Chapter 3

Section 3.1

3.1.1 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc:c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right], \text{ so that } x_1 = x_2 = 0.$$

$\ker(A) = \{\vec{0}\}$.

3.1.2 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$, or $x_1 + 2x_2 + 3x_2 = 0$.

The solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t - 3r \\ t \\ r \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, so that

$\ker(A) = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right)$.

3.1.3 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$; note that all $\vec{x}$ in $\mathbb{R}^2$ satisfy the equation, so that $\ker(A) = \mathbb{R}^2 = \operatorname{span}(\vec{e}_1, \vec{e}_2)$.

3.1.4 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc:c} 2 & 3 & 0 \\ 6 & 9 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ so that } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3t}{2} \\ t \end{bmatrix}$$

Setting $t = 2$ we find $\ker(A) = \operatorname{span}\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

3.1.5 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]; \begin{array}{rl} x_1 & = x_3 \\ x_2 & = -2x_3 \end{array}; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$$

$\ker(A) = \operatorname{span}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

3.1.6 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$. Solving this system yields $\ker(A) = \operatorname{span}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

3.1.7 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$. Since $\operatorname{rref}(A) = I_3$ we have $\ker(A) = \{\vec{0}\}$.

3.1.8 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]; x_1 + x_2 + x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -r-t \\ r \\ t \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\ker(A) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right).$$

3.1.**9** Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$. Solving this system yields $\ker(A) = \{\vec{0}\}$.

3.1.**10** Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right)$.

3.1.**11** Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span}\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$.

3.1.**12** Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span}\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

3.1.**13** Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

3.1.**14** By Theorem 3.1.3, the image of A is the span of the column vectors of A:

$$\text{im}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}\right).$$

Since these three vectors are parallel, we need only one of them to span the image:

$$\text{im}(A) = \text{span}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

3.1.**15** By Theorem 3.1.3, the image of A is the span of the columns of A:

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}1\\4\end{bmatrix}\right).$$

Since any two of these vectors span all of $\mathbb{R}^2$ already, we can write

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right).$$

3.1.**16** By Theorem 3.1.3, the image of A is the span of the column vectors of A:

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right).$$

3.1.**17** By Theorem 3.1.3, $\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right) = \mathbb{R}^2$ (the whole plane).

3.1.**18** By Theorem 3.1.3, $\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}4\\12\end{bmatrix}\right) = \operatorname{span}\begin{bmatrix}1\\3\end{bmatrix}$ (a line in $\mathbb{R}^2$).

3.1.**19** Since the four column vectors of A are parallel, we have $\operatorname{im}(A) = \operatorname{span}\begin{bmatrix}1\\-2\end{bmatrix}$, a line in $\mathbb{R}^2$.

3.1.**20** Compare with the solution to Exercise 21.

$$\left[\begin{array}{cc|c}2 & 1 & 3\\3 & 4 & 2\\6 & 5 & 7\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 0 & 2\\0 & 1 & -1\\0 & 0 & 0\end{array}\right]$$

This computation shows that the third column vector of A, $\vec{v}_3$, is a linear combination of the first two, Thus, only the first two vectors are independent, and the image is a plane in $\mathbb{R}^3$.

3.1.**21** By Theorem 3.1.3, $\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}4\\1\\5\end{bmatrix}, \begin{bmatrix}7\\9\\6\end{bmatrix}, \begin{bmatrix}3\\2\\8\end{bmatrix}\right)$.

We must simply find out how many of the column vectors are not redundant to determine a basis of the image. We can detemine this by taking the rref of the matrix:

$\left[\begin{array}{cc|c}4 & 7 & 3\\1 & 9 & 2\\5 & 6 & 8\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right]$, which shows us that all three column vectors are independent: the span is all of $\mathbb{R}^3$.

3.1.**22** Since the three column vectors of A are parallel, we have $\operatorname{im}(A) = \operatorname{span}\begin{bmatrix}1\\1\\1\end{bmatrix}$, a line in $\mathbb{R}^3$.

3.1.**23** $\text{im}(T) = \mathbb{R}^2$ and $\ker(T) = \{\vec{0}\}$, since T is invertible (see Summary 3.1.8).

3.1.**24** $\text{im}(T)$ is the plane $x + 2y + 3z = 0$, and $\ker(T)$ is the line perpendicular to this plane, spanned by the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (compare with Examples 5 and 9).

3.1.**25** $\text{im}(T) = \mathbb{R}^2$ and $\ker(T) = \{\vec{0}\}$, since T is invertible (see Summary 3.1.8).

3.1.**26** Since $\lim_{t\to\infty} f(t) = \infty$ and $\lim_{t\to-\infty} f(t) = -\infty$, we have $\text{im}(f) = \mathbb{R}$.

A careful proof involves the intermediate value theorem (see Exercise 2.2.47),

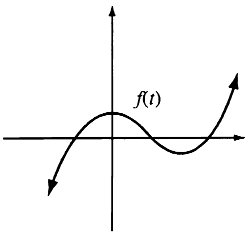


Figure 3.1: for Problem 3.1.26.

Any horizontal line intersects this graph at least once (compare with Example 3 and see Figure 3.1).

3.1.**27** Let $f(x) = x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$.

Then $\text{im}(f) = \mathbb{R}$, since

$$\lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to-\infty} f(x) = -\infty$$

but the function fails to be invertible since the equation $f(x) = 0$ has three solutions, $x = 0$, 1, and -1.

3.1.**28** This ellipse can be obtained from the unit circle by means of the linear transformation with matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, as shown in Figure 3.2 (compare with Exercise 2.2.53).

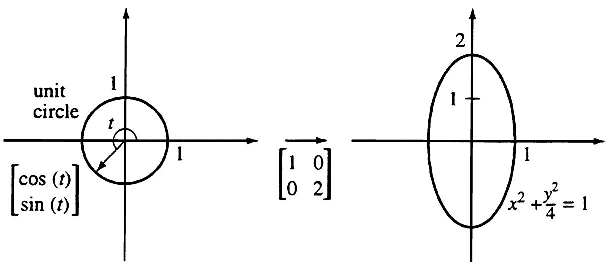


Figure 3.2: for Problem 3.1.28.

We obtain the parametrization $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix}$ for the ellipse.

We can check that $x^2 + \frac{y^2}{4} = \cos^2(t) + \frac{4\sin^2(t)}{4} = 1$.

3.1.**29** Use spherical coordinates (see any good text on multivariable calculus): $f\begin{bmatrix}\phi\\ \theta\end{bmatrix} = \begin{bmatrix}\sin(\phi)\cos(\theta)\\ \sin(\phi)\sin(\theta)\\ \cos(\phi)\end{bmatrix}$

3.1.**30** By Theorem 3.1.3, $A = \begin{bmatrix}1\\5\end{bmatrix}$ does the job. There are many other possible answers: any nonzero $2\times n$ matrix A whose column vectors are scalar multiples of vector $\begin{bmatrix}1\\5\end{bmatrix}$.

3.1.**31** The plane $x + 3y + 2z = 0$ is spanned by the two vectors $\begin{bmatrix}-2\\0\\1\end{bmatrix}$ and $\begin{bmatrix}-3\\1\\0\end{bmatrix}$, for example. Therefore, $A = \begin{bmatrix}-2 & -3\\ 0 & 1\\ 1 & 0\end{bmatrix}$ does the job. There are many other correct answers.

3.1.**32** By Theorem 3.1.3, $A = \begin{bmatrix}7\\6\\5\end{bmatrix}$ does the job. There are many other correct answers: any nonzero $3\times n$ matrix A whose column vectors are scalar multiples of $\begin{bmatrix}7\\6\\5\end{bmatrix}$.

3.1.**33** The plane is the kernel of the linear transformation $T\begin{bmatrix}x\\y\\z\end{bmatrix} = x + 2y + 3z$ from $\mathbb{R}^3$ to $\mathbb{R}$.

3.1.**34** To describe a subset of $\mathbb{R}^3$ as a kernel means to describe it as an intersection of planes (think about it). By inspection, the given line is the intersection of the planes

$$\begin{aligned} x + y &= 0 \quad \text{and}\\ 2x + z &= 0. \end{aligned}$$

This means that the line is the kernel of the linear transformation $T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x+y\\2x+z\end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

3.1.**35** $\ker(T) = \{\vec{x} : T(\vec{x}) = \vec{v}\cdot\vec{x} = 0\}$ = the plane with normal vector $\vec{v}$.

$\text{im}(T) = \mathbb{R}$, since for every real number k there is a vector $\vec{x}$ such that $T(\vec{x}) = k$, for example, $\vec{x} = \frac{k}{\vec{v}\cdot\vec{v}}\vec{v}$.

3.1.**36** $\ker(T) = \{\vec{x} : T(\vec{x}) = \vec{v}\times\vec{x} = \vec{0}\}$ = the line spanned by $\vec{v}$

(see Theorem A.10d in the Appendix)

$\text{im}(T)$ = the plane with normal vector $\vec{v}$

By Definition A.9, $T(\vec{x}) = \vec{v} \times \vec{x}$ is in this plane, for all $\vec{x}$ in $\mathbb{R}^3$. Conversely, for every vector $\vec{w}$ in this plane there is an $\vec{x}$ in $\mathbb{R}^3$ such that $T(\vec{x}) = \vec{w}$, namely $\vec{x} = -\frac{1}{\vec{v}\cdot\vec{v}}T(\vec{w})$ (verify this!).

3.1.**37** $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so that

$\ker(A) = \text{span}(\vec{e}_1)$, $\ker(A^2) = \text{span}(\vec{e}_1, \vec{e}_2)$, $\ker(A^3) = \mathbb{R}^3$, and

$\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2)$, $\text{im}(A^2) = \text{span}(\vec{e}_1)$, $\text{im}(A^3) = \{\vec{0}\}$.

3.1.**38** a If a vector $\vec{x}$ is in $\ker(A^k)$, that is, $A^k\vec{x} = \vec{0}$, then $\vec{x}$ is also in $\ker(A^{k+1})$, since $A^{k+1}\vec{x} = AA^k\vec{x} = A\vec{0} = \vec{0}$.

Therefore, $\ker(A) \subseteq \ker(A^2) \subseteq \ker(A^3) \subseteq \dots$

Exercise 37 shows that these kernels need not be equal.

b If a vector $\vec{y}$ is in $\text{im}(A^{k+1})$, that is, $\vec{y} = A^{k+1}\vec{x}$ for some $\vec{x}$, then $\vec{y}$ is also in $\text{im}(A^k)$, since we can write $\vec{y} = A^k(A\vec{x})$. Therefore, $\text{im}(A) \supseteq \text{im}(A^2) \supseteq \text{im}(A^3) \supseteq \dots$.

Exercise 37 shows that these images need not be equal.

3.1.**39** a If a vector $\vec{x}$ is in $\ker(B)$, that is, $B\vec{x} = \vec{0}$, then $\vec{x}$ is also in $\ker(AB)$, since $AB(\vec{x}) = A(B\vec{x}) = A\vec{0} = \vec{0}$:

$\ker(B) \subseteq \ker(AB)$.

Exercise 37 (with $A = B$) illustrates that these kernels need not be equal.

b If a vector $\vec{y}$ is in $\text{im}(AB)$, that is, $\vec{y} = AB\vec{x}$ for some $\vec{x}$, then $\vec{y}$ is also in $\text{im}(A)$, since we can write

$\vec{y} = A(B\vec{x})$:

$\text{im}(AB) \subseteq \text{im}(A)$.

Exercise 37 (with $A = B$) illustrates that these images need not be equal.

3.1.**40** For any $\vec{x}$ in $\mathbb{R}^m$, the vector $B\vec{x}$ is in $\text{im}(B) = \ker(A)$, so that $AB\vec{x} = \vec{0}$. If we apply this fact to $\vec{x} = \vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$, we find that all the columns of the matrix AB are zero, so that $AB = 0$.

3.1.**41** a $\text{rref}(A) = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 0 \end{bmatrix}$, so that $\ker(A) = \text{span}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

$\text{im}(A) = \text{span}\begin{bmatrix} 0.36 \\ 0.48 \end{bmatrix} = \text{span}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Note that $\text{im}(A)$ and $\ker(A)$ are perpendicular lines.

b $A^2 = A$

If $\vec{v}$ is in $\text{im}(A)$, with $\vec{v} = A\vec{x}$, then $A\vec{v} = A^2\vec{x} = A\vec{x} = \vec{v}$.

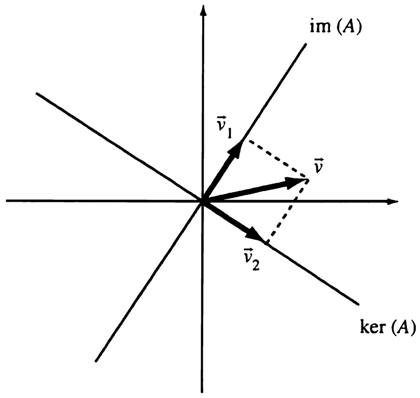


Figure 3.3: for Problem 3.1.41c.

c Any vector $\vec{v}$ in $\mathbb{R}^2$ can be written uniquely as $\vec{v} = \vec{v}_1 + \vec{v}_2$, where $\vec{v}_1$ is in im(A) and $\vec{v}_2$ is in ker(A). (See Figure 3.3.) Then $A\vec{v} = A\vec{v}_1 + A\vec{v}_2 = \vec{v}_1(A\vec{v}_1 = \vec{v}_1$ by part b, $A\vec{v}_2 = \vec{0}$ since $\vec{v}_2$ is in ker(A)), so that A represents the *orthogonal projection* onto $\text{im}(A) = \text{span}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

3.1.**42** Using the hint, we see that the vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ is in the image of A if

$$\begin{array}{llllll} y_1 & & -3y_3 & +2y_4 & = 0 & \text{and} \\ & y_2 & -2y_3 & +y_4 & = 0. & \end{array}$$

This means that im(A) is the kernel of the matrix $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$.

3.1.**43** Using our work in Exercise 42 as a guide, we come up with the following procedure to express the image of an $n \times m$ matrix A as the kernel of a matrix B:

If rank$(A) = n$, let B be the $n \times n$ zero matrix.

If $r = \text{rank}(A) < n$, let B be the $(n-r) \times n$ matrix obtained by omitting the first r rows and the first m columns of rref$[A \vdots I_n]$.

3.1.**44** a Yes; by construction of the echelon form, the systems $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solutions (it is the whole point of Gaussian elimination not to change the solutions of a system).

b No; as a counterexample, consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, with im$(A) = \text{span}(\vec{e}_2)$, but $B = \text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, with im$(B) = \text{span}(\vec{e}_1)$.

3.1.**45** As we solve the system $A\vec{x} = \vec{0}$, we obtain r leading variables and $m - r$ free variables. The "general vector" in ker(A) can be written as a linear combination of $m - r$ vectors, with the free variables as coefficients. (See Example 11, where $m - r = 5 - 3 = 2$.)

3.1.**46** If $\text{rank}(A) = r$, then $\text{im}(A) = \text{span}(\vec{e}_1, \ldots, \vec{e}_r)$. See Figure 3.4.

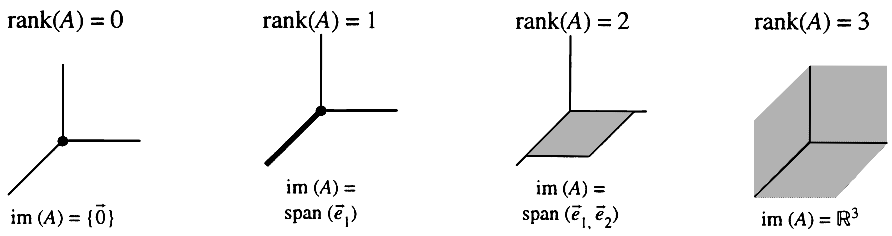


Figure 3.4: for Problem 3.1.46.

3.1.**47** $\text{im}(T) = L_2$ and $\ker(T) = L_1$.

3.1.**48** a $\vec{w} = A\vec{x}$, for some $\vec{x}$, so that $A\vec{w} = A^2\vec{x} = A\vec{x} = \vec{w}$.

b If $\text{rank}(A) = 2$, then A is invertible, and the equation $A^2 = A$ implies that $A = I_2$ (multiply by A^{-1}).

If $\text{rank}(A) = 0$ then $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

c First note that $\text{im}(A)$ and $\ker(A)$ are lines (there is one nonleading variable).

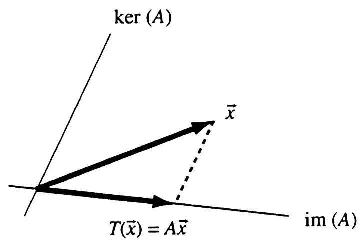


Figure 3.5: for Problem 3.1.48c.

By definition of a projection, we need to verify that $\vec{x} - A\vec{x}$ is in $\ker(A)$. This is indeed the case, since

$A(\vec{x} - A\vec{x}) = A\vec{x} - A^2\vec{x} = A\vec{x} - A\vec{x} = \vec{0}$ (we are told that $A^2 = A$). See Figure 3.5.

3.1.**49** If $\vec{v}$ and $\vec{w}$ are in $\ker(T)$, then $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) = \vec{0} + \vec{0} = \vec{0}$, so that $\vec{v} + \vec{w}$ is in $\ker(T)$ as well.

If $\vec{v}$ is in $\ker(T)$ and k is an arbitrary scalar, then $T(k\vec{v}) = kT(\vec{v}) = k\vec{0} = \vec{0}$, so that $k\vec{v}$ is in $\ker(T)$ as well.

3.1.**50** From Exercise 38 we know that $\ker(A^3) \subseteq \ker(A^4)$. Conversely, if $\vec{x}$ is in $\ker(A^4)$, then $A^4\vec{x} = A^3(A\vec{x}) = \vec{0}$, so that $A\vec{x}$ is in $\ker(A^3) = \ker(A^2)$, which implies that $A^2(A\vec{x}) = A^3\vec{x} = \vec{0}$, that is, $\vec{x}$ is in $\ker(A^3)$. We have shown that $\ker(A^3) = \ker(A^4)$.

3.1.**51** We need to find all $\vec{x}$ such that $AB\vec{x} = \vec{0}$. If $AB\vec{x} = \vec{0}$, then $B\vec{x}$ is in $\ker(A) = \{\vec{0}\}$, so that $B\vec{x} = \vec{0}$.

Since $\ker(B) = \{\vec{0}\}$, we can conclude that $\vec{x} = \vec{0}$. It follows that $\ker(AB) = \{\vec{0}\}$.

3.1.**52** Since $C\vec{x} = \begin{bmatrix} A \\ B \end{bmatrix} \vec{x} = \begin{bmatrix} A\vec{x} \\ B\vec{x} \end{bmatrix}$, we can conclude that $C\vec{x} = \vec{0}$ if (and only if) both $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$. It follows that $\ker(C)$ is the intersection of $\ker(A)$ and $\ker(B)$: $\ker(C) = \ker(A) \cap \ker(B)$.

3.1.**53** a Using the equation $1 + 1 = 0$ (or $-1 = 1$), we can write the general vector $\vec{x}$ in $\ker(H)$ as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} p+r+s \\ p+q+s \\ p+q+r \\ p \\ q \\ r \\ s \end{bmatrix}$$

$$= p \underset{\vec{v}_1}{\underset{\uparrow}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}} + q \underset{\vec{v}_2}{\underset{\uparrow}{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}} + r \underset{\vec{v}_3}{\underset{\uparrow}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}} + s \underset{\vec{v}_4}{\underset{\uparrow}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}}$$

b $\ker(H) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ by part (a), and $\text{im}(M) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ by Theorem 3.1.3, so that $\text{im}(M) = \ker(H)$. $M\vec{x}$ is in $\text{im}(M) = \ker(H)$, so that $H(M\vec{x}) = \vec{0}$.

3.1.**54** a If no error occurred, then $\vec{w} = \vec{v} = M\vec{u}$, and $H\vec{w} = H(M\vec{u}) = \vec{0}$, by Exercise 53b.

If an error occurred in the ith component, then $\vec{w} = \vec{v} + \vec{e}_i = M\vec{u} + \vec{e}_i$, so that

$H\vec{w} = H(M\vec{u}) + H\vec{e}_i = i$th column of H.

Since the columns of H are all different, this method allows us to find out where an error occurred.

b $H\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ = seventh column of H: an error occurred in the seventh component of $\vec{v}$.

Therefore $\vec{v} = \vec{w} + \vec{e}_7 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Section 3.2

3.2.**1** Not a subspace, since W does not contain the zero vector.

3.2.2 Not a subspace, since W contains the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ but not the vector $(-1)\vec{v} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$.

3.2.3 $W = \text{im} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a subspace of $\mathbb{R}^3$, by Theorem 3.2.2.

3.2.4 $\text{span}(\vec{v}_1, \ldots, \vec{v}_m) = \text{im}[\vec{v}_1 \ \ldots \ \vec{v}_m]$ is a subspace of $\mathbb{R}^n$, by Theorem 3.2.2.

3.2.5 We have subspaces $\{\vec{0}\}$, $\mathbb{R}^3$, and all lines and planes (through the origin). To prove this, mimic the reasoning in Example 2.

3.2.6 a Yes!

- The zero vector is in $V \cap W$, since $\vec{0}$ is in both V and W.
- If $\vec{x}$ and $\vec{y}$ are in $V \cap W$, then both $\vec{x}$ and $\vec{y}$ are in V, so that $\vec{x} + \vec{y}$ is in V as well, since V is a subspace of $\mathbb{R}^n$. Likewise, $\vec{x} + \vec{y}$ is in W, so that $\vec{x} + \vec{y}$ is in $V \cap W$.
- If $\vec{x}$ is in $V \cap W$ and k is an arbitrary scalar, then $k\vec{x}$ is in both V and W, since they are subspaces of $\mathbb{R}^n$. Therefore, $k\vec{x}$ is in $V \cap W$.

b No; as a counterexample consider $V = \text{span}(\vec{e}_1)$ and $W = \text{span}(\vec{e}_2)$ in $\mathbb{R}^2$.

3.2.7 Yes; we need to show that W contains the zero vector. We are told that W is nonempty, so that it contains some vector $\vec{v}$. Since W is closed under scalar multiplication, it will contain the vector $0\vec{v} = \vec{0}$, as claimed.

3.2.8 We need to solve the system $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The general solution is $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$.

Picking $t = 1$ we find the nontrivial relation $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

3.2.9 These vectors are linearly dependent, since $\vec{v}_m = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_{m-1}$.

3.2.10 Linearly dependent, since $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 7 \\ 11 \end{bmatrix}$. Thus, the vector $\vec{0}$ is redundant.

3.2.11 Linearly independent, since the two vectors are not parallel, and therefore not redundant.

3.2.12 Linearly dependent, since $\begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Thus, the vector $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ is redundant.

3.2.**13** Linearly dependent, since the second vector is redundant $\left(\begin{bmatrix}1\\2\end{bmatrix} = 1\begin{bmatrix}1\\2\end{bmatrix}\right)$.

3.2.**14** Certainly $\begin{bmatrix}3\\2\\1\end{bmatrix}$ is not a multiple of $\begin{bmatrix}1\\1\\1\end{bmatrix}$, so it is not redundant. However, $\begin{bmatrix}6\\5\\4\end{bmatrix} = 3\begin{bmatrix}1\\1\\1\end{bmatrix} + 1\begin{bmatrix}3\\2\\1\end{bmatrix}$, so $\begin{bmatrix}6\\5\\4\end{bmatrix}$ is redundant. Thus, these vectors are linearly dependent.

3.2.**15** Linearly dependent. By Theorem 3.2.8, since we have three vectors in $\mathbb{R}^2$, at least one must be redundant. We can perform a straightforward computation to reveal that $\vec{v}_3 = -\vec{v}_1 + 2\vec{v}_2$.

3.2.**16** Linearly independent, since ref $\begin{bmatrix}1 & 1 & 1\\0 & 2 & 2\\0 & 0 & 3\end{bmatrix} = I_3$ (use Theorem 3.2.6).

3.2.**17** Linearly independent. The first two vectors are clearly not redundant, and since rref $\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 3 & 6\end{bmatrix} = I_3$, the last vector is also not redundant. Thus, the three vectors turn out to be linearly independent.

3.2.**18** $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is redundant, simply because it is the zero vector.

$\begin{bmatrix}1\\0\\0\end{bmatrix}$ is our first non-zero vector, and thus, is not redundant.

$\begin{bmatrix}3\\0\\0\end{bmatrix} = 3\begin{bmatrix}1\\0\\0\end{bmatrix}$ and is redundant.

$\begin{bmatrix}0\\1\\0\end{bmatrix}$ is not a multiple of $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and is not redundant.

$\begin{bmatrix}4\\5\\0\end{bmatrix} = 4\begin{bmatrix}1\\0\\0\end{bmatrix} + 5\begin{bmatrix}0\\1\\0\end{bmatrix}$ and is redundant.

Similarly, $\begin{bmatrix}6\\7\\0\end{bmatrix} = 6\begin{bmatrix}1\\0\\0\end{bmatrix} + 7\begin{bmatrix}0\\1\\0\end{bmatrix}$ and is also redundant.

However, by inspection, $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is not a linear combination of $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$, meaning that this last vector is not redundant. Thus, the seven vectors are linearly dependent.

3.2.**19** Linearly dependent. First we see that $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ is not redundant, because it is first, and non-zero. However, $\begin{bmatrix}2\\0\\0\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, so it is redundant.

$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ are clearly not redundant, but $\begin{bmatrix}3\\4\\5\\0\end{bmatrix} = 3\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + 5\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$, so it is redundant.

3.2.**20** Linearly dependent, since rref $\begin{bmatrix}1&1&1\\1&2&4\\1&3&7\\1&4&10\end{bmatrix} = \begin{bmatrix}1&0&-2\\0&1&3\\0&0&0\\0&0&0\end{bmatrix}$. So, we find that the vector $\begin{bmatrix}1\\4\\7\\10\end{bmatrix}$ turns out to be redundant.

3.2.**21** Certainly, since the second vector equals the first, the second is redundant. So $\vec{v}_1 = \vec{v}_2$, $1\vec{v}_1 - 1\vec{v}_2 = \vec{0}$, revealing that $\begin{bmatrix}1\\-1\end{bmatrix}$ is in $\ker(A)$.

3.2.**22** $\begin{bmatrix}3\\6\end{bmatrix}$ is redundant, because $\begin{bmatrix}3\\6\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix}$. So, $3\begin{bmatrix}1\\2\end{bmatrix} - 1\begin{bmatrix}3\\6\end{bmatrix} = \vec{0}$. Thus, $\begin{bmatrix}3\\-1\end{bmatrix}$ is in the kernel of $\begin{bmatrix}1&3\\2&6\end{bmatrix}$.

3.2.**23** The first column is $\vec{0}$, so it is redundant. $1\vec{v}_1 = \vec{0}$, so $\begin{bmatrix}1\\0\end{bmatrix}$ is in $\ker(A)$.

3.2.**24** $\begin{bmatrix}2\\3\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix}$, so $\begin{bmatrix}2\\3\\0\end{bmatrix}$ is redundant. Now, $2\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} - 1\begin{bmatrix}2\\3\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\end{bmatrix} = \vec{0}$, revealing that $\begin{bmatrix}2\\3\\-1\\0\end{bmatrix}$ is in the kernel of $\begin{bmatrix}1&0&2&0\\0&1&3&0\\0&0&0&1\end{bmatrix}$.

3.2.**25** The third column equals the first, so it is redundant and $\vec{v}_1 = \vec{v}_3$, or $1\vec{v}_1 + 0\vec{v}_2 - 1\vec{v}_3 = \vec{0}$. Thus, $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ is in $\ker(A)$.

3.2.**26** $\begin{bmatrix}6\\5\\4\end{bmatrix}$ is redundant, because $\begin{bmatrix}6\\5\\4\end{bmatrix} = 3\begin{bmatrix}1\\1\\1\end{bmatrix} + 1\begin{bmatrix}3\\2\\1\end{bmatrix}$. Thus, $3\begin{bmatrix}1\\1\\1\end{bmatrix} + 1\begin{bmatrix}3\\2\\1\end{bmatrix} - 1\begin{bmatrix}6\\5\\4\end{bmatrix} = \vec{0}$ and $\begin{bmatrix}3\\1\\-1\end{bmatrix}$ is in the kernel of $\begin{bmatrix}1&3&6\\1&2&5\\1&1&4\end{bmatrix}$.

3.2.27 A basis of im(A) is $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\end{bmatrix}$, by Theorem 3.2.4.

3.2.28 $\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2)$

We can choose $\vec{e}_1, \vec{e}_2$ as a basis of im(A).

3.2.29 The three column vectors of A span all of $\mathbb{R}^2$, so that $\text{im}(A) = \mathbb{R}^2$. We can choose any two of the columns of A to form a basis of im(A); another sensible choice is $\vec{e}_1, \vec{e}_2$.

3.2.30 The three column vectors are linearly independent, since $\text{rref}\begin{bmatrix}1&1&1\\1&2&5\\1&3&7\end{bmatrix} = I_3$.

Therefore, the three columns form a basis of $\text{im}(A)(=\mathbb{R}^3)$:

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\5\\7\end{bmatrix}.$$

Another sensible choice for a basis of im(A) is $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

3.2.31 The two column vectors of the given matrix A are linearly independent (they are not parallel), so that they form a basis of im(A).

3.2.32 By inspection, the first, third and sixth columns are redundant. Thus, a basis of the image consists of the remaining column vectors: $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$.

3.2.33 $\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, so that $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is a basis of im(A).

3.2.34 The fact that $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ is in $\ker(A)$ means that

$$A\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4]\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4 = \vec{0}, \text{ so that } \vec{v}_4 = -\tfrac{1}{4}\vec{v}_1 - \tfrac{1}{2}\vec{v}_2 - \tfrac{3}{4}\vec{v}_3.$$

3.2.35 If $\vec{v}_i$ is a linear combination of the other vectors in the list, $\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_n\vec{v}_n$, then we can subtract $\vec{v}_i$ from both sides to generate a nontrivial relation (the coefficient of $\vec{v}_i$ will be -1).

Conversely, if there is a nontrivial relation $c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_n\vec{v}_n = \vec{0}$, with $c_i \neq 0$, then we can solve for vector $\vec{v}_i$ and thus express $\vec{v}_i$ as a linear combination of the other vectors in the list.

3.2.36 Yes; we know that there is a nontrivial relation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0}$.

Now apply the transformation T to the vectors on both sides, and use linearity:

$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m) = T(\vec{0})$, so that $c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_mT(\vec{v}_m) = \vec{0}$.

This is a nontrivial relation among the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_m)$, so that these vectors are linearly dependent, as claimed.

3.2.**37** No; as a counterexample, consider the extreme case when T is the zero transformation, that is, $T(\vec{x}) = \vec{0}$ for all $\vec{x}$. Then the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_m)$ will all be zero, so that they are linearly dependent.

3.2.**38** a Using the terminology introduced in the exercise, we need to show that any vector $\vec{v}$ in V is a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$. Choose a specific vector $\vec{v}$ in V. Since we can find no more than m linearly independent vectors in V, the $m+1$ vectors $\vec{v}_1, \ldots, \vec{v}_m$, $\vec{v}$ will be linearly dependent. Since the vectors $\vec{v}_1, \ldots, \vec{v}_m$ are independent, $\vec{v}$ must be redundant, meaning that $\vec{v}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$, as claimed.

b With the terminology introduced in part a, we can let $V = \operatorname{im}\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix}$.

3.2.**39** Yes; the vectors are linearly independent. The vectors in the list $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent (and therefore non-redundant), and $\vec{v}$ is non-redundant since it fails to be in the span of $\vec{v}_1, \ldots, \vec{v}_m$.

3.2.**40** Yes; by Theorem 3.2.8, $\ker(A) = \{\vec{0}\}$ and $\ker(B) = \{\vec{0}\}$. Then $\ker(AB) = \{\vec{0}\}$ by Exercise 3.1.51, so that the columns of AB are linearly independent, by Theorem 3.2.8.

3.2.**41** To show that the columns of B are linearly independent, we show that $\ker(B) = \{\vec{0}\}$. Indeed, if $B\vec{x} = \vec{0}$, then $AB\vec{x} = A\vec{0} = \vec{0}$, so that $\vec{x} = \vec{0}$ (since $AB = I_m$).

By Theorem 3.2.8, $\operatorname{rank}(B) = \#$ columns $= m$, so that $m \leq n$ and in fact $m < n$ (we are told that $m \neq n$). This implies that the rank of the $m \times n$ matrix A is less than n, so that the columns of A are linearly dependent (by Theorem 3.2.8).

3.2.**42** We can use the hint and form the dot product of $\vec{v}_i$ and both sides of the relation

$c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m = \vec{0}$:

$(c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m) \cdot \vec{v}_i = \vec{0} \cdot \vec{v}_i$, so that $c_1(\vec{v}_1 \cdot \vec{v}_i) + \cdots + c_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + c_m(\vec{v}_m \cdot \vec{v}_i) = 0$.

Since $\vec{v}_i$ is perpendicular to all the other $\vec{v}_j$, we will have $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $j \neq i$; since $\vec{v}_i$ is a unit vector, we will have $\vec{v}_i \cdot \vec{v}_i = 1$. Therefore, the equation above simplifies to $c_i = 0$.

Since this reasoning applies to all $i = 1, \ldots, m$, we have only the trivial relation among the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$, so that these vectors are linearly independent, as claimed.

3.2.**43** Consider a linear relation $c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) + c_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{0}$, or, $(c_1 + c_2 + c_3)\vec{v}_1 + (c_2 + c_3)\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Since there is only the trivial relation among the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, we must have $c_1 + c_2 + c_3 = c_2 + c_3 = c_3 = 0$, so that $c_3 = 0$ and then $c_2 = 0$ and then $c_1 = 0$, as claimed.

3.2.**44** Yes; this is a special case of Exercise 40 (recall that $\ker(A) = \{\vec{0}\}$, by Theorem 3.1.7b).

3.2.**45** Yes; if A is invertible, then $\ker(A) = \{\vec{0}\}$, so that the columns of A are linearly independent, by Theorem 3.2.8.

3.2.46 Solve the system $\begin{bmatrix} x_1 & + & 2x_2 & & & + & 3x_4 & + & 5x_5 & = 0 \\ & & & & x_3 & + & 4x_4 & + & 6x_5 & = 0 \end{bmatrix}$.

The solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s-3t-5r \\ s \\ -4t-6r \\ t \\ r \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}$ span the kernel, by construction, and they are linearly independent, by Theorem 3.2.5. Therefore, the three vectors form a basis of the kernel.

3.2.47 By Theorem 3.2.8, the rank of A is 3. Thus, $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

3.2.48 We can write $3x_1 + 4x_2 + 5x_3 = [3\ 4\ 5]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$, so that $V = \ker[3\ 4\ 5]$.

To express V as an image, choose a basis of V, for example, $\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -4 \end{bmatrix}$.

Then, $V = \text{im}\begin{bmatrix} 4 & 0 \\ -3 & 5 \\ 0 & -4 \end{bmatrix}$.

There are other solutions.

3.2.49 $L = \text{im}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

To write L as a kernel, think of L as the intersection of the planes $x = y$ and $y = z$, that is, as the solution set of the system $\begin{bmatrix} x & - & y & & & = 0 \\ & & y & - & z & = 0 \end{bmatrix}$.

Therefore, $L = \ker\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

There are other solutions.

3.2.50 The verification of the three properties listed in Definition 3.2.1 is straightforward. Alternatively, we can choose a basis $\vec{v}_1, \ldots, \vec{v}_p$ of V and a basis $\vec{w}_1, \ldots, \vec{w}_q$ of W (see Exercise 38a) and show that $V + W = \text{span}(\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q)$ (compare with Exercise 4).

Indeed, if $\vec{v}+\vec{w}$ is in $V+W$, then $\vec{v}$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_p$ and $\vec{w}$ is a linear combination of $\vec{w}_1, \dots, \vec{w}_q$, so that $\vec{v}+\vec{w}$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$. Conversely, if $\vec{x}$ is in $\text{span}(\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q)$, then $\vec{x} = (c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) + (d_1\vec{w}_1 + \cdots + d_q\vec{w}_q)$, so that $\vec{x}$ is in $V+W$.

If V and W are distinct lines in $\mathbb{R}^3$ (spanned by $\vec{v}$ and $\vec{w}$, respectively), then $V+W$ is the plane spanned by $\vec{v}$ and $\vec{w}$.

3.2.**51** a Consider a relation $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p + d_1\vec{w}_1 + \cdots + d_q\vec{w}_q = \vec{0}$.

Then the vector $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = -d_1\vec{w}_1 - \cdots - d_q\vec{w}_q$ is both in V and in W, so that this vector is $\vec{0}$: $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$ and $d_1\vec{w}_1 + \cdots + d_q\vec{w}_q = \vec{0}$.

Now the c_i are all zero (since the $\vec{v}_i$ are linearly independent) and the d_j are zero (since the $\vec{w}_j$ are linearly independent).

Since there is only the trivial relation among the vectors $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$, they are linearly independent.

b In Exercise 50 we show that $V+W = \text{span}(\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q)$, and in part (a) we show that these vectors are linearly independent.

3.2.**52** If a, c and f are nonzero, then $\text{rref}\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and the three vectors are linearly independent, by Theorem 3.2.6. If at least one of the constants a, c or f is zero, then at least one column of rref will not contain a leading one, so that the three vectors are linearly dependent.

3.2.**53** The zero vector is in $V^\perp$, since $\vec{0}\cdot\vec{v} = 0$ for all $\vec{v}$ in V.

If $\vec{w}_1$ and $\vec{w}_2$ are both in $V^\perp$, then $(\vec{w}_1+\vec{w}_2)\cdot\vec{v} = \vec{w}_1\cdot\vec{v} + \vec{w}_2\cdot\vec{v} = 0+0 = 0$ for all $\vec{v}$ in V, so that $\vec{w}_1+\vec{w}_2$ is in $V^\perp$ as well.

If $\vec{w}$ is in $V^\perp$ and k is an arbitrary constant, then $(k\vec{w})\cdot\vec{v} = k(\vec{w}\cdot\vec{v}) = k0 = 0$ for all $\vec{v}$ in V, so that $k\vec{w}$ is in $V^\perp$ as well.

3.2.**54** We need to find all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x+2y+3z = 0$.

These vectors have the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2s-3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

Therefore, $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ is a basis of $L^\perp$.

3.2.**55** We need to find all vectors $\vec{x}$ in $\mathbb{R}^5$ such that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = x_1+2x_2+3x_3+4x_4+5x_5 = 0$.

These vectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2a-3b-4c-5d \\ a \\ b \\ c \\ d \end{bmatrix} = a\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The four vectors to the right form a basis of $L^\perp$; they span $L^\perp$, by construction, and they are linearly independent, by Theorem 3.2.5.

3.2.**56** Consider a linear relation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$ among the four given vectors. The last component of the vector on the left hand side is c_3, so that $c_3 = 0$. Now the fifth component on the left is c_1, so that $c_1 = 0$. The third component is now c_4, so $c_4 = 0$. It follows that $c_2 = 0$ as well.

We have shown that there is only the trivial relation among the given vectors, so that they are linearly independent, regardless of the values of the constants $a, b \ldots, m$.

3.2.**57** We will begin to go through the possibilties for j until we see a pattern:

$j = 1$: Yes, because $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is in $\ker(A)$ (the first column is $\vec{0}$).

$j = 2$: No, this would just be a multiple of the second column, and only $\vec{0}$ if the jth component is zero.

$j = 3$: Yes, since $\begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is in $\ker(A)$.

At this point, we realize that we are choosing the redundant columns. Thus, j can also be 6 and 7, because $\begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \\ 5 \\ -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are in $\ker(A)$.

3.2.**58** This occurs for each column, j, that is redundant. If $\vec{x}$ is in the kernel, and the j^{th} component of $\vec{x}$ is the last non-zero component, then

$x_1\vec{v}_1 + \cdots + x_j\vec{v}_j + x_{j+1}\vec{v}_{j+1} + \cdots + x_m\vec{v}_m = \vec{0}$, but $x_{j+1} = \cdots = x_m = 0$, so

$x_1\vec{v}_1 + \cdots + x_j\vec{v}_j = \vec{0}$.

Thus, since $x_j \neq 0, \vec{v}_j = -\frac{x_1\vec{v}_1+\cdots+x_{j-1}\vec{v}_{j-1}}{x_j}$ and $\vec{v}_j$ is redundant. Conversely, if $\vec{v}_j$ is redundant, with $\vec{v}_j = c_1\vec{v}_1+\cdots+c_{j-1}\vec{v}_{j-1}$, then the vector $\vec{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_{j-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is in the kernel of A. The last non-zero component of $\vec{x}$ is the j^{th}, as required.

Section 3.3

3.3.1 Clearly the second column is just three time the first, and thus is redundant. Applying the notion of Kyle Numbers, we see:

$\begin{array}{c} \begin{matrix} 3 & -1 \end{matrix} \\ \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \end{array}$, so $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is in the ker(A). No other vectors belong in our list, so a basis of the kernel is $\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)$, and a basis of the image is $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$.

3.3.2 The first column is redundant. We use the following Kyle Numbers:

$\begin{array}{c} \begin{matrix} 1 & 0 \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \end{array}$, so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in the ker(A). No other vectors belong in our list, so a basis of the kernel is $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, and a basis of the image is $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$.

3.3.3 The two columns here are independent, so there are no redundant vectors. Thus, $\emptyset$ is a basis of the kernel, and the two columns form a basis of the image: $\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$.

3.3.4 Using Kyle Numbers, we see that the second column is redundant:

$\begin{array}{c} \begin{matrix} 4 & -1 \end{matrix} \\ \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \end{array}$, so $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is in the ker(A). No other vectors belong in our list, so a basis of the kernel is $\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right)$, and a basis of the image is $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$.

3.3.5 The first two vectors are non-redundant, but the third is a multiple of the first. We see:

$\begin{array}{c} \begin{matrix} 3 & 0 & -1 \end{matrix} \\ \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \end{array}$, so a basis of the kernel is $\left(\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}\right)$, and a basis of the image consists of the non-redundant columns, or $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix}\right)$.

3.3.6 The first two vectors are non-redundant, but the third is a combination of the first two:

$\begin{array}{c} \begin{matrix} 1 & 2 & -1 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix} \end{array}$, so a basis of the kernel is $\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right)$, and a basis of the image is $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$.

3.3.7 We immediately see fitting Kyle numbers for one relation:

$\begin{array}{c} \begin{matrix} 2 & -1 & 0 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \end{array}$. Now, since the second column is redundant, we remove it from further inspection and keep a zero above it:

$\begin{array}{c} \begin{matrix} & 0 & \end{matrix} \\ \begin{bmatrix} 1 & \cancel{2} & 3 \\ 1 & \cancel{2} & 4 \end{bmatrix} \end{array}$. However, in this case, there are no more redundant vectors. Thus, a basis of the kernel is $\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right)$, and a basis of the image is $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right)$.

3.3.8 The first column is redundant, and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is in the kernel:

$\begin{array}{c} \begin{matrix} 1 & 0 & 0 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \end{array}$. No other columns are redundant, however, meaning that a basis of the kernel is $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$, while a basis of the image is $\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$.

3.3.9 The second column is redundant, and we can choose Kyle numbers as follows:

$\begin{array}{c} \begin{matrix} 2 & -1 & 0 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \end{array}$, but the third column is non-redundant. Thus, a basis of the kernel is $\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right)$, while a basis of the image is $\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$.

3.3.10 Here the second column is redundant, with Kyle Numbers as:

$\begin{array}{c} \begin{matrix} 3 & 1 \end{matrix} \\ \begin{bmatrix} 1 & -3 \\ 2 & -6 \\ 3 & -9 \end{bmatrix} \end{array}$. This reveals a basis of our kernel as $\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$ and a basis of the image to be $\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$.

3.3.11 Here it is clear that only the third column is redundant, since it is equal to the first. Thus, a basis of the kernel is $\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$, and $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$ is a basis of the image.

3.3.**12** The first and the third columns are redundant, as the Kyle Numbers show us:

$\begin{array}{ccc} 1 & 0 & 0 \\ \multicolumn{3}{c}{\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}} \end{array}$, then $\begin{array}{ccc} 0 & 2 & -1 \\ \multicolumn{3}{c}{\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}} \end{array}$, so that a basis of the kernel is $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right)$. This leaves $([1])$ to be a basis of the image.

3.3.**13** Here we first see $\begin{array}{ccc} 2 & -1 & 0 \\ \multicolumn{3}{c}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \end{array}$, then $\begin{array}{ccc} 3 & 0 & -1 \\ \multicolumn{3}{c}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \end{array}$,

so both the second and third columns are redundant, and a basis of the kernel is $\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right)$. This leaves $([1])$ to be a basis of the image.

3.3.**14** The third vector is the only redundant vector here, shown by:

$\begin{array}{c} \begin{matrix} 0 & 1 & -1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$. No other columns are redundant, however, meaning that a basis of the kernel is $\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$, while a basis of the image is $\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

3.3.**15** We quickly find that the third column is redundant, with the Kyle numbers

$\begin{array}{c} \begin{matrix} 2 & 2 & -1 & 0 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$, then see that the fourth column is also redundant,

$\begin{array}{c} \begin{matrix} 0 & 0 & 0 & 1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$.

Thus, a basis of our kernel is $\left(\begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$, while $\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$ is a basis of our image.

3.3.**16** This matrix is already in rref, and we see that there are two columns without leading ones. These will be our redundant columns. Thus we see

$\begin{array}{c} \begin{matrix} 2 & 1 & 0 & 0 & 0 \end{matrix} \\ \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$, and $\begin{array}{c} \begin{matrix} -1 & 0 & 5 & -1 & 0 \end{matrix} \\ \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$.

Then $\left(\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\5\\-1\\0\end{bmatrix}\right)$ is a basis of the kernel, and $\left(\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$ is a basis of the image.

3.3.**17** For this problem, we again successively use Kyle Numbers to find our kernel, investigating the columns from left to right. We initially see that the first column is redundant:

$\begin{array}{c}\begin{matrix}1 & 0 & 0 & 0 & 0\end{matrix}\\ \begin{bmatrix}0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 4\end{bmatrix}\end{array}$, then the third column: $\begin{array}{c}\begin{matrix}0 & 2 & -1 & 0 & 0\end{matrix}\\ \begin{bmatrix}0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 4\end{bmatrix}\end{array}$, followed by the fifth column: $\begin{array}{c}\begin{matrix}0 & 3 & 0 & 4 & -1\end{matrix}\\ \begin{bmatrix}0 & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & 1 & 4\end{bmatrix}\end{array}$.

Thus, $\left(\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\2\\-1\\0\\0\end{bmatrix},\begin{bmatrix}0\\3\\0\\4\\-1\end{bmatrix}\right)$ is a basis of the kernel, and $\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)$ is a basis of the image.

3.3.**18** The third column is redundant, as we find with

$\begin{array}{c}\begin{matrix}3 & 2 & -1 & 0\end{matrix}\\ \begin{bmatrix}1 & 1 & 5 & 1\\ 0 & 1 & 2 & 2\\ 0 & 1 & 2 & 3\\ 0 & 1 & 2 & 4\end{bmatrix}\end{array}$. The fourth column, however, fails to be redundant.

Thus, a basis of our kernel is $\left(\begin{bmatrix}3\\2\\-1\\0\end{bmatrix}\right)$, while $\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right)$ is a basis of our image.

3.3.**19** We see that the third column is redundant, and choose Kyle numbers as follows:

$\begin{array}{c}\begin{matrix}5 & 4 & -1 & 0 & 0\end{matrix}\\ \begin{bmatrix}1 & 0 & 5 & 3 & 0\\ 0 & 1 & 4 & 2 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\end{array}$, then we see that the fourth column is also redundant,

$\begin{array}{c}\begin{matrix}3 & 2 & 0 & -1 & 0\end{matrix}\\ \begin{bmatrix}1 & 0 & 5 & 3 & 0\\ 0 & 1 & 4 & 2 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\end{array}$.

Thus, $\left(\begin{bmatrix}5\\4\\-1\\0\\0\end{bmatrix},\begin{bmatrix}3\\2\\0\\-1\\0\end{bmatrix}\right)$ is a basis of the kernel, and $\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right)$ is a basis of the image.

3.3.**20** Although this matrix is not quite in rref, we can still quickly see that columns 2, 3, and 5 are the redundant columns:

$$\begin{array}{c} \begin{matrix} 0 & 1 & 0 & 0 & 0 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad \begin{array}{c} \begin{matrix} 5 & 0 & -1 & 0 & 0 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad \begin{array}{c} \begin{matrix} -12 & 0 & 0 & 3 & -1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

So, $\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 0 \\ 0 \\ 3 \\ -1 \end{bmatrix}\right)$ is a basis of the kernel, and $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$ is a basis of the image.

3.3.**21** $\operatorname{rref}\begin{bmatrix} 1 & 3 & 9 \\ 4 & 5 & 8 \\ 7 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, which we can use to "spot" a vector in the kernel: $\begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$. Since the third column is the only redundant one, this forms a basis of the kernel, and implies that the third column of A is also redundant. Thus, a basis of $\operatorname{im}(A)$ is $\left(\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}\right)$.

3.3.**22** $\operatorname{rref}\begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$. It is clear that the third vector is redundant, and we quickly see that the vector $\begin{bmatrix} -6 \\ 5 \\ -1 \end{bmatrix}$ is in the kernel. Since this is the only redundant column, $\left(\begin{bmatrix} -6 \\ 5 \\ -1 \end{bmatrix}\right)$ is a basis of the kernel. Thus, a basis of $\operatorname{im}(A)$ is $\left(\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}\right)$.

3.3.**23** $\operatorname{rref}\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Using the method of Exercises 17 and 19, we find the kernel:

$$\begin{array}{c} \begin{matrix} 2 & -3 & -1 & 0 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \text{ then } \begin{array}{c} \begin{matrix} 4 & -1 & 0 & -1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

So a basis of $\ker(A)$ is $\left(\begin{bmatrix} 2 \\ -3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ -1 \end{bmatrix}\right)$. The non-redundant column vectors of A form a basis of $\operatorname{im}(A)$: $\left(\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ -1 \end{bmatrix}\right)$.

3.3.**24** $\text{rref}\begin{bmatrix} 4 & 8 & 1 & 1 & 6 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Here our kernel is the span of only one vector:

$\left(\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$, while a basis of the image of A is $\left(\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 10 \\ 0 \end{bmatrix}\right)$.

3.3.**25** $\text{rref}\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. We will emulate Exercise 23 to find the

kernel: $\begin{array}{c} \begin{matrix} 2 & -1 & 0 & 0 & 0 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$, then $\begin{array}{c} \begin{matrix} 5 & 0 & -1 & -1 & 0 \end{matrix} \\ \begin{bmatrix} 1 & \cancel{2} & 0 & 5 & 0 \\ 0 & \cancel{0} & 1 & -1 & 0 \\ 0 & \cancel{0} & 0 & 0 & 1 \\ 0 & \cancel{0} & 0 & 0 & 0 \end{bmatrix} \end{array}$.

So a basis of $\ker(A)$ is $\left(\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}\right)$ and a basis of $\text{im}(A)$ is $\left(\begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}\right)$.

3.3.**26** a We notice that each of the six matrices has two identical columns. In matrices C and L, the second column is identical to the third, so that $\ker(C) = \ker(L) = \text{span}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. In matrices H, T, X and Y, the first column is identical to the third, so that $\ker(H) = \ker(T) = \ker(X) = \ker(Y) = \text{span}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Thus, only L has the same kernel as C.

b We observe that each of the six matrices in the list has two identical rows. For example, the first and the last row of matrix C are identical, so that any vector $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ in $\text{im}(C)$ will satisfy the equation $y_1 = y_3$. We can conclude that $\text{im}(C) = \text{im}(H) = \text{im}(X) = \left\{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_1 = y_3\right\}$, $\text{im}(L) = \left\{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_1 = y_2\right\}$, and $\text{im}(T) = \text{im}(Y) = \left\{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_2 = y_3\right\}$.

c Our discussion in part *b* shows that the answer is matrix L.

3.3.**27** Form a 4×4 matrix A with the given vectors as its columns. We find that $\text{rref}(A) = I_4$, so that the vectors do indeed form a basis of $\mathbb{R}^4$, by Summary 3.3.10.

3.3.**28** Form a 4×4 matrix A with the given vectors as its columns. The matrix A reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k-29 \end{bmatrix}.$$

This matrix can be reduced further to I_4 if (and only if) $k - 29 \neq 0$, that is, if $k \neq 29$.

By Summary 3.3.10, the four given vectors form a basis of $\mathbb{R}^4$ unless $k = 29$.

3.3.**29** $x_1 = -\frac{3}{2}x_2 - \frac{1}{2}x_3$; let $x_2 = s$ and $x_3 = t$. Then the solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}.$$

Multiplying the two vectors by 2 to simplify, we obtain the basis $\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

3.3.**30** Proceeding as in Exercise 29, we find the basis $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

3.3.**31** Proceeding as in Exercise 29, we can find the following basis of V: $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Now let A be the 4×3 matrix with these three vectors as its columns. Then $\text{im}(A) = V$ by Theorem 3.1.3, and $\ker(A) = \{\vec{0}\}$ by Theorem 3.2.8, so that A does the job.

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.3.**32** We need to find all vectors $\vec{x}$ in $\mathbb{R}^4$ such that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 0$.

This amounts to solving the system $\begin{bmatrix} x_1 & & - & x_3 & + & x_4 & = & 0 \\ & x_2 & + & 2x_3 & + & 3x_4 & = & 0 \end{bmatrix}$, which in turn amounts to finding the kernel of $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

Using Kyle Numbers, we find the basis $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

3.3.**33** We can write $V = \ker(A)$, where A is the $1 \times n$ matrix $A = [c_1 \ c_2 \ \cdots \ c_n]$.

Since at least one of the c_i is nonzero, the rank of A is 1, so that $\dim(V) = \dim(\ker(A)) = n - \text{rank}(A) = n - 1$, by Theorem 3.3.7.

A "hyperplane" in $\mathbb{R}^2$ is a line, and a "hyperplane" in $\mathbb{R}^3$ is just a plane.

3.3.**34** We can write $V = \ker(A)$, where A is the $n \times m$ matrix with entries a_{ij}. Note that $\text{rank}(A) \leq n$. Therefore, $\dim(V) = \dim(\ker(A)) = m - \text{rank}(A) \geq m - n$, by Theorem 3.3.7.

3.3.**35** We need to find all vectors $\vec{x}$ in $\mathbb{R}^n$ such that $\vec{v} \cdot \vec{x} = 0$, or $v_1x_1 + v_2x_2 + \cdots + v_nx_n = 0$, where the v_i are the components of the vector $\vec{v}$. These vectors form a hyperplane in $\mathbb{R}^n$ (see Exercise 33), so that the dimension of the space is $n - 1$.

3.3.**36** No; if $\text{im}(A) = \ker(A)$ for an $n \times n$ matrix A, then $n = \dim(\ker(A)) + \dim(\text{im}(A)) = 2\dim(\text{im}(A))$, so that n is an even number.

3.3.**37** Since $\dim(\ker(A)) = 5 - \text{rank}(A)$, any 4×5 matrix with rank 2 will do; for example,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.3.**38** a The rank of a 3×5 matrix A is 0,1,2, or 3, so that $\dim(\ker(A)) = 5 - \text{rank}(A)$ is 2,3,4, or 5.

b The rank of a 7×4 matrix A is at most 4, so that $\dim(\text{im}(A)) = \text{rank}(A)$ is 0,1,2,3, or 4.

3.3.**39** Note that $\ker(C) \neq \{\vec{0}\}$, by Theorem 3.1.7a, and $\ker(C) \subseteq \ker(A)$. Therefore, $\ker(A) \neq \{\vec{0}\}$, so that A is not invertible.

3.3.**40** Substituting a point $(x, y) = (x_i, y_i)$ into the equation $c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 = 0$ produces a linear equation in the six unknows $c_1, ..., c_6$. Fitting a conic through m points $P_i\,(x_i, y_i)$, for $i = 1, ..., m$, amounts to solving a system of m homogenous linear equations in six unknowns. This system can be written in matrix form as $A\vec{x} = \vec{0}$, where A is an $m \times 6$ matrix. The i^{th} row of A is $\begin{bmatrix} 1 & x_i & y_i & x_i^2 & x_iy_i & y_i^2 \end{bmatrix}$.

3.3.**41** The kernel of a 4×6 matrix is at least two-dimensional. Since every one-dimensional subspace of this kernel defines a conic through the four given points, there will be infinitely many such conics.

3.3.**42** The kernel of a 5×6 matrix is at least one-dimensional, so that there is at least one conic passing through five given points. Exercise 1.2.54 provides an example with exactly one solution, while there are infinitely many solutions in Exercise 1.2.56.

3.3.**43** Here, "anything can happen":

1. If $\ker A = \left\{\vec{0}\right\}$, then there is no solution, as in Exercise 1.2.61.
2. If $\dim\ker A = 1$, then there is a unique solution. For example, the only conic through the six points $(0,0), (1,0), (2,0), (3,0), (0,1), (0,2)$ is $xy = 0$.
3. If $\dim\ker A > 1$, then there are infinitely many solutions. For example, any conic consisting of the x axis and some other line runs through the six points
$(0,0), (1,0), (2,0), (3,0), (4,0), (5,0)$.

3.3.**44** The conic runs through the point $(0,0)$ if $c_1 = 0$.

Now the conic runs through (1,0) and (2,0) if $c_2 + c_4 + c_7 = 0$ and $2c_2 + 4c_4 + 8c_7 = 0$. We find that $c_2 = 2c_7$ and $c_4 = -3c_7$.

Likewise, the conic runs through (0,1) and (0,2) if $c_3 = 2c_{10}$ and $c_6 = -3c_{10}$.

It follows that the equation of the cubic is of the given form.

3.3.**45** The conic runs through the point $(0,0)$ if $c_1 = 0$.

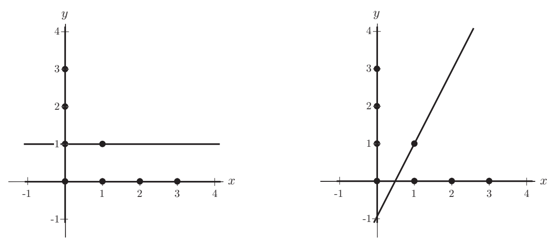
Now the conic runs through (1,0), (2,0), and (3,0) if $c_2+c_4+c_7 = 0$, $2c_2+4c_4+8c_7 = 0$, and $3c_2+9c_4+27c_7 = 0$. Solving this system, we find that $c_2 = c_4 = c_7 = 0$.

Likewise, the conic runs through (0,1) (0,2), and (0,3) if $c_3 = c_6 = c_{10} = 0$.

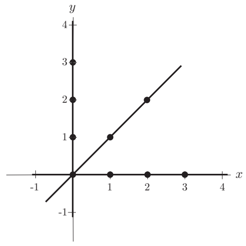
It follows that the equation of the cubic is of the given form.

Each such cubic is the union of the x axis, the y axis, and an arbitrary line through the origin.

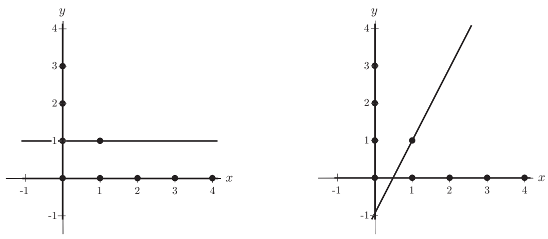
3.3.**46** To run through the points $(0,0), (1,0), (2,0)$, and $(3,0)$, the cubic must satisfy the equations $c_1 = 0$, $c_1 + c_2 + c_4 + c_7 = 0$, $c_1 + 2c_2 + 4c_4 + 8c_7 = 0$, and $c_1 + 3c_2 + 9c_4 + 27c_7 = 0$. This means that $c_1 = c_2 = c_4 = c_7 = 0$. Likewise, the cubic runs through the points $(0,0), (0,1), (0,2)$ and $(0,3)$ if (and only if) $c_1 = c_3 = c_6 = c_{10} = 0$. Therefore, to run through the first seven given points, the equation must be of the form $c_5xy + c_8x^2y + c_9xy^2 = 0$. The last point, $(1,1)$ imposes the condition $c_5 + c_8 + c_9 = 0$, or $c_5 = -c_8 - c_9$. Setting $c_8 = a$ and $c_9 = b$, we find the family of cubics $(-a-b)\,xy + ax^2y + bxy^2 = xy\,(ax + by - a - b) = 0$, where $a \neq 0$ or $b \neq 0$. Such a cubic is the union of the x axis, the y axis, and some line through the point $(1,1)$. Two sample solutions are shown in the accompanying figures.



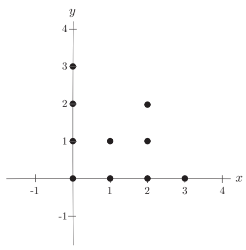
3.3.**47** To run through the points $(0,0), (1,0), (2,0)$, and $(3,0)$, the cubic must satisfy the equations $c_1 = 0$, $c_1 + c_2 + c_4 + c_7 = 0$, $c_1 + 2c_2 + 4c_4 + 8c_7 = 0$, and $c_1 + 3c_2 + 9c_4 + 27c_7 = 0$. This means that $c_1 = c_2 = c_4 = c_7 = 0$. Likewise, the cubic runs through the points $(0,0), (0,1), (0,2)$ and $(0,3)$ if (and only if) $c_1 = c_3 = c_6 = c_{10} = 0$. Therefore, to run through the first seven given points, the equation must be of the form $c_5xy + c_8x^2y + c_9xy^2 = 0$. The last two point, $(1,1)$ and $(2,2)$, impose the conditions $c_5 + c_8 + c_9 = 0$ and $4c_5 + 8c_8 + 8c_9 = 0$, implying that $c_5 = 0$ and $c_8 = -c_9$. Letting $c_9 = 1$, we find that there is one such cubic, $-x^2y + xy^2 = xy\,(y - x) = 0$, the union of the x axis, the y axis, and the line $y = x$.



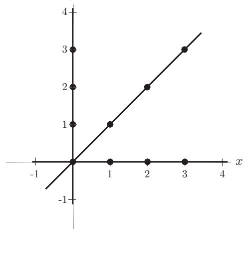
3.3.**48** Here we add the point $(4,0)$ to the list of points in Exercise 46. In Exercise 46 we found the family of cubics $xy\,(ax+by-a-b)=0$, where $a\neq 0$ or $b\neq 0$. Since all these cubics run through the point $(4,0)$, we find the same solutions here: $xy\,(ax+by-a-b)=0$, where $a\neq 0$ or $b\neq 0$. Two sample solutions are shown in the accompanying figures.



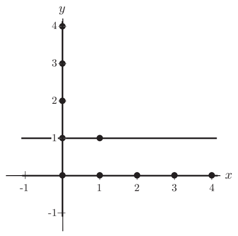
3.3.**49** Here we add the point $(2,1)$ to the list of points in Exercise 47. In Exercise 47 we found the cubic $xy\,(y-x)=0$. Since this cubic fails to run through the point $(2,1)$, there are no solutions here.

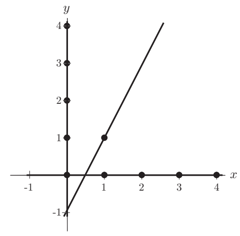


3.3.**50** Here we add the point $(3,3)$ to the list of points in Exercise 47. In Exercise 47 we found the cubic $xy\,(y-x)=0$. Since this cubic runs though the point $(3,3)$, we get the same solution here, $xy\,(y-x)=0$.

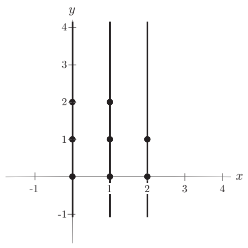


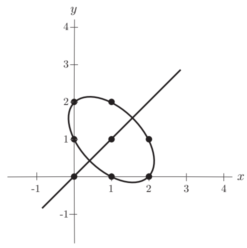
3.3.**51** Here we add the point $(0,4)$ to the list of points in Exercise 48. In Exercise 48 we found the family of cubics $xy\,(ax+by-a-b)=0$, where $a\neq 0$ or $b\neq 0$. Since all these cubics run through the point $(0,4)$, we find the same solutions here: $xy\,(ax+by-a-b)=0$, where $a\neq 0$ or $b\neq 0$. Each such cubic is the union of the x axis, the y axis, and some line through the point (1,1). Two sample solutions are shown in the accompanying figures.



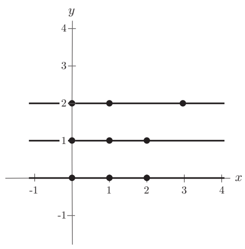


3.3.**52** To run through the points $(0,0)\,,(1,0)\,,$ and $(2,0)$, the cubic must satisfy the equations $c_1=0,\quad c_1+c_2+c_4+c_7=0,$ and $\quad c_1+2c_2+4c_4+8c_7=0$. This means that $c_1=0,\ c_2=2c_7$ and $c_4=-3c_7$. Likewise, the cubic runs through the points $(0,0)\,,(0,1)\,,$ and $(0,2)$ if (and only if) $c_1=0,\ c_3=2c_{10}$ and $c_6=-3c_{10}$. Therefore, to run through the points $(0,0),\ (1,0),\ (2,0),\ (0,1),\ (0,2)$ the cubic must be of the form $c_5xy+c_7\left(x^3-3x^2+2x\right)+c_8x^2y+c_9xy^2+c_{10}\left(y^3-3y^2+2y\right)=0$. The three additional points, $(1,1)\,,\ (2,1)\,,$ and $(1,2)$ impose the conditions $c_5+c_8+c_9=0,\ 2c_5+4c_8+2c_9=0$ and $2c_5+2c_8+4c_9=0$, implying that $c_5=c_8=c_9=0$. Letting $c_7=a$ and $c_{10}=b$, we find the family of cubics $a\left(x^3-3x^2+2x\right)+b\left(y^3-3y^2+2y\right)=0$, where $a\neq 0$ or $b\neq 0$. Two sample solutions are shown in the accompanying figures.

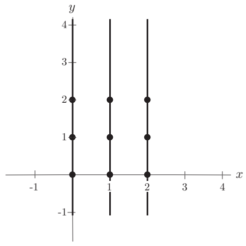


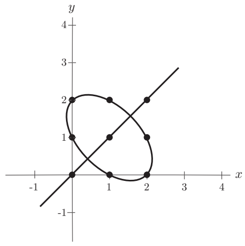


3.3.**53** Here we add the point $(3,2)$ to the list of points in Exercise 52. In Exercise 52 we found the family of cubics $a\left(x^3-3x^2+2x\right)+b\left(y^3-3y^2+2y\right)=0$, where $a\neq 0$ or $b\neq 0$. The point $(3,2)$ now imposed the condition $a=0$, so that the only solution here is $y^3-3y^2+2y=0$, the union of the lines $y=0,\ y=1$, and $y=2$.

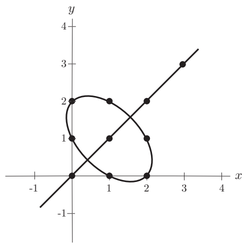


3.3.**54** Here we add the point $(2,2)$ to the list of points in Exercise 52. In Exercise 52 we found the family of cubics $a\left(x^3-3x^2+2x\right)+b\left(y^3-3y^2+2y\right)=0$, where $a\neq 0$ or $b\neq 0$. Since all these cubics run through the point $(2,2)$, we find the same solutions here: $a\left(x^3-3x^2+2x\right)+b\left(y^3-3y^2+2y\right)=0$, where $a\neq 0$ or $b\neq 0$. Two sample solutions are shown in the accompanying figures.





3.3.**55** Here we add the point $(3,3)$ to the list of points in Exercise 54 In Exercise 54 we found the family of cubics $a\left(x^3-3x^2+2x\right)+b\left(y^3-3y^2+2y\right)=0$, where $a \neq 0$ or $b \neq 0$. The point $(3,3)$ now imposes the condition $a=-b$, so that the only cubic here (with $b=1$) is $-2x+2y+3x^2-3y^2-x^3+y^3=0$, or $x(x-1)(x-2)=y(y-1)(y-2)$.



3.3.**56** Substituting a point $(x,y)=(x_i,y_i)$ into the equation $c_1+c_2x+...+c_9xy^2+c_{10}y^3=0$ produces a linear equation in the ten unknows $c_1,...,c_{10}$. Fitting a cubic through m points $P_i\left(x_i,y_i\right)$, for $i=1,...,m$, amounts to solving a system of m homogenous linear equations in ten unknowns. This system can be written in matrix form as $A\vec{x}=\vec{0}$, where A is an $m\times 10$ matrix. The i^{th} row of A is $\begin{bmatrix} 1 & x_i & y_i & x_i^2 & x_iy_i & y_i^2 & x_i^3 & x_i^2y_i & x_iy_i^2 & y_i^3 \end{bmatrix}$.

3.3.**57** The kernel of a 8×10 matrix is at least two-dimensional. Since every one-dimensional subspace of this kernel defines a cubic through the eight given points, there will be infinitely many such cubics.

3.3.**58** The kernel of a 9×10 matrix is at least one-dimensional, so that there is at least one cubic passing through nine given points. Exercise 53 provides an example with exactly one solution, while there are infinitely many solutions in Exercise 54.

3.3.**57** Here, "anything can happen":

If $\ker A=\left\{\vec{0}\right\}$, then there is no solution, as in Exercise 49

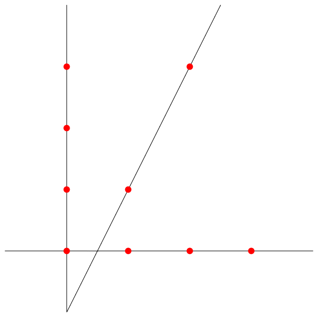
If $\dim\ker A=1$, then there is a unique solution, as in Exercise 50

If $\dim\ker A>1$, then there are infinitely many solutions, as in Exercise 51

3.3.**60** It is not true that "Any nine distinct points determine a unique cubic," as illustrated in Exercises 48 and 54, although "most" sets of nine distinct points do determine a unique cubic.

3.3.**61** Here we add the point P to the list of points in Exercise 46. In Exercise 46 we found the family of cubics $xy(ax+by-a-b)=0$, where $a\neq 0$ or $b\neq 0$. As long as we choose a point P on the x or on the y axis (with

$y = 0$ or $x = 0$), all these cubics will run through P, so that there are infinitely many cubics through the nine given points. (Another, "cheap" solution is $P = (1, 1)$). However, if P is neither $(1, 1)$ nor located on one of the axes, then the only cubic through the nine points is the union of the two axes and the line through $(1, 1)$ and P.



3.3.**62** We can choose a basis $\vec{v}_1, \dots, \vec{v}_p$ in V, where $p = \dim(V)$. Then $\vec{v}_1, \dots, \vec{v}_p$ are linearly independent vectors in W, so that $\dim(V) = p \leq \dim(W)$, by Theorem 3.3.4a, as claimed.

3.3.**63** We can choose a basis $\vec{v}_1, \dots, \vec{v}_p$ of V, where $p = \dim(V) = \dim(W)$. Then $\vec{v}_1, \dots, \vec{v}_p$ is a basis of W as well, by Theorem 3.3.4c, so that $V = W = \text{span}(\vec{v}_1, \dots, \vec{v}_p)$, as claimed.

3.3.**64** Consider a basis $\vec{v}_1, \dots, \vec{v}_n$ of V. Since the $\vec{v}_i$ are n linearly independent vectors in $\mathbb{R}^n$, they form a basis of $\mathbb{R}^n$ (by parts (vii) and (ix) of Summary 3.3.10), so that $V = \text{span}(\vec{v}_1, \dots, \vec{v}_n) = \mathbb{R}^n$, as claimed. (Note that Exercise 3.3.64 is a special case of Exercise 3.3.63.)

3.3.**65** $\dim(V + W) = \dim(V) + \dim(W)$, by Exercise 3.2.51b.

3.3.**66** Suppose that $V \cap W = \{\vec{0}\}$ and $\dim(V) + \dim(W) = n$.

Choose a basis $\vec{v}_1, \dots, \vec{v}_p$ of V and a basis $\vec{w}_1, \dots, \vec{w}_q$ in W; note that $p + q = n$. By Exercise 3.2.51b, the n vectors $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ in $\mathbb{R}^n$ are linearly independent, so that they form a basis of $\mathbb{R}^n$ (by parts (vii) and (ix) of Summary 3.3.10). By Theorem 3.2.10, any vector $\vec{x}$ can be written uniquely as

$\vec{x} = (c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) + (d_1\vec{w}_1 + \cdots + d_q\vec{w}_q)$, with $\vec{v} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$ in V and $\vec{w} = d_1\vec{w}_1 + \cdots + d_q\vec{w}_q$ in W, which gives the desired representation.

Conversely, suppose V and W are complements. Let us first show that $V \cap W = \{\vec{0}\}$ in this case.

Indeed, if $\vec{x}$ is in $V \cap W$, then we can write

$$\vec{x} = \underset{\substack{\uparrow \\ \text{in} \\ V}}{\vec{x}} + \underset{\substack{\uparrow \\ \text{in} \\ W}}{\vec{0}} = \underset{\substack{\uparrow \\ \text{in} \\ V}}{\vec{0}} + \underset{\substack{\uparrow \\ \text{in} \\ W}}{\vec{x}}$$

Since this representation is unique (by definition of complements), we must have $\vec{x} = \vec{0}$, so that $V \cap W = \{\vec{0}\}$. By definition of complements, we have $\mathbb{R}^n = V + W$, so that $n = \dim(V + W) = \dim(V) + \dim(W)$, by Exercise 65.

3.3.**67** Note that $\text{im}(A) = \text{span}(\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q) = V$, since the $\vec{w}_j$ alone span V.

To find a basis of $V = \text{im}(A)$, we omit the redundant vectors from the list $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots \vec{w}_q$, by Theorem 3.2.4. Since the vectors $\vec{v}_1, \dots, \vec{v}_p$ are linearly independent, none of them are redundant, so that our basis of V contains all vectors $\vec{v}_1, \dots, \vec{v}_p$ and some of the vectors from the list $\vec{w}_1, \dots, \vec{w}_q$.

3.3.68 Use Exercise 67 with $\vec{v}_1 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\4\\6\\8\end{bmatrix}$, and $\vec{w}_i = \vec{e}_i$ for $i = 1, 2, 3, 4$.

Now $\text{rref}\begin{bmatrix}1&1&1&0&0&0\\2&4&0&1&0&0\\3&6&0&0&1&0\\4&8&0&0&0&1\end{bmatrix} = \begin{bmatrix}1&0&2&0&0&-\frac{1}{4}\\0&1&-1&0&0&\frac{1}{4}\\0&0&0&1&0&-\frac{1}{2}\\0&0&0&0&1&-\frac{3}{4}\end{bmatrix}$.

Picking the non-redundant columns gives the basis $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\4\\6\\8\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$.

3.3.69 Using the terminology suggested in the hint, we need to show that

$$\vec{u}_1, \ldots, \vec{u}_m,\ \vec{v}_1, \ldots, \vec{v}_p,\ \vec{w}_1, \ldots, \vec{w}_q$$

is a basis of $V + W$. Then $\dim(V+W) + \dim(V \cap W) = (m+p+q)+m = (m+p)+(m+q) = \dim(V)+\dim(W)$, as claimed. Any vector $\vec{x}$ in $V + W$ can be written as $\vec{x} = \vec{v} + \vec{w}$, where $\vec{v}$ is in V and $\vec{w}$ is in W. Since $\vec{v}$ is a linear combination of the $\vec{u}_i$ and the $\vec{v}_j$, and $\vec{w}$ is a linear combination of the $\vec{u}_i$ and $\vec{w}_j$, $\vec{x}$ will be a linear combination of the $\vec{u}_i, \vec{v}_j$, and $\vec{w}_k$; this shows that the vectors $\vec{u}_1, \ldots, \vec{u}_m,\ \vec{v}_1, \ldots, \vec{v}_p,\ \vec{w}_1, \ldots, \vec{w}_q$ span $V + W$.

To show linear independence, consider the relation $a_1\vec{u}_1 + \cdots + a_m\vec{u}_m + b_1\vec{v}_1 + \cdots + b_p\vec{v}_p + c_1\vec{w}_1 + \cdots + c_q\vec{w}_q = \vec{0}$. Then the vector $a_1\vec{u}_1 + \cdots + a_m\vec{u}_m + b_1\vec{v}_1 + \cdots + b_p\vec{v}_p = -c_1\vec{w}_1 - \cdots - c_q\vec{w}_q$ is in $V \cap W$, so that it can be expressed uniquely as a linear combination of $\vec{u}_1, \ldots, \vec{u}_m$ alone; this implies that the b_i are all zero. Now our relation simplifies to $a_1\vec{u}_1 + \cdots + a_m\vec{u}_m + c_1\vec{w}_1 + \cdots + c_q\vec{w}_q = \vec{0}$, which implies that the a_i and the c_j are zero as well (since the vectors $\vec{u}_1, \ldots, \vec{u}_m, \vec{w}_1, \ldots, \vec{w}_q$ are linearly independent).

3.3.70 By Exercise 3.3.69, $\dim(V \cap W) = \dim(V) + \dim(W) - \dim(V + W) = 13 - \dim(V + W)$.

The dimension of $V + W$ is at least 7 (since $W \subseteq V + W$) and at most 10 (since $V + W \subseteq \mathbb{R}^{10}$); therefore the dimension of $V \cap W$ is at least 3 and at most 6.

3.3.71 The nonzero rows of E span the row space, and they are linearly independent (consider the leading ones), so that they form a basis of the row space: $[0\ 1\ 0\ 2\ 0]$, $[0\ 0\ 1\ 3\ 0]$, $[0\ 0\ 0\ 0\ 1]$.

3.3.72 As in Exercise 3.3.71, we observe that the nonzero rows of E form a basis of the row space, so that dim(row space of E) = rank(E).

3.3.73 a All elementary row operations leave the row space unchanged, so that A and rref(A) have the same row space.

b By part (a) and Exercise 3.3.72, dim(row space of A) = dim(row space of rref(A)) = rank(rref(A)) = rank(A).

3.3.74 $\text{rref}(A) = \begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{bmatrix}$

By Exercises 3.3.72 and 3.3.73a, $[1\ 0\ -1\ -2]$, $[0\ 1\ 2\ 3]$ is a basis of the row space of A.

3.3.**75** Using the terminology suggested in the hint, we observe that the vectors $\vec{v}, A\vec{v}, \dots, A^n\vec{v}$ are linearly dependent (by Theorem 3.2.8), so that there is a nontrivial relation $c_0\vec{v} + c_1A\vec{v} + \cdots + c_nA^n\vec{v} = \vec{0}$.

We can rewrite this relation in the form $(c_0I_n + c_1A + \cdots + c_nA^n)\vec{v} = \vec{0}$.

The nonzero vector $\vec{v}$ is in the kernel of the matrix $c_0I_n + c_1A + \cdots + c_nA^n$, so that this matrix fails to be invertible.

3.3.**76** We can use the approach outlined in Exercise 3.3.75, with $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, say.

Then $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $A^2\vec{v} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

We find the relation $5\vec{v} - 2A\vec{v} + A^2\vec{v} = \vec{0}$, so that the matrix $5I_2 - 2A + A^2$ does the job.

3.3.**77** If $\text{rank}(A) = n$, then the n non-redundant columns of A form a basis of $\text{im}(A) = \mathbb{R}^n$, so that the matrix formed by the non-redundant columns is invertible (by Summary 3.3.10).

Conversely, if A has an invertible $n \times n$ submatrix B, then the columns of B form a basis of $\mathbb{R}^n$ (again by Summary 3.3.10), so that $\text{im}(A) = \mathbb{R}^n$ and therefore $\text{rank}(A) = \dim(\text{im}(A)) = n$.

3.3.**78** Using the terminology suggested in the Exercise, we multiply the relation $c_0\vec{v} + c_1A\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$ with A^{m-1} and obtain $c_0A^{m-1}\vec{v} = \vec{0}$ (all other terms vanish since $A^m = 0$).

Since the vector $A^{m-1}\vec{v}$ is nonzero (by construction), the scalar c_0 must be zero, and our relation simplifies to $c_1A\vec{v} + c_2A^2\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$.

Now we multiply both sides with A^{m-2} and obtain $c_1A^{m-1}\vec{v} = \vec{0}$, so that $c_1 = 0$ as above. Continuing like this we conclude that all the c_i must be zero, as claimed.

3.3.**79** As in Exercise 78, let m be the smallest positive integer such that $A^m = 0$. In Exercise 78 we construct m linearly independent vectors $\vec{v}, A\vec{v}, \dots, A^{m-1}\vec{v}$ in $\mathbb{R}^n$; now $m \leq n$ by Theorem 3.2.8.

Therefore $A^n = A^mA^{n-m} = 0A^{n-m} = 0$, as claimed.

3.3.**80** If the vectors $\vec{w}_1, \dots, \vec{w}_q$ span an m-dimensional space V (with basis $\vec{v}_1, \dots, \vec{v}_m$), then $m \leq q$ by Theorem 3.3.1 (since the vectors $\vec{v}_i$ are linearly independent).

3.3.**81** Prove Theorem 3.3.4d: If m vectors $\vec{v}_1, \dots, \vec{v}_m$ span an m-dimensional space V, then they form a basis of V. We need to show that the vectors $\vec{v}_i$ are linearly independent. We will argue indirectly, assuming that the vectors are linearly dependent; this means that at least one of the vectors $\vec{v}_i$ is redundant, say $\vec{v}_p$.

But then $V = \text{span}(\vec{v}_1, \dots, \vec{v}_p, \dots, \vec{v}_m) = \text{span}(\vec{v}_1, \dots, \vec{v}_{p-1}, \vec{v}_{p+1}, \dots, \vec{v}_m)$, contradicting Theorem 3.3.4b.

3.3.**82** $\text{im}(A)$ is the plane onto which we project, so that $\text{rank}(A) = \dim(\text{im}(A)) = 2$.

3.3.**83** a Note that $\text{rank}(B) \leq 2$, so that $\dim(\ker(B)) = 5 - \text{rank}(B) \geq 3$ and $\dim(\ker(AB)) \geq 3$ since $\ker(B) \subseteq \ker(AB)$. Since $\ker(AB)$ is an subspace of $\mathbb{R}^5$, $\dim(\ker(AB))$ could be 3,4, or 5. It is easy to give an example for each case; for example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\dim(\ker(AB)) = 3$.

b Since $\dim(\text{im}(AB)) = 5 - \dim(\ker(AB))$, the possible values of $\dim(\text{im}(AB))$ are 0,1, and 2, by part a.

3.3.**84** Write $A = [\vec{v}_1 \ \dots \ \vec{v}_m]$ and $B = [\vec{w}_1 \ \dots \ \vec{w}_m]$, so that $A+B = [\vec{v}_1 + \vec{w}_1 \ \cdots \vec{v}_m + \vec{w}_m]$. Any linear combination of the columns of $A + B$, $\vec{y} = c_1(\vec{v}_1 + \vec{w}_1) + \cdots + c_m(\vec{v}_m + \vec{w}_m)$, can be written as

$$\vec{y} = \underbrace{(c_1\vec{v} + \cdots + c_m\vec{v}_m)}_{\text{in im}(A)} + \underbrace{(c_1\vec{w}_1 + \cdots + c_m\vec{w}_m)}_{\text{in im}(B)}$$

so that $\text{im}(A + B) \subseteq \text{im}(A) + \text{im}(B)$ (see Exercise 3.2.50). Since $\dim(V + W) \leq \dim(V) + \dim(W)$, by Exercise 3.3.69, we can conclude that $\text{rank}(A+B) = \dim(\text{im}(A+B)) \leq \dim(\text{im}(A)) + \dim(\text{im}(B)) = \text{rank}(A) + \text{rank}(B)$.

Summary: $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

3.3.**85** a By Exercise 3.1.39b, $\text{im}(AB) \subseteq \text{im}(A)$, and therefore $\text{rank}(AB) \leq \text{rank}(A)$.

b Write $B = [\vec{v}_1 \ \cdots \ \vec{v}_m]$ and $AB = [A\vec{v}_1 \ \cdots \ A\vec{v}_m]$. If $r = \text{rank}(B)$, then the r non-redundant columns of B will span $\text{im}(B)$, and the corresponding r columns of AB will span $\text{im}(AB)$, by linearity of A. By Theorem 3.3.4b, $\text{rank}(AB) = \dim(\text{im}(AB)) \leq r = \text{rank}(B)$.

Summary: $\text{rank}(AB) \leq \text{rank}(A)$, and $\text{rank}(AB) \leq \text{rank}(B)$.

3.3.**86** Same answer as Exercise 3.3.87.

3.3.**87** Let $\vec{v}_1, \dots, \vec{v}_6$ be the columns of matrix A. Following the hint, we observe that $\vec{v}_5 = 4\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_4$, which gives the relation $4\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_4 - \vec{v}_5 = \vec{0}$. Thus the vector

$$\vec{x} = \begin{bmatrix} 4 \\ 5 \\ 0 \\ 6 \\ -1 \\ 0 \end{bmatrix}$$

is in the kernel of matrix A. Since $\vec{x}$ fails to be in the kernel of matrix B, the two kernels are different, as claimed.

3.3.**88** We will freely use the terminology introduced in the hint. First we need to show that at least one of the column vectors $\vec{a}_k$ and $\vec{b}_k$ fails to contain a leading 1. If $\text{rank}[\vec{a}_1 \ \cdots \ \vec{a}_{k-1}] = \text{rank}\left[\vec{b}_1 \ \cdots \ \vec{b}_{k-1}\right] = r$, and if $\vec{a}_k$ contains a leading 1, then $\vec{a}_k$ is the standard vector $\vec{e}_{r+1}$; likewise for $\vec{b}_k$. Since $\vec{a}_k$ and $\vec{b}_k$ are different vectors, they cannot both contain a leading 1. Without loss of generality, we can assume that $\vec{a}_k$ fails to contain a leading 1, so that $\vec{a}_k$ is redundant: We can write $\vec{a}_k = c_1\vec{a}_1 + \cdots + c_{k-1}\vec{a}_{k-1}$. Then the vector

$\vec{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_{k-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is in the kernel of A. We will show that $\vec{x}$ fails to be in the kernel of matrix B, so that $\ker(A) \neq \ker(B)$, as claimed. Indeed, $B\vec{x} = c_1\vec{b}_1 + \cdots + c_{k-1}\vec{b}_{k-1} - \vec{b}_k = c_1\vec{a}_1 + \cdots + c_{k-1}\vec{a}_{k-1} - \vec{b}_k = \vec{a}_k - \vec{b}_k \neq \vec{0}$. We have used the fact that the first $k-1$ columns of B are identical to those of A, while $\vec{b}_k \neq \vec{a}_k$.

3.3.**89** Exercise 88 shows that if two matrices A and B of the same size are both in rref and have the same kernel, then $A = B$. Apply this fact to A and $B = \text{rref}(M)$.

3.3.**90** We have $T(\vec{x}) = \vec{v} \times \vec{x} = \vec{0}$ if (and only if) the vector $\vec{x}$ is parallel to $\vec{v}$ (by Definition A.9). This means that $\ker(T) = \text{span}(\vec{v})$, a line.

By the rank-nullity theorem (Theorem 3.3.7), $\dim(\text{im}T) = 3 - \dim(\ker T) = 2$, so that the image of T is a plane. Since $T(\vec{x}) = \vec{v} \times \vec{x}$ is perpendicular to $\vec{v}$, for all $\vec{x}$ (by Definition A.9), $\text{im}(T)$ is the plane of all vectors perpendicular to $\vec{v}$.

Section 3.4

3.4.**1** $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

3.4.**2** $\begin{bmatrix} 23 \\ 29 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 46 \\ 58 \end{bmatrix} + 0\begin{bmatrix} 61 \\ 67 \end{bmatrix}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$.

3.4.**3** $\begin{bmatrix} 31 \\ 37 \end{bmatrix} = 0\begin{bmatrix} 23 \\ 29 \end{bmatrix} + 1\begin{bmatrix} 31 \\ 37 \end{bmatrix}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

3.4.**4** $\begin{bmatrix} 3 \\ -4 \end{bmatrix} = -4\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

3.4.**5** $\begin{bmatrix} 7 \\ 16 \end{bmatrix} = -4\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 5 \\ 12 \end{bmatrix}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

This may not be as obvious as Exercises 1 and 3, but we can find our coefficients simply by reducing the matrix $\left[\begin{array}{cc|c} 2 & 5 & 7 \\ 5 & 12 & 16 \end{array}\right]$.

3.4.**6** We need to find the scalars c_1 and c_2 such that $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Attempting to solve the linear system reveals an inconsistency; $\vec{x}$ is not in the span of $\vec{v}_1$ and $\vec{v}_2$.

3.4.**7** We need to find the scalars c_1 and c_2 such that $\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Solving a linear system gives $c_1 = 3,\ c_2 = 4$. Thus $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

3.4.**8** $\begin{bmatrix} -4 \\ 4 \end{bmatrix} = 11 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$.

We arrive at this solution by reducing the matrix $\left[\begin{array}{cc|c} 1 & 5 & -4 \\ 2 & 6 & 4 \end{array}\right]$.

3.4.**9** We can solve this by inspection: Note that our first coefficient must be 3 because of the first terms of the vectors. Also, the second coefficient must be 2 due to the last terms.

However, $3\vec{v}_1 + 2\vec{v}_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$. Thus, we reason that $\vec{x}$ is not in the span of $\vec{v}_1$ and $\vec{v}_2$.

We can also see this by attempting to solve $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 1 & -1 & 3 \\ 0 & 2 & 4 \end{array}\right]$, which turns out to be inconsistent. Thus, $\vec{x}$ is not in V.

3.4.**10** Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

3.4.**11** Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

3.4.**12** Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

3.4.**13** Here, we quickly see that since $x_1 = 1 = 1c_1 + 0c_2 + 0c_3$, c_1 must equal 1. We find $c_2 = -1$ similarly, since $x_2 = 1 = 2(1) + 1c_2 + 0c_3$. Finally, now that $x_3 = 3(1) + 2(-1) + 1c_3$, c_3 must be zero.

So $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

3.4.**14** We reduce $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 6 & 3 \end{array}\right]$ to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 21 \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & 8 \end{array}\right]$,

revealing that $\vec{x} = 21\vec{v}_1 - 22\vec{v}_2 + 8\vec{v}_3$, and $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 21 \\ -22 \\ 8 \end{bmatrix}$.

3.4.**15** This may be a bit too difficult to do by inspection. Instead we reduce $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 0 \\ 1 & 4 & 8 & 0 \end{array}\right]$ to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & 5 \end{array}\right]$, revealing that $\vec{x} = 8\vec{v}_1 - 12\vec{v}_2 + 5\vec{v}_3$, and $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -12 \\ 5 \end{bmatrix}$.

3.4.**16** We proceed by inspection here, noting that we need $c_1 = 3$, then see that c_2 must be 4. Finally, c_3 must be 6.

Thus, $\begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$.

3.4.**17** By inspection, we see that in order for $\vec{x}$ to be in V, $\vec{x} = 1\vec{v}_1 + 1\vec{v}_2 - 1\vec{v}_3$ (by paying attention to the first, second, and fourth terms). Now we need to verify that the third terms "work out". So, $1(2)+1(3)-1(4) = 5-4 = 1 = x_3$.

Thus $\vec{x}$ is in V, and $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

3.4.**18** Here, $\vec{x}$ is not in V, as we find an inconsistency while attempting to solve the system.

3.4.**19** a $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and we find the inverse S^{-1} to be equal to $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Then $B = S^{-1}AS = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

b Our commutative diagram:

$$\begin{array}{ccc}
\vec{x} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} & \xrightarrow[T]{} & \begin{aligned} T(\vec{x}) &= A\vec{x} = c_1 A\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 A\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} - c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned} \\
\downarrow & & \downarrow \\
[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow[B]{} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix}
\end{array}$$

So, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix}$, and we quickly find $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

c $B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}] = \left[\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]_{\mathcal{B}}\right] = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{\mathcal{B}}\right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

3.4.**20** a $S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, and we find the inverse S^{-1} to be equal to $\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

Then $B = S^{-1}AS = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 & 10 \\ 10 & -5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 25 & 0 \\ 0 & -25 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$.

b Our commutative diagram:

$$\begin{tikzcd}
\vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \arrow[r, "T"] \arrow[d] & T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 A \begin{bmatrix} -2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ 10 \end{bmatrix} + c_2 \begin{bmatrix} 10 \\ -5 \end{bmatrix} = 5c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 5c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \arrow[d] \\
[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \arrow[r, "B"] & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 5c_1 \\ -5c_2 \end{bmatrix}
\end{tikzcd}$$

So, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5c_1 \\ -5c_2 \end{bmatrix}$, and we quickly find $B = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$.

c $B = [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}] = \left[\left[\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right]$

$= \left[\begin{bmatrix} 5 \\ 10 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 10 \\ -5 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$.

3.4.**21** a $S = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$, and we find the inverse S^{-1} to be equal to $\frac{1}{7} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$.

Then $B = S^{-1}AS = \frac{1}{7} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7 & 14 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 49 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$.

b Our commutative diagram:

$$\begin{tikzcd}
\vec{x} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \arrow[r, "T"] \arrow[d] & T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 A \begin{bmatrix} -2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 7 \\ 21 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 7c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \arrow[d] \\
[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \arrow[r, "B"] & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 7c_1 \\ 0 \end{bmatrix}
\end{tikzcd}$$

So, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7c_1 \\ 0 \end{bmatrix}$, and we quickly find $B = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$.

c $B = [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}] = \left[\left[\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right]$

$= \left[\begin{bmatrix} 7 \\ 21 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$.

3.4.**22** a $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and we find the inverse S^{-1} to be equal to $\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Then $B = S^{-1}AS = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

b Our commutative diagram:

$$\begin{array}{ccc}
\vec{x} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} & \xrightarrow[T]{} & T(\vec{x}) = A\vec{x} = c_1A\begin{bmatrix}1\\1\end{bmatrix} + c_2A\begin{bmatrix}1\\-1\end{bmatrix} \\
 & & = c_1\begin{bmatrix}2\\2\end{bmatrix} + c_2\begin{bmatrix}0\\0\end{bmatrix} = 2c_1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}1\\-1\end{bmatrix} \\
\downarrow & & \downarrow \\
[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix} & \xrightarrow[B]{} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix}2c_1\\0\end{bmatrix}
\end{array}$$

So, $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2c_1\\0\end{bmatrix}$, and we quickly find $B = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$.

c $B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}] = \begin{bmatrix}\left[\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right]_{\mathcal{B}} & \left[\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\right]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}_{\mathcal{B}} & \begin{bmatrix}0\\0\end{bmatrix}_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$.

3.4.**23** a $S = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$, and we find the inverse S^{-1} to be equal to $\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$.

Then $B = S^{-1}AS = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}5 & -3\\6 & -4\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} = \begin{bmatrix}4 & -2\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}$.

b Our commutative diagram:

$$\begin{array}{ccc}
\vec{x} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix} & \xrightarrow[T]{} & T(\vec{x}) = A\vec{x} = c_1A\begin{bmatrix}1\\1\end{bmatrix} + c_2A\begin{bmatrix}1\\2\end{bmatrix} \\
 & & = c_1\begin{bmatrix}2\\2\end{bmatrix} + c_2\begin{bmatrix}-1\\-2\end{bmatrix} = 2c_1\begin{bmatrix}1\\1\end{bmatrix} - 1c_2\begin{bmatrix}1\\2\end{bmatrix} \\
\downarrow & & \downarrow \\
[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix} & \xrightarrow[B]{} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix}2c_1\\-c_2\end{bmatrix}
\end{array}$$

So, $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2c_1\\-c_2\end{bmatrix}$, and we quickly find $B = \begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}$.

c $B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}] = \begin{bmatrix}\left[\begin{bmatrix}5 & -3\\6 & -4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right]_{\mathcal{B}} & \left[\begin{bmatrix}5 & -3\\6 & -4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\right]_{\mathcal{B}}\end{bmatrix}$

$= \begin{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}_{\mathcal{B}} & \begin{bmatrix}-1\\-2\end{bmatrix}_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}$.

3.4.**24** a $S = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$, and we find the inverse S^{-1} to be equal to $\begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$.

Then $B = S^{-1}AS = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}\begin{bmatrix}13 & -20\\6 & -9\end{bmatrix}\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix} = \begin{bmatrix}9 & -15\\-1 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix} = \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}$.

b Our commutative diagram:

$$\begin{tikzcd}
\vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} \arrow[r, "T"] \arrow[d] & T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 A \begin{bmatrix} 5 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 6 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 3c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1c_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} \arrow[d] \\
{[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} \arrow[r, "B"] & {[T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 3c_1 \\ c_2 \end{bmatrix}}
\end{tikzcd}$$

So, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3c_1 \\ c_2 \end{bmatrix}$, and we quickly find $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

c $B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}] = \left[\left[\begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[\begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right]_{\mathcal{B}} \right]$

$= \left[\begin{bmatrix} 6 \\ 3 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

3.4.**25** We will use the commutative diagram method here (though any method suffices).

$$\begin{tikzcd}
\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \arrow[r, "T"] \arrow[d] & T(\vec{x}) = A\vec{x} = c_1 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 11 \end{bmatrix} = c_1 \left(-1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) + c_2 \left(-1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 6 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = (-c_1 - c_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (4c_1 + 6c_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \arrow[d] \\
{[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} \arrow[r, "B"] & {[T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} -c_1 - c_2 \\ 4c_1 + 6c_2 \end{bmatrix}}
\end{tikzcd}$$

$B \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 \\ 4c_1 + 6c_2 \end{bmatrix}$, so $B = \begin{bmatrix} -1 & -1 \\ 4 & 6 \end{bmatrix}$.

3.4.**26** Let's build B "column-by-column":

$B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}]$

$= \left[\left[\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right]$

$= \left[\begin{bmatrix} 2 \\ 8 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 1 \\ 5 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 6 & 4 \\ -4 & -3 \end{bmatrix}$.

3.4.**27** We use a commutative diagram:

$$\vec{x} = c_1\begin{bmatrix}2\\1\\-2\end{bmatrix} + c_2\begin{bmatrix}0\\2\\1\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\end{bmatrix} \quad \xrightarrow[T]{} \quad T(\vec{x}) = A\vec{x}$$

$$= c_1A\begin{bmatrix}2\\1\\-2\end{bmatrix} + c_2A\begin{bmatrix}0\\2\\1\end{bmatrix} + c_3A\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$= c_1\begin{bmatrix}18\\9\\-18\end{bmatrix} + \vec{0} + \vec{0} = 9c_1\begin{bmatrix}2\\1\\-2\end{bmatrix}$$

$$\downarrow \qquad\qquad \downarrow$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} \quad \xrightarrow[B]{} \quad [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix}9c_1\\0\\0\end{bmatrix}$$

$$B\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}9c_1\\0\\0\end{bmatrix}, \text{ so } B = \begin{bmatrix}9&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

3.4.**28** Let's build B "column-by-column":

$$B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}[T(\vec{v}_3)]_{\mathcal{B}}]$$

$$= \left[\left[\begin{bmatrix}5&-4&-2\\-4&5&-2\\-2&-2&8\end{bmatrix}\begin{bmatrix}2\\2\\1\end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix}5&-4&-2\\-4&5&-2\\-2&-2&8\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix}5&-4&-2\\-4&5&-2\\-2&-2&8\end{bmatrix}\begin{bmatrix}0\\1\\-2\end{bmatrix}\right]_{\mathcal{B}}\right]$$

$$= \left[\begin{bmatrix}0\\0\\0\end{bmatrix}_{\mathcal{B}} \begin{bmatrix}9\\-9\\0\end{bmatrix}_{\mathcal{B}} \begin{bmatrix}0\\9\\-18\end{bmatrix}_{\mathcal{B}}\right] = \begin{bmatrix}0&0&0\\0&9&0\\0&0&9\end{bmatrix}.$$

3.4.**29** Let's build B "column-by-column":

$$B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}[T(\vec{v}_3)]_{\mathcal{B}}]$$

$$= \left[\left[\begin{bmatrix}-1&1&0\\0&-2&2\\3&-9&6\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix}-1&1&0\\0&-2&2\\3&-9&6\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix}-1&1&0\\0&-2&2\\3&-9&6\end{bmatrix}\begin{bmatrix}1\\3\\6\end{bmatrix}\right]_{\mathcal{B}}\right]$$

$$= \left[\begin{bmatrix}0\\0\\0\end{bmatrix}_{\mathcal{B}} \begin{bmatrix}1\\2\\3\end{bmatrix}_{\mathcal{B}} \begin{bmatrix}2\\6\\12\end{bmatrix}_{\mathcal{B}}\right] = \begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}.$$

3.4.**30** Let's build B "column-by-column":

$$B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}[T(\vec{v}_3)]_{\mathcal{B}}]$$

$$= \left[\left[\begin{bmatrix}0&2&-1\\2&-1&0\\4&-4&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix}0&2&-1\\2&-1&0\\4&-4&1\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix}0&2&-1\\2&-1&0\\4&-4&1\end{bmatrix}\begin{bmatrix}1\\2\\4\end{bmatrix}\right]_{\mathcal{B}}\right]$$

$$= \left[\begin{bmatrix}1\\1\\1\end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix}0\\-1\\-2\end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix}0\\0\\0\end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix}.$$

3.4.**31** We can use a commutative diagram to see how this works:

$$\begin{array}{ccc}
\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 & \xrightarrow{T} & T(\vec{x}) = \vec{v}_2 \times \vec{x} = c_1(\vec{v}_2 \times \vec{v}_1) + c_2(\vec{v}_2 \times \vec{v}_2) + c_3(\vec{v}_2 \times \vec{v}_3) \\
 & & = c_1(-\vec{v}_3) + c_2(\vec{0}) + c_3(\vec{v}_1) = c_3\vec{v}_1 - c_1\vec{v}_3 \\
\downarrow & & \downarrow \\
[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} & \xrightarrow{B} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix}c_3\\0\\-c_1\end{bmatrix}
\end{array}$$

$B\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}c_3\\0\\-c_1\end{bmatrix}$, so $B = \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\-1 & 0 & 0\end{bmatrix}$.

3.4.**32** Here we will build B column-by-column:

$B = [\,[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}\,]$

$= [\,[\vec{v}_1 \times \vec{v}_3]_{\mathcal{B}} \quad [\vec{v}_2 \times \vec{v}_3]_{\mathcal{B}} \quad [\vec{v}_3 \times \vec{v}_3]_{\mathcal{B}}\,] = [\,[-\vec{v}_2]_{\mathcal{B}} \quad [\vec{v}_1]_{\mathcal{B}} \quad \vec{0}\,]$, since all three are perpendicular unit vectors.

So, $B = \begin{bmatrix}0 & 1 & 0\\-1 & 0 & 0\\0 & 0 & 0\end{bmatrix}$.

3.4.**33** Here we will build B column-by-column:

$B = [\,[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}\,]$

$= [\,[(\vec{v}_2 \cdot \vec{v}_1)\vec{v}_2]_{\mathcal{B}} \quad [(\vec{v}_2 \cdot \vec{v}_2)\vec{v}_2]_{\mathcal{B}} \quad [(\vec{v}_2 \cdot \vec{v}_3)\vec{v}_2]_{\mathcal{B}}\,] = [\,\vec{0} \quad [1\vec{v}_2]_{\mathcal{B}} \quad \vec{0}\,]$, since all three are perpendicular unit vectors.

So, $B = \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$.

3.4.**34** Here we will build B column-by-column:

$B = [\,[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}\,]$

$= [\,[\vec{v}_1 - 2(\vec{v}_3 \cdot \vec{v}_1)\vec{v}_3]_{\mathcal{B}} \quad [\vec{v}_2 - 2(\vec{v}_3 \cdot \vec{v}_2)\vec{v}_3]_{\mathcal{B}} \quad [\vec{v}_3 - 2(\vec{v}_3 \cdot \vec{v}_3)\vec{v}_3]_{\mathcal{B}}\,] = [\,[\vec{v}_1]_{\mathcal{B}} \quad [\vec{v}_2]_{\mathcal{B}} \quad [-\vec{v}_3]_{\mathcal{B}}\,]$.

So, $B = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{bmatrix}$. This is the reflection about the plane spanned by $\vec{v}_1$ and $\vec{v}_2$.

3.4.**35** Using another commutative diagram:

$$\begin{array}{ccc}
\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 & \xrightarrow{T} & \begin{aligned} T(\vec{x}) &= c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) \\ &= c_1(\vec{v}_1 - 2(\vec{v}_1\cdot\vec{v}_1)\vec{v}_2) + c_2(\vec{v}_2 - 2(\vec{v}_1\cdot\vec{v}_2)\vec{v}_2) + \\ &\quad c_3(\vec{v}_3 - 2(\vec{v}_1\cdot\vec{v}_3)\vec{v}_2) \\ &= c_1(\vec{v}_1 - 2\vec{v}_2) + c_2(\vec{v}_2 - \vec{0}) + c_3(\vec{v}_3 - \vec{0}) \\ &= c_1\vec{v}_1 + (-2c_1 + c_2)\vec{v}_2 + c_3\vec{v}_3 \end{aligned} \\
\downarrow & & \downarrow \\
[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} & \xrightarrow{B} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ -2c_1 + c_2 \\ c_3 \end{bmatrix}
\end{array}$$

So $B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This is a shear along the second term.

3.4.**36** Here we will build B column-by-column:

$B = [\,[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}\,]$

$= [\,[\vec{v}_1 \times \vec{v}_1 + (\vec{v}_1\cdot\vec{v}_1)\vec{v}_1]_{\mathcal{B}} \quad [\vec{v}_1 \times \vec{v}_2 + (\vec{v}_1\cdot\vec{v}_2)\vec{v}_1]_{\mathcal{B}} \quad [\vec{v}_1 \times \vec{v}_3 + (\vec{v}_1\cdot\vec{v}_3)\vec{v}_1]_{\mathcal{B}}\,]$

$= [\,[\vec{v}_1]_{\mathcal{B}} \quad [\vec{v}_3]_{\mathcal{B}} \quad [-\vec{v}_2]_{\mathcal{B}}\,]$.

So, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. This is a 90-degree rotation about the line spanned by $\vec{v}_1$. The rotation is counterclockwise when looking from the positive $\vec{v}_1$ direction.

3.4.**37** By Theorem 3.4.7, we want a basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$ such that $T(\vec{v}_1) = a\vec{v}_1$ and $T(\vec{v}_2) = b\vec{v}_2$ for some scalars a and b. Then the $\mathcal{B}$-matrix of T will be $B = [\,[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}}\,] = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, which is a diagonal matrix as required. Note that $T(\vec{v}) = \vec{v} = 1\vec{v}$ for vectors parallel to the line L onto which we project, and $T(\vec{w}) = \vec{0} = 0\vec{w}$ for vectors perpendicular to L. Thus, we can pick a basis where $\vec{v}_1$ is parallel to L and $\vec{v}_2$ is perpendicular, for example, $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

3.4.**38** By Theorem 3.4.7, we want a basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$ such that $T(\vec{v}_1) = a\vec{v}_1$ and $T(\vec{v}_2) = b\vec{v}_2$ for some scalars a and b. Then the $\mathcal{B}$-matrix of T will be $B = [\,[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}}\,] = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, which is a diagonal matrix as required. Note that $T(\vec{v}) = \vec{v} = 1\vec{v}$ for vectors parallel to the line L about which we reflect, and $T(\vec{w}) = -\vec{w} = (-1)\vec{w}$ for vectors perpendicular to L. Thus, we can pick a basis where $\vec{v}_1$ is parallel to L and $\vec{v}_2$ is perpendicular, for example, $\mathcal{B} = \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right)$.

3.4.**39** Using the same approach as in Exercise 37, we want a basis, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ such that $T(\vec{v}_1) = a\vec{v}_1, T(\vec{v}_2) = b\vec{v}_2$ and $T(\vec{v}_3) = c\vec{v}_3$. First we see that if $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $T(\vec{v}_1) = \vec{v}_1$. Next we notice that if $\vec{v}_2$ and $\vec{v}_3$ are perpendicular to $\vec{v}_1$, then $T(\vec{v}_2) = -\vec{v}_2$ and $T(\vec{v}_3) = -\vec{v}_3$. So we can pick $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, for example.

3.4.**40** From Exercise 37, we see that we want one of our basis vectors to be parallel to the line, while the others must be perpendicular the line. We can easily find such a basis: $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$.

3.4.**41** We will use the same approach as in Exercises 37 and 39. Any basis with 2 vectors in the plane and one perpendicular to it will work nicely here! So, let $\vec{v}_1, \vec{v}_2$ be in the plane. $\vec{v}_1$ can be $\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$, and $\vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ (note that these must be independent). Then $\vec{v}_3$ should be perpendicular to the plane. We will use $\vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$—the coefficient vector. This is perpendicular to the plane because all vectors perpendicular to $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ lie in the plane.

So, our basis is: $\left(\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right)$.

3.4.**42** From Exercise 38, we deduce that one of our vectors should be perpendicular to this plane, while two should fall inside it. Finding the perpendicular is not difficult: we simply take the coefficient vector: $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$. Then we add two linearly independent vectors on the plane, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, for instance. These three vectors form one possible basis.

3.4.**43** By definition of coordinates (Definition 3.4.1), $\vec{x} = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + (-3) \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$.

3.4.**44** By definition of coordinates, $\vec{x} = 2 \begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \\ -1 \end{bmatrix}$.

3.4.**45** If $\vec{v}_1, \vec{v}_2$ is a basis with the desired property, then $\vec{x} = 2\vec{v}_1 + 3\vec{v}_2$, or $\vec{v}_2 = \frac{1}{3}\vec{x} - \frac{2}{3}\vec{v}_1$. Thus we can make $\vec{v}_1$ any vector in the plane that is not parallel to $\vec{x}$, and then let $\vec{v}_2 = \frac{1}{3}\vec{x} - \frac{2}{3}\vec{v}_1$. For example, if we choose $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$, then $\vec{v}_2 = \frac{1}{3} \begin{bmatrix} -4 \\ -4 \\ -1 \end{bmatrix}$.

3.4.**46** As in Exercise 3.4.45, we can make $\vec{v}_1$ any vector in the plane that is not parallel to $\vec{x}$, and then let $\vec{v}_2 = 2\vec{v}_1 - \vec{x}$. For example, if we choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, then $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$.

3.4.**47** By Theorem 3.4.4, we have $A = SBS^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$.

3.4.**48** $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ means that $\vec{x} = -\vec{v} + 2\vec{w}$. See Figure 3.6

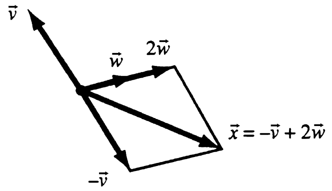


Figure 3.6: for Problem 3.4.48.

3.4.**49** $\vec{u} + \vec{v} = -\vec{w}$, so that $\vec{w} = -\vec{u} - \vec{v}$, i.e., $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

3.4.**50** a $\overrightarrow{OP} = \vec{w} + 2\vec{v}$, so that $[\overrightarrow{OP}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\overrightarrow{OQ} = \vec{v} + 2\vec{w}$, so that $[\overrightarrow{OQ}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

b $\overrightarrow{OR} = 3\vec{v} + 2\vec{w}$. See Figure 3.7.

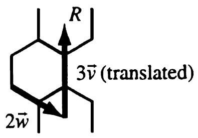


Figure 3.7: for Problem 3.4.50.

c If the tip of $\vec{u}$ is a vertex, then so is the tip of $\vec{u} + 3\vec{v}$ and also the tip of $\vec{u} + 3\vec{w}$ (draw a sketch!). We know that the tip P of $2\vec{v} + \vec{w}$ is a vertex (see part a.). Therefore, the tip S of $\overrightarrow{OS} = 17\vec{v} + 13\vec{w} = (2\vec{v} + \vec{w}) + 5(3\vec{v}) + 4(3\vec{w})$ is a vertex as well.

3.4.**51** Let $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m)$. Then, let $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m$ and $\vec{y} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_m\vec{v}_m$. Then $[\vec{x}+\vec{y}]_{\mathcal{B}} = [a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m + b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_m\vec{v}_m]_{\mathcal{B}} = [(a_1+b_1)\vec{v}_1 + (a_2+b_2)\vec{v}_2 + \cdots + (a_m+b_m)\vec{v}_m]_{\mathcal{B}}$

$$= \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_m + b_m \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$$

3.4.**52** Yes; $T(\vec{x}) = [\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}$, so that T is "given by a matrix." (See Definition 2.1.1.)

3.4.**53** By Definition 3.4.1, we have $\vec{x} = S[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 40 \\ 58 \end{bmatrix}$.

3.4.54 Let Q be the matrix whose columns are the vectors of the basis $\mathcal{T}$. Then $[[\vec{v}_1]_\mathcal{T} \dots [\vec{v}_n]_\mathcal{T}] = [Q^{-1}\vec{v}_1 \dots Q^{-1}\vec{v}_n] = Q^{-1}[\vec{v}_1 \dots \vec{v}_n]$ is an invertible matrix, so that the vectors $[\vec{v}_1]_\mathcal{T} \dots [\vec{v}_n]_\mathcal{T}$ form a basis of $\mathbb{R}^n$.

3.4.55 By Definition 3.4.1, we have $\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} [\vec{x}]_\mathcal{B}$ and $\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} [\vec{x}]_\mathcal{R}$, so that $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} [\vec{x}]_\mathcal{B} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} [\vec{x}]_\mathcal{R}$ and $[\vec{x}]_\mathcal{R} = \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{P} [\vec{x}]_\mathcal{B}$, i.e., $P = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$.

3.4.56 Let $S = [\vec{v}_1 \vec{v}_2]$ where $\vec{v}_1, \vec{v}_2$ is the desired basis. Then by Theorem 3.4.1, $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = S \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = S \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, i.e. $S \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. Hence $S = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 12 & -7 \\ 14 & -8 \end{bmatrix}$. The desired basis is $\begin{bmatrix} 12 \\ 14 \end{bmatrix}, \begin{bmatrix} -7 \\ -8 \end{bmatrix}$.

3.4.57 If we can find a basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ such that the $\mathcal{B}$-matrix of A is

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, then A must be similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Because of the entries in the matrix B, it is required that $A\vec{v}_1 = \vec{v}_1, A\vec{v}_2 = \vec{v}_2$ and $A\vec{v}_3 = -\vec{v}_3$. So, all we need for our basis is to pick independent $\vec{v}_1, \vec{v}_2$ in the plane, and $\vec{v}_3$ perpendicular to the plane.

3.4.58 a Consider a linear relation $c_1 A^2\vec{v} + c_2 A\vec{v} + c_3\vec{v} = \vec{0}$.

Multiplying A^2 with the vectors on both sides and using that $A^3\vec{v} = \vec{0}$ and $A^4\vec{v} = \vec{0}$, we find that $c_3 A^2\vec{v} = \vec{0}$ and therefore $c_3 = 0$, since $A^2\vec{v} \neq \vec{0}$.

Therefore, our relation simplifies to $c_1 A^2\vec{v} + c_2 A\vec{v} = \vec{0}$.

Multiplying A with the vectors on both sides we find that $c_2 A^2\vec{v} = \vec{0}$ and therefore $c_2 = 0$. Then $c_1 = 0$ as well. We have shown that there is only the trivial relation among the vectors $A^2\vec{v}, A\vec{v}$, and $\vec{v}$, so that these three vectors from a basis of $\mathbb{R}^3$, as claimed.

b $T(A^2\vec{v}) = A^3\vec{v} = \vec{0}$ so $[T(A^2\vec{v})]_\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$T(A\vec{v}) = A^2\vec{v}$ so $[T(A\vec{v})]_\mathcal{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$T(\vec{v}) = A\vec{v}$ so $[T(\vec{v})]_\mathcal{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Hence, by Theorem 3.4.3, the desired matrix is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

3.4.59 First we find the matrices $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, or

$\begin{bmatrix} 2x & 2y \\ 3z & 3t \end{bmatrix} = \begin{bmatrix} 2x & x+3y \\ 2z & z+3t \end{bmatrix}$. The solutions are of the form $S = \begin{bmatrix} -y & y \\ 0 & t \end{bmatrix}$, where y and t are arbitrary constants. Since there are *invertible* solutions S (for example, let $y = t = 1$), the matrices $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ are indeed similar.

3.4.**60** First we find the matrices $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, or, $\begin{bmatrix} x & y \\ -z & -t \end{bmatrix} = \begin{bmatrix} y & x \\ t & z \end{bmatrix}$. The solutions are of the form $S = \begin{bmatrix} y & y \\ -t & t \end{bmatrix}$, where y and t are arbitrary constants. Since there are *invertible* solutions S (for example, let $y = t = 1$), the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are indeed similar.

3.4.**61** We seek a basis $\vec{v}_1 = \begin{bmatrix} x \\ z \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} y \\ t \end{bmatrix}$ such that the matrix $S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ satisfies the equation $\begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Solving the ensuing linear system gives $S = \begin{bmatrix} -\frac{3z}{2} & \frac{z}{4} - \frac{3t}{2} \\ z & t \end{bmatrix}$.

We need to choose z and t so that S will be invertible. For example, if we let $z = 6$ and $t = 1$, then $S = \begin{bmatrix} -9 & 0 \\ 6 & 1 \end{bmatrix}$, so that $\vec{v}_1 = \begin{bmatrix} -9 \\ 6 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

3.4.**62** We seek a basis $\vec{v}_1 = \begin{bmatrix} x \\ z \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} y \\ t \end{bmatrix}$ such that the matrix $S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ satisfies the equation $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$. Solving the ensuing linear system gives $S = \begin{bmatrix} \frac{z}{2} & -t \\ z & t \end{bmatrix}$. We need to choose both z and t nonzero to make S invertible. For example, if we let $z = 2$ and $t = 1$, then $S = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, so that $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

3.4.**63** First we find the matrices $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that $\begin{bmatrix} p & -q \\ q & p \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$, or, $\begin{bmatrix} px - qz & py - qt \\ qx + pz & qy + pt \end{bmatrix} = \begin{bmatrix} px - qy & qx + py \\ pz - qt & qz + pt \end{bmatrix}$. If $q \neq 0$, then the solutions are of the form $S = \begin{bmatrix} -t & z \\ z & t \end{bmatrix}$, where z and t are arbitrary constants. Since there are *invertible* solutions S (for example, let $z = t = 1$), the matrices $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ and $\begin{bmatrix} p & q \\ -q & p \end{bmatrix}$ are indeed similar. (If $q = 0$, then the two matrices are equal.)

3.4.**64** If b and c are both zero, then the given matrices are equal, so that they are similar, by Theorem 3.4.6.a. Let's now assume that at least one of the scalars b and c is nonzero; reversing the roles of b and c if necessary, we can assume that $c \neq 0$.

Let's find the matrices $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, or

$\begin{bmatrix} ax + bz & ay + bt \\ cx + bz & cy + dt \end{bmatrix} = \begin{bmatrix} ax + by & cx + dy \\ az + bt & cz + dt \end{bmatrix}$. The solutions are of the form

$S = \begin{bmatrix} \frac{(a-d)z+b}{c} & z \\ z & t \end{bmatrix}$, where z and t are arbitrary constants. Since there are *invertible* solutions S (for example, let $z = 1, t = 0$), the matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ are indeed similar.

3.4.**65** a If $S = I_n$, then $S^{-1}AS = A$.

b If $S^{-1}AS = B$, then $SBS^{-1} = A$. If we let $R = S^{-1}$, then $R^{-1}BR = A$, showing that B is similar to A.

3.4.**66** We build B "column-by-column":

$$B = \left[\left[T \begin{bmatrix} b \\ 1-a \end{bmatrix} \right]_{\mathcal{B}} \left[T \begin{bmatrix} a-1 \\ b \end{bmatrix} \right]_{\mathcal{B}} \right] = \left[\begin{bmatrix} ab+b-ba \\ b^2+a^2-a \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} a^2+b^2-a \\ ba-b-ab \end{bmatrix}_{\mathcal{B}} \right]$$

$$= \left[\begin{bmatrix} b \\ 1-a \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 1-a \\ -b \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, this matrix represents the reflection about the line spanned by $\begin{bmatrix} b \\ 1-a \end{bmatrix}$. Note that the two vectors $\begin{bmatrix} b \\ 1-a \end{bmatrix}$ and $\begin{bmatrix} a-1 \\ b \end{bmatrix}$ are perpendicular.

3.4.**67** The matrix we seek is $\left[\left[T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} \left[T \begin{bmatrix} a \\ c \end{bmatrix} \right]_{\mathcal{B}} \right] = \left[\begin{bmatrix} a \\ c \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} a^2+bc \\ ac+cd \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 0 & bc-ad \\ 1 & a+d \end{bmatrix}$.

3.4.**68** Using Exercise 67 as a guide, consider the basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and let $S = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$.

3.4.**69** The matrix of the transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is $D = \left[\begin{bmatrix} 3 \\ 6 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} -2 \\ -1 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$. Thus $S^{-1}AS = D$ for $S = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

3.4.**70** Suppose such a basis $\vec{v}_1, \vec{v}_2$ exists. If $B = [[T(\vec{v}_1)]_{\mathcal{B}}\, [T(\vec{v}_2)]_{\mathcal{B}}]$ is upper triangular, of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, then $[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} a \\ 0 \end{bmatrix}$, so that $T(\vec{v}_1) = a\vec{v}_1$, that is, $T(\vec{v}_1)$ is parallel to $\vec{v}_1$. But this is impossible, since T is a rotation through $\frac{\pi}{2}$.

3.4.**71** a Note that $AS = SB$. If $\vec{x}$ is in $\ker(B)$, then $A(S\vec{x}) = SB\vec{x} = S\vec{0} = \vec{0}$, so that $S\vec{x}$ is in $\ker(A)$, as claimed.

b We use the hint and observe that nullity $(B) = \dim(\ker B) = p \leq \dim(\ker A) = \text{nullity}(A)$, since $S\vec{v}_1, \ldots, S\vec{v}_p$ are p linearly independent vectors in $\ker(A)$. Reversing the roles of A and B shows that, conversely, $\text{nullity}(A) \leq \text{nullity}(B)$, so that the equation $\text{nullity}(A) = \text{nullity}(B)$ holds, as claimed.

3.4.**72** If A and B are similar $n \times n$ matrices, then $\text{rank}(A) = n - \text{nullity}(A) = n - \text{nullity}(B) = \text{rank}(B)$, by Exercise 71 and the rank nullity theorem (Theorem 3.3.7).

3.4.**73** a By inspection, we can find an orthonormal basis $\vec{v}_1 = \vec{v}, \vec{v}_2, \vec{v}_3$ of $\mathbb{R}^3$, $\vec{v}_1 = \vec{v} = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0.8 \\ -0.6 \\ 0 \end{bmatrix}$.

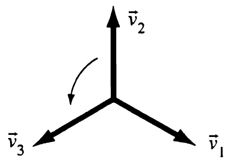


Figure 3.8: for Problem 3.4.73b.

b Now $T(\vec{v}_1) = \vec{v}_1, T(\vec{v}_2) = \vec{v}_3$ and $T(\vec{v}_3) = -\vec{v}_2$ (see Figure 3.8), so that the matrix B of T with respect to the basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.Then $A = SBS^{-1} = \begin{bmatrix} 0.36 & 0.48 & 0.8 \\ 0.48 & 0.64 & -0.6 \\ -0.8 & 0.6 & 0 \end{bmatrix}$.

3.4.**74** a $\vec{v}_0 + \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

b If $\mathcal{B}$ is the basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$, then $\vec{v}_0 + \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$ (by part a) so $\vec{v}_0 = -\vec{v}_1 - \vec{v}_2 - \vec{v}_3$, i.e. $[\vec{v}_0]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$.

c $T(\vec{v}_2) = T(-\vec{v}_0 - \vec{v}_1 - \vec{v}_3) = -T(\vec{v}_0) - T(\vec{v}_1) - T(\vec{v}_3) = -\vec{v}_3 - \vec{v}_0 - \vec{v}_1 = \vec{v}_2$

Hence, T is a rotation through 120° about the line spanned by $\vec{v}_2$. Its matrix, B, is given by

$[[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}\ [T(\vec{v}_3)]_{\mathcal{B}}]$ where

$T(\vec{v}_1) = \vec{v}_0 = -\vec{v}_1 - \vec{v}_2 - \vec{v}_3$ so $[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$

$T(\vec{v}_2) = \vec{v}_2$ so $[T(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$T(\vec{v}_3) = \vec{v}_1$ so $[T(\vec{v}_3)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and $B = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

$B^3 = I_3$ since if the tetrahedron rotates through 120° three times, it returns to the original position.

3.4.**75** $B = S^{-1}AS$, where $S = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Thus $B = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

3.4.**76** $B = S^{-1}AS$, where $S = A = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$. Thus $B = A = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$.

3.4.**77** Let S be the $n \times n$ matrix whose columns are $\vec{e}_n, \vec{e}_{n-1}, \ldots, \vec{e}_1$. Note that S has all 1's on "the other diagonal" and 0's elsewhere:

$$s_{ij} = \begin{cases} 1 & \text{if } i + j = n + 1 \\ 0 & \text{otherwise} \end{cases}$$

Also, $S^{-1} = S$.

Now, $B = S^{-1}AS = SAS$; the entries of B are $b_{ij} = s_{i,n+1-i}a_{n+1-i,n+1-j}s_{n+1-j,j} = a_{n+1-i,n+1-j}$.

Answer: $b_{ij} = a_{n+1-i,n+1-j}$

B is obtained from A by reversing the order of the rows and of the columns.

3.4.**78** Note first that the diagonal entry s_{ij} of S gives the unit price of good i.

If a_{ij} tells us how many dollars' worth of good i are required to produce one dollar's worth of good j, then $a_{ij}s_{jj}$ tells us how many dollars' worth of good i are required to produce one unit of good j, and $s_{ii}^{-1}a_{ij}s_{jj}$ is the number of units of good i required to produce one unit of good j. Thus $b_{ij} = s_{ii}^{-1}a_{ij}s_{jj}$, and $B = S^{-1}AS$.

3.4.**79** By Theorem 3.4.7, we are looking for a basis $\vec{v}_1, \vec{v}_2$ such that $A\vec{v}_1 = \vec{v}_1$ and $A\vec{v}_2 = -\vec{v}_2$. Solving the linear systems $A\vec{x} = \vec{x}$ and $A\vec{x} = -\vec{x}$, we find $\vec{v}_1 = \begin{bmatrix} 3t \\ t \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 5r \\ 2r \end{bmatrix}$, where both t and r are nonzero. Letting $t = r = 1$, for example, produces the basis $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

3.4.**80** We are looking for a basis $\vec{v}_1, \vec{v}_2$ such that $A\vec{v}_1 = \vec{v}_1$ and $A\vec{v}_2 = \vec{v}_1 + \vec{v}_2$. Solving the linear systems $A\vec{x} = \vec{x}$, we find $\vec{v}_1 = \begin{bmatrix} 3t \\ t \end{bmatrix}$, where t is nonzero. With $t = 1$, for example, we have $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. To find $\vec{v}_2$, we need to solve the linear system $(A - I_2)\vec{x} = \vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. This gives $\vec{v}_2 = \begin{bmatrix} 3r - 1 \\ r \end{bmatrix}$, where r is arbitrary. With $r = 0$, for example, we have $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Thus one possible basis is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

3.4.**81** a. We seek the real numbers x_2, x_3, and c such that $T\begin{bmatrix} 1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_2 + x_3 \end{bmatrix} = c\begin{bmatrix} 1 \\ x_2 \\ x_3 \end{bmatrix}$. Examining the components of this vector equation, we find $x_2 = c$, $x_3 = x_2^2$ and $x_2 + x_2^2 = x_2^3$. Writing the last equation as $x_2\left(x_2^2 - x_2 - 1\right) = 0$, we find the three solutions $x_2 = 0$, $x_2 = a = \frac{1+\sqrt{5}}{2}$, and $x_2 = b = \frac{1-\sqrt{5}}{2}$. [Note that $a = \frac{1+\sqrt{5}}{2}$ is the "golden ratio", often denoted by ϕ]. The vectors we seek are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix}$,where $a = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $b = \frac{1-\sqrt{5}}{2} \approx -0.618$.

b. By Theorem 3.4.7, the three vectors we found in part a will do the job. (Check that these vectors are linearly independent.)

3.4.**82** a. We seek the real numbers $x_2,\ x_3,$ and c such that $T\begin{bmatrix}1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_2\\x_3\\3x_3-2x_2\end{bmatrix}=c\begin{bmatrix}1\\x_2\\x_3\end{bmatrix}$. Examining the components of this vector equation, we find $x_2=c$, $x_3=x_2^2$ and $3x_2^2-2x_2=x_2^3$. Writing the last equation as $x_2\left(x_2^2-3x_2+2\right)=x_2(x_2-1)(x_2-2)=0$, we find the three solutions $x_2=0,\ 1,2$. The vectors we seek are $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\2\\4\end{bmatrix}$. b. By Theorem 3.4.7, the three vectors we found in part a will do the job. (Check that these vectors are linearly independent.)

True or False

Ch 3.TF.**1** T, by Theorem 3.3.2.

Ch 3.TF.**2** F; The nullity is $6-4=2$, by Theorem 3.3.7.

Ch 3.TF.**3** F; It's a subspace of $\mathbb{R}^3$.

Ch 3.TF.**4** T; by Definition 3.1.2.

Ch 3.TF.**5** T, by Summary 3.3.10.

Ch 3.TF.**6** F, by Theorem 3.3.7.

Ch 3.TF.**7** T, by Summary 3.3.10.

Ch 3.TF.**8** F; The identity matrix is similar only to itself.

Ch 3.TF.**9** T; We have the nontrivial relation $3\vec{u}+3\vec{v}+3\vec{w}=\vec{0}$.

Ch 3.TF.**10** F; The columns could be $\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4$ in $\mathbb{R}^5$, for example.

Ch 3.TF.**11** T, by Theorem 3.4.6, parts b and c.

Ch 3.TF.**12** F; Let $V=\text{span}\begin{bmatrix}1\\1\end{bmatrix}$ in $\mathbb{R}^2$, for example.

Ch 3.TF.**13** T, by Definition 3.2.3.

Ch 3.TF.**14** T, by Definition 3.2.1.

Ch 3.TF.**15** T; Check that $\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&1\\-1&1\end{bmatrix}=\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

Ch 3.TF.**16** T, by Theorem 3.3.9.

Ch 3.TF.**17** T, by Theorem 3.2.8.

Ch 3.TF.**18** T, by Summary 3.3.10.

Ch 3.TF.**19** F; The number n may exceed 4.

Ch 3.TF.**20** T, by Definition 3.2.1 (V is closed under linear combinations)

Ch 3.TF.**21** T, since $A^{-1}(AB)A = BA$.

Ch 3.TF.**22** T, since both kernels consist of the zero vector alone.

Ch 3.TF.**23** F; There is no invertible matrix S as required in the definition of similarity.

Ch 3.TF.**24** F; Five vectors in $\mathbb{R}^4$ must be dependent, by Theorem 3.2.8.

Ch 3.TF.**25** T, by Definition 3.2.1 (all vectors in $\mathbb{R}^3$ are linear combinations of $\vec{e}_1, \vec{e}_2, \vec{e}_3$).

Ch 3.TF.**26** T; Use a basis with one vector on the line and the other perpendicular to it.

Ch 3.TF.**27** T, since $AB\vec{v} = A\vec{0} = \vec{0}$.

Ch 3.TF.**28** T, by Definition 3.2.3.

Ch 3.TF.**29** F; Suppose $\vec{v}_2 = 2\vec{v}_1$. Then $T(\vec{v}_2) = 2T(\vec{v}_1) = 2\vec{e}_1$ cannot be $\vec{e}_2$.

Ch 3.TF.**30** F; Consider $\vec{u} = \vec{e}_1,\ \vec{v} = 2\vec{e}_1$, and $\vec{w} = \vec{e}_2$.

Ch 3.TF.**31** F; Note that $\mathbb{R}^2$ isn't even a subset of $\mathbb{R}^3$. A vector in $\mathbb{R}^2$, with two components, does not belong to $\mathbb{R}^3$.

Ch 3.TF.**32** T; If $B = S^{-1}AS$, then $B + 7I_n = S^{-1}(A + 7I_n)S$.

Ch 3.TF.**33** T; for any $\vec{v}_1, \vec{v}_2, \vec{v}_3$ of V also $k\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a basis too, for any nonzero scalar k.

Ch 3.TF.**34** F; The identity matrix is similar only to itself.

Ch 3.TF.**35** F; Consider $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$.

Ch 3.TF.**36** F; Let $A = I_2,\ B = -I_2$ and $\vec{v} = \vec{e}_1$, for example.

Ch 3.TF.**37** F; Let $V = \text{span}(\vec{e}_1)$ and $W = \text{span}(\vec{e}_2)$ in $\mathbb{R}^2$, for example.

Ch 3.TF.**38** T; If $A\vec{v} = A\vec{w}$, then $A(\vec{v} - \vec{w}) = \vec{0}$, so that $\vec{v} - \vec{w} = \vec{0}$ and $\vec{v} = \vec{w}$.

Ch 3.TF.**39** T; Consider the linear transformation with matrix $[\vec{w}_1\ \ldots\ \vec{w}_n][\vec{v}_1\ \ldots\ \vec{v}_n]^{-1}$.

Ch 3.TF.**40** F; Suppose A were similar to B. Then $A^4 = I_2$ were similar to $B^4 = -I_2$, by Example 7 of Section 3.4. But this isn't the case: I_2 is similar only to itself.

Ch 3.TF.**41** T; Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, for example, with $\ker(A) = \operatorname{im}(A) = \operatorname{span}(\vec{e}_1)$.

Ch 3.TF.**42** F; Consider I_n and $2I_n$, for example.

Ch 3.TF.**43** T; Matrix $B = S^{-1}AS$ is invertible, being the product of invertible matrices.

Ch 3.TF.**44** T; Note that $\operatorname{im}(A)$ is a subspace of $\ker(A)$, so that

$\dim(\operatorname{im} A) = \operatorname{rank}(A) \leq \dim(\ker A) = 10 - \operatorname{rank}(A)$.

Ch 3.TF.**45** T; Pick three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ that span V. Then $V = \operatorname{im}[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$.

Ch 3.TF.**46** T; Check that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$.

Ch 3.TF.**47** T; Pick a vector $\vec{v}$ that is neither on the line nor perpendicular to it. Then the matrix of the linear transformation $T(\vec{x}) = R\vec{x}$ with respect to the basis $\vec{v}$, $R\vec{v}$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, since $R(R\vec{v}) = \vec{v}$.

Ch 3.TF.**48** F; If $B = S^{-1}AS$, then $B = (2S)^{-1}A(2S)$ as well.

Ch 3.TF.**49** T; Note that $A(B - C) = 0$, so that all the columns of matrix $B - C$ are in the kernel of A. Thus $B - C = 0$ and $B = C$, as claimed.

Ch 3.TF.**50** T; Suppose $\vec{v}$ is in both $\ker(A)$ and $\operatorname{im}(A)$, so that $\vec{v} = A\vec{w}$ for some vector $\vec{w}$. Then $\vec{0} = A\vec{v} = A^2\vec{w} = A\vec{w} = \vec{v}$, as claimed.

Ch 3.TF.**51** F; Suppose such a matrix A exists. Then there is a vector $\vec{v}$ in $\mathbb{R}^2$ such that $A^2\vec{v} \neq \vec{0}$ but $A^3\vec{v} = \vec{0}$. As in Exercise 3.4.58a we can show that vectors $\vec{v}, A\vec{v}, A^2\vec{v}$ are linearly independent, a contradiction (we are looking at three vectors in $\mathbb{R}^2$).

Ch 3.TF.**52** T; The ith column $\vec{a}_i$ of A, being in the image of A, is also in the image of B, so that $\vec{a}_i = B\vec{c}_i$ for some $\vec{c}_i$ in $\mathbb{R}^m$. If we let $C = [\vec{c}_1\ \cdots\ \vec{c}_m]$, then $BC = [B\vec{c}_1\ \cdots\ B\vec{c}_m] = [\vec{a}_1\ \cdots\ \vec{a}_m] = A$, as required.

Ch 3.TF.**53** F; Think about this problem in terms of "building" such an invertible matrix column by column. If we wish the matrix to be invertible, then the first column can be any column other than $\vec{0}$ (7 choices). Then the second column can be any column other than $\vec{0}$ or the first column (6 choices). For the third column, we have *at most* 5 choices (not $\vec{0}$ or the first or second columns, as well as possibly some other columns). For some choices of the first two columns there will be other columns we have to exclude (the sum or difference of the first two), but not for others. Thus, in total, fewer than $7 \times 6 \times 5 = 210$ matrices are invertible, out of a total $2^9 = 512$ matrices. Thus, most are not invertible.