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{3} & \frac{2}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{3} & -\frac{4}{3\sqrt{5}} \\ 0 & \frac{2}{3} & -\frac{\sqrt{5}}{3} \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

8.1.**11** $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an orthonormal eigenbasis, with $\lambda_1 = 2$, $\lambda_2 = 0$, and $\lambda_3 = 1$, so $S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

8.1.**12** a $E_1 = \text{span} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $E_{-1} = (E_1)^\perp$. An orthonormal eigenbasis is $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

b Use Theorem 7.4.1: $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

c $A = SBS^{-1} = \begin{bmatrix} -0.6 & 0 & 0.8 \\ 0 & -1 & 0 \\ 0.8 & 0 & 0.6 \end{bmatrix}$, where $S = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{bmatrix}$.

8.1.**13** Yes; if $\vec{v}$ is an eigenvector of A with eigenvalue λ, then $\vec{v} = I_3\vec{v} = A^2\vec{v} = \lambda^2\vec{v}$, so that $\lambda^2 = 1$ and $\lambda = 1$ or $\lambda = -1$. Since A is symmetric, E_1 and E_{-1} will be orthogonal complements, so that A represents the reflection about E_1.

8.1.**14** Let S be as in Example 3. Then $S^{-1}AS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

a. This matrix is $2A$ so that $S^{-1}(2A)S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

b. This is $A - 3I_3$, so that $S^{-1}(A - 3I_3)S = S^{-1}AS - 3I_3 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

c. This is $\frac{1}{2}(A - I_3)$, so that $S^{-1}\left(\frac{1}{2}(A - I_3)\right)S = \frac{1}{2}(S^{-1}AS - I_3) = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

8.1.**15** Yes, if $A\vec{v} = \lambda\vec{v}$, then $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$, so that an orthonormal eigenbasis for A is also an orthonormal eigenbasis for A^{-1} (with reciprocal eigenvalues).

8.1.**16** a $\ker(A)$ is four-dimensional, so that the eigenvalue 0 has multiplicity 4, and the remaining eigenvalue is $\operatorname{tr}(A) = 5$.

b $B = A + 2I_5$, so that the eigenvalues are 2, 2, 2, 2, 7.

c $\det(B) = 2^4 \cdot 7 = 112$ (product of eigenvalues)

8.1.**17** If A is the $n \times n$ matrix with all 1's, then the eigenvalues of A are 0 (with multiplicity $n-1$) and n. Now $B = qA + (p-q)I_n$, so that the eigenvalues of B are $p-q$ (with multiplicity $n-1$) and $qn + p - q$. Thus $\det(B) = (p-q)^{n-1}(qn + p - q)$.

8.1.**18** By Theorem 6.3.6, the volume is $|\det A| = \sqrt{\det(A^T A)}$. Now $\vec{v}_i \cdot \vec{v}_j = \|\vec{v}_i\|\|\vec{v}_j\|\cos(\theta) = \frac{1}{2}$, so that $A^T A$ has all 1's on the diagonal and $\frac{1}{2}$'s outside. By Exercise 17 $\left(\text{with } p = 1 \text{ and } q = \frac{1}{2}\right)$, $\det(A^T A) = \left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}n + \frac{1}{2}\right) = \left(\frac{1}{2}\right)^n (n+1)$, so that the volume is $\sqrt{\det(A^T A)} = \left(\frac{1}{2}\right)^{n/2}\sqrt{n+1}$.

8.1.**19** Let $L(\vec{x}) = A\vec{x}$. Then $A^T A$ is symmetric, since $(A^T A)^T = A^T (A^T)^T = A^T A$, so that there is an orthonormal eigenbasis $\vec{v}_1, \ldots, \vec{v}_m$ for $A^T A$. Then the vectors $A\vec{v}_1, \ldots, A\vec{v}_m$ are orthogonal, since $A\vec{v}_i \cdot A\vec{v}_j = (A\vec{v}_i)^T A\vec{v}_j = \vec{v}_i^T A^T A\vec{v}_j = \vec{v}_i \cdot (A^T A\vec{v}_j) = \vec{v}_i \cdot (\lambda_j \vec{v}_j) = \lambda_j(\vec{v}_i \cdot \vec{v}_j) = 0$ if $i \neq j$.

8.1.**20** By Exercise 19, there is an orthonormal basis $\vec{v}_1, \ldots, \vec{v}_m$ of $\mathbb{R}^m$ such that $T(\vec{v}_1), \ldots, T(\vec{v}_m)$ are orthogonal. Suppose that $T(\vec{v}_1), \ldots, T(\vec{v}_r)$ are nonzero and $T(\vec{v}_{r+1}), \ldots, T(\vec{v}_m)$ are zero. Then let $\vec{w}_i = \frac{1}{\|T(\vec{v}_i)\|}T(\vec{v}_i)$ for $i = 1, \ldots, r$ and choose an orthonormal basis $\vec{w}_{r+1}, \ldots, \vec{w}_n$ of $[\operatorname{span}(\vec{w}_1, \ldots, \vec{w}_r)]^\perp$. Then $\vec{w}_1, \ldots, \vec{w}_n$ does the job.

8.1.**21** For each eigenvalue there are two unit eigenvectors: $\pm\vec{v}_1$, $\pm\vec{v}_2$, and $\pm\vec{v}_3$. We have 6 choices for the first column of S, 4 choices remaining for the second column, and 2 for the third.

Answer: $6 \cdot 4 \cdot 2 = 48$.

8.1.**22** a If we let $k = 2$ then A is symmetric and therefore (orthogonally) diagonalizable.

b If we let $k = 0$ then 0 is the only eigenvalue (but $A \neq 0$), so that A fails to be diagonalizable.

8.1.**23** The eigenvalues are real (by Theorem 8.1.3), so that the only possible eigenvalues are ± 1. Since A is symmetric, E_1 and E_{-1} are orthogonal complements. Thus A represents a *reflection* about E_1.

8.1.**24** Note that A is symmetric and orthogonal, so that the eigenvalues are 1 and -1 (see Exercise 23).

$E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$ and $E_{-1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right)$, so that

$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ is an orthonormal eigenbasis.

8.1.**25** Note that A is symmetric an orthogonal, so that the eigenvalues of A are 1 and -1.

$$E_1 = \text{span}\left(\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}\right), E_{-1} = \text{span}\left(\begin{bmatrix}1\\0\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-1\\0\end{bmatrix}\right)$$

The columns of S must form an eigenbasis for $A : S = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&1&0\\0&1&0&0&1\\0&0&\sqrt{2}&0&0\\0&1&0&0&-1\\1&0&0&-1&0\end{bmatrix}$ is one possible choice.

8.1.**26** Since J_n is both orthogonal and symmetric, the eigenvalues are 1 and -1. If n is even, then both have multiplicity $\frac{n}{2}$ (as in Exercise 24). If n is odd, then the multiplicities are $\frac{n+1}{2}$ for 1 and $\frac{n-1}{2}$ for -1 (as in Exercise 25). One way to see this is to observe that $\text{tr}(J_n)$ is 0 for even n, and 1 for odd n (recall that the trace is the sum of the eigenvalues).

8.1.**27** If n is even, then this matrix is $J_n + I_n$, for the J_n introduced in Exercise 26, so that the eigenvalues are 0 and 2, with multiplicity $\frac{n}{2}$ each. E_2 is the span of all $\vec{e}_i + \vec{e}_{n+1-i}$, for $i = 1, \ldots, \frac{n}{2}$, and E_0 is spanned by all $\vec{e}_i - \vec{e}_{n+1-i}$. If n is odd, then E_2 is spanned by all $\vec{e}_i + \vec{e}_{n+1-i}$, for $i = 1, \ldots, \frac{n-1}{2}$; E_0 is spanned by all $\vec{e}_i - \vec{e}_{n+1-i}$, for $i = 1, \ldots, \frac{n-1}{2}$, and E_1 is spanned by $\vec{e}_{\frac{n+1}{2}}$.

8.1.**28** For $\lambda \neq 0$

$$f_A(\lambda) = \det\left[\begin{array}{cccc|c}-\lambda & & & 0 & 1\\ & -\lambda & & & 1\\ & & \ddots & & \vdots\\ 0 & & & -\lambda & 1\\ 1 & 1 & \cdots & 1 & 1-\lambda\end{array}\right] = \frac{1}{\lambda}\det\left[\begin{array}{cccc|c}-\lambda & & & 0 & 1\\ & -\lambda & & & 1\\ & & \ddots & & \vdots\\ & 0 & & -\lambda & 1\\ \lambda & \lambda & \cdots & \lambda & \lambda-\lambda^2\end{array}\right]$$

$$= \frac{1}{\lambda}\det\left[\begin{array}{cccc|c}-\lambda & & & 0 & 1\\ & -\lambda & & & 1\\ & & \ddots & & \vdots\\ & 0 & & -\lambda & 1\\ 0 & 0 & \cdots & 0 & -\lambda^2+\lambda+12\end{array}\right]$$

$= -\lambda^{11}(\lambda^2 - \lambda - 12) = -\lambda^{11}(\lambda - 4)(\lambda + 3)$

Eigenvalues are 0 (with multiplicity 11), 4 and -3.

Eigenvalues for 0 are $\vec{e}_1 - \vec{e}_i (i = 2, \ldots, 12)$,

$$E_4 = \text{span}\begin{bmatrix}1\\1\\\vdots\\1\\4\end{bmatrix} \text{(12 ones)}, E_{-3} = \text{span}\begin{bmatrix}1\\1\\\vdots\\1\\-3\end{bmatrix} \text{(12 ones)}$$

so

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -3 \end{bmatrix}$$

diagonalizes A, and $D = S^{-1}AS$ will have all zeros as entries except $d_{12,\ 12} = 4$ and $d_{13,\ 13} = -3$.

8.1.**29** By Theorem 5.4.1 $(\text{im } A)^{\perp} = \ker(A^T) = \ker(A)$, so that $\vec{v}$ is orthogonal to $\vec{w}$.

8.1.**30** The columns $\vec{v}, \vec{v}_2, \ldots, \vec{v}_n$ of R form an orthogonal eigenbasis for $A = \vec{v}\,\vec{v}^T$, with eigenvalues $1, 0, 0, \ldots, 0 (n-1$ zeros), since

$A\vec{v} = \vec{v}v^T\vec{v} = \vec{v}(\vec{v}\cdot\vec{v}) = \vec{v}$, (since $\vec{v}\cdot\vec{v} = 1$) and $A\vec{v}_i = \vec{v}\,v^T\,\vec{v}_i = \vec{v}(\vec{v}\cdot\vec{v}_i) = \vec{0}$ (since $\vec{v}\cdot\vec{v}_i = 0$).

Therefore we can let $S = R$, and $D = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}$

8.1.**31** True; A is diagonalizable, that is, A is similar to a diagonal matrix D; then A^2 is similar to D^2. Now $\text{rank}(D) = \text{rank}(D^2)$ is the number of nonzero entries on the diagonal of D (and D^2). Since similar matrices have the same rank (by Theorem 7.3.6b) we can conclude that $\text{rank}(A) = \text{rank}(D) = \text{rank}(D^2) = \text{rank}(A^2)$.

8.1.**32** By Exercise 17, $\det(A) = (1-q)^{n-1}(qn+1-q)$. A is invertible if $\det(A) \neq 0$, that is, if $q \neq 1$ and $q \neq \frac{1}{1-n}$.

8.1.**33** The angles must add up to 2π, so $\theta = \frac{2\pi}{3} = 120°$. (See Figure 8.1.)

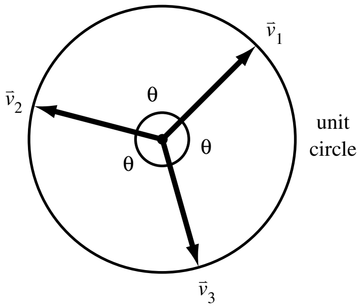

Figure 8.1: for Problem 8.1.33.

Algebraically, we can see this as follows: let $A = [\,\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3\,]$, a 2×3 matrix.

Then $A^TA = \begin{bmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{bmatrix}$ is a noninvertible 3×3 matrix, so that $\cos\theta = \frac{1}{1-3} = -\frac{1}{2}$, by Exercise 32, and $\theta = \frac{2\pi}{3} = 120°$.

8.1.**34** Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ be such vectors. Form $A = [\,\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4\,]$, a 3×4 matrix.

Then $A^TA = \begin{bmatrix} 1 & \cos\theta & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta & \cos\theta \\ \cos\theta & \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & \cos\theta & 1 \end{bmatrix}$ is noninvertible, so that $\cos\theta = \frac{1}{1-4} = -\frac{1}{3}$, by Exercise 32, and $\theta = \arccos\left(-\frac{1}{3}\right) \approx 109.5°$. See Figure 8.2.

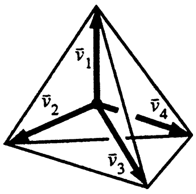

Figure 8.2: for Problem 8.1.34.

The tips of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ form a regular tetrahedron.

8.1.**35** Let $\vec{v}_1, \ldots, \vec{v}_{n+1}$ be these vectors. Form $A = [\,\vec{v}_1 \quad \cdots \quad \vec{v}_{n+1}\,]$, an $n \times (n+1)$ matrix.

Then $A^TA = \begin{bmatrix} 1 & \cos\theta & \cdots & \cos\theta \\ \cos\theta & 1 & \cdots & \cos\theta \\ \vdots & & \ddots & \\ \cos\theta & & \cdots & 1 \end{bmatrix}$ is a noninvertible $(n+1) \times (n+1)$ matrix with 1's on the diagonal and $\cos\theta$ outside, so that $\cos\theta = \frac{1}{1-n}$, by Exercise 32, and $\theta = \arccos\left(\frac{1}{1-n}\right)$.

8.1.**36** If $\vec{v}$ is an eigenvector with eigenvalue λ, then $\lambda\vec{v} = A\vec{v} = A^2\vec{v} = \lambda^2\vec{v}$, so that $\lambda = \lambda^2$ and therefore $\lambda = 0$ or $\lambda = 1$. Since A is symmetric, E_0 and E_1 are orthogonal complements, so that A represents the orthogonal projection onto E_1.

8.1.**37** In Example 4 we see that the image of the unit circle is an ellipse with semi-axes 2 and 3. Thus $||A\vec{u}||$ takes all values in the interval $[2, 3]$.

8.1.**38** The spectral theorem tells us that there exists an orthonormal eigenbasis $\vec{v}_1, \vec{v}_2$ for A, with associated eigenvalues -2 and 3. Consider a unit vector $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$ in $\mathbb{R}^2$, with $c_1^2 + c_2^2 = 1$. Then $\vec{u} \cdot A\vec{u} = (c_1\vec{v}_1 + c_2\vec{v}_2) \cdot (-2c_1\vec{v}_1 + 3c_2\vec{v}_2) = -2c_1^2 + 3c_2^2$, which takes all values on the interval $[-2, 3]$ since $-2 = -2c_1^2 - 2c_2^2 \le -2c_1^2 + 3c_2^2 \le 3c_1^2 + 3c_2^2 = 3$.

8.1.**39** The spectral theorem tells us that there exists an orthonormal eigenbasis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ for A, with associated eigenvalues -2, 3 and 4. Consider a unit vector $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ in $\mathbb{R}^3$, with $c_1^2 + c_2^2 + c_3^2 = 1$. Then

$\vec{u} \cdot A\vec{u} = (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) \cdot (-2c_1\vec{v}_1 + 3c_2\vec{v}_2 + 4c_3\vec{v}_3) = -2c_1^2 + 3c_2^2 + 4c_3^2$, which takes all values on the interval $[-2, 4]$ since $-2 = -2c_1^2 - 2c_2^2 - 2c_3^2 \leq -2c_1^2 + 3c_2^2 + 4c_3^2 \leq 4c_1^2 + 4c_2^2 + 4c_3^2 = 4$.

8.1.**40** Using the terminology introduced in Exercise 8.1.39, we have
$\|A\vec{u}\| = \|-2c_1\vec{v}_1 + 3c_2\vec{v}_2 + 4c_3\vec{v}_3\| = \sqrt{4c_1^2 + 9c_2^2 + 16c_2^2}$, which takes all values on the interval $[2, 4]$. Geometrically, the image of the unit sphere under A is the ellipsoid with semi-axes 2, 3, and 4.

8.1.**41** The spectral theorem tells us that there exists an orthogonal matrix S such that $S^{-1}AS = D$ is diagonal. Let D_1 be the diagonal matrix such that $D_1^3 = D$; the diagonal entries of D_1 are the cube roots of those of D. Now $B = SD_1S^{-1}$ does the job, since $B^3 = \left(SD_1S^{-1}\right)^3 = SD_1^3S^{-1} = SDS^{-1} = A$.

8.1.**42** We will use the strategy outlined in Exercise 8.1.41. An orthogonal matrix that diagonalizes $A = \frac{1}{5}\begin{bmatrix} 12 & 14 \\ 14 & 33 \end{bmatrix}$ is $S = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, with $S^{-1}AS = D = \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix}$. Now $D_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = SD_1S^{-1} = \frac{1}{5}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$.

8.1.**43** There is an orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \vec{v}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with associated eigenvalues -9, -9, 24. We are looking for a nonzero vector $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ such that $\vec{v} \cdot A\vec{v} = (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) \cdot (-9c_1\vec{v}_1 - 9c_2\vec{v}_2 + 24c_3\vec{v}_3) = -9c_1^2 - 9c_2^2 + 24c_3^2 = 0$ or $-3c_1^2 - 3c_2^2 + 8c_3^2 = 0$. One possible solution is $c_1 = \sqrt{8} = 2\sqrt{2},\ c_2 = 0,\ c_3 = \sqrt{3}$, so that $\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

8.1.**44** Use Exercise 8.1.43 as a guide. Consider an orthonormal eigenbasis $\vec{v}_1, ..., \vec{v}_n$ for A, with associated eigenvalues $\lambda_1 \leq \lambda_2 \leq .... \leq \lambda_n$, listed in ascending order. If $\vec{v} = c_1\vec{v}_1 + ... + c_n\vec{v}_n$ is any nonzero vector in $\mathbb{R}^n$, then $\vec{v} \cdot A\vec{v} = (c_1\vec{v}_1 + ... + c_n\vec{v}_n) \cdot (\lambda_1 c_1\vec{v}_1 + ... + \lambda_n c_n\vec{v}_n) = \lambda_1 c_1^2 + ... + \lambda_n c_n^2$. If all the eigenvalues are positive, then $\vec{v} \cdot A\vec{v}$ will be positive. Likewise, if all the eigenvalues are negative, then $\vec{v} \cdot A\vec{v}$ will be negative. However, if A has positive as well as negative eigenvalues, meaning that $\lambda_1 < 0 < \lambda_n$ (as in Example 8.1.43) , then there exist nonzero vectors $\vec{v}$ with $\vec{v} \cdot A\vec{v} = 0$, for example, $\vec{v} = \sqrt{\lambda_n}\vec{v}_1 + \sqrt{-\lambda_1}\vec{v}_n$.

8.1.**45** a If $S^{-1}AS$ is upper triangular then the first column of S is an eigenvector of A. Therefore, any matrix without real eigenvectors fails to be triangulizable over $\mathbb{R}$, for example, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

b Proof by induction on n: For an $n \times n$ matrix A we can choose a complex invertible $n \times n$ matrix P whose first column is an eigenvector for A. Then $P^{-1}AP = \begin{bmatrix} \lambda & \vec{v} \\ 0 & B \end{bmatrix}$. B is triangulizable, by induction hypothesis, that is, there is an invertible $(n-1) \times (n-1)$ matrix Q such that $Q^{-1}BQ = T$ is upper triangular.
Now let $R = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$. Then $R^{-1}\begin{bmatrix} \lambda & \vec{v} \\ 0 & B \end{bmatrix}R = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix}\begin{bmatrix} \lambda & \vec{v} \\ 0 & B \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \lambda & \vec{v}Q \\ 0 & T \end{bmatrix}$ is upper triangular.
$R^{-1}\begin{bmatrix} \lambda & \vec{v} \\ 0 & B \end{bmatrix}R = R^{-1}P^{-1}APR = S^{-1}AS$, where $S = PR$, proving our claim.

8.1.**46** a By definition of an upper triangular matrix, $\vec{e}_1$ is in ker U, $\vec{e}_2$ is in $\ker(U^2), \ldots, \vec{e}_n$ is in $\ker(U^n)$, so that all $\vec{x}$ in $\mathbb{C}^n$ are in $\ker(U^n)$, that is, $U^n = 0$.

b By Exercise 45b, there exists an invertible S such that $S^{-1}AS = U$ is upper triangular. The diagonal entries of U are all zero, since A and U have the same eigenvalues; therefore $U^n = 0$ by part a. Now $A = SUS^{-1}$ and $A^n = SU^nS^{-1} = 0$, as claimed.

8.1.**47** a For all i, j, $\left[\sum_{k=1}^{n} a_{ik}b_{kj}\right] \le \sum_{k=1}^{n} |a_{ik}b_{kj}| = \sum_{k=1}^{n} |a_{ik}||b_{kj}|$

$\uparrow$

triangle inequality

b By induction on t: $|A^t| = |A^{t-1}A| \le |A^{t-1}||A| \le |A|^{t-1}|A| = |A|^t$

$\uparrow$ $\uparrow$

part a by induction hypothesis

8.1.**48** If $t \ge n-1$ then $(I_n + U)^t = I_n + \binom{t}{1}U + \binom{t}{2}U^2 + \cdots + \binom{t}{n-1}U^{n-1}$, since $U^n = 0$. Now $\binom{t}{k} \le t^n$ for $k = 1, \ldots, n-1$, so that $(I_n + U)^t \le t^n(I_n + U + \cdots + U^{n-1})$, as claimed. Check that the formula holds for $t < n-1$ as well.

8.1.**49** Let λ be the largest $|r_{ii}|$; note that $\lambda < 1$. Then $|R| = \begin{bmatrix} |r_{11}| & & * \\ & \ddots & \\ 0 & & |r_{nn}| \end{bmatrix} \le \begin{bmatrix} \lambda & & * \\ & \ddots & \\ 0 & & \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} =$ $\lambda(I_n + U)$, and $|R^t| \le |R|^t \le \lambda^t(I_n + U)^t \le \lambda^t t^n(I_n + U + \cdots + U^{n-1})$.

We learn in Calculus that $\lim_{t\to\infty}(\lambda^t t^n) = 0$, so that $\lim_{t\to\infty}(R^t) = 0$.

8.1.**50** a From Exercise 45b we know that there is an invertible S and an upper triangular R such that $S^{-1}AS = R$, and $|r_{ii}| < 1$ for all i, since the diagonal entries of R are the eigenvalues of A. Now $\lim_{t\to\infty} R^t = 0$ by Exercise 49. Note that $A = SRS^{-1}$ and $A^t = SR^tS^{-1}$, so that $\lim_{t\to\infty} A^t = 0$, as claimed.

b See the remark after Definition 7.6.1.

Section 8.2

8.2.**1** We have $a_{11} =$ coefficient of $x_1^2 = 6, a_{22} =$ coefficient of $x_2^2 = 8, a_{12} = a_{21} = \frac{1}{2}($ coefficient of $x_1x_2) = -\frac{7}{2}$. So, $A = \begin{bmatrix} 6 & -\frac{7}{2} \\ -\frac{7}{2} & 8 \end{bmatrix}$

8.2.**2** $A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

8.2.**3** $A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 4 & \frac{7}{2} \\ 3 & \frac{7}{2} & 5 \end{bmatrix}$

8.2.4 $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$, positive definite

8.2.5 $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, indefinite (since $\det(A) < 0$)

8.2.6 $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, indefinite (since $\det(A) < 0$)

8.2.7 $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, indefinite (eigenvalues 2, -2, 3)

8.2.8 If $S^{-1}AS = D$ is diagonal, then $S^{-1}A^2S = D^2$, so that all eigenvalues of A^2 are ≥ 0. So A^2 is positive semi-definite; it is positive definite if and only if A is invertible.

8.2.9 a $(A^2)^T = (A^T)^2 = (-A)^2 = A^2$, so that A^2 is symmetric.

b $q(\vec{x}) = \vec{x}^T A^2 \vec{x} = \vec{x}^T AA\vec{x} = -\vec{x}^T A^T A\vec{x} = -(A\vec{x}) \cdot (A\vec{x}) = -\|A\vec{x}\|^2 \leq 0$ for all $\vec{x}$, so that A^2 is negative semi-definite. The eigenvalues of A^2 will be ≥ 0.

c If $\vec{v}$ is a complex eigenvector of A with eigenvalue λ, then $A^2\vec{v} = \lambda^2\vec{v}$, and $\lambda^2 \leq 0$, by part b. Therefore, λ is *imaginary*, that is, $\lambda = bi$ for a real b. Thus, the zero matrix is the only skew-symmetric matrix that is diagonalizable over $\mathbb{R}$.

8.2.10 $L(\vec{x}) = (\vec{x}+\vec{v})^T A(\vec{x}+\vec{v}) - \vec{x}^T A\vec{x} - \vec{v}^T A\vec{v} = \vec{x}^T A\vec{x} + \vec{x}^T A\vec{v} + \vec{v}^T A\vec{x} + \vec{v}^T A\vec{v} - \vec{x}^T A\vec{x} - \vec{v}^T A\vec{v} = \vec{x}^T A\vec{v} + \vec{v}^T A\vec{x} = \vec{v}^T A\vec{x} + \vec{v}^T A\vec{x} = (2\vec{v}^T A)\vec{x}$,

$\uparrow$

note that $\vec{x}^T A\vec{v}$ is a scalar so that $\vec{x}^T A\vec{v} = (\vec{x}^T A\vec{v})^T = \vec{v}^T A^T \vec{x} = \vec{v}^T A\vec{x}$ if A is symmetric.

So L is linear with matrix $2\vec{v}^T A$.

8.2.11 The eigenvalues of A^{-1} are the reciprocals of those of A, so that A and A^{-1} have the same definiteness.

8.2.12 $\det(A)$ is the product of the two (real) eigenvalues. q is indefinite if an only if those have different signs, that is, their product is negative.

8.2.13 $q(\vec{e}_i) = \vec{e}_i \cdot A\vec{e}_i = a_{ii} > 0$

8.2.14 If $\det(A)$ is positive then both eigenvalues have the same sign, so that A is positive definite or negative definite. Since $\vec{e}_1 \cdot A\vec{e}_1 = a > 0$, A is in fact positive definite.

8.2.15 $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$; eigenvalues $\lambda_1 = 7$ and $\lambda_2 = 2$

orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\lambda_1 c_1^2 + \lambda_2 c_2^2 = 1$ or $7c_1^2 + 2c_2^2 = 1$. (See Figure 8.3.)

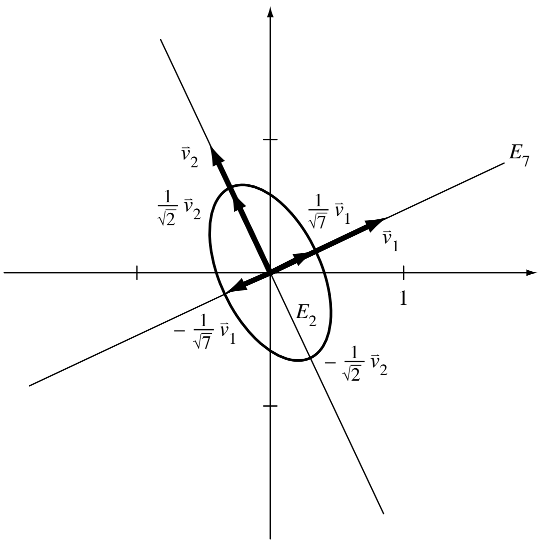

Figure 8.3: for Problem 8.2.15.

8.2.**16** $A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$; eigenvalues $\lambda_1 = \frac{1}{2}$, and $\lambda_2 = -\frac{1}{2}$

orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\frac{1}{2}c_1^2 - \frac{1}{2}c_2^2 = 1$. (See Figure 8.4.)

8.2.**17** $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$, eigenvalues $\lambda_1 = 4$, $\lambda_2 = -1$

orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$4c_1^2 - c_2^2 = 1$ (hyperbola) (See Figure 8.5.)

8.2.**18** $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$, eigenvalues $\lambda_1 = 10$, $\lambda_2 = 5$

orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$10c_1^2 + 5c_2^2 = 1$. This is an ellipse, as shown in Figure 8.6.

8.2.**19** $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$; eigenvalues $\lambda_1 = 5$, $\lambda_2 = 0$

eigenvectors $\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

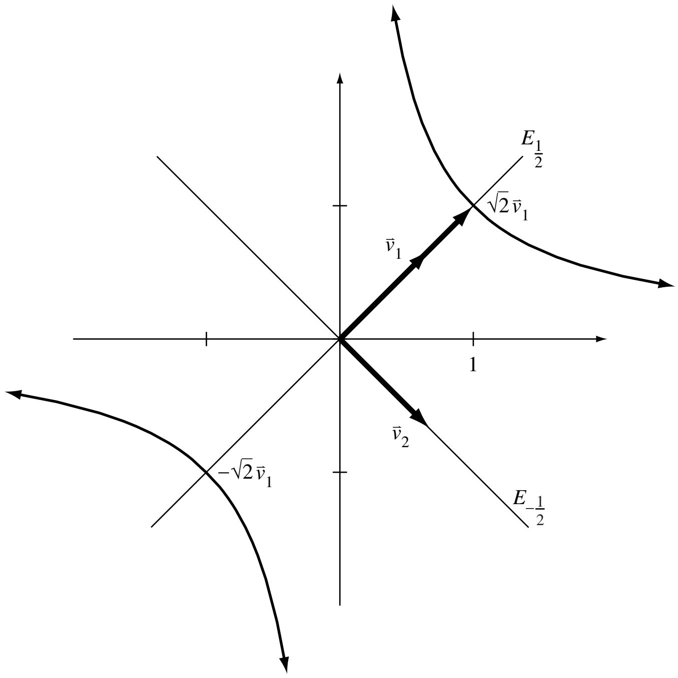

Figure 8.4: for Problem 8.2.16.

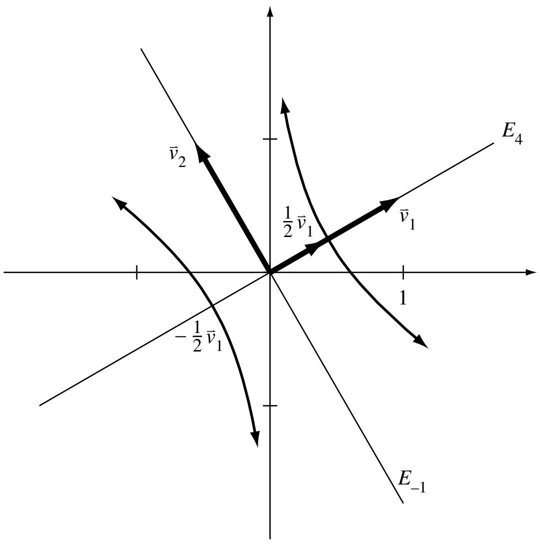

Figure 8.5: for Problem 8.2.17.

$5c_1^2 = 1$ (a pair of lines) (See Figure 8.7.)

Note that $(x_1^2 + 4x_1x_2 + 4x_2^2) = (x_1 + 2x_2)^2 = 1$, so that $x_1 + 2x_2 = \pm 1$, and the two lines are $x_2 = \frac{1-x_1}{2}$ and $x_2 = \frac{-1-x_1}{2}$.

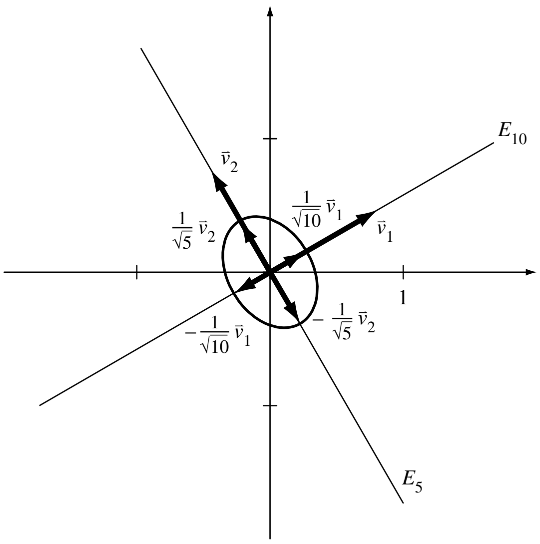

Figure 8.6: for Problem 8.2.18.

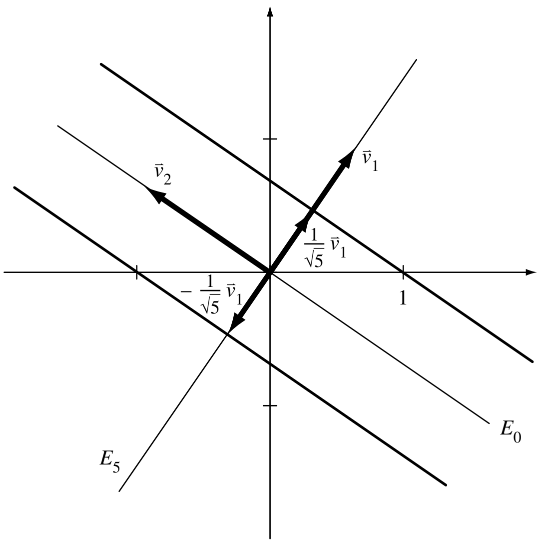

Figure 8.7: for Problem 8.2.19.

8.2.**20** $A = \begin{bmatrix} -3 & 3 \\ 3 & 5 \end{bmatrix}$; eigenvalues $\lambda_1 = 6$ and $\lambda_2 = -4$

orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$6c_1^2 - 4c_2^2 = 1$. This is a hyperbola, as shown in Figure 8.8.

8.2.**21** a In each case, it is informative to think about the intersections with the three coordinate planes: $x_1 - x_2$, $x_1 - x_3$, and $x_2 - x_3$.

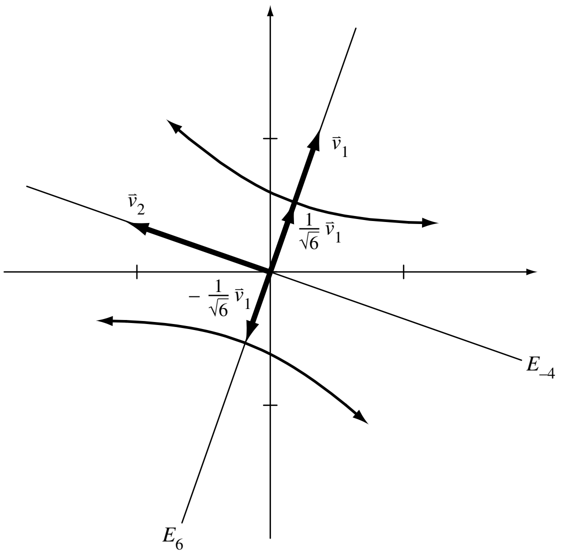

Figure 8.8: for Problem 8.2.20.

•] For the surface $x_1^2 + 4x_2^2 + 9x_3^2 = 1$, all these intersections are *ellipses*, and the surface itself is an *ellipsoid*.

This surface is connected and bounded; the points closest to the origin are $\pm\begin{bmatrix}0\\0\\\frac{1}{3}\end{bmatrix}$, and those farthest $\pm\begin{bmatrix}1\\0\\0\end{bmatrix}$. (See Figure 8.9.)

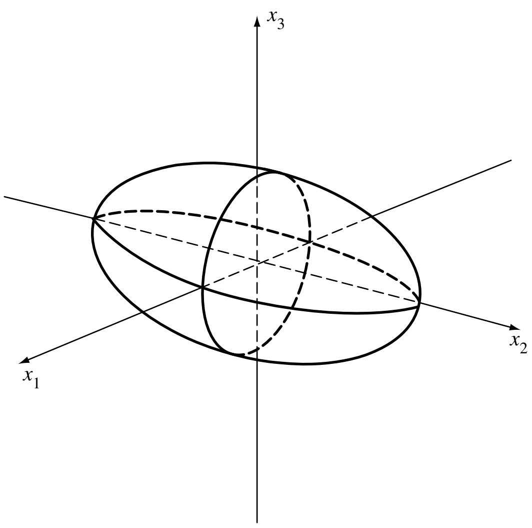

Figure 8.9: for Problem 8.2.21a: $x_1^2 + 4x_2^2 + 9x_3^2 = 1$, an *ellipsoid* (not to scale).

•] In the case of $x_1^2 + 4x_2^2 - 9x_3^2 = 1$, the intersection with the $x_1 - x_2$ plane is an ellipse, and the two other intersections are hyperbolas. The surface is connected and not bounded; the points closest to the origin are

$\pm \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$. (See Figure 8.10.)

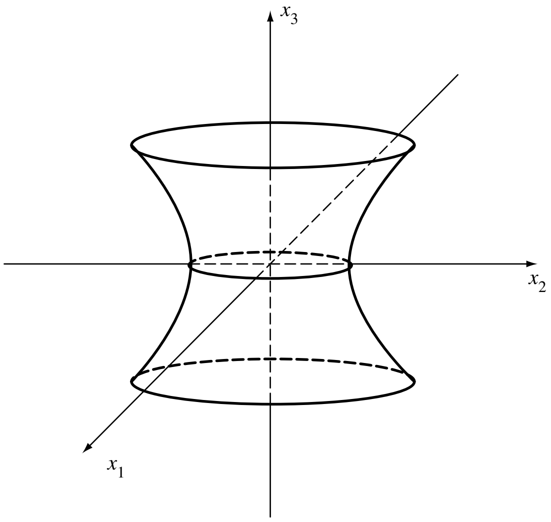

Figure 8.10: for Problem 8.2.21a: $x_1^2 + 4x_2^2 - 9x_3^2 = 1$, a *hyperboloid of one sheet* (not to scale).

•] In the case $-x_1^2-4x_2^2+9x_3^2 = 1$, the intersection with the x_1-x_2 plane is empty, and the two other intersections are hyperbolas. The surface consists of two pieces and is unbounded. The points closest to the origin are $\pm \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$. (See Figure 8.11.)

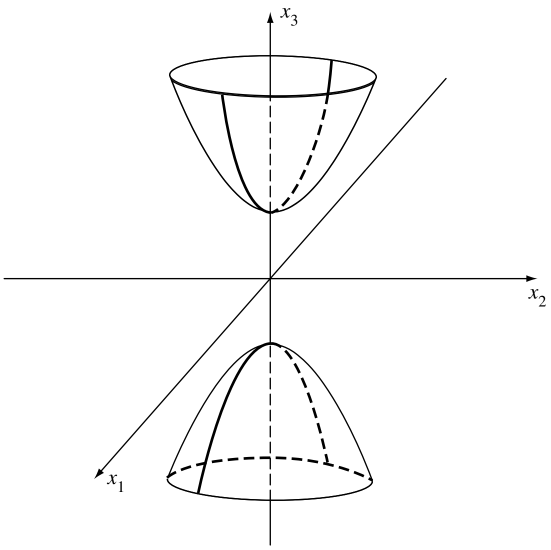

Figure 8.11: for Problem 8.2.21a: $-x_1^2 - 4x_2^2 + 9x_3^2 = 1$, a *hyperboloid of two sheets* (not to scale).

b $A = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 2 & \frac{3}{2} \\ 1 & \frac{3}{2} & 3 \end{bmatrix}$ is positive definite, with three positive eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

The surface is given by $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 1$ with respect to the principal axis, an *ellipsoid*. To find the points closest to and farthest from the origin, use technology to find the eigenvalues and eigenvectors:

eigenvalues: $\lambda_1 \approx 0.56, \lambda_2 \approx 4.44, \lambda_3 = 1$

unit eigenvectors: $\vec{v}_1 \approx \begin{bmatrix} 0.86 \\ 0.19 \\ -0.47 \end{bmatrix}$, $\vec{v}_2 \approx \begin{bmatrix} 0.31 \\ 0.54 \\ 0.78 \end{bmatrix}$, $\vec{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Equation: $0.56c_1^2 + 4.44c_2^2 + c_3^2 = 1$

Farthest points when $c_1 = \pm\frac{1}{\sqrt{0.56}}$ and $c_2 = c_3 = 0$

Closest points when $c_2 = \pm\frac{1}{\sqrt{4.44}}$ and $c_1 = c_3 = 0$

Farthest points $\approx \pm\frac{1}{\sqrt{0.56}} \begin{bmatrix} 0.86 \\ 0.19 \\ -0.47 \end{bmatrix} \approx \pm \begin{bmatrix} 1.15 \\ 0.26 \\ -0.63 \end{bmatrix}$

Closest points $\approx\pm\frac{1}{\sqrt{4.44}} \begin{bmatrix} 0.31 \\ 0.54 \\ 0.78 \end{bmatrix} \approx \pm \begin{bmatrix} 0.15 \\ 0.26 \\ 0.37 \end{bmatrix}$

8.2.**22** $A = \begin{bmatrix} -1 & 0 & 5 \\ 0 & 1 & 0 \\ 5 & 0 & -1 \end{bmatrix}$; eigenvalues $\lambda_1 = 4$, $\lambda_2 = -6$, $\lambda_3 = 1$

Equation with respect to principal axis: $4c_1^2 - 6c_2^2 + c_3^2 = 1$, a hyperboloid of one sheet (see Figure 8.10).

Closest to origin when $c_1 = \pm\frac{1}{2}$, $c_2 = c_3 = 0$.

A unit eigenvector for eigenvalue 4 is $\vec{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, so that the desired points are $\pm\frac{1}{2}\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \approx \pm \begin{bmatrix} 0.35 \\ 0 \\ 0.35 \end{bmatrix}$.

8.2.**23** Yes; $M = \frac{1}{2}(A + A^T)$ is symmetric, and

$\vec{x}^T M \vec{x} = \frac{1}{2}\vec{x}^T A\vec{x} + \frac{1}{2}\vec{x}^T A^T \vec{x} = \frac{1}{2}\vec{x}^T A\vec{x} + \frac{1}{2}\vec{x}^T A\vec{x} = \vec{x}^T A\vec{x}$

Note that $\vec{x}^T A\vec{x}$ is a 1×1 matrix, so that $\vec{x}^T A\vec{x} = (\vec{x}^T A\vec{x})^T = \vec{x}^T A^T \vec{x}$.

8.2.**24** $q(\vec{e}_1) = \vec{e}_1 \cdot A\vec{e}_1 = \vec{e}_1 \cdot$ (first column of A) $= a_{11}$

8.2.**25** $q(\vec{v}) = \vec{v} \cdot A\vec{v} = \vec{v} \cdot (\lambda\vec{v}) = \lambda(\vec{v} \cdot \vec{v}) \underset{\uparrow}{=} \lambda$

$\vec{v}$ is a unit vector

8.2.**26** False; If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then $q\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$.

8.2.**27** Let $\vec{v}_1, \dots, \vec{v}_n$ be an orthonormal eigenbasis for A with $A\vec{v}_i = \lambda_i \vec{v}_i$. We know that $q(\vec{v}_i) = \lambda_i$ (see Exercise 25), so that $q(\vec{v}_1) = \lambda_1$ and $q(\vec{v}_n) = \lambda_n$ are in the image.

We claim that all numbers between λ_n and λ_1 are in the image as well. To see this, apply the Intermediate Value Theorem to the continuous function $f(t) = q((\cos t)\vec{v}_n + (\sin t)\vec{v}_1)$ on $\left[0, \frac{\pi}{2}\right]$ (note that $f(0) = q(\vec{v}_n) = \lambda_n$ and $f\left(\frac{\pi}{2}\right) = q(\vec{v}_1) = \lambda_1$). (See Figure 8.12.)

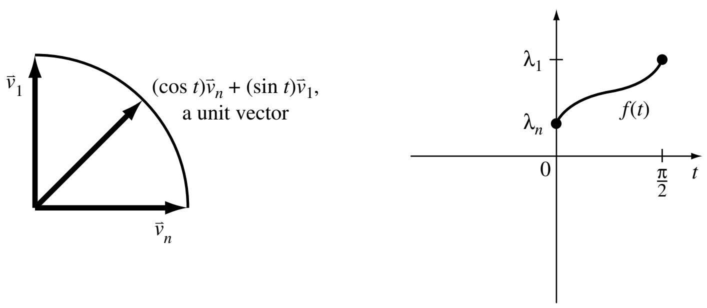

Figure 8.12: for Problem 8.2.27.

The Intermediate Value Theorem tells us that for any c between λ_n and λ_1, there is a t_0 such that $f(t_0) = q((\cos t_0)\vec{v}_n + (\sin t_0)\vec{v}_1) = c$. Note that $(\cos t_0)\vec{v}_n + (\sin t_0)\vec{v}_1$ is a unit vector. Now we will show that, conversely, $q(\vec{v})$ is on $[\lambda_n, \lambda_1]$ for all unit vectors $\vec{v}$. Write $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ and note that $\|\vec{v}\|^2 = c_1^2 + \cdots + c_n^2 = 1$. Then $q(\vec{v}) = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_n c_n^2 \le \lambda_1 c_1^2 + \lambda_1 c_2^2 + \cdots + \lambda_1 c_n^2 = \lambda_1$. Likewise, $q(\vec{v}) \ge \lambda_n$. We have shown that the image of S^{n-1} under q is the closed interval $[\lambda_n, \lambda_1]$.

8.2.**28** The hint almost gives it away. Since D is a diagonal matrix with positive diagonal entries, we can write $D = D_1^2$, where D_1 is diagonal with positive diagonal entries (the square roots of the entries of D). Now $A = SDS^T = SD_1D_1S^T = SD_1(SD_1)^T = BB^T$ where $B = SD_1$. The columns of B are scalar multiples of the corresponding columns of S, so that they are orthogonal.

8.2.**29** From Example 1 we have $S = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$. Let $D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = SD_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 6 & 2 \\ -3 & 4 \end{bmatrix}$.

8.2.**30** Define D_1 as in Exercise 28. Then $A = SDS^{-1} = SD_1D_1S^{-1} = (SD_1S^{-1})(SD_1S^{-1}) = B^2$, where $B = SD_1S^{-1}$. B is positive definite, since $S^{-1}BS = D_1$ is diagonal with positive diagonal entries.

8.2.**31** $S = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ (see Exercise 29), so that $B = SD_1S^{-1} = \begin{bmatrix} 2.8 & -0.4 \\ -0.4 & 2.2 \end{bmatrix}$.

8.2.**32** Recall that $a = q(\vec{e}_1) > 0$ and $\det A = ac - b^2 = \lambda_1\lambda_2 > 0$.

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} x & 0 \\ y & z \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x^2 & xy \\ xy & y^2 + z^2 \end{bmatrix} \text{ means that } \begin{aligned} x^2 &= a \\ xy &= b \\ y^2 + z^2 &= c \end{aligned}$$

It is required that x and z be positive. This system has the unique solution

$x = \sqrt{a}$

$y = \frac{b}{x} = \frac{b}{\sqrt{a}}$

$z = \sqrt{c - y^2} = \sqrt{c - \frac{b^2}{a}} = \sqrt{\frac{ac-b^2}{a}}$

8.2.**33** Use the formulas for x, y, z derived in Exercise 32.

$x = \sqrt{a} = \sqrt{8} = 2\sqrt{2}$

$y = \frac{b}{\sqrt{a}} = -\frac{2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}$

$z = \sqrt{\frac{ac-b^2}{a}} = \sqrt{\frac{36}{8}} = \frac{3}{\sqrt{2}}$, so that

$L = \begin{bmatrix} 2\sqrt{2} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix}$.

8.2.**34** (i) implies (ii): See the hint at the end of the exercise.
(ii) implies (iii): det $A^{(m)}$ is the product of the (positive) eigenvalues.
(iii) implies (iv):

$$A = \begin{bmatrix} A^{(n-1)} & \vec{v} \\ \vec{v}^T & k \end{bmatrix} = \begin{bmatrix} B & 0 \\ \vec{x}^T & 1 \end{bmatrix} \begin{bmatrix} B^T & \vec{x} \\ 0 & t \end{bmatrix} = \begin{bmatrix} BB^T & B\vec{x} \\ \vec{x}^T B^T & \vec{x}^T\vec{x} + t \end{bmatrix}$$

The system $\begin{bmatrix} B\vec{x} = \vec{v} \\ \vec{x}^T\vec{x} + t = k \end{bmatrix}$ has the unique solution

$\vec{x} = B^{-1}\vec{v}$

$t = k - \vec{x}^T\vec{x} = k - \|B^{-1}\vec{v}\|^2$.

Note that t is positive since $0 < \det(A^{(n)}) = \det(A) = \det \begin{bmatrix} B & 0 \\ \vec{x}^T & 1 \end{bmatrix} \det \begin{bmatrix} B^T & \vec{x} \\ 0 & t \end{bmatrix} = (\det B)^2 \cdot t$.

(iv) $\Rightarrow$ (i)

$\vec{x}^T A \vec{x} = \vec{x}^T L L^T \vec{x} = (L^T\vec{x})^T (L^T\vec{x}) = \|L^T\vec{x}\|^2 > 0$ if $\vec{x} \neq \vec{0}$, since L is invertible.

8.2.**35** Solve the system $\begin{bmatrix} 4 & -4 & 8 \\ -4 & 13 & 1 \\ 8 & 1 & 26 \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ y & w & 0 \\ z & t & s \end{bmatrix} \begin{bmatrix} x & y & z \\ 0 & w & t \\ 0 & 0 & s \end{bmatrix}$

$$\left.\begin{array}{c} x^2 = 4, \text{ so } x = 2 \\ 2y = -4, \text{ so } y = -2 \\ 2z = 8, \text{ so } z = 4 \\ 4 + w^2 = 13, \text{ so } w = 3 \\ -8 + 3t = 1, \text{ so } t = 3 \\ 16 + 9 + s^2 = 26, \text{ so } s = 1 \end{array}\right\} L = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 3 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

8.2.**36** If $A = QR$, then $A^T A = (QR)^T QR = R^T Q^T Q R = R^T R = L L^T$, $L = R^T$.

8.2.**37** $\frac{\partial q}{\partial x_1} = 2ax_1 + bx_2$ and $\frac{\partial q}{\partial x_2} = bx_1 + 2cx_2$, so that $q_{11} = \frac{\partial^2 q}{\partial x_1^2} = 2a$, $q_{22} = \frac{\partial^2 q}{\partial x_2^2} = 2c$, and $q_{12} = \frac{\partial^2 q}{\partial x_1 \partial x_2} = b$, and $D = \det\begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} = \det\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} = 4ac - b^2 > 0$.

The matrix $A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}$ of q is positive definite, since $a > 0$ and $\det(A) = \frac{1}{4}D > 0$. This means, by definition, that q has a minimum at $\vec{0}$, since $q(\vec{x}) > 0 = q(\vec{0})$ for all $\vec{x} \neq \vec{0}$.

8.2.**38** The eigenvalues of B are $p - q$ and $nq + p - q = p + (n-1)q$, so that B is positive definite if $p - q > 0$ and $p + (n-1)q > 0$.

8.2.**39** If $\vec{v}_1, \ldots, \vec{v}_n$ is such a basis consisting of unit vectors, and we let $A = [\vec{v}_1 \cdots \vec{v}_n]$, then

$A^T A = \begin{bmatrix} 1 & \cos\theta & \cdots & \cos\theta \\ \cos\theta & 1 & \ddots & \cos\theta \\ \vdots & \ddots & \ddots & \vdots \\ \cos\theta & \cos\theta & \cdots & 1 \end{bmatrix}$ is positive definite, so that, by Exercise 38, $1 - \cos\theta > 0$ and $1 + (n-1)\cos\theta > 0$ or $1 > \cos\theta > \frac{1}{1-n}$ or $0 < \theta < \arccos\left(\frac{1}{1-n}\right)$.

Conversely, if θ is in this range, then the matrix $\begin{bmatrix} 1 & \cos\theta & \cdots & \cos\theta \\ \cos\theta & 1 & \ddots & \cos\theta \\ \vdots & \ddots & \ddots & \vdots \\ \cos\theta & \cos\theta & \cdots & 1 \end{bmatrix}$ is positive definite, so that it has a Cholesky factorization LL^T. The columns of L^T give us a basis with the desired property.

8.2.**40** Let λ be the smallest eigenvalue of A. If we let $k = 1 - \lambda$, then the smallest eigenvalue of the matrix $A + kI_n$ will be $\lambda + k = 1$, so that all the eigenvalues of $A + kI_n$ will be positive. Thus matrix $A + kI_n$ will be positive definite, by Theorem 8.2.4.

8.2.**41** The functions x_1^2, x_1x_2, x_2^2 form a basis of Q_2, so that $\dim(Q_2) = 3$.

8.2.**42** The functions x_ix_j form a basis of Q_n, where $1 \le i \le j \le n$. A little combinatorics shows that there are $1 + 2 + 3 + \cdots + n = n(n+1)/2$ such functions, so that $\dim(Q_n) = n(n+1)/2$

8.2.**43** Note that $T(ax_1^2 + bx_1x_2 + cx_2^2) = ax_1^2$ (we let $x_2 = 0$). Thus $\text{im}(T) = \text{span}(x_1^2)$, $\text{rank}(T) = 1$, $\ker(T) = \text{span}(x_1x_2, x_2^2)$, $\text{nullity}(T) = 2$.

8.2.**44** Note that $T(ax_1^2 + bx_1x_2 + cx_2^2) = ax_1^2 + bx_1 + c$ (we let $x_2 = 1$). Thus $\text{im}(T) = P_2$, $\text{rank}(T) = 3$, $\ker(T) = \{0\}$, $\text{nullity}(T) = 0$ (T is an isomorphism).

8.2.**45** Note that $T(ax_1^2 + bx_2^2 + cx_3^2 + dx_1x_2 + ex_1x_3 + fx_2x_3) = ax_1^2 + b + c + dx_1 + ex_1 + f$ (we let $x_2 = x_3 = 1$). Thus $\text{im}(T) = P_2$ and $\text{rank}(T) = 3$. The kernel of T consists of the quadratic forms with $a = 0$, $d + e = 0$, and $b + c + f = 0$ (consider the coefficients of x_1^2, x_1, and 1). The general element of the kernel is $q(x_1, x_2, x_3) = (-c - f)x_2^2 + cx_3^2 - ex_1x_2 + ex_1x_3 + fx_2x_3 = c(x_3^2 - x_2^2) + e(x_1x_3 - x_1x_2) + f(x_2x_3 - x_2^2)$. Thus $\ker(T) = \text{span}(x_3^2 - x_2^2, x_1x_3 - x_1x_2, x_2x_3 - x_2^2)$ and $\text{nullity}(T) = 3$.

8.2.**46** Note that $T(ax_1^2 + bx_2^2 + cx_3^2 + dx_1x_2 + ex_1x_3 + fx_2x_3) = ax_1^2 + bx_2^2 + cx_1^2 + dx_1x_2 + ex_1^2 + fx_1x_2$ (we let $x_3 = x_1$). Thus $\text{im}(T) = Q_2$ and $\text{rank}(T) = 3$. The kernel of T consists of the quadratic forms with

$a+c+e=0, b=0$, and $d+f=0$ (consider the coefficients of x_1^2, x_2^2, and x_1x_2). The general element of the kernel is $q(x_1,x_2,x_3)=(-c-e)x_1^2+cx_3^2-fx_1x_2+ex_1x_3+fx_2x_3=c(x_3^2-x_1^2)+e(x_1x_3-x_1^2)+f(x_2x_3-x_1x_2)$. Thus $\ker(T)=\text{span}(x_3^2-x_1^2, x_1x_3-x_1^2, x_2x_3-x_1x_2)$ and $\text{nullity}(T)=3$.

8.2.**47** $T(A+B)(\vec{x})=\vec{x}^T(A+B)\vec{x}=\vec{x}^TA\vec{x}+\vec{x}^TB\vec{x}$ equals $(T(A)+T(B))(\vec{x})$

$=T(A)(\vec{x})+T(B)(\vec{x})=\vec{x}^TA\vec{x}+\vec{x}^TB\vec{x}$. The verification of the second axiom of linearity is analogous.

By definition of a quadratic form, $\text{im}(T)=Q_n$: For every quadratic form q in Q_n there is a *symmetric* $n\times n$ matrix A such that $q=T(A)$. Thus, the rank of T is $\dim(Q_n)=n(n+1)/2$ (see Exercise 42).

By the rank nullity theorem,

$$\text{nullity}(T)=\dim(\mathbb{R}^{n\times n})-\text{rank}(T)=n^2-\frac{n(n+1)}{2}=\frac{n(n-1)}{2}$$

Next, let's think about the kernel of T. In our solution to Exercise 23 we observed that $T(A)=T(\frac{1}{2}(A+A^T))$; note that matrix $\frac{1}{2}(A+A^T)$ is symmetric. Now $\frac{1}{2}(A+A^T)=0$ if (and only if) $A^T=-A$, that is, if A is skew-symmetric. Thus the skew-symmetric matrices are in the kernel of T. Since the space of skew-symmetric matrices has the same dimension as $\ker(T)$, namely, $n(n-1)/2$, we can conclude that $\ker(T)$ consists of all skew-symmetric $n\times n$ matrices.

8.2.**48** The matrix of T with respect to the basis x_1^2, x_1x_2, x_2^2 is $A=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$, with the eigenvalues 1,1, -1 and corresponding eigenvectors $\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}$. Thus x_1x_2 and $x_1^2+x_2^2$ are eigenfunctions with eigenvalue 1, and $x_1^2-x_2^2$ has eigenvalue -1. Yes, T is diagonalizable, since there is an eigenbasis.

8.2.**49** The matrix of T with respect to the basis x_1^2, x_1x_2, x_2^2 is $A=\begin{bmatrix}1&0&0\\0&2&0\\0&0&4\end{bmatrix}$, with the eigenvalues 1,2,4 and corresponding eigenvectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$. Thus x_1^2, x_1x_2, x_2^2 are eigenfunctions with eigenvalues 1, 2, and 4, respectively. Yes, T is diagonalizable, since there is an eigenbasis.

8.2.**50** The matrix of T with respect to the basis x_1^2, x_1x_2, x_2^2 is $A=\begin{bmatrix}0&1&0\\2&0&2\\0&1&0\end{bmatrix}$, with the eigenvalues 0,2, -2 and corresponding eigenvectors $\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}$.

Thus $x_1^2-x_2^2, x_1^2+2x_1x_2+x_2^2, x_1^2-2x_1x_2+x_2^2$ are eigenfunctions with eigenvalues 0, 2, and -2, respectively. Yes, T is diagonalizable, since there is an eigenbasis.

8.2.**51** If B is negative definite, then $A=-B$ is positive definite, so that the determinants of all principal submatrices $A^{(m)}$ are positive. Thus $\det(B^{(m)})=\det(-A^{(m)})=(-1)^m\det(A^{(m)})$ is positive for even m and negative for odd m.

8.2.**52** Because $a_{ij} = \vec{e}_j^T A\vec{e}_i$, we have $q(\vec{e}_i) = a_{ii}$. Further, using linearity $q(\vec{e}_i + \vec{e}_j) = (\vec{e}_i + \vec{e}_j)^T A(\vec{e}_i + \vec{e}_j) = \vec{e}_i^T A\vec{e}_i + \vec{e}_i^T A\vec{e}_j + \vec{e}_j^T A\vec{e}_i + \vec{e}_j^T A\vec{e}_j = q(\vec{e}_i) + q(\vec{e}_j) + 2a_{ij}$. Solving for a_{ij} gives $a_{ij} = \frac{1}{2}(q(\vec{e}_i + \vec{e}_j) - q(\vec{e}_i) - q(\vec{e}_j))$.

8.2.**53** a. Because $p(x, y) = q(x\vec{e}_i + y\vec{e}_j) = (x\vec{e}_i + y\vec{e}_j)^T A(x\vec{e}_i + y\vec{e}_j) = a_{ii}x^2 + a_{ij}xy + a_{ji}yx + a_{jj}y^2$, this is a quadratic form with matrix B.

b. If q is positive definite and $(x, y) \neq (0, 0)$, then $p(x, y) = q(x\vec{e}_i + y\vec{e}_j) > 0$.

c. If q is positive semidefinite, then $p(x, y) = q(x\vec{e}_i + y\vec{e}_j) \geq 0$ for all x, y.

d. If $q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$ and we let $i = 1, j = 2$, then $p(x, y) = q(x, y, 0) = x^2 + y^2$ is positive definite.

8.2.**54** The entries $a_{1j} = a_{j1}$ must all be 0. To see that $a_{1j} = 0$, consider the function $p(x, y) = q(x\vec{e}_1 + y\vec{e}_j)$ defined in Exercise 8.2.53. By Exercise 53a, the symmetric matrix of p will be $\begin{bmatrix} a_{11} & a_{1j} \\ a_{j1} & a_{jj} \end{bmatrix} = \begin{bmatrix} 0 & a_{1j} \\ a_{j1} & a_{jj} \end{bmatrix}$. This matrix is positive semidefinite, by Exercise 53c, implying that $\det \begin{bmatrix} 0 & a_{1j} \\ a_{j1} & a_{jj} \end{bmatrix} = -a_{1j}^2 \geq 0$. Thus $a_{1j} = 0$, as claimed.

8.2.**55** As the hint suggests, it suffices to prove that $a_{ij} < a_{ii}$ or $a_{ij} < a_{jj}$, implying that for every entry off the diagonal there exists a larger entry on the diagonal. Now $q(\vec{e}_i - \vec{e}_j) = (\vec{e}_i - \vec{e}_j)^T A(\vec{e}_i - \vec{e}_j) = a_{ii} - 2a_{ij} + a_{jj} > 0$, or, $a_{ii} + a_{jj} > 2a_{ij}$, proving the claim.

8.2.**56** Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be the eigenvalues of A. In Exercise 8.2.27 we see that the range of q on unit vectors is the interval $[\lambda_n, \lambda_1]$. Since $a_{11} = q(\vec{e}_1)$ is in that range, we must have $a_{11} \leq \lambda_1$, as claimed.

8.2.**57** Working in coordinates with respect to an orthonormal eigenbasis for A, we can write the equation $q(\vec{x}) = 1$ as $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 1$, where the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are positive. This level surface is an ellipsoid.

8.2.**58** Working in coordinates with respect to an orthonormal eigenbasis of A, we can write the equation $q(\vec{x}) = 1$ as $\lambda_1 c_1^2 + \lambda_2 c_2^2 = 1$, where the eigenvalues λ_1 and λ_2 are positive. This level surface is a cylinder.

8.2.**59** Working in coordinates with respect to an orthonormal eigenbasis of A, we can write the equation $q(\vec{x}) = 1$ as $\lambda_1 c_1^2 = 1$, where the eigenvalue λ_1 is positive. This level surface is a pair of parallel planes, $c_1 = \pm 1/\sqrt{\lambda_1}$

8.2.**60** Working in coordinates with respect to an orthonormal eigenbasis of A, we can write the equation $q(\vec{x}) = 1$ as $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 1$, where the eigenvalue λ_1 is positive, while λ_2 and λ_3 are negative. This level surface is a hyperboloid of two sheets.

8.2.**61** Working in coordinates with respect to an orthonormal eigenbasis of A, we can write the equation $q(\vec{x}) = 1$ as $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 1$, where the eigenvalues λ_1 and λ_2 are positive, while λ_3 is negative. This level surface is a hyperboloid of one sheet.

8.2.**62** Working in coordinates with respect to an orthonormal eigenbasis of A, we can write the equation $q(\vec{x}) = 0$ as $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 0$, where the eigenvalues λ_1 and λ_2 are positive, while λ_3 is negative. This level surface is a cone.

8.2.**63** Note that $\vec{w}_i \cdot \vec{w}_i = \frac{1}{\lambda_i}$. Now $q(c_1\vec{w}_1 + ... + c_n\vec{w}_n)$
$= (c_1\vec{w}_1 + ... + c_n\vec{w}_n) \cdot A(c_1\vec{w}_1 + ... + c_n\vec{w}_n) = (c_1\vec{w}_1 + ... + c_n\vec{w}_n) \cdot (\lambda_1 c_1\vec{w}_1 + ... + \lambda_n c_n\vec{w}_n) = \lambda_1 c_1^2 \frac{1}{\lambda_1} + ... +$

$\lambda_n c_n^2 \frac{1}{\lambda_n} = c_1^2 + ... + c_n^2$, as claimed.

8.2.**64** We will use the strategy outlined in Exercise 8.2.63. The symmetric matrix of q is $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$, with an orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, with associated eigenvalues $\lambda_1 = 9$ and $\lambda_2 = 4$. Thus the orthogonal basis $\vec{w}_1 = \frac{1}{\sqrt{\lambda_1}} \vec{v}_1 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{w}_2 = \frac{1}{\sqrt{\lambda_2}} \vec{v}_2 = \frac{1}{2\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has the required property. (See Figure 8.13.)

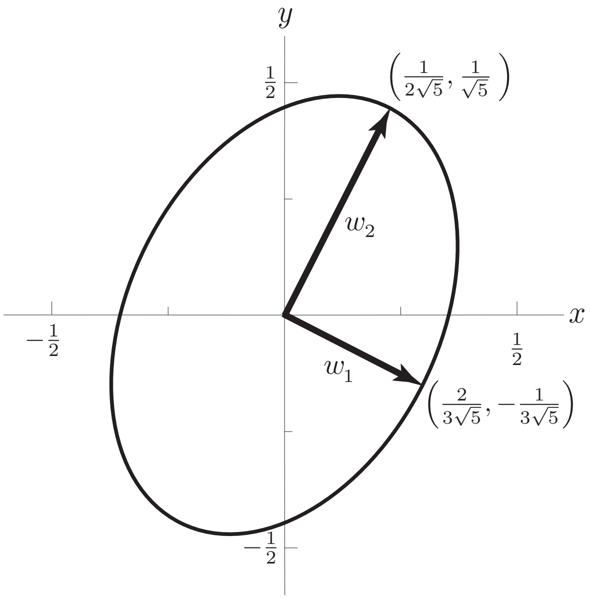

Figure 8.13: for Problem 8.2.64.

8.2.**65** Working in coordinates c_1, c_2 with respect to an orthonormal eigenbasis $\vec{v}_1, \vec{v}_2$ for A, we can write $q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2$, where the eigenvalue λ_1 is positive while λ_2 is negative. We define the orthogonal vectors $\vec{w}_1 = \frac{1}{\sqrt{\lambda_1}} \vec{v}_1$ and $\vec{w}_2 = \frac{1}{\sqrt{-\lambda_2}} \vec{v}_2$. Note that $\vec{w}_1 \cdot \vec{w}_1 = \frac{1}{\lambda_1}$ and $\vec{w}_2 \cdot \vec{w}_2 = \frac{1}{(-\lambda_2)}$. Now $q(c_1\vec{w}_1 + c_2\vec{w}_2) = (c_1\vec{w}_1 + c_2\vec{w}_2) \cdot (\lambda_1 c_1 \vec{w}_1 + \lambda_2 c_2 \vec{w}_2) = \lambda_1 c_1^2 \frac{1}{\lambda_1} + \lambda_2 c_2^2 \frac{1}{(-\lambda_2)} = c_1^2 - c_2^2$, as claimed.

8.2.**66** We will use the strategy outlined in Exercise 8.2.65. The symmetric matrix of q is $A = \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix}$, with an orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, with associated eigenvalues $\lambda_1 = 8$ and $\lambda_2 = -2$. Thus the orthogonal basis $\vec{w}_1 = \frac{1}{\sqrt{\lambda_1}} \vec{v}_1 = \frac{1}{4} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \vec{w}_2 = \frac{1}{\sqrt{-\lambda_2}} \vec{v}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has the required property that $q(c_1\vec{w}_1 + c_2\vec{w}_2) = c_1^2 - c_2^2$. (See Figure 8.14.)

8.2.**67** Consider an orthonormal eigenbasis $\vec{v}_1, ..., \vec{v}_n$ for A with associated eigenvalues $\lambda_1, ..., \lambda_n$, such that the eigenvalues $\lambda_1, ..., \lambda_p$ are positive, $\lambda_{p+1}, ..., \lambda_r$ are negative, and the remaining eigenvalues are 0. Define a

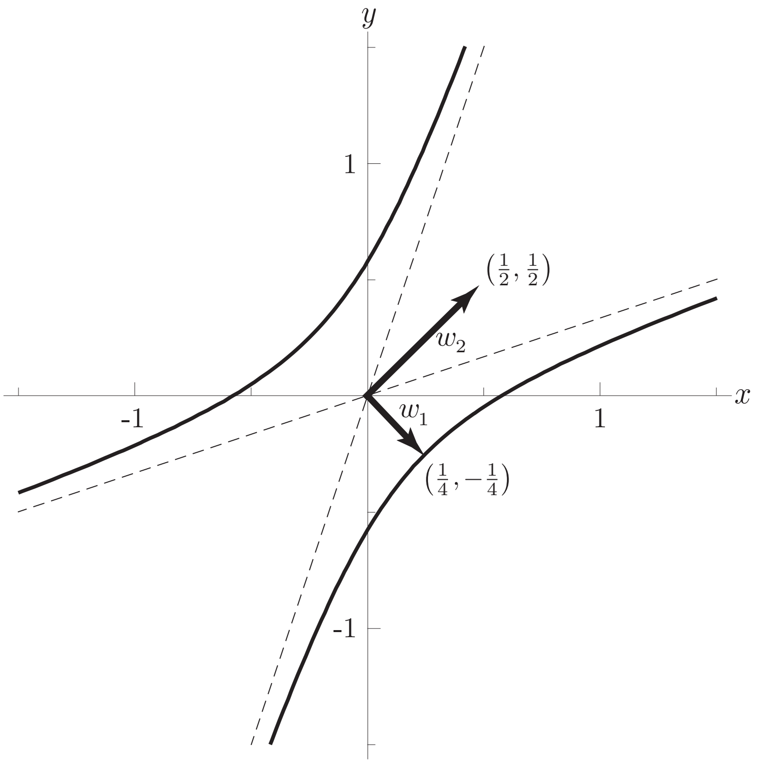

Figure 8.14: for Problem 8.2.66.

new orthogonal eigenbasis $\vec{w}_1, ..., \vec{w}_n$ by setting $\vec{w}_i = \left(1\Big/\sqrt{|\lambda_i|}\right)\vec{v}_i$ for $i = 1, ..., r$ and $\vec{w}_i = \vec{v}_i$ for $i = r+1, ..., n$. Note that $\vec{w}_i \cdot \vec{w}_i = 1/\lambda_i$ for $i = 1, ..., p$ and $\vec{w}_i \cdot \vec{w}_i = 1/(-\lambda_i)$ for $i = p+1, ..., r$. Now $q(c_1\vec{w}_1 + ... + c_p\vec{w}_p + ... + c_r\vec{w}_r + ... + c_n\vec{w}_n) = (c_1\vec{w}_1 + ... + c_p\vec{w}_p + ... + c_r\vec{w}_r + ... + c_n\vec{w}_n) \cdot (\lambda_1 c_1\vec{w}_1 + ... + \lambda_p c_p\vec{w}_p + ... +$ $= \lambda_1 c_1^2 \frac{1}{\lambda_1} + ... + \lambda_p c_p^2 \frac{1}{\lambda_p} - ... - \lambda_r c_r^2 \frac{1}{(-\lambda_r)} = c_1^2 + ... + c_p^2 - c_{p+1}^2 - ... - c_r^2$, as claimed.

8.2.**68** $p(\vec{x}) = q(L(\vec{x})) = q(R\vec{x}) = (R\vec{x})^T A (R\vec{x}) = \vec{x}^T \left(R^T A R\right)\vec{x}$, proving that p is a quadratic form with symmetric matrix $R^T AR$.

8.2.**69** If A is positive definite, then $\vec{x}^T \left(R^T AR\right)\vec{x} = (R\vec{x})^T A(R\vec{x}) \geq 0$ for all $\vec{x}$, meaning that $R^T AR$ is positive semidefinite. If A is positive definite and $\ker R = \left\{\vec{0}\right\}$, then $\vec{x}^T \left(R^T AR\right)\vec{x} = (R\vec{x})^T A(R\vec{x}) > 0$ for all nonzero $\vec{x}$, meaning that $R^T AR$ is positive definite. Conclusion: $R^T AR$ is always positive semidefinite, and $R^T AR$ is positive definite if (and only if) $\ker R = \left\{\vec{0}\right\}$, meaning that the rank of R is m.

8.2.**70** Since A is indefinite, there exist vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^n$ such that $\vec{v}_1^T A\vec{v}_1 > 0$ and $\vec{v}_2^T A\vec{v}_2 < 0$. Since the $n \times m$ matrix R has rank n, we know that the image of R is all of $\mathbb{R}^n$, so that there exist vectors $\vec{w}_1$ and $\vec{w}_2$ in $\mathbb{R}^m$ with $R\vec{w}_1 = \vec{v}_1$ and $R\vec{w}_2 = \vec{v}_2$. Now $\vec{w}_1^T R^T AR\vec{w}_1 = (R\vec{w}_1)^T A(R\vec{w}_1) = \vec{v}_1^T A\vec{v}_1 > 0$ and $\vec{w}_2^T R^T AR\vec{w}_2 = \vec{v}_2^T A\vec{v}_2 < 0$, proving that matrix $R^T AR$ is indefinite.

8.2.**71** Anything can happen. Consider the example $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $R_1 = I_2$, $R_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $R_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$R_1^T A R_1 = A$ is indefinite, $R_2^T A R_2 = [1]$ is positive definite, and $R_3^T A R_3 = [-1]$ is negative definite.

Section 8.3

8.3.**1** $\sigma_1 = 2, \sigma_2 = 1$

8.3.**2** The image of the unit circle is the unit circle, since the transformation defined by A preserves length. Thus $\sigma_1 = \sigma_2 = 1$ by Theorem 8.3.2.

8.3.**3** $A^T A = I_n$; the eigenvalues of $A^T A$ are all 1, so that the singular values of A are all 1.

8.3.**4** $A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, with eigenvalues $\lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}$. The singular values of A are $\sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2} \approx 1.62$ and $\sigma_2 = \sqrt{\frac{3-\sqrt{5}}{2}} = \frac{-1+\sqrt{5}}{2} \approx 0.62$.

8.3.**5** $A^T A = \begin{bmatrix} p^2+q^2 & 0 \\ 0 & p^2+q^2 \end{bmatrix}$, with eigenvalues $\lambda_1 = \lambda_2 = p^2+q^2$. The singular values of A are $\sigma_1 = \sigma_2 = \sqrt{p^2+q^2}$. A represents a rotation combined with a scaling, with a scaling factor of $\sqrt{p^2+q^2}$, so that the image of the unit circle is a circle with radius $\sqrt{p^2+q^2}$.

8.3.**6** The eigenvalues of $A^T A$ are $\lambda_1 = 25$ and $\lambda_2 = 0$, so that the singular values of A are $\sigma_1 = 5$ and $\sigma_2 = 0$ (these are also the eigenvalues of A; compare with Exercise 24).

$E_5 = \text{span} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so that $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ works. The image of the unit circle is the line segment connecting the tips of $A\vec{v}_1 = 5\vec{v}_1$ and $A(-\vec{v}_1) = -5\vec{v}_1$. See Figure 8.15.

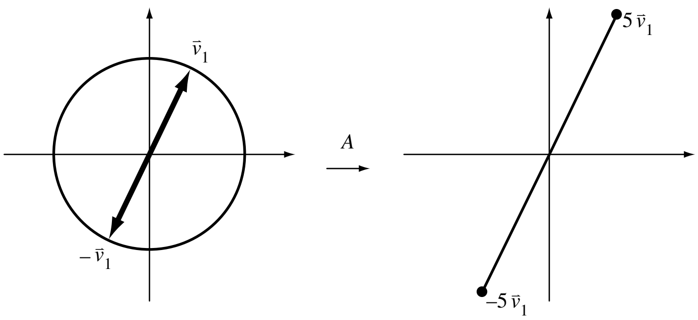

Figure 8.15: for Problem 8.3.6.

8.3.**7** $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$\lambda_1 = 4, \lambda_2 = 1$; $\sigma_1 = 2$, $\sigma_2 = 1$

eigenvectors of A^TA : $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{u}_1 = \frac{1}{\sigma_1}(A\vec{v}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{\sigma_2}(A\vec{v}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so that $U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

8.3.**8** $A^TA = \begin{bmatrix} p^2+q^2 & 0 \\ 0 & p^2+q^2 \end{bmatrix}$; $\lambda_1 = \lambda_2 = p^2+q^2$; $\sigma_1 = \sigma_2 = \sqrt{p^2+q^2}$

eigenvectors of A^TA : $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{\sqrt{p^2+q^2}}\begin{bmatrix} p \\ q \end{bmatrix}$, $\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{\sqrt{p^2+q^2}}\begin{bmatrix} -q \\ p \end{bmatrix}$, so that $U = \frac{1}{\sqrt{p^2+q^2}}\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$, $\Sigma = (\sqrt{p^2+q^2})I_2$, $V = I_2$.

8.3.**9** $A^TA = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$ (See Exercise 6)

$\lambda_1 = 25$, $\lambda_2 = 0$; $\sigma_1 = 5$, $\sigma_2 = 0$; eigenvectors of A^TA :

$\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{u}_2 =$ a unit vector orthogonal to

$\vec{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ so that $U = V = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$.

8.3.**10** In Example 4 we found $\begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right)$; now take the transpose of both sides:

$$\begin{bmatrix} 6 & -7 \\ 2 & 6 \end{bmatrix} = \underset{\substack{\uparrow \\ U}}{\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}}\underset{\substack{\uparrow \\ \Sigma}}{\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}}\underset{\substack{\uparrow \\ V^T}}{\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\right)}.$$

8.3.**11** $A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$; $\lambda_1 = 4$, $\lambda_2 = 1$; $\sigma_1 = 2$, $\sigma_2 = 1$ eigenvectors of A^TA :

$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

$U = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

8.3.**12** In Example 5 we see that $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

Now take the transpose of both sides.

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \underset{\substack{\uparrow \\ U}}{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}} \underset{\substack{\uparrow \\ \Sigma}}{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} \underset{\substack{\uparrow \\ V^T}}{\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right)}$$

8.3.**13** $A^TA = \begin{bmatrix} 37 & 16 \\ 16 & 13 \end{bmatrix}$; $\lambda_1 = 45$, $\lambda_2 = 5$; $\sigma_1 = 3\sqrt{5}$, $\sigma_2 = \sqrt{5}$ eigenvectors of A^TA :

$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so that

$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$, $V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

8.3.**14** $A^TA = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$; $\lambda_1 = 16$, $\lambda_2 = 1$; $\sigma_1 = 4$, $\sigma_2 = 1$

eigenvectors of A^TA : $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2$

$= \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, so that

$U = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$, $V = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

8.3.**15** If $A\vec{v}_1 = \sigma_1 \vec{u}_1$ and $A\vec{v}_2 = \sigma_2 \vec{u}_2$, then $A^{-1}\vec{u}_1 = \frac{1}{\sigma_1}\vec{v}_1$ and $A^{-1}\vec{u}_2 = \frac{1}{\sigma_2}\vec{v}_2$, so that the singular values of A^{-1} are the reciprocals of the singular values of A.

8.3.**16** If $A = U\Sigma V^T$ then $A^{-1} = V\Sigma^{-1}U^T$ and $(A^{-1})^T A^{-1} = U\left(\Sigma^{-1}\right)^2 U^{-1}$. Thus $(A^{-1})^T A^{-1}$ is similar to $\left(\Sigma^{-1}\right)^2$, so that the eigenvalues of $(A^{-1})^T A^{-1}$ are the squares of the reciprocals of the singular values of A. It follows that the singular values of A^{-1} are the reciprocals of those of A.

8.3.**17** We need to check that $A\left(\frac{\vec{b}\cdot\vec{u}_1}{\sigma_1}\vec{v}_1 + \cdots + \frac{\vec{b}\cdot\vec{u}_m}{\sigma_m}\vec{v}_m\right) = \text{proj}_{\text{im}A}\vec{b}$. (see Page 239).

But $A\left(\frac{\vec{b}\cdot\vec{u}_1}{\sigma_1}\vec{v}_1 + \cdots + \frac{\vec{b}\cdot\vec{u}_m}{\sigma_m}\vec{v}_m\right) = \frac{\vec{b}\cdot\vec{u}_1}{\sigma_1}A\vec{v}_1 + \cdots + \frac{\vec{b}\cdot\vec{u}_m}{\sigma_m}A\vec{v}_m = (\vec{b}\cdot\vec{u}_1)\vec{u}_1 + \cdots + (\vec{b}\cdot\vec{u}_m)\vec{u}_m$

$= \text{proj}_{\text{im}A}\vec{b}$, since $\vec{u}_1, \ldots, \vec{u}_m$ is an orthonormal basis of $\text{im}(A)$ (see Theorem 5.1.5).

8.3.**18** $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, $\vec{v}_1 = \frac{1}{5} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{5} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $\sigma_1 = 2, \sigma_2 = 1$, so that

$\vec{x}^* = \frac{\vec{b}\cdot\vec{u}_1}{\sigma_1}\vec{v}_1 + \frac{\vec{b}\cdot\vec{u}_2}{\sigma_2}\vec{v}_2 = \begin{bmatrix} -0.1 \\ -3.2 \end{bmatrix}$.

8.3.**19** $\vec{x} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$ is a least-squares solution if $A\vec{x} = c_1A\vec{v}_1 + \cdots + c_mA\vec{v}_m = c_1\sigma_1\vec{u}_1 + \cdots + c_r\sigma_r\vec{u}_r = \text{proj}_{\text{im}A}\vec{b}$. But $\text{proj}_{\text{im}A}\vec{b} = (\vec{b}\cdot\vec{u}_1)\vec{u}_1 + \cdots + (\vec{b}\cdot\vec{u}_r)\vec{u}_r$, since $\vec{u}_1, \ldots, \vec{u}_r$ is an orthonormal basis of $\text{im}(A)$. Comparing the coefficients of $\vec{u}_i$ above we find that it is required that $c_i\sigma_i = \vec{b}\cdot\vec{u}_i$ or $c_i = \frac{\vec{b}\cdot\vec{u}_i}{\sigma_i}$, for $i = 1, \ldots, r$, while no condition is imposed on $c_{r+1}, \ldots, c_m$. The least-squares solutions are of the form $\vec{x}^* = \frac{b\cdot\vec{u}_1}{\sigma_1}\vec{v}_1 + \cdots + \frac{\vec{b}\cdot\vec{u}_r}{\sigma_r}\vec{v}_r + c_{r+1}\vec{v}_{r+1} + \cdots c_m\vec{v}_m$, where $c_{r+1}, \ldots, c_m$ are arbitrary (see Exercise 17 for a special case).

8.3.**20** a $A = U\Sigma V^T = UV^TV\Sigma V^T = QS$, where $Q = UV^T$ and $S = V\Sigma V^T$. Note that Q is orthogonal, being the product of orthogonal matrices; S is symmetric as $S^T = (V^T)^T\Sigma^TV^T = V\Sigma V^T = S$; and S is similar to Σ, so that the eigenvalues of S are the (nonnegative) diagonal entries of Σ.

b Yes, write $A = U\Sigma V^T = U\Sigma U^TUV^T = S_1Q_1$ where $S_1 = U\Sigma U^T$ and $Q_1 = UV^T$.

8.3.**21** $$A = \underbrace{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}}_{\Sigma}\underbrace{\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right)}_{V^T}$$

$$= \underbrace{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}_{U}\underbrace{\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right)}_{V^T}\underbrace{\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\right)}_{V}\underbrace{\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}}_{\Sigma}\underbrace{\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right)}_{V^T}$$

$$= \underbrace{\frac{1}{5}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}}_{Q}\underbrace{\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}}_{S}$$

8.3.**22** a. T_1 is the orthogonal projection onto the plane perpendicular to the vector $\vec{v}$. T_2 scales by the length of the vector $\vec{v}$ and T_3 is a rotation about the line through the origin spanned by $\vec{v}$ by a rotation angle $\pi/2$. Because $Q = A_3$ is orthogonal and $S = A_2A_1$ is symmetric this is a polar decomposition: $A = QS$.
b. Here, A_1 represents the orthogonal projection onto the xz plane, A_2 represents a scaling by a factor of 2, and A_3 represents a rotation about the y axis through an angle of $\pi/2$, counterclockwise as viewed from the positive y axis:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Thus $Q = A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, $S = A_2A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, and $A = QS = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$.

8.3.**23** $AA^TU = U\Sigma V^TV\Sigma^TU^TU = U\Sigma Sigma^T$, since $V^TV = I_m$ and $U^TU = I_n$, so that

$$AA^T\vec{u}_i = \begin{cases} \sigma_i^2\vec{u}_i & \text{for } i = 1, \ldots, r \\ \vec{0} & \text{for } i = r+1, \ldots, n \end{cases}$$

The *nonzero* eigenvalues of A^TA and AA^T are the same.

8.3.**24** The eigenvalues of $A^TA = A^2$ are the squares of the eigenvalues of A, so that the singular values of A are the absolute values of the eigenvalues of A.

8.3.**25** See Figure 8.16.

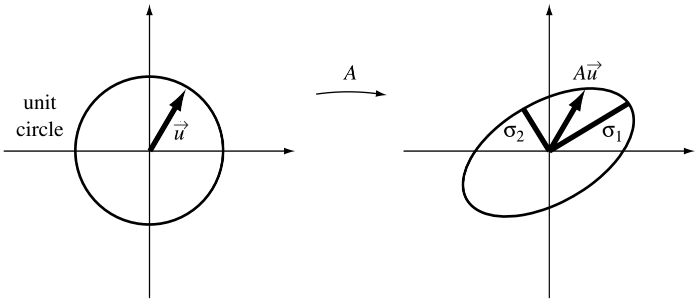

Figure 8.16: for Problem 8.3.25.

Algebraically: Write $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$ and note that $\|\vec{u}\|^2 = c_1^2 + c_2^2 = 1$.

Then $A\vec{u} = c_1\sigma_1\vec{u}_1 + c_2\sigma_2\vec{u}_2$, so that $\|A\vec{u}\|^2 = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 \geq c_1^2\sigma_2^2 + c_2^2\sigma_2^2 = \sigma_2^2$ and $\|A\vec{u}\| \geq \sigma_2$. Likewise $\|A\vec{u}\| \leq \sigma_1$.

8.3.**26** Write $\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$ and note that $\|\vec{v}\|^2 = c_1^2 + \cdots + c_m^2$. Then $A\vec{v} = c_1\sigma_1\vec{u}_1 + \cdots + c_r\sigma_r\vec{u}_r$ and $\|A\vec{v}\|^2 = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_r^2\sigma_r^2 \leq c_1^2\sigma_1^2 + c_2^2\sigma_1^2 + \cdots + c_r^2\sigma_1^2 \leq \sigma_1^2\|v\|^2$ so that $\|A\vec{v}\| \leq \sigma_1\|\vec{v}\|$. Likewise, $\|A\vec{v}\| \geq \sigma_m\|\vec{v}\|$.

8.3.**27** Let $\vec{v}$ be a unit eigenvector with eigenvalue λ and use Exercise 26.

8.3.**28** If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A^TA, then $(\det A)^2 = \det(A^TA) = \lambda_1 \cdots \lambda_n = \sigma_1^2 \cdots \sigma_n^2$, so that $|\det A| = \sigma_1 \cdots \sigma_n$. For a 2×2 matrix:

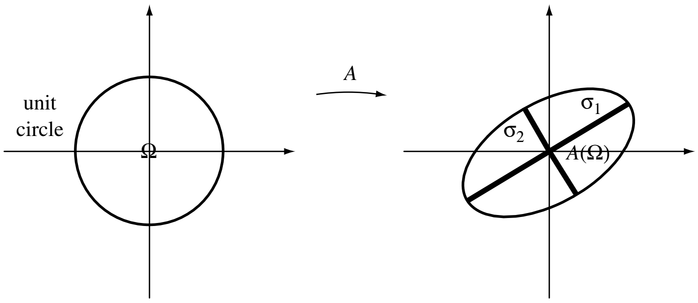

Figure 8.17: for Problem 8.3.28.

$|\det(A)| = \text{expansion factor} = \dfrac{\text{area of ellipse } A(\Omega)}{\text{area of unit circle } \Omega} = \frac{\pi\sigma_1\sigma_2}{\pi} = \sigma_1\sigma_2$. See Figure 8.17.

8.3.**29** $A = U\Sigma V^T = [\vec{u}_1 \cdots \vec{u}_r \cdots] \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_r^T \\ \vdots \end{bmatrix} = [\vec{u}_1 \cdots \vec{u}_r \cdots] \begin{bmatrix} \sigma_1\vec{v}_1^T \\ \vdots \\ \sigma_r\vec{v}_r^T \\ 0 \end{bmatrix}$

$= \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$

8.3.**30** $\begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = 10 \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \left(\frac{1}{\sqrt{5}} [2 - 1] \right)$

$+5 \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left(\frac{1}{\sqrt{5}} [1 \quad 2] \right) = \begin{bmatrix} 4 & -2 \\ -8 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

8.3.**31** The formula $A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$ gives such a representation.

8.3.**32** $(SAR)^T SAR = R^T A^T S^T SAR = R^T A^T AR$ is similar to $A^T A$, so that the matrices $A^T A$ and $(SAR)^T SAR$ have the same eigenvalues. Thus A and SAR have the same singular values.

8.3.**33** Yes; since $A^T A$ is diagonalizable and has only 1 as an eigenvalue, we must have $A^T A = I_n$.

8.3.**34** $A = U \Sigma U^T$ means that $U^T AU = U^{-1} AU = \Sigma$, i.e., A is orthogonally diagonalizable, with eigenvalues ≥ 0. This is the case if and only if A is *symmetric* and *positive semidefinite*.

8.3.**35** We will freely use the diagram on Page 393 (with $r = m$). We have $A^T A \vec{v}_i = A^T (\sigma_i \vec{u}_i) = \sigma_i^2 \vec{v}_i$ and therefore $(A^T A)^{-1} \vec{v}_i = \frac{1}{\sigma_i^2} \vec{v}_i$ for $i = 1, \ldots, m$. Then $(A^T A)^{-1} A^T \vec{u}_i = (A^T A)^{-1} (\sigma_i \vec{v}_i) = \frac{1}{\sigma_i} \vec{v}_i$ for $i = 1, \ldots, m$ and $(A^T A)^{-1} A^T \vec{u}_i = \vec{0}$ for $i = m+1, \ldots, n$ since $\vec{u}_i$ is in $\ker(A^T)$ in this case. Note that $(A^T A)^{-1} A^T \vec{u}_i$ is the least-squares solution of the equation $A\vec{x} = \vec{u}_i$; for $i = 1, \ldots, m$ this is the exact solution since $\vec{u}_i$ is in $\operatorname{im}(A)$.

8.3.**36** We will freely use the diagram on Page 411. By construction of the $\vec{v}_i$ as eigenvectors of $A^T A$ we have $A^T A \vec{v}_i = \lambda_i \vec{v}_i = \sigma_i^2 \vec{v}_i$, or $(A^T A)^{-1} \vec{v}_i = \frac{1}{\sigma_i^2} \vec{v}_i$. Then $A(A^T A)^{-1} A^T \vec{u}_i = A(A^T A)^{-1} (\sigma_i \vec{v}_i) = A \left(\frac{1}{\sigma_i} \vec{v}_i \right) = \frac{1}{\sigma_i} A \vec{v}_i = \vec{u}_i$ for $i = 1, \ldots, m$ and $A(A^T A)^{-1} A^T \vec{u}_i = \vec{0}$ for $i = m+1, \ldots, n$ since $\vec{u}_i$ is in $\ker(A^T)$ in this case. The fact that $A(A^T A)^{-1} A^T \vec{u}_i = \begin{cases} \vec{u}_i & \text{if } i = 1, \ldots, m \\ \vec{0} & \text{if } i = m+1, \ldots, n \end{cases}$ means that the matrix $A(A^T A)^{-1} A^T$ represents the orthogonal projection onto $\operatorname{im}(A) = \operatorname{span}(\vec{u}_1, \ldots, \vec{u}_m)$.

True or False

Ch 8.TF.**1** T. If $D = \begin{bmatrix} \lambda_1 & . & 0 \\ . & . & . \\ 0 & . & \lambda_n \end{bmatrix}$, then $D^T D = D^2 = \begin{bmatrix} \lambda_1^2 & . & 0 \\ . & . & . \\ 0 & . & \lambda_n^2 \end{bmatrix}$. The eigenvalues of $D^T D$ are $\lambda_1^2, \ldots, \lambda_n^2$, and the singular values of D are $\sqrt{\lambda_1^2} = |\lambda_1|, \ldots, \sqrt{\lambda_n^2} = |\lambda_n|$.

Ch 8.TF.**2** F, since $\det \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 3 \end{bmatrix} = -\frac{1}{4} < 0$ (see Theorem 8.2.7).

Ch 8.TF.**3** T, by the spectral theorem (Theorem 8.1.1)

Ch 8.TF.**4** T. Note that $[1 \quad 0] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a > 0$, by Definition 8.2.3.

Ch 8.TF.**5** F. The orthogonal matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ fails to be diagonalizable (over $\mathbb{R}$).

Ch 8.TF.**6** T. If $A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, then the eigenvalue of $A^T A = [\,3 \quad 4\,] \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [25]$ is $\lambda = 25$, so that the singular value of A is $\sigma = \sqrt{\lambda} = 5$.

Ch 8.TF.**7** F. The last term, $5x_2$, does not have the form required in Definition 8.2.1

Ch 8.TF.**8** F. The singular values of A are *the square roots of* the eigenvalues of $A^T A$, by Definition 8.3.1.

Ch 8.TF.**9** T, by Theorem 8.2.4.

Ch 8.TF.**10** T, by Definition 8.2.1

Ch 8.TF.**11** F. Consider the shear matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The unit circle isn't mapped into itself, so that the singular values fail to be 1, 1.

Ch 8.TF.**12** F. In general, $(A^T A)^T = A^T A \neq AA^T$

Ch 8.TF.**13** T, by Theorem 8.3.2

Ch 8.TF.**14** T. All four eigenvalues are negative, so that their product, the determinant, is positive.

Ch 8.TF.**15** T, by Theorem 8.1.2

Ch 8.TF.**16** F, since the determinant is 0, so that 0 is an eigenvalue.

Ch 8.TF.**17** F. Consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Ch 8.TF.**18** T, since AA^T is symmetric (use the spectral theorem)

Ch 8.TF.**19** T, by Theorem 8.2.4: all the eigenvalues are positive.

Ch 8.TF.**20** T, since the matrix is symmetric.

Ch 8.TF.**21** T. The eigenvalues $\lambda_1, \dots, \lambda_n$ of A are nonzero, since A is invertible, so that the eigenvalues $\lambda_1^2, \dots, \lambda_n^2$ of A^2 are positive. Now use Theorem 8.2.4.

Ch 8.TF.**22** T, by Theorem 8.3.2, since $\vec{v} = A\vec{e}_1$ and $\vec{w} = A\vec{e}_2$ are the principal semi-axis of the image of the unit circle.

Ch 8.TF.**23** F. As a counterexample, consider $A = S = 2I_n$.

Ch 8.TF.**24** T, since $\vec{e}_i^{\,T} A\vec{e}_i = a_{ii} < 0$.

Ch 8.TF.**25** T. By Theorem 7.3.6, matrices A and B have the same eigenvalues. Now use Theorem 8.2.4.

Ch 8.TF.**26** T. The spectral theorem guarantees that there is an orthogonal R such that R^TAR is diagonal. Now let $S = R^T$.

Ch 8.TF.**27** F. Let $A = I_2$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Ch 8.TF.**28** T. Consider the singular value decomposition $A = U\sum V^T$, or $AV = U\sum$, where V is orthogonal (see Theorem 8.3.5). We can let $S = V$, since the columns of $AS = AV = U\sum$ are orthogonal, by construction.

Ch 8.TF.**29** T. By the spectral theorem, A is diagonalizable: $S^{-1}AS = D$ for some invertible S and a diagonal D. Now $D^n = S^{-1}A^nS = S^{-1}0S = 0$, so that $D = 0$ (since D is diagonal). Finally, $A = SDS^{-1} = S0S^{-1} = 0$, as claimed.

Ch 8.TF.**30** F. If k is negative, then $kq(\vec{x})$ will be negative definite.

Ch 8.TF.**31** F. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then 1 is a singular value of BA but not of AB.

Ch 8.TF.**32** T, since $A + A^{-1} = A + A^T$ is symmetric.

Ch 8.TF.**33** F. For example, $(x_1^2)(x_2x_3)$ fails to be a quadratic form.

Ch 8.TF.**34** T. We can write $q(\vec{x}) = \vec{x}^T \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{bmatrix} \vec{x}$.

Ch 8.TF.**35** F. Consider $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is indefinite.

Ch 8.TF.**36** T, by Definition 8.2.3: $\vec{x}^T(A + B)\vec{x} = \vec{x}^TA\vec{x} + \vec{x}^TB\vec{x} > 0$ for all nonzero $\vec{x}$.

Ch 8.TF.**37** T, since $\vec{x} \cdot A\vec{x}$ is positive, so that $\cos\theta$ is positive, where θ is the angle enclosed by $\vec{x}$ and $A\vec{x}$.

Ch 8.TF.**38** T. Preliminary remark: If σ is the largest singular value of an $n \times m$ matrix M, then $\|M\vec{v}\| \leq \sigma\|\vec{v}\|$ for all $\vec{v}$ in $\mathbb{R}^m$ (see Exercise 8.3.26). Now let σ_1, σ_2 be the singular values of matrix AB, with $\sigma_1 \geq \sigma_2$, and let $\vec{v}_1$ be a unit vector in $\mathbb{R}^2$ such that $\|AB\vec{v}_1\| = \sigma_1$ (see Theorem 8.3.3). Now $\sigma_2 \leq \sigma_1 = \|A(B\vec{v}_1)\| \leq 3\|B\vec{v}_1\| \leq 3 \cdot 5\|\vec{v}_1\| = 15$, proving our claim; note that we have used the preliminary remark twice.

Ch 8.TF.**39** F. Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Ch 8.TF.**40** T. If λ is the smallest eigenvalue of A, let $k = 1 - \lambda$. Then the smallest eigenvalue of $A + kI_n$ is $\lambda + k = 1$, so that all the eigenvalues of $A + kI_n$ are positive. Now use Theorem 8.2.4.

Ch 8.TF.**41** T. The quadratic form $q(x_1,x_2) = [\,x_1 \quad 0 \quad x_2\,]\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}\begin{bmatrix} x_1 \\ 0 \\ x_2 \end{bmatrix} = ax_1^2 + 2cx_1x_2 + fx_2^2$ is positive definite. The matrix of this quadratic form is $A = \begin{bmatrix} a & c \\ c & f \end{bmatrix}$, and $\det(A) = af - c^2 > 0$ since A is positive definite. Thus $af > c^2$, as claimed.

Ch 8.TF.**42** F. Consider the positive definite matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

Ch 8.TF.**43** F. Consider the indefinite matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Ch 8.TF.**44** T. By Theorem 8.3.2., the continuous function $f(x) = A\begin{bmatrix} \cos x \\ \sin x \end{bmatrix}$ has the global maximum 5 and the global minimum 3. Note that the image of the unit circle consists of all vectors of the form $A\begin{bmatrix} \cos x \\ \sin x \end{bmatrix}$. By the intermediate value theorem, $f(c) = 4$ for some c. Let $\vec{u} = \begin{bmatrix} \cos c \\ \sin c \end{bmatrix}$ (draw a sketch!).

Ch 8.TF.**45** T, since $\vec{x}^T A^2 \vec{x} = -\vec{x}^T A^T A\vec{x} = -(A\vec{x})^T A\vec{x} = -\|A\vec{x}\|^2 \le 0$ for all $\vec{x}$.

Ch 8.TF.**46** T. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A^TA, then $\lambda_1\lambda_2\ldots\lambda_n = \det(A^TA) = (\det A)^2$. If

$\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_n = \sqrt{\lambda_n}$ are the singular values of A, then

$\sigma_1\sigma_2\ldots\sigma_n = \sqrt{\lambda_1\lambda_2\ldots\lambda_n} = |\det A|$, as claimed.

Ch 8.TF.**47** F. Note that the columns of S must be unit eigenvectors of A. There are two distinct real eigenvalues, λ_1, λ_2, and for each of them there are two unit eigenvectors, $\pm\vec{v}_1$ (for λ_1) and $\pm\vec{v}_2$ (for λ_2). (Draw a sketch!) Thus there are 8 matrices S, namely $S = [\,\pm\vec{v}_1 \quad \pm\vec{v}_2\,]$ and $S = [\,\pm\vec{v}_2 \quad \pm\vec{v}_1\,]$

Ch 8.TF.**48** T. See the remark following Definition 8.2.1.

Ch 8.TF.**49** F. Some eigenvalues of A may be negative.

Ch 8.TF.**50** F. Consider the similar matrices $A = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ 0 & 3 \end{bmatrix}$. Matrix A has the singular values 0 and 3, while those of B are 0 and 5.

Ch 8.TF.**51** T. Let $\vec{v}_1, \vec{v}_2$ be an orthonormal eigenbasis, with $A\vec{v}_1 = \vec{v}_1$ and $A\vec{v}_2 = 2\vec{v}_2$. Consider a nonzero vector $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$; then $A\vec{x} = c_1\vec{v}_1 + 2c_2\vec{v}_2$. If $c_1 = 0$, then $\vec{x} = c_2\vec{v}_2$ and $A\vec{x} = 2c_2\vec{v}_2$ are parallel, and we are all set. Now consider the case when $c_1 \ne 0$. Then the angle between $\vec{x}$ and $A\vec{x}$ is $\arctan(2c_2/c_1) - \arctan(c_2/c_1)$; to see this, subtract the angle between $\vec{v}_1$ and $\vec{x}$ from the angle between $\vec{v}_1$ and $A\vec{x}$ (draw a sketch). Let $m = c_2/c_1$ and use calculus to see that the function $f(m) = \arctan(2m) - \arctan(m)$ assumes its global maximum at $m = \frac{1}{\sqrt{2}}$. The maximal angle between $\vec{x}$ and $A\vec{x}$ is $\arctan(\sqrt{2}) - \arctan(1/\sqrt{2}) < 0.34 < \pi/6$.

Ch 8.TF.**52** T. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. By Theorem 8.3.2, $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \sqrt{a^2 + c^2} < 5$ (since the length of the semi-major axis of the image of the unit circle is less than 5). Thus $a < 5$ and $c < 5$. Likewise, $b < 5$ and $d < 5$.

Ch 8.TF.**53** T. We need to show that each entry $a_{ij} = a_{ji}$ off the diagonal is smaller than some entry on the diagonal. Now $(\vec{e}_i - \vec{e}_j)^T A(\vec{e}_i - \vec{e}_j) = a_{ii} + a_{jj} - 2a_{ij} > 0$, so that $a_{ii} + a_{jj} > 2a_{ij}$. Thus the larger of the diagonal entries a_{ii} and a_{jj} must exceed a_{ij}.

Ch 8.TF.**54** T. Let $\lambda_1, ..., \lambda_m$ be the distinct eigenvalues of A, with the associated eigenspaces E_{A,λ_i}. Since A is diagonalizable, we know that $\sum_{k=1}^{m} \dim\left(E_{A,\lambda_i}\right) = n$. By definition of an eigenvector ,$E_{A,\lambda_i}$is a subspace of E_{A^3,λ_i^3}, Since $\sum_{k=1}^{m} \dim\left(E_{A^3,\lambda_i^3}\right)$ cannot exceed n, we must have E_{A,λ_i}=E_{A^3,λ_i^3} for all eigenvalues. Applying the same reasoning to B and B^3, we can conclude that E_{A,λ_i}=$E_{A^3,\lambda_i^3} = E_{B^3,\lambda_i^3} = E_{B,\lambda_i}$. Since the diagonalizable matrices A and B have the same eigenvectors with the same eigenvalues, they must be equal.