PROBABILITY

RANDOM EXPERIMENTS :

In our day to day life, we perform many activities which have a fixed result no matter any number of times they are repeated. For example given any triangle, without knowing the three angles, we can definitely say that the sum of measure of angles is 180°.

We also perform many experimental activities, where the result may not be same, when they are repeated under identical conditions. For example, when a coin is tossed it may turn up a head or a tail, but we are not sure which one of these results will actually be obtained. Such experiments are called random experiments.

An experiment is called **random experiment** if it satisfies the following to conditions :

(i) It has more than one possible outcome.

(ii) It is not possible to predict the outcome in advance.

Example :

(i) Tossing a coin is a random experiment.

(ii) Throwing a dice is a random experiment.

(iii) Drawing a card from a well shuffled deck of 52 playing card is also a random experiment.

OUT COMES AND SAMPLE SPACE :

A possible result of a random experiment is called its **outcome**.
Consider the experiment of rolling a die. The outcomes of this experiment are 1, 2, 3, 4, 5 or 6, if we are interested in the number of dots on the upper face of the die.

The set of outcomes {1, 2, 3, 4, 5, 6} is called the sample space of the experiment.
Thus, the set of all possible outcomes of a random experiment is called the **sample space** associated with the experiment. Sample space is denoted by the symbol S.

Each element of the sample space is called a **sample point**. In other words, each outcome of the random experiment is also called sample point.

Illustration :

Two coins (a one rupee coin and a two rupee coin) are tossed once. Find a sample space.

Sol. *Clearly the coins are distinguishable in the sense that we can speak of the first coin and the second coin Since either coin can turn up Head (H) or Tail (T), the possible outcomes may be*
Heads on both coins = (H, H) = HH
Head on first coin and Tail on the other = (H, T) = HT
Tail on first coin and Head on the other = (T, H) = TH
Tail on both coins = (T, T) = TT
Thus, the sample space is S = {HH, HT, TH, TT}

Illustration :

When a coin is tossed twice if head appears in the second throw then a dice is thrown. Write down the sample space of the experiment.

Sol. *When a coin is tossed two times then possible outcomes are {(TT), (HT), (TH), (HH)}*
If head appears in the second throw then dice is thrown.
$\therefore$ *All possible outcomes of the experiment are*
S = {(TT), (HT), (TH1), (TH2), (TH3), (TH4), (TH5), (TH6), (HH1),(HH2), (HH3), (HH4), (HH5), (HH6)}

EVENT :

Consider the experiment of tossing a coin two times. An associated sample space is

S = {HH, HT, TH, TT}

Now suppose that we are intersected in those outcomes which correspond to the occurrence of exactly one head. We find that HT and TH are the only elements of S corresponding to the occurrence of this happening (event). These two elements form the set E = {HT, TH}.

We know that the set E is a subset of the sample space S. Similarly, we find the following correspondence between events and subsets of S.

Description of events	Corresponding subset of 'S'
Number of tail is exactly 2	A = {TT}
Number of tails is atleast one	B = {HT, TH, TT}
Number of heads is atmost one	C = {HT, TH, TT}
Second toss is not head	D = {HT, TT}
Number of tails is atmost two	S = {HH, HT, TH, TT}
Number of tails is more than two	ϕ

Definition :

Any subset E of a sample space S is called an event.

Note : The maximum number of events which can be associated with an experiment is 2^n, where n is the number of elements in the sample space.

i.e., ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$

Illustration :

In throwing a pair of dice write down two possible events.

E_1 = sum of the numbers appear on both the dice is 7.

E_2 = The sum of the numbers appear on both the dice is divisible by 3.

Sol. $E_1 = \{(6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)\}$

$E_2 = \{(2, 1), (1, 2), (5, 1), (4, 2), (3, 3), (2, 4) (1, 5), (6, 3), (5, 4), (4,5), (3, 6), (6, 6)\}$

Occurrence of an event :

Consider the experiment of throwing a die. Let E denotes the event " a number less than 4 appears". If actually '1' had appeared on the die then we say that event E has occurred. As a matter of fact if outcomes are 2 or 3, we say that even E has occurred.

Thus, the event E of a sample space S is said to have occurred if the outcome ω of the experiment is such that $\omega \in E$. If the outcome ω is such that $\omega \notin E$, we say that the event E has not occurred.

Impossible and Sure Events :

The empty set ϕ and the sample space S describe events. In fact ϕ is called an **impossible event** and S, i.e., the whole sample space is called the **sure event**.

To understand these let us consider the experiment of rolling a die. The associated sample space is

$S = \{1, 2, 3, 4, 5, 6\}$

Let E be the event" the number appears on the die is a multiple of 7".

Clearly no outcome satisfies the condition given in the event, i.e., no element of the sample space ensure the occurrence of the event E. Thus, we say that the empty set only correspond to the event E. In other words we can say that it is impossible to have a multiple of 7 on the upper face of the die. Thus, the event $E = \phi$ is an impossible event.

Now let us take up another event F " the number turns up is odd or even". Clearly $F = \{1, 2, 3, 4, 5, 6\} = S$, i.e., all outcomes of the experiment ensure the occurrence of the event F. Thus, the event $F = S$ is a sure event.

Simple Event :

If an event E has only one sample point of a sample space, it is called a simple (or elementary) event.

In a sample space containing n distinct elements, there are exactly n simple events.

$S = \{HH, HT, TH, TT\}$

There are four simple events corresponding of this sample space. These are

$E_1 = \{HH\}, E_2 = \{HT\}, E_3 = \{TH\}$ and $E_4 = \{TT\}$.

Compound Event :

If an event has more than one sample point, it is called a compound event.

For example, in the experiment of "tossing a coin thrice" the events
E : 'Exactly one head appeared'
F : ' Atleast one head appeared'
G : 'Atmost one head appeared' etc.

are all compound events. The subsets of associated with these events are
$E = \{HTT, THT, TTH\}$
$F = \{HTT, THT, TTH, HHT, HTH, THH, HHH\}$
$G = \{TTT, THT, HTT, TTH\}$

Each of the above subsets contain more than one sample point, hence they are all compound events.

ALGEBRA OF EVENTS :

In the Chapter on Sets, we have studied about different ways of combining two or more sets, viz, union, intersection, difference, complement of a set etc. Like-wise we can combine two or more events by using the analogous set notations.
Let A, B, C be events associated with an experiment whose sample space is S.

Complementary Event :

For every event A, there corresponds another event A' or $\overline{A}$ called the complementary event to A. It is also called the event 'not A'.

For example, take the experiment 'of tossing three coins'. An associated sample space is

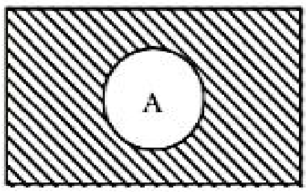
$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Let A = {HTH, HHT, THH} be the event 'only one tail appears'.

Clearly for the outcome HTT, the event A has not occurred. But we may say that the event 'not A' has occurred. Thus, with every outcome which is not in A, we say that 'not A' occurs.
Thus the complementary event 'not A' to the event A is

$$A' = \{HHH, HTT, THT, TTH, TTT\}$$

or $\quad A' = \{\omega : \omega \in S \text{ and } \omega \notin A\} = S - A$

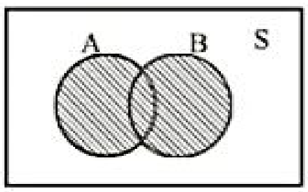


A' or $\overline{A}$

The Event 'A or B' :

Recall that union of two sets A and B denoted by $A \cup B$ contains all those elements which are either in A or in B or in both.

When the sets A and B are two events associated with a sample space, then '$A \cup B$' is the event 'either A or B or both'. This event '$A \cup B$' is also called 'A or B'.



Therefore Event 'A or B' = $A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$

$A \cup B$

The Event 'A and B' :

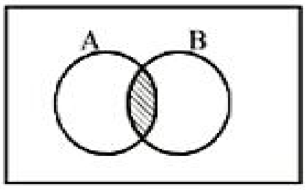
We know that intersection of two sets $A \cap B$ is the set of those elements which are common to both A and B. i.e., which belong to both 'A and B'.
If A and B are two events, then the set $A \cap B$ denotes the event 'A and B'.
Thus, $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

For example, in the experiment of 'throwing a die twice'
Let A be the event 'score on the first thrown is six' and
B is the even 'sum of two scores is atleast 11' then

$A = \{(6, 1), (6, 2), (6, 3), (6, 4), (6, 5) (6, 6)\}$

and $B = \{(5, 6), (6, 5), (6, 6)\}$

so $A \cap B = \{(6, 5), (6, 6)\}$



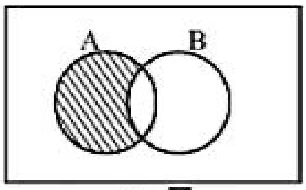
$A \cap B$

Note that the set $A \cap B = \{(6, 5), (6, 6)\}$ may represent the vent ' the score on the first throw is six and the sum of the scores is atleast 11.'

The Event 'A but not B' :

We know that A – B is the set of all those elements which are in A but not in B. Therefore, the set A – B may denote the event 'A but not B'. We know that

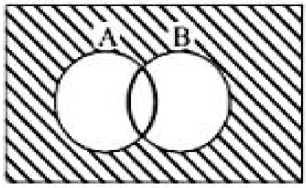
$A - B = A \cap B'$



$A \cup \overline{B}$

The Event 'neither A nor B' :

The set of the elements which are neither in set A nor in set B. i.e. $S - (A \cup B)$ and which is denoted on $\overline{A} \cap \overline{B}$.



$\overline{A} \cap \overline{B}$

Illustration :

Consider the experiment of rolling a die. Let A be the event 'getting a prime number', B be the event 'getting an odd number'. Write the sets representing the events

(i) A or B *(ii) A and B* *(iii) A but not B* *(iv) 'not A'.*

Sol. *Here* $S = \{1, 2, 3, 4, 5, 6\}, A = \{2, 3, 5\}$

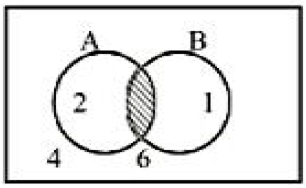
Obviously

(i) 'A or B' $= A \cup B = \{1, 2, 3, 5\}$

(ii) 'A and B' $= A \cap B = \{3, 5\}$

(iii) 'A but not B' $= A - B = \{2\}$

(iv) 'not A' $= A' = \{1, 4, 6\}$



Note : $\left.\begin{array}{l} (i)\ \overline{A \cup B} = \overline{A} \cap \overline{B} \\ (ii)\ \overline{A \cap B} = \overline{A} \cup \overline{B} \end{array}\right\}$ *De Morgan's Law*

THREE MOST IMPORTANT EVENTS :

(1) Equally Likely Events :

Events are said to be equally likely when no particular event preference to occur in relation to the other event.

Example :

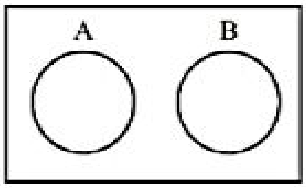
(i) The outcomes as result throwing a die are equally likely, as no particular face is more likely to occur as compared to other faces. That is why we normally write as fair die or unbiased die.

(ii) The outcomes as result of drawing a card from a well shuffled pack of 52 playing cards are equally likely to occur. Each card is as likely to be withdrawn as any other card.

(iii) How ever getting of a total of 7 is not as equally likely as getting of a total of 12 when a pair of dice are rolled once. It is also to be noted that it is 6 times more likely to get a total of 7 than to get a total of 12 in a single throw with the pair of dice.

(2) Mutually Exclusive / Disjoint / Incompatible Events :

Two events A and B are said to be mutually exclusive events if their simultaneous occurrence is impossible, i.e. both the events can not occur together.



$A \cap B = \phi$

Example :

(i) In throwing a fair die, to events A and B are such that

A : getting an odd number

B : getting an even number

then A & B are mutually exclusive events.

(ii) In drawing a card from a well shuffled pack of 52 playing card two events A and B are such that

A : getting an ace

B : getting a red card

then A and B are not mutually exclusive events.

(3) Exhaustive Events :

If $E_1, E_2, ..., E_n$ are n events associated with an experiment whose sample space is S and if

$$E_1 \cup E_2 \cup E_3 \cup ......\cup E_n = \bigcup_{i=1}^{n} E_i = S$$

then $E_1, E_2,......., E_n$ are called exhaustive events. In other words, events $E_1, E_2, ..., E_n$ are said to be exhaustive if atleast one of them necessarily occurs whenever the experiment is performed.

Further, if $E_i \cap E_j = \phi$ for $i \neq j$ i.e., events E_i and E_j are pairwise disjoint and $\bigcup_{i=1}^{n} E_i = S$, then events $E_1, E_2, \ldots, E_n$ are called mutually exclusive and exhaustive events.

Example :

Consider the experiment of throwing a die. We have

$S = \{1, 2, 3, 4, 5, 6\}$. Let us define the following events

A : 'a number less than 4 appears'.

B : 'a number greater than 2 but less than 5 appears'

and C : 'a number greater than 4 appears'.

Then $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{5, 6\}$. We observe that

$A \cup B \cup C = \{1, 2, 3\} \cup \{3, 4\} \cup \{5, 6\} = S.$

Such events A, B and C are called exhaustive events.

Practice Problem

Q.1 Suppose 3 bulbs are selected at random from a lot. Each bulb is tested and classified as defective (D) or non-defective (N). Write the sample space of this experiment.

Q.2 A coin is tossed. If the out come is a head, a die is thrown. If the die shows up and even number, the die is thrown again. What is the sample space for the experiment?

Q.3 A die is thrown. Describe the following events :

(i)	A : a number less than 7	(ii)	B : a number greater than 7
(iii)	C : a multiple of 3	(iv)	D : a number less than 4
(v)	E : an even number greater than 4	(vi)	F : a number not less than 3

Also find $A \cup B, A \cap B, B \cup C, E \cap F, D \cap E, A - C, D - E, E \cap F', F'$

Q.4 Two dice are thrown. The events A, B and C are as follows :

A : getting an even number on the first die.

B : getting an odd number on the first die.

C : getting the sum of the numbers on the dice ≤ 5.

Describe the events

(i)	A'	(ii)	not B	(iii)	A or B
(iv)	A and B	(v)	A but not C	(vi)	B or C
(vii)	B and C	(viii)	$A \cap B' \cap C'$		

Q.5 Refer to question 4 above, state true or false; (give reason for your answer)

(i) A and B are mutually exclusive

(ii) A and B are mutually exclusive and exhaustive

(iii) $A = B'$

(iv) A and C are mutually exclusive

(v) A and B' are mutually exclusive

(vi) A', B', C are mutually exclusive and exhaustive.

Answer key

Q.1 DDD, DDN, DND, NDD, DNN, NDN, NND, NNN

Q.2 T, H1, H3, H5, H21, H22, H23, H24, H25, H26, H41, H42, H43, H44, H45, H46, H61, H62, H63, H64, H65, H66

Q.3 (i) $\{1, 2, 3, 4, 5, 6\}$ (ii) ϕ (iii) $\{3, 6\}$ (iv) $\{1, 2, 3\}$ (v) $\{6\}$
(vi) $\{3, 4, 5, 6\}$, $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $B \cup C = \{3, 6\}$, $E \cap F = \{6\}$, $D \cap E = \phi$,
$A - C = \{1, 2, 4, 5\}$, $D - E = \{1, 2, 3\}$, $E \cap F' = \phi$, $F' = \{1, 2\}$

Q.4 $A = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

$B = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$

$C = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)\}$

(i) $A' = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\} = B$

(ii) $B' = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 4), (6, 6)\} = A$

(iii) $A \cup B = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (2, 1), (2, 2), (2, 3), (2, 5), (2, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\} = S$

(iv) $A \cap B = \phi$

(v) $A - C = \{(2, 4), (2, 5), (2, 6), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

(vi) $B \cup C = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$

(vii) $B \cap C = \{(1, 1), (1, 2), (1, 3), (1, 4), (3, 1), (3, 2)\}$

(viii) $A \cap B' \cap C' = \{(2, 4), (2, 5), (2, 6), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

Q.5 (i) True ; (ii) True ; (iii) True ; (iv) False ; (v) False ; (vi) False

CLASSICAL (A PRIORI) DEFINITION OF PROBABILITY :

If an experiment results in a total of $(m + n)$ outcomes which are equally likely and mutually exclusive with one another and if 'm' outcomes are favourable to an event 'A' while 'n' are unfavourable, then the probability of occurrence of the event 'A', denoted by P(A), is defined by

$$\frac{m}{m+n} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

i.e. $\quad P(A) = \dfrac{m}{m+n}$.

Note that $P(\bar{A})$ or $P(A')$ or $P(A^C)$, i.e. probability of non-occurrence of $A = \dfrac{n}{m+n} = 1 - P(A)$

In the above we shall denote the number of out comes favourable to the event A by n(A) and the total number of out comes in the sample space S by n(S).

$\therefore \quad P(A) = \dfrac{n(A)}{n(S)}$.

If $\quad P(A) = 0 \quad$ Event is impossible

$\quad P(A) = 1 \quad$ Event is sure

$\quad P(A) \ngtr 1$ and $P(A) \nless 0$

Note :

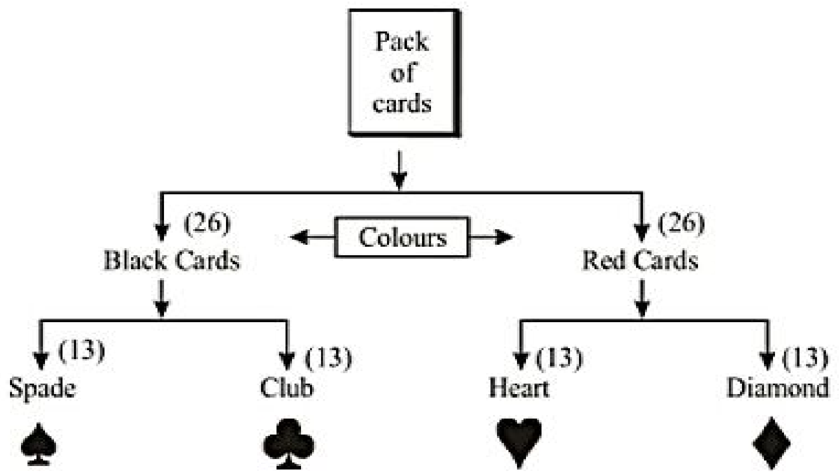
(i) More is the probability of an event, more are chances of its happening.

(ii) $P(\phi) = 0$ & $P(S) = 1$ i.e. nothing outside sample space can occur.

Designation of Cards :

Colours : There are two colours. Red & Block

Suits : There are four (4) suits (types).
Each suit contains 13 cards



Recognition of Cards :

	K King	Q Queen	J Jack	A Ace
♥	1	1	1	1
♦	1	1	1	1
♠	1	1	1	1
♣	1	1	1	1
	4	4	4	4

(i) Face Cards :

Face cards contain 12 cards all of K, Q and J having designed a figure of a person.
i.e., Face cards = 4 + 4 + 4 = 12,

(ii) Honours Cards :

It contains all face cards and also a card marked A.
i.e. Honours cards = (4 + 4 + 4) + 4 = 16 cards.

(iii) Knave Cards :

(10, J, Q) = 4 + 4 + 4 = 12 cards

Illustration :

An old man while dialing a seven digit telephone number, after having dialed the first five digits, suddenly forgets the last two. But he remembered that the last two digits were different. On this assumption he randomly dials the last two digits. What is the probability that the correct telephone number is dialed.

Sol. *Note that total number of ways in which the last two digits (different) can be dialed is* $10 \times 9 = 90$. *Out of these 90 EL/ME/ and exhaustive outcomes only one of them favours happening of the event "correct telephone is dialed". Hence* $P(E) = \frac{1}{90}$.

What the probability would have been if he did not even remember the last two digits were different:
Here $n(S) = 10 \times 10 = 100$

Hence $P(E) = \frac{1}{100}$.

Illustration :

4 Apples and 3 Oranges are randomly placed in a line. Find the chances that the extreme fruits are both oranges.

Sol. $n(S) = \frac{7!}{4!.3!}$; $n(A) = \frac{5!}{4!} \Rightarrow P = \frac{5!}{4!} \cdot \frac{4!3!}{7!} = \frac{1}{7}$

Note whether fruits the same species are different or alike that probability of the required event remains the same.

Illustration :

Two natural are randomly selected from the set of first 20 natural numbers. Find the probability that (A) their sum is odd (B) sum is even (C) selected pair is twin prime.

Sol. $S = \{1, 2, 3, ..., 19, 20\}$; $n(S) = {}^{20}C_2$

$n(A) = {}^{10}C_1 . {}^{10}C_1 = 100 \Rightarrow P(A) = \frac{100}{190} = \frac{10}{19}$ (sum odd $\Rightarrow$ one odd and one even)

$n(B) = {}^{10}C_2 + {}^{10}C_2 = 2.{}^{10}C_2 = 90 \Rightarrow P(B) = \frac{90}{190} = \frac{9}{19}$

(sum even $\Rightarrow$ both odd or both even)

$n(C) = \{(3, 5), (5, 7), (11, 13), (17, 19)\} \Rightarrow P(C) = \frac{4}{190} = \frac{2}{95}$

Illustration :

What is the chance that the fourth power of an integer chosen randomly ends in the digit six.

Sol. *Any integer randomly selected can end in 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. These are EL/ME and Exhaustive cases. Out of these 10 case only four cases, when the integer ends in 2, 4, 6 and 8 favours happening of the required event. Hence*

$$P \text{ (required event)} = \frac{4}{10} = 40\%$$

It will be incorrect to think this problem as :

4th power of an integer can end in 0, 1, 5 and 6. Hence the probability $= \frac{1}{4}$ *which is wrong. Note that four events are ME and exhaustive but not equally likely hence the definition of probability can not be based on them. In factor 4th power of an integer.*

ending in '0' is favoured by only 1 case {0}
ending in '1' is favoured by only 4 cases {1, 3, 7, 9}
ending in '5' is favoured by only 1 case {5}
ending in '6' is favoured by only 4 cases {2, 4, 6, 8}

$$\Rightarrow \quad P(0) = \frac{1}{10}; P(1) = \frac{4}{10}; P(5) = \frac{1}{10}; P(6) = \frac{4}{10}$$

Illustration :

Pair of dice has been rolled/thrown/cast once. Find the probability that atleast one of the dice shows up the face one.

Sol. *There are four ME and Exhaustive cases*
E_1 : 1^{st} dice only shows up the face one.
E_2 : 2^{nd} dice only shows up the face one.
E_3 : both dice shows up the face one.
E_4 : None of the dice shows up the face one.
Out of these, first 3 cases favours happening of the required event. Hence

$$P\text{ (required event) } = 1 - P(E_4) = 1 - \frac{5\times 5}{36} = \frac{11}{36}$$

Note that E_1, E_2, E_3, E_4 are not equally likely.

Illustration :

A leap year is selected at random. Find the probability that it has
(A) 53 Sundays (B) 53 Sundays and Mondays (C) 53 Sundays or 53 Mondays

Sol. *Leap year means which is divisible by 4 if it not a century year. If it is a century year it must be divisible by 400 as well. A leap year has 366 days out of this 364 days are consumed for 52 weeks i.e. 52 times*
S, M, T, W, Th, F and Sat. For remaining 2 days of the leap year can begin with SM, MT, TW, W Th., Th. F, F Sat and Sat Sun.

$$\Rightarrow P(A) = \frac{2}{7}\,;\; P(B) = \frac{1}{7}\,;\; P(C) = \frac{3}{7}$$

Illustration :

A card is drawn randomly from a well shuffled pack of 52 cards. The probability that the drawn card is "neither a heart nor a face card".

Sol. *Note that there are 22 cards which either H or Face cards (All K, Q and J) hence*

$$P(\text{either a H or Face card}) = \frac{22}{52} = \frac{11}{26}$$

$$\therefore \quad P(\text{neither a H nor FC}) = 1 - \frac{11}{26} = \frac{15}{26}$$

It is to be noted that

$P(\text{not } A \text{ or } \overline{A} \text{ or } A^c) = 1 - P(A)$

Note that A and A^c makes an event a sure event and probability of a sure event is one.

ODDS IN FAVOUR AND ODDS AGAINST OF AN EVENT :

If an experiment has (m + n) as a total number of outcomes which are equally likely, mutually exclusive and exhaustive, and if 'm' outcomes are in favour of an event 'A' and n outcomes are not in favour of that event A means n outcomes are in against of event A then we can say –

$$\text{Odds in favour of event A} = \frac{m}{n} = \frac{\text{No. of outcomes which are in favour of event A}}{\text{No. of outcomes which are not in favour of event A}}$$

$$\text{Odds in against of event A} = \frac{n}{m} = \frac{\text{No. of outcomes which are not in favour of event A}}{\text{No. of outcomes which are in favour of event A}}$$

Note : If $P(A) = \frac{a}{b}$ then

(i) odds in favour of event A = a : b – a.

(ii) odds against of event = b – a: a.

Illustration :

5 different marbles are placed in 5 different boxes randomly. Find the odds in favour that exactly two boxes remain empty. Given each box an hold any number of marbles.

Sol. *$n(S) = 5^5$; For computing favourable outcomes.*

2 boxes which are remain empty, can be selected in ${}^5C_2$ ways and 5 marbles can be placed in the remaining 3 boxes in groups of 221 or 311 in $3!\times\left[\frac{5!}{2!\,2!\,2!}+\frac{5!}{3!\,2!}\right] = 150$ *ways*

$$\therefore \quad P(E) = {}^5C_2 \times \frac{150}{5^5} = \frac{12}{25}$$

Hence, odds in favour of event E = 12 : 13 ***Ans.***

Practice Problem

Q.1 If three cards are drawn from a well shuffled pack of 52 cards randomly. What is the probability that is has

(i)	all three Kings?	(ii)	one King and two Queens?
(iii)	all three of same colour?	(iv)	all three of different suits?
(v)	all three of same denomination?	(vi)	at least one King?

Q.2 The first twelve letters of the alphabet are written down at random. What is the probability that there are exactly 4 letters between A and B?

Q.3 If n biscuits are distributed at random among N beggars. Find the probability that a particular beggar receives r (< n) biscuits.

Q.4 If k is chosen at random from the interval [0, 5]. Find the probability of the equation $x^2 + kx + \frac{1}{4}(k+2) = 0$ to have real roots.

Q.5 The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. Find the chance of each.

Answer key

Q.1 (i) $\frac{{}^4C_3}{{}^{52}C_3}$; (ii) $\frac{{}^4C_1 \times {}^4C_2}{{}^{52}C_3}$; (iii) $\frac{{}^2C_1 \times {}^{26}C_3}{{}^{52}C_3}$; (iv) $\frac{{}^4C_3 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_3}$;

(v) $\frac{{}^{13}C_1 \times {}^4C_3}{{}^{52}C_3}$; (vi) $\frac{{}^4C_1 \times {}^{48}C_2 + {}^4C_2 \times {}^{48}C_1 + {}^4C_3 \times {}^{48}C_0}{{}^{52}C_3}$

Q.2 $\frac{7}{66}$ Q.3 $\frac{{}^nC_r(1)^r.(N-1)^{n-r}}{N^n}$ Q.4 $\frac{3}{5}$ Q.5 1/9

DEPENDENT AND INDEPENDENT EVENTS :

Independent events – Events A and B are said to be independent if occurrences or non occurrence of one does not affect the probability of occurrence or non-occurrence of the other.

(i) Two people holding a normal dice and the other a coin, throw them once, then getting a 6 on normal dice and getting a head on the coin are the examples of events which are independent.

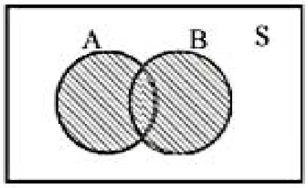
(ii) From an urn containing 2R, 3G and 4W balls, a ball is drawn its colour is noted, the ball is replaced in the urn and another ball is drawn. Getting a red and a red ball on both the occasion are the examples of events which are independent.

(iii) Similar example can be given in playing cards 'getting an ace' and 'an ace' in two successive draws from a well shuffled pack of 52 cards when the first drawn card is replaced in the pack before the second is drawn. If it is not replaced the events become dependent or contingent.

Note : Dependent/Independent events come from two different experiments while mutually exclusive events come form the same experiment.

ADDITION THEOREM ON PROBABILITY :

If A and B are two events associated with an experiment then $P(A \cup B)$ is called the sum of the probabilities of all the sample points in $A \cup B$ or probability of occurrence of atleast one of the events from A and B and the expression for $P(A \cup B)$ is called the addition theorem on probability



From the Venn diagram it is clear that

P(Occurence atleast one of the events from A and B)

P(A or B or both)

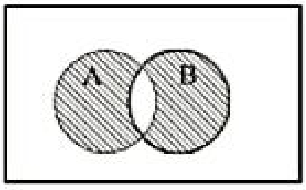
or

P(A + B)

$$\Rightarrow P(A \cup B) = \begin{cases} = P(A) + P(B) - P(A \cap B) \\ = P(A) + P(\bar{A} \cap \bar{B}) \\ = P(B) + (A \cap \bar{B}) \\ = P(A \cap \bar{B}) + P(A\ B) + P(\bar{A} \cap B) \\ = 1 - P(\bar{A} \cap \bar{B}) \\ = 1 - P(\overline{A \cap B}) \end{cases}$$

P(occurrence of exactly one of the events)
or
P(A or B but not both)

$$\begin{cases} P(A \cap \bar{B}) + P(\bar{A} \cap B) \\ P(A) + P(B) - P(A \cap B) \end{cases}$$



Note :

(i) If A and B are mutually exclusive events then –
$P(A \cup B) = P(A) + P(B)$ $\{\because P(A \cap B) = 0\}$

(ii) If A and B are exhaustive events then $P(A \cup B) = 1$

(iii) $P(A \cup B) = 1 - P(\overline{A \cup B})$

Illustration :

Two students Anil and Ashima appeared in an examination. The probability that Anil will qualify the examination is 0.05 and that Ashima will qualify the examination is 0.10. The probability that both will qualify the examination is 0.02. Find the probability that

(a) Both Anil and Ashima will not qualify the examination

(b) Atleast one of them will not qualify the examination and

(c) Only one of them will qualify the examination.

Sol. *Let E and F denote the events that Anil and Ashima will qualify the examination, respectively. Given that*

$$P(E) = 0.05, \ P(F) = 0.10 \ \text{ and } \ P(E \cap F) = 0.02$$

Then

(a) The event 'both Anil and Ashima will not qualify the examination' may be expressed as $E' \cap F'$. Since, E' is not E, i.e., Anil will not qualify the examination and F' is 'not F, i.e., Ashima will not qualify the examination.

Also $E' \cap F' = (E \cup F)'$ *(By Demorgan's Law)*

Now $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

or $P(E \cup F) = 0.05 + 0.10 - 0.02 = 0.13$

Therefore $P(E' \cap F') = P(E \cup F)' = 1 - P(E \cup F) = 1 - 0.13 = 0.87$

(b) *P(atleast one of them will not qualify)*

$= 1 -$ *P(both of them will qualify)*

$= 1 - 0.02 = 0.98$

(c) *The event only one of them will qualify the examination is same as the event either (Anil will qualify, and Ashima will not qualify) or (Anil will not qualify and Ashima will qualify) i.e.,* $E \cap F'$ *or* $E' \cap F$, *where* $E \cap F'$ *and* $E' \cap F$ *are mutually exclusive.*

Therefore, P(only one of them will qualify) $= P(E \cap F'$ *or* $E' \cap F)$

$= P(E \cap F') + P(E' \cap F) = P(E) - P(E \cap F) + P(F) - P(E \cap F)$

$= 0.05 - 0.02 + 0.10 - 0.02 = 0.11$

Illustration :

A and B are any two events such that $P(A) = 0.3$, $P(B) = 0.1$ *and* $P(A \cap B) = 0.16$. *Find the probability that exactly one of the events happens.*

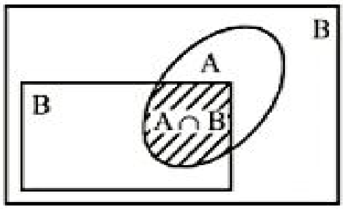
Sol. *Exactly one of the events happens* $= P(A \cap B')$ *or* $P(A' \cap B)$

$P(A \cap B') + P(A' \cap B) = P(A) + P(B) - 2P(A \cap B)$

$= 0.3 + 0.1 - 2 \times 0.16 = 0.08$

CONDITIONAL PROBABILITY :

Let A and B be any two events associated with a random experiment. The probability of occurrence of event A when the event B has already occurred is called the conditional probability of A when B is given and is denoted as $P(A/B)$. The conditional probability $P(A/B)$ is meaningful only when $P(B) \neq 0$, i.e., when B is not an impossible event.



By definition,

$$P\left(\frac{A}{B}\right) = \text{Probability of occurrence of event A when the event B as already occurred}$$

$$= \frac{\text{Number of cases favourable to B which are also favourable to A}}{\text{Number of cases favourable to B}}$$

$$\therefore \quad P\left(\frac{A}{B}\right) = \frac{\text{Number of cases favourable to } A \cap B}{\text{Number of cases favourable to B}}$$

Also, $P\left(\frac{A}{B}\right) = \dfrac{\dfrac{\text{Number of cases favourable to } A \cap B}{\text{Number of cases in the sample space}}}{\dfrac{\text{Number of cases favourable to B}}{\text{Number of cases in the sample space}}}$

$$\therefore \quad P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0.$$

Similarly, we have

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0.$$

Illustration :

Roll a fair die twice. Let A be the event that the sum of the two rolls equals six, and let B be the event that the same number comes up twice. What is P(A/B)?

(A) 1/6 *(B) 5/36* *(C) 1/5* *(D) none*

Sol. $A = \{(1, 5), (4, 4), (3, 3), (4, 2), (5, 1)\}$

$B = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

$$\therefore P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)} = \frac{1}{6}$$

Illustration :

In a class, 30% of the students failed in Physics, 25% failed in Mathematics and 15% failed in both Physics and Mathematics. If a student is selected at random failed in Mathematics, find the probability that he failed in Physics also.

Sol. *Let A be the event "failed in Physics" and B be the event "failed in Mathematics". We want to find* $P\left(\frac{A}{B}\right)$. *It is given that*

$$P(A) = \frac{30}{100} \quad \text{and} \quad P(B) = \frac{25}{100}$$

Also $P(A \cap B) = \frac{15}{100}$

Therefore $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{15/100}{25/100} = \frac{15}{25} = \frac{3}{5}$

Illustration :

Let A and B be two events such that P(A) = 0.3, P(B) = 0.6 and $P\left(\frac{B}{A}\right) = 0.5$. *Then* $P\left(\frac{\overline{A}}{\overline{B}}\right)$ *equals*

(A) $\frac{3}{4}$ (B) $\frac{5}{8}$ (C) $\frac{9}{40}$ (D) $\frac{1}{4}$

Sol. $P(A \cap B) = P(A)\, P\left(\frac{B}{A}\right) = (0.3)\,(0.5) = 0.15$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.6 - 0.15 = 0.75$

Also $P\left(\frac{\overline{A}}{\overline{B}}\right) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \frac{1-P(A \cup B)}{1-P(B)} = \frac{1-0.75}{1-0.6} = \frac{0.25}{0.4} = \frac{250}{400} = \frac{5}{8}$

MULTIPLICATION THEOREM ON PROBABILITY :

Let A and B be two events associated with a sample space S. Clearly, the set $A \cap B$ denotes the event that both A and B have occurred. In other words, $A \cap B$ denotes the simultaneous occurrence of the events E and F. The event $A \cap B$ is also written as AB.

We know that the conditional probability of event A given that B has occurred is denoted by $P(A \mid B)$ and is given by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},\ P(B) \neq 0$$

From this result, we can write

$$P(A \cap B) = P(B) \cdot P(A \mid B) \qquad \ldots(i)$$

Also, we know that

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)},\ P(A) \neq 0$$

or $$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \quad (\text{since } A \cap B = B \cap A)$$

Thus, $$P(A \cap B) = P(A) \cdot P(B \mid A) \qquad \ldots(ii)$$

Combining (i) and (ii), we find that

$$P(A \cap B) = P(A)\,P(B \mid A) = P(B)\,P(A \mid B) \text{ provided } P(A) \neq 0 \text{ and } P(B) \neq 0$$

The above result is known as the multiplication rule of probability.

Note : If A & B are independent events then $P\left(\frac{A}{B}\right) = P(A)$ and $P\left(\frac{B}{A}\right) = P(B)$ and in this case multiplication theorem $P(A \cap B) = P(A) \cdot P(B)$.

Theorem-I :

Let A and B be events associated with a random experiment. If A and B are independent, then show that the events (i) $\overline{A}, B$ (ii) $A, \overline{B}$ (iii) $\overline{A}, \overline{B}$ are also independent.

Proof : The events A and B are independent.

$\therefore \quad P(A \cap B) = P(A)\, P(B) \qquad \text{......(i)}$

(i) $\quad (A \cap B) \cap (\overline{A} \cap B) = (A \cap \overline{A}) \cap (B \cap B) = \phi \cap B = \phi$

and $\quad (A \cap B) \cup (\overline{A} \cap B) = (A \cup \overline{A}) \cap B = S \cap B = B$

$\therefore$ The events $A \cap B$ and $\overline{A} \cap B$ are mutually exclusive and their union is B.

$\therefore$ By addition theorem, we have $P(B) = P(A \cap B) + P(\overline{A} \cap B) \qquad \text{......(i)}$

$\Rightarrow \quad P(\overline{A} \cap B) = P(B) - P(A \cap B) = P(B) - P(A)\, P(B)$

$\qquad = (1 - P(A))\, P(B) = P(\overline{A})\, P(B) \qquad \text{(Using (i))}$

$\therefore \quad P(\overline{A} \cap B) = P(\overline{A})\, P(B)$ i.e., $\overline{A}$ and B are independent.

(ii) $\quad (A \cap B) \cap (A \cap \overline{B}) = (A \cap A) \cap (B \cap \overline{B}) = A \cap \phi = \phi$

and $\quad (A \cap B) \cup (A \cap \overline{B}) = A \cap (B \cup \overline{B}) = A \cap S = A$

$\therefore$ The events $A \cap B$ and $A \cap \overline{B}$ are mutually exclusive and their union is A.

$\therefore$ By addition theorem, we have $P(A) = P(A \cap B) + P(A \cap \overline{B}) \qquad \text{......(i)}$

$\Rightarrow \quad P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) - P(A)\, P(B)$

$\qquad = P(A)(1 - P(B)) = P(A)\, P(\overline{B}) \qquad \text{(Using (i))}$

$\therefore \quad P(A \cap \overline{B}) = P(A)\, P(\overline{B})$ i.e., A and $\overline{B}$ are independent.

(iii) $\quad (\overline{A} \cap B) \cap (\overline{A} \cap \overline{B}) = (\overline{A} \cap \overline{A}) \cap (B \cap \overline{B}) = \overline{A} \cap \phi = \phi$

and $\quad (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) = \overline{A} \cap (B \cup \overline{B}) = \overline{A} \cap S = \overline{A}$

$\therefore$ The events $\overline{A} \cap B$ and $\overline{A} \cap (B \cap \overline{B})$ are mutually exclusive and their union is $\overline{A}$.

$\therefore$ By addition theorem, we have $P(\overline{A}) = P(\overline{A} \cap B) + P(\overline{A} \cap \overline{B}) \qquad \text{......(i)}$

$\Rightarrow \quad P(\overline{A} \cap \overline{B}) = P(\overline{A}) - P(\overline{A} \cap B) = P(\overline{A}) - P(\overline{A})\, P(B)$

$\qquad = P(\overline{A})(1 - P(B)) = P(\overline{A})\, P(\overline{B}) \qquad \text{(Using (i))}$

$\therefore \quad P(\overline{A} \cap \overline{B}) = P(\overline{A})\, P(\overline{B})$ i.e., $\overline{A}$ and $\overline{B}$ are independent.

Illustration :

A die is thrown. If E is the event 'the number appearing is a multiple of 3' and F be the event 'the number appearing is even' then find whether E and F are independent?

Sol. We know that the sample space is $S = \{1, 2, 3, 4, 5, 6\}$

Now $E = \{3, 6\}$, $F = \{2, 4, 6\}$ and $E \cap F = \{6\}$

Then $P(E) = \frac{2}{6} = \frac{1}{3}$, $P(F) = \frac{3}{6} = \frac{1}{2}$ and $P(E \cap F) = \frac{1}{6}$

Clearly $P(E \cap F) = P(E) \cdot P(F)$

Hence E and F are independent events.

Illustration :

Three coins are tossed simultaneously. Consider the event E 'three heads or three tails', F 'at least two heads' and G 'at most two heads'. Of the pairs (E, F), (E, G) and (F, G), which are independent? which are dependent?

Sol. The sample space of the experiment is given by

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Clearly $E = \{HHH, TTT\}$, $F = \{HHH, HHT, HTH, THH\}$

and $G = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Also $E \cap F = \{HHH\}$, $E \cap G = \{TTT\}$, $F \cap G = \{HHT, HTH, THH\}$

Therefore $P(E) = \frac{2}{8} = \frac{1}{4}$, $P(F) = \frac{4}{8} = \frac{1}{2}$, $P(G) = \frac{7}{8}$

and $P(E \cap F) = \frac{1}{8}$, $P(E \cap G) = \frac{1}{8}$, $P(F \cap G) = \frac{3}{8}$

Also $P(E) \cdot P(F) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$, $P(E) \cdot P(G) = \frac{1}{4} \times \frac{7}{8} = \frac{7}{32}$

and $P(F) \cdot P(G) = \frac{1}{2} \times \frac{7}{8} = \frac{7}{16}$

Thus $P(E \cap F) = P(E) \cdot P(F)$

$P(E \cap G) \neq P(E) \cdot P(G)$ and $P(F \cap G) \neq P(F) \cdot P(G)$

Hence, the events (E and F) are independent, and the events (E and G) and (F and G) are dependent.

Illustration :

A pair of fair dice is thrown. Find the probability that either of the dice shows 2 if the sum is 6.

Sol. *The sample space of the experiment "throwing a pair of fair dice" consists of 36(= 6 × 6) ordered pair (a, b), where a and b can be any integers from 1 to 6. Let A be the event "2 appears on either of the dice" and B be the event "sum is 6". We want to find* $P\left(\frac{A}{B}\right)$. *Note that*

$A = [(2, b) \mid 1 \le b \le 6] \cup [(a, 2) \mid 1 \le a \le 6]$ and $B = [(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)]$

Also, $A \cap B = [(2, 4), (4, 2)]$

Therefore $P(B) = \frac{5}{36}$ and $P(A \cap B) = \frac{2}{36}$

So $$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{5/36} = \frac{2}{5}$$

Illustration :

A jar contains 10 white balls and 6 blue balls, all are of equal size. Two balls are drawn without replacement. Find the probability that the second ball is white if it is known that the first is white.

Sol. *Let E_1 be the event " the first ball drawn is white" and E_2 be the event" the second ball drawn is white again. Then*

$$P(E_1) = \frac{10}{16}$$

since 10 out of 10 + 6 balls are white. But, after one ball is chosen, there remain 9 white balls and 6 blue balls. Therefore the required probability is

$$P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{\frac{10}{16} \cdot \frac{9}{15}}{\frac{10}{16}} = \frac{9}{15} = \frac{3}{5}$$

Illustration :

There are four machines and it is known that exactly two of them are faulty. They are tested one by one, in a random order till both the faulty machines are identified. The probability that only two tests are needed is

(A) $\frac{1}{3}$ (B) $\frac{1}{6}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Sol. *The procedure ends in first two tests if either both are faulty or both are good. Therefore the probability is*

$$= P(G \cap G) + P(F \cap F) = P(G) \cdot P\left(\frac{G}{G}\right) + P(F) \cdot P\left(\frac{F}{F}\right) = \frac{2}{4}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{3} = \frac{1}{3} \qquad \textit{Ans.}$$

Practice Problem

Q.1 If A and B are any two events with P(A)=3/8 ; P(B) = 1/2 and $P(A \cap B) = 1/4$. Find
(i) $P(A \cup B)$ (ii) $P(A^C)$ and $P(B^C)$ (iii) $P(A^C \cap B^C)$ (iv) $P(A^C \cup B^C)$
(v) $P(A \cap B^C)$ (vi) $P(B \cap A^C)$

Q.2 A problem in mathematics is given to 2 children who solve it independently. If probability of A solving it is 1/2 and probability of B solving it is 2/3. Find the probability that the problem is solved.

Q.3 A pair of dice is rolled until a total of 5 or 7 is obtained. Find the probability that the total of 5 comes before a total of 7

Q.4 A box contains 5 tubes, 2 of them defective and 3 good one. Tubes are tested by one-by-one till the 2 defective tubes are discovered. What is the probability that the testing procedure comes to an end at the end of
(i) second testing (ii) 3rd testing

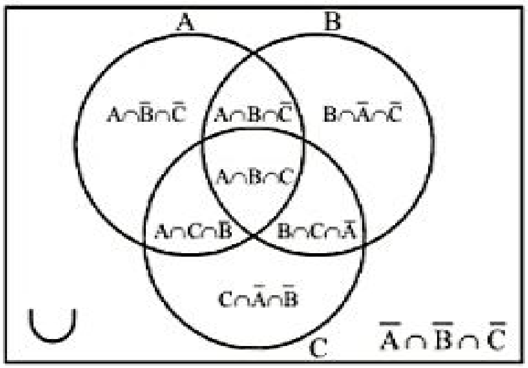
Q.5 In the following experiment, we roll a fair die 5 times
(i) What is the probability of the sequence "1, 2, 3, 4, 5".
(ii) What is the probability that the sequence starts with a "1"
(iii) What is the probability that the number "2" appears exactly twice.
(iv) Let E be the event that we find the sequence "1, 2, 3, 4, 5" and let F be the event that the sequence starts with a "1".
What are the probabilities P(E/F) and P(F/E)

Answer key

Q.1 (i) 5/8 ; (ii) 5/8 & 1/2 ; (iii) 3/8 ; (iv) 3/4 ; (v) 1/8 ; (vi) 1/4
Q.2 5/6 Q.3 2/5 Q.4 (i) 1/10 ; (ii) 3/10

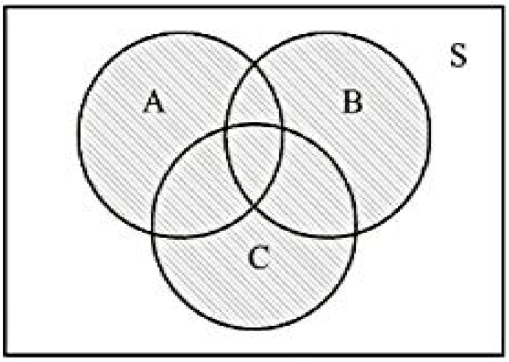
Q.5 (i) $\frac{1}{6^5}$; (ii) $\frac{1\cdot 6^4}{6^5} = \frac{1}{6}$; (iii) $\frac{{}^5C_2 \cdot 1 \cdot 5^3}{6^5} = \frac{2}{3}\left(\frac{5}{6}\right)^4$; (iv) $P(E/F) = \frac{1}{6^4}$; $P(F/E) = 1$

THREE EVENTS ASSOCIATED WITH AN EXPERIMENTAL PERFORMANCE :



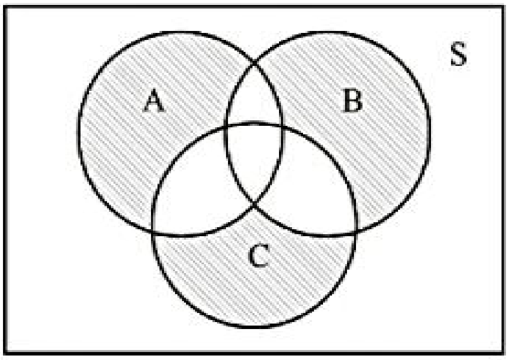
The addition theorem can be extended when three events are associated with the experiment.
If A, B and C are three events then
$P(A \cup B \cup C)$ denotes the sum of probabilities of all the sample points in $(A \cup B \cup C)$ or probability of occurrence of atleast one of the events.

(i) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$



$(A \cup B \cup C)$

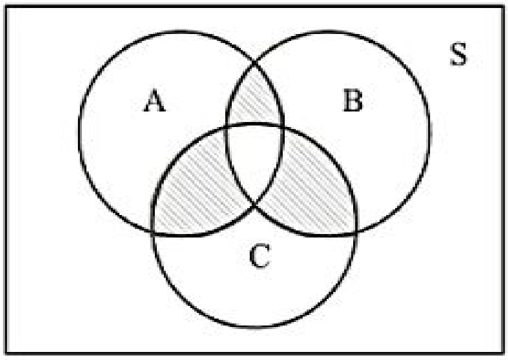
(ii) P (occurrence of exactly one of the events) =
$P(A) + P(B) + P(C) - 2[P(A \cap B) + P(B \cap C) + P(C \cap A)] + 3P(A \cap B \cap C)$



$(A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$

(iii) P (occurrence of exactly two of the events) =

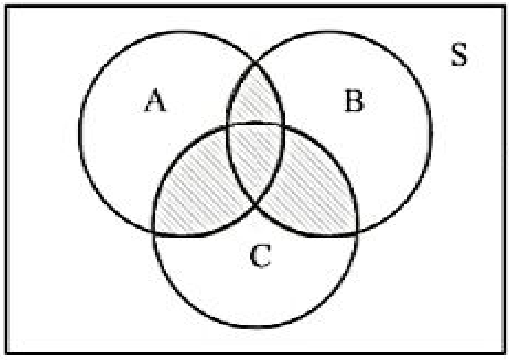
$$P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C)$$



$(A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap C) \cup (A \cap \overline{B} \cap C)$

(iv) P (occurrence of atleast two of the events) =

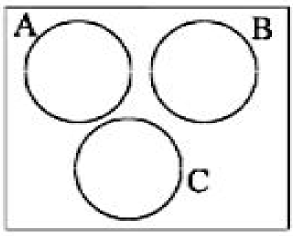
$$P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$$



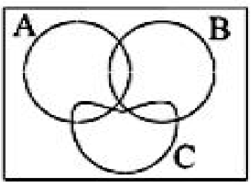
$(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C) \cup (A \cap B \cap C)$

Note :

(a) If A, B, C are three pair wise mutually exclusive $\Rightarrow$ they are mutually exclusive however if A, B, C are mutually exclusive $\not\Rightarrow$ they are pair wise mutually exclusive



ME $\not\Rightarrow$ pair wise ME



Pair wise ME $\Rightarrow$ ME

(b) However, if A,B,C are pair wise independent $\Rightarrow$ they are independent. Infact for 3 events A, B and C to be independent they must be

(i) pair wise (ii) mutually independent, mathematically

$P(A \cap B) = P(A) . P(B)$; $P(B \cap C) = P(B).P(C)$; $P(C \cap A) = P(C).P(A)$

and $P(A \cap B \cap C) = P(A).P(B).P(C)$

for n independent events, the total number of conditions would be

${}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n - n - 1$

Illustration :

A, B and C are three newspapers from a city. 25% of the population reads A, 20% reads B, 15% reads C, 16% reads both A and B, 10% reads both B and C, 8% reads both A and C and 4% reads all the three. Find the percentage of the population who read atleast one of A, B and C.

Sol. *We are given that*

$$P(A) = \frac{25}{100}, P(B) = \frac{20}{100}, P(C) = \frac{15}{100}$$

$$P(A \cap B) = \frac{16}{100}, P(B \cap C) = \frac{10}{100}, P(C \cap A) = \frac{8}{100} \quad \text{and} \quad P(A \cap B \cap C) = \frac{4}{100}$$

We have to find $P(A \cap B \cap C)$. *We can use the formula*

$$P(A \cap B \cap C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{1}{100}(25 + 20 + 15 - 16 - 10 - 8 + 4) = \frac{30}{100}$$

Thus 30% of the people read atleast one of the newspapers.

Illustration :

Let A, B and C be three events such that

$p = P$ *(exactly one of A or B) = P(exactly one of B or C) = P(exactly one of C or A)*

and $P(A, B, C \text{ simultaneously}) = p^2$

where $0 < p < \frac{1}{2}$. *Then P (at least one of A, B or C) is equal to*

(A) $\frac{3p + 2p^2}{2}$ (B) $\frac{2p + 3p^2}{2}$ (C) $\frac{2p + 3p^2}{4}$ (D) $\frac{3p + 2p^2}{4}$

Sol. *Exactly one of A or B means*

So $P(A) + P(B) - 2P(A \cap B) = p$ ...(i)

Similarly P (exactly one of B or C)

$P(B) + P(B \cap C) - 2P(B \cap C) = p$ ...(ii)

and $P(C) + P(A) - P(C \cap A) = p$ ...(iii)

Adding equation (i) – (iii), we have

$$2[P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A)] = 3p \quad ...(iv)$$

Now P(atleast one A, B or C) is given by [see part (3), theorem 7.2]

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{3p}{2} + p^2 \qquad [\text{Equation (iv) and } P(A \cap B \cap C) = p^2]$$

$$= \frac{3p + 2p^2}{2}$$

BINOMIAL PROBABILITY :

Let an experiment has n-independent trials, and each of the trial has two possible outcomes

(i) success (ii) failure

If probability of getting success, $P(S) = p$ and probability getting failure, $P(F) = q$ such that $p + q = 1$.

Then, $P(r \text{ successes}) = {}^nC_r\, p^r\, q^{n-r}$

Proof :

Consider the compound event where r successes are in succession and $(n - r)$ failures are in succession.

$$P\left(\underbrace{SSS......S}_{r}\ \underbrace{FFF......F}_{(n-r)}\right) = \underbrace{P(S).P(S)......P(S)}_{r \text{ times}}\ \underbrace{P(F).P(F)......P(F)}_{(n-r)\text{times}} = P^r \,.\, q^{n-r}$$

But these r successes and $(n - r)$ failures can be arranged in $\dfrac{n!}{r!(n-r)!} = {}^nC_r$ ways and in each arrangement the probability will be $p^r \,.\, q^{n-r}$

Hence total pr. $= P(r) = {}^nC_r\, p^r\, q^{n-r}$ ......(1)

Recurrence relation

$$p(r + 1) = {}^nC_{r+1}\, p^{r+1} \,.\, q^{n-r-1}$$

$$\therefore \quad \frac{P(r+1)}{P(r)} = \frac{{}^nC_{r+1}}{{}^nC_r}\frac{p}{q} = \frac{n-r}{r+1}\,\frac{p}{1-p}$$

$$\therefore \quad P(r+1) = \frac{n-r}{r+1}\cdot\frac{p}{1-p}P(r) \qquad ......(2)$$

Equation (2) is used for completely the probabilities of P(1); P(2); P(3); ...... etc. once P(0) is determined.

Illustration :

A pair of dice is thrown 6 times, getting a doublet is considered a success. Compute the probability of

(i) no success (ii) exactly one success

(iii) at least one success (iv) at most one success

Sol. *Total sample spaces are = 36*

In which six doublets then

$$p = \frac{3}{36} = \frac{1}{6}; \quad q = 1 - \frac{1}{6} = \frac{5}{6}$$

(i) *No success for r = 0*

$$\therefore \quad p(0) = {}^6C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^6 = \left(\frac{5}{6}\right)^6$$

(ii) *Exactly one success for r = 1*

$$\therefore \quad p(1) = {}^6C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^5 = \left(\frac{5}{6}\right)^5$$

(iii) *For at least are success for r = 1, 2, 3, 4, 5, 6.*

$$\therefore \quad \sum_{r=1}^{6} {}^6C_r p^r q^{6-r} = {}^6C_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^5 + {}^6C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4 + {}^6C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 + {}^6C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2$$

$$+ {}^6C_5\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^1 + {}^6C_6\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^0$$

(iv) *For at most one success for r = 0, 1*

$$\sum_{r=0}^{1} {}^6C_r\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^{6-r} = {}^6C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^6 + {}^6C_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^5$$

Illustration :

In a hurdle race a man has to clear 9 hurdles. Probability that he clears a hurdle 2/3 and the probability that he knocks down the hurdle is 1/3. Find the probability that he knocks down fewer than 2 hurdles.

Sol. *For probability that he knocks down fewer than two hurdles for r = 0, 1*

where $p = \frac{1}{3}, \quad q = \frac{2}{3}$

$$\therefore \quad \sum_{r=0}^{1} {}^9C_r\left(\frac{1}{3}\right)^r\left(\frac{2}{3}\right)^{9-r} = {}^9C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^9 + {}^9C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^8$$

Illustration :

A drunkard takes a step forward or backward. The probability that he takes a step forward is 0.4. Find the probability that at the end of 11 steps he is one step away from the starting point.

Sol. *At the end of 11 steps he is one step away from the starting point by two ways*

(i) *Man has taken 6 steps forward and 5 steps backward*

(ii) *Man has taken 6 steps backward and 6 steps forward*

here $p = \text{probability of forward} = \dfrac{2}{5}$

$q = \text{probability of backward} = \dfrac{3}{5}$

$$\therefore \quad \text{Probability} = {}^{11}C_6\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^5 + {}^{11}C_5\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^6$$

Practice Problem

Q.1 A die is thrown 7 times. What is the chance that an odd number turns up (i) exactly 4 times (ii) at least 4 times?

Q.2 A fair coin is tossed a fixed number of times. If the probability of getting 7 heads is equal to getting 9 heads, the probability of getting 2 heads is ,

(A) $\dfrac{15}{2^5}$ (B) $\dfrac{2}{15}$ (C) $\dfrac{15}{2^{13}}$ (D) None of these

Q.3 A coin is twice as likely to land heads as tails. In a sequence of five independent trials, find the probability that the third head occurs on the fifth toss.

Q.4 A fair coin is flipped n times. Let E be the event "a head is obtained on the first flip", and let F_k be the event "exactly k heads are obtained". For which one of the following pairs (n, k) are E and F_k independent?
(A) (12, 4) (B) (20, 10) (C) (40, 10) (D) (100, 51)

Answer key

Q.1 $\dfrac{35}{128}, \dfrac{1}{2}$ Q.2 C Q.3 $\dfrac{16}{81}$ Q.4 B

TOTAL PROBABILITY THEOREM :

Let $E_1, E_2, \ldots\ldots, E_n$ be n mutually exclusive and exhaustive events, with non-zero probabilities, of a random experiment. If A be any arbitrary event of the sample space of the above random experiment with $P(A) > 0$, then

$$P(A) = P(E_1)\, P\left(\frac{A}{E_1}\right) + P(E_2)\, P\left(\frac{A}{E_2}\right) + \ldots\ldots + P(E_n)\, P\left(\frac{A}{E_n}\right).$$

Proof : Let S be the sample space of the random experiment.

Since $E_1, E_2, \ldots\ldots, E_n$ are exhaustive, we have $S = E_1 \cup E_2 \cup \ldots\ldots \cup E_n$.

Now $A = S \cap A = (E_1 \cup E_2 \cup \ldots\ldots \cup E_n) \cap A$

$\Rightarrow$ $A = (E_1 \cap A) \cup (E_2 \cap A) \cup \ldots\ldots \cup (E_n \cap A)$ ......(i)

Since $E_1, E_2, \ldots\ldots, E_n$ are mutually exclusive, we have $E_i \cap E_j = \phi$ for $i \neq j$

Now $(E_i \cap A) \cap (E_j \cap A) = (E_i \cap E_j) \cap A = \phi \cap A = \phi$

$\therefore$ $E_1 \cap A,\ E_2 \cap A,\ \ldots\ldots, E_n \cap A$ are also mutually exclusive.

By using addition theorem, (i) implies

$$P(A) = P(E_1 \cap A) + P(E_2 \cap A) + \ldots\ldots + P(E_n \cap A)$$

$$\Rightarrow \quad P(A) = P(E_1)\, P\left(\frac{A}{E_1}\right) + P(E_2)\, P\left(\frac{A}{E_2}\right) + \ldots\ldots + P(E_n)\, P\left(\frac{A}{E_n}\right).$$

Remark : In practical problems, it is found convenient to write as follows :

$$P(A) = P(E_1A \text{ or } E_2A \text{ or } \ldots\ldots E_nA) = P(E_1A) + P(E_2A) + \ldots\ldots + P(E_nA)$$

$$P(A) = P(E_1)\, P\left(\frac{A}{E_1}\right) + P(E_2)\left(\frac{A}{E_2}\right).$$

Illustration :

A box contains three coins, one coin is fair, one coin is two-headed, and one coin is weighted so that the probability of head appearing is 1/3. A coin is selected at random and tossed. Find the probability that (i) head (ii) tail appears.

Sol. *Let E_1, E_2 and E_3 be the events of selecting at random first coin, second coin and third coin respectively.*

$$\therefore \quad P(E_1) = \frac{1}{3},\ P(E_2) = \frac{1}{3} \text{ and } P(E_3) = \frac{1}{3}$$

Let H and T be events of getting head and tail respectively.

$$\therefore \quad P\left(\frac{H}{E_1}\right) = \frac{1}{2},\ P\left(\frac{T}{E_1}\right) = \frac{1}{2} \qquad (\because \text{ First coin is fair})$$

$$P\left(\frac{H}{E_2}\right) = 1,\ P\left(\frac{T}{E_2}\right) = 0 \qquad (\because \text{ Second coin is two-headed})$$

(i) $P(\text{getting head}) = P(H) = P(E_1H \text{ or } E_2H \text{ or } E_3H)$

$$= P(E_1H) + P(E_2H) + P(E_3H)$$

$$= P(E_1)\, P\left(\frac{H}{E_1}\right) + P(E_2)\, P\left(\frac{H}{E_2}\right) + P(E_3)\, P\left(\frac{H}{E_3}\right)$$

$$= \frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times 1+\frac{1}{3}\times\frac{1}{3}=\frac{11}{18}$$

(ii) $P(\text{getting tail}) = P(T)$

$$= P(E_1T \text{ or } E_3T) = P(E_1T) + P(E_3T) = P(E_1)\, P\left(\frac{T}{E_1}\right) + P(E_3)\, P\left(\frac{T}{E_3}\right)$$

$$= \frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{2}{3}=\frac{7}{18}$$

Illustration :

There are two bags. The first bag contains 5 white and 3 black balls and the second bag contains 3 white and 5 black balls. Two balls are drawn at random from the first bag and are put into the second bag, without nothing their colours. Then two balls are drawn from the second bag. Find the probability that the balls drawn are white and black.

Sol.

5 White 3 Black	3 White 5 Black
Bag-I	Bag-II

Let E_1, E_2 and E_3 be the events of transferring 2 white, 1 white and 1 black, 2 black balls respectively from the fist bag to the second bag.

$$\therefore \quad P(E_1) = \frac{{}^5C_2}{{}^8C_2} = \frac{10}{28} = \frac{5}{14}$$

$$P(E_2) = \frac{{}^5C_1 \times {}^8C_1}{{}^8C_2} = \frac{5\times 3}{28} = \frac{15}{28}$$

$$P(E_3) = \frac{{}^3C_2}{{}^8C_2} = \frac{3}{28}$$

Let A be the event of drawing one white and one black ball from the second bag.

$$P(A) = P(E_1A \text{ or } E_2A \text{ or } E_3A)$$

$$= P(E_1A) + P(E_2A) + P(E_3A)$$

$$= P(E_1)\, P\left(\frac{A}{E_1}\right) + P(E_2)\, P\left(\frac{A}{E_2}\right) + P(E_3)\, P\left(\frac{A}{E_3}\right)$$

$$= \frac{5}{14} \times \frac{{}^5C_1 \times {}^5C_1}{{}^{10}C_2} + \frac{15}{28} \times \frac{{}^4C_1 \times {}^6C_1}{{}^{10}C_2} + \frac{3}{28} \times \frac{{}^3C_1 \times {}^7C_1}{{}^{10}C_2}$$

$$= \frac{5}{14} \times \frac{5}{9} + \frac{15}{28} \times \frac{8}{15} + \frac{3}{28} \times \frac{7}{15} = \frac{673}{12600}$$

Illustration :

Two machines A and B produce respectively 60% and 40% of the total numbers of items of a factory. The percentages of defective output of these machines are respectively 2% and 5%. If an item is selected at random, what is the probability that the item is (i) defective (ii) non-defective?

Sol. Let E_1, E_2 be the events of drawing an item produced by machine A and machine B respectively. Let A be the event of selecting a defective item.

$\therefore$ $\bar{A}$ represent the event of selecting a non-defective item.

We have

$P(E_1) = 60\%$; $P(E_2) = 40\%$

$P\left(\frac{A}{E_1}\right)$ = Probability that an item produced A is defective

$= 2\%$

$P\left(\frac{A}{E_2}\right)$ = Probability that an item produced by B is defective

$= 5\%$

(i) P(selected item is defective)

$= P(A) = P(E_1 A \text{ or } E_2 A) = P(E_1 A) + P(E_2 A)$

$= P(E_1)\, P\left(\frac{A}{E_1}\right) + P(E_2)\left(\frac{A}{E_2}\right)$

$= (60\%)\ (2\%) + (40\%)\ (5\%)$

$= \frac{60}{100} \times \frac{2}{100} + \frac{40}{100} \times \frac{5}{100} = \frac{320}{1000} = 0.032$

(ii) P(selected item is non-defective)

$= P(\bar{A}) = P(E_1 \bar{A} \text{ or } E_2 \bar{A}) = P(E_1 \bar{A}) + P(E_2 \bar{A})$

$= P(E_1)\, P\left(\frac{\bar{A}}{E_1}\right) + P(E_2)\, P\left(\frac{\bar{A}}{E_2}\right)$

$= (60\%)\ (98\%) + (40\%)\ (95\%)$

$= \frac{60}{100} \times \frac{98}{100} + \frac{40}{100} \times \frac{95}{100} = \frac{9680}{10000} = 0.968$

BAYE'S THEOREM :

If an event A can occur only with one of the n pair wise mutually exclusive and exhaustive events $B_1, B_2, ... B_n$ & if the conditional probabilities of the events.

$P(A/B_1), P(A/B_2) ....P(A/B_n)$ are known then, $P(B_1/A) = \dfrac{P(B_i)P(A/B_i)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)}$

Proof :

The event A occurs with one of the 'n' mutually exclusive e exhaustive events $B_1, B_2, B_3, ..., B_n$

$$A = AB_1 + AB_2 + AB_3 + ... + AB_n$$

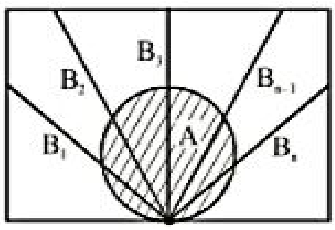
$$P(A) = P(AB_1) + P(AB_2) + ... + P(AB_n) = \sum_{i=1}^{n} P(AB_i)$$

Note :

A = event what we have,
B_1 = event what we want,
$B_1, B_2, ..., B_n$ are alternative events.

Now,

$$P(AB_i) = P(A) \cdot P\left(\frac{B_i}{A}\right) = P(B_i) \cdot P\left(\frac{A}{B_i}\right)$$



$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i)P\left(\frac{A}{B_i}\right)}{P(A)} = \frac{P(B_i) \cdot P\left(\frac{A}{B_i}\right)}{\sum_{i=1}^{n} P(AB_i)} = \frac{P(B_i) \cdot P\left(\frac{A}{B_i}\right)}{\sum_{i=1}^{n} P(B_i) \cdot P\left(\frac{A}{B_i}\right)}$$

Illustration :

Bag A contains 3 white and 2 black balls. Bag B contains 2 white and 2 black balls. One ball is drawn at random from A and transferred to B. One ball is selected at random from B and is found to be white. The probability that the transferred ball is white is

(A) $\frac{8}{13}$ *(B)* $\frac{5}{13}$ *(C)* $\frac{4}{13}$ *(D)* $\frac{9}{13}$

Sol. *Let E_1 and E_2 denote the events of the transferred ball being white and black, respectively. W denotes the drawn ball from B is white. By hypothesis,*

$$P(E_1) = \frac{{}^3C_1}{{}^5C_1} = \frac{3}{5} \quad , \quad P(E_2) = \frac{{}^2C_1}{{}^5C_1} = \frac{2}{5}$$

$$P\left(\frac{W}{E_1}\right) = \frac{{}^3C_1}{{}^5C_1} = \frac{3}{5}\ , \quad P\left(\frac{W}{E_2}\right) = \frac{{}^2C_1}{{}^5C_1} = \frac{2}{5}$$

By Boyes' theorem

$$P\left(\frac{E_1}{W}\right) = \frac{P(E_1)P\left(\frac{W}{E_1}\right)}{P(E_1)P\left(\frac{W}{E_1}\right) + P(E_2)P\left(\frac{W}{E_2}\right)} = \frac{\frac{3}{5}\times\frac{3}{5}}{\frac{3}{5}\times\frac{3}{5}+\frac{2}{5}\times\frac{2}{5}} = \frac{9}{13}$$

Illustration :

A letter is to come from either LONDON or CLIFTON. The postal mark on the letter legibly shows consecutive letters "ON". The probability that the letter has come from LONDON is

(A) $\frac{12}{17}$ (B) $\frac{13}{17}$ (C) $\frac{5}{17}$ (D) $\frac{4}{17}$

Sol. *Let the events be defined as*

E_1 : *Letter coming from LONDON*
E_2 : *Letter coming from CLIFTON*
E_3 : *Two consecutive letters ON.*

The word LONDON contains 5 types of consecutive letters (LO, ON, ND, DO, ON) of which there are two ON's. The word CLIFTON contains 6 types of consecutive letters (CL, LI, IF, FT, TO, ON) of which there is one "ON". Now

$$P(E_1) = \frac{1}{2} = P(E_2) \quad \Rightarrow \quad P\left(\frac{E_3}{E_2}\right) = \frac{2}{5} \quad \text{and} \quad P\left(\frac{E_3}{E_2}\right) = \frac{1}{6}$$

By Boyes' theorem

$$P\left(\frac{E_1}{E_3}\right) = \frac{\frac{1}{2}\times\frac{2}{5}}{\frac{1}{2}\times\frac{2}{5}+\frac{1}{2}\times\frac{1}{6}} = \frac{12}{17}$$

Illustration :

In a factory which manufactures bolts, machines A, B and C manufacture respectively 25%, 35% and 40% of the bolts. Of their outputs, 5, 4 and 2 percent are respectively defective bolt. A bolts is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B?

Sol. *Let events B_1, B_2, B_3 be the following*

B_1 : *the bolt is manufactured by machine A*
B_2 : *the bolt is manufactured by machine B*
B_3 : *the bolt is manufactured by machine C*

Clearly, B_1, B_2, B_3 are mutually exclusive and exhaustive events and hence, they represent a partition of the sample space.

Let the event E be 'the bolt is defective'.

The event E occurs with B_1 or with B_2 or with B_3. Given that,

$$P(B_1) = 25\% = 0.25,\ \ P(B_2) = 0.35\ \text{ and } P(B_3) = 0.40$$

Again $P(E|B_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine

$$A = 5\% = 0.05$$

Similarly, $\quad P(E|B_2) = 0.04,\ \ P(E\,|\,B_3) = 0.02$

Hence, by Bayes' Theorem, we have

$$P(B_2|E) = \frac{P(B_2)\ P(E\,|\,B_2)}{P(B_1)\ P(E\,|\,B_1) + P(B_2)\ P(E\,|\,B_2)\ +\ P(B_3)\ P(E\,|\,B_3)}$$

$$= \frac{0.35\times 0.04}{0.25\times 0.05\times 0.35\times 0.04 + 0.40\times 0.02} = \frac{0.0140}{0.0345} = \frac{28}{69}$$

Illustration :

In a test, an examinee either guesses or copies or knows the answer for a multiple choice question having FOUR choices of which exactly one is correct. The probability that he makes a guess is 1/3 and the probability for copying is 1/6. The probability that his answer is correct, given that he copied it is 1/8. The probability that he knew the answer, given that his answer is correct is

(A) $\frac{5}{29}$ (B) $\frac{9}{29}$ (C) $\frac{24}{29}$ (D) $\frac{20}{29}$

Sol. *Let the events be defined as*

E_1 : *Guessing*
E_2 : *Copying*
E_3 : *Knowing*
E : *Correct answer*

By hypothesis,

$$P(E_1) = \frac{1}{3},\ P(E_2) = \frac{1}{6},\ P(E_3) = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$$

$$P\left(\frac{E}{E_1}\right) = \frac{1}{4} \quad \textit{(out of four choices only one is correct)}$$

$$P\left(\frac{E}{E_2}\right) = \frac{1}{8}$$

$$P\left(\frac{E}{E_3}\right) = 1$$

Therefore by Bayes' theorem

$$P\left(\frac{E_3}{E}\right)=\frac{P(E_3)P\left(\frac{E}{E_3}\right)}{P(E_1)P\left(\frac{E}{E_1}\right)+P(E_2)P\left(\frac{E}{E_2}\right)+P(E_3)P\left(\frac{E}{E_3}\right)}=\frac{\frac{1}{2}\times 1}{\frac{1}{3}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{8}+\frac{1}{2}\times 1}=\frac{24}{29}$$

Illustration :

A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{12}$, if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

Sol. *Let E be the event that the doctor visits the patient late and let T_1, T_2, T_3, T_4 be the events that the doctor comes by train, bus scooter, and other means of transport respectively.*

Then $P(T_1) = \frac{3}{10}$, $P(T_2) = \frac{1}{5}$, $P(T_3) = \frac{1}{10}$ *and* $P(T_4) = \frac{2}{5}$ *(given)*

$P(E|T_1)$ = *Probability that the doctor arriving late comes by train* $= \frac{1}{4}$

Similarly, $P(E|T_2) = \frac{1}{3}$, $P(E|T_3) = \frac{1}{12}$ *and* $P(E|T_4) = 0$*, since he is not late if he comes by other means by other means of transport.*

Therefore, by Bayes' Theorem, we have

$P(T_1|E)$ = *Probability that the doctor arriving late comes by train*

$$=\frac{P(T_1)\,P(E|T_1)}{P(T_1)\,P(E|T_1)+P(T_2)\,P(E|T_2)+P(T_3)\,P(E|T_3)+P(T_4)\,P(E|T_4)}$$

$$=\frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0}=\frac{3}{40}\times\frac{120}{8}=\frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$.

Illustration :

Suppose that the reliability of a HIV test is specified as follows :

Of people having HIV, 90% of the test detect the disease but 10% go undetected. Of people free of HIV, 99% of the test are judged HIV –ive but 1% are diagnosed as showing HIV +ve. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV +ve. What is the probability that the person actually has HIV?

Sol. *Let E denote the event that the person selected is actually having HIV and A the event that the person's HIV test is diagnosed as +ve. We need to find P(E|A). Also E' denotes the event that the person selected is actually not having HIV.*

Clearly, {E, E'} is a partition of the sample space of all people in the population. We are given that

$$P(E) = 0.1\% = \frac{0.1}{100} = 0.001$$

$$P(E') = 1 - P(E) = 0.999$$

$P(A|E) = P$ (Person tested as HIV +ve given that he/she is actually having HIV)

$$= 90\% = \frac{90}{100} = 0.9$$

and $P(A|E') = P$(Person tested as HIV +ve given that he/she is actually not having HIV)

$$= 1\% = \frac{1}{100} = 0.01$$

Now, by Bayes' theorem

$$P(E|A) = \frac{P(E)\,P(A|E)}{P(E)\,P(A|E) + P(E')\,P(A|E')} = \frac{0.001 \times 0.9}{0.001 \times 0.9 + 0.999 \times 0.01} = \frac{90}{1089}$$

Thus, the probability that a person selected at random is actually having HIV given that he/she is tested HIV +ve is $\frac{90}{1089}$.

Illustration :

A bag contains 4 balls of unknown colours. A ball is drawn at random from it and is found to be white. The probability that all the balls in the bag are white is

(A) 4/5 (B) 1/5 (C) 3/5 (D) 2/5

Sol. *Let* W_j $(j = 1, 2, 3, 4)$ *denote 1, 2, 3 and 4 white balls are in the bag. Let W be the ball drawn is white. Then* $P(W_1) = P(W_2) = P(W_3) = P(W_4) = \frac{1}{4}$

$$P\left(\frac{W}{W_1}\right) = \frac{1}{4},\quad P\left(\frac{W}{W_2}\right) = \frac{2}{4},\quad P\left(\frac{W}{W_3}\right) = \frac{3}{4},\quad P\left(\frac{W}{W_4}\right) = 1$$

Therefore by Bayes' theorem

$$P\left(\frac{W}{W_4}\right) = \frac{P(W_4)\,P\left(\frac{W}{W_4}\right)}{\sum_{j=1}^{4} P(W_j)\,P\left(\frac{W}{W_j}\right)} = \frac{\frac{1}{4}\times 1}{\frac{1}{4}\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}+\frac{4}{4}\right)} = \frac{4}{10} = \frac{2}{5}$$

PROBABILITIES THROUGH STATISTICAL (STOCHASTIC) TREE DIAGRAM:

Illustration :

A : box contains three coins A, B and C

A : Normal coin; B : Double Headed (DH) coin ; C : a weighted coin so that $P(H) = \frac{1}{3}$

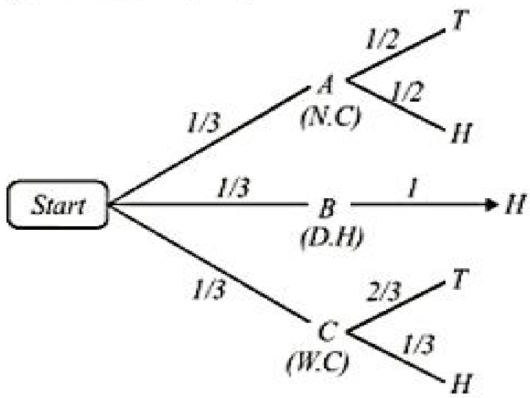
A coin is randomly selected & tossed

(A) Find the probability that head appears.

(B) If head appear find the probability that it is a normal coin P(A/H)

(C) Find the probability that tail appears.

(D) If tail appears, find the probability that it is a weighted coin P(C/T)



Sol. (A) $P(H) = \frac{1}{3}\cdot\frac{1}{2}+\frac{1}{3}\cdot 1+\frac{1}{3}\cdot\frac{1}{3} = \frac{11}{18}$

(B) $P\left(\frac{A}{H}\right) = \frac{P(A\cap B)}{P(H)} = \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{11}{18}} = \frac{3}{11}$

(C) $P(T) = \frac{1}{3}\cdot\frac{1}{2}+\frac{1}{3}\cdot 0+\frac{1}{3}\cdot\frac{2}{3} = \frac{7}{18}$

or $1 - P(H) = 1 - \frac{11}{18} = \frac{7}{18}$

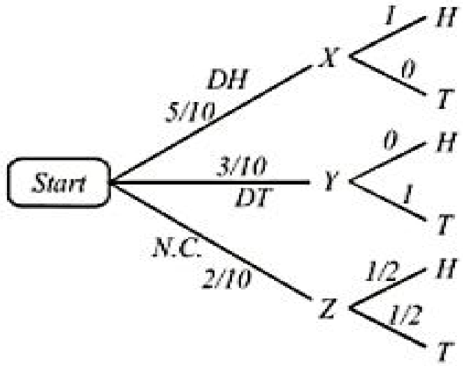
(D) $P\left(\frac{C}{T}\right) = \frac{P(C\cap T)}{P(T)} = \frac{\frac{1}{3}\cdot\frac{2}{3}}{\frac{7}{18}} = \frac{4}{7}$

Illustration :

A box contains 10 coins ⟨ *5 coins DH denoted by say X* / *3 coins DT denoted by say Y* / *2 coins normal denoted by Z*

A coin is dream at random from the box and tossed, fall headwise. Find the probability that it was a normal coin.

Sol. $P\left(\frac{Z}{H}\right)=\frac{P(H\cap Z)}{P(H)}$



$$P(H)=\frac{5}{10}\cdot 1+\frac{2-1}{10-2}=\frac{6}{10}=\frac{\frac{2}{10}\cdot\frac{1}{2}}{\frac{6}{10}}=\frac{1}{6}$$

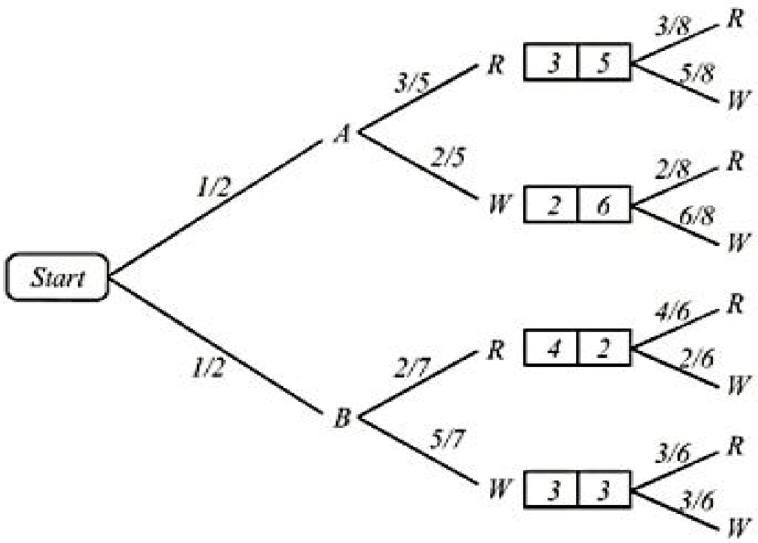
Illustration :

Let the contents of the two boxes A and B with respect to number of R and W marbles is as given below:

Bag	R	W
A	3	2
B	2	5

A bag is selected at random; a marble is drawn and put into the other box; then a marble is drawn from the second box. Find the probability the both marbles drawn the of same colour.

Sol.



$$P(E)=\frac{1}{2}\cdot\frac{3}{5}\cdot\frac{3}{8}+\frac{1}{2}\cdot\frac{2}{5}\cdot\frac{6}{8}+\frac{1}{2}\cdot\frac{2}{7}\cdot\frac{4}{6}+\frac{1}{2}\cdot\frac{5}{7}\cdot\frac{3}{6}=\frac{901}{1680}$$

Practice Problem

Q.1 Three political parties namely the Congress, the B.J.P. and the Janta Dal are contesting for a state legislative assembly elections. The state does not have a common entrance test after the 12^{th} standard, for the admissions to the medical or engineering colleges. The probabilities of these parties winning the elections are $\frac{1}{3}$, $\frac{4}{9}$ & $\frac{2}{9}$ respectively. If the Congress comes to the power, the probability of its introducing the common entrance test for the state is 0.6 and the corresponding probabilities for the B.J.P. and Janta Dal are 0.7 and 0.5 respectively. Find the probability that the common entrance test is introduced.

Q.2 Three persons Mr. Iyyengar, Dr. Singh and Prof. Mukherjee and competing for the post of the principal of a degree college exclusively meant for boys. Their chances are, respectively, 0.5, 0.3 and 0.2. If Mr. Iyyengar is selected, he will introduce co-education with probability 0.5 while the probabilities are 0.7 and 0.6 with regard to Dr. Singh and Prof. Mukherjee, respectively. Co-education is introduced in the college. The probability that Dr. Singh is selected as principal is

(A) $\frac{31}{58}$ (B) $\frac{21}{58}$ (C) $\frac{27}{58}$ (D) $\frac{37}{58}$

Q.3 In a city 60% are males and 40% are females. Suppose 50% of males and 30% of females have colour blindness. One is selected at random and is found to be colour blind. The probability that the selected person is male is

(A) $\frac{10}{19}$ (B) $\frac{12}{87}$ (C) $\frac{9}{19}$ (D) $\frac{5}{7}$

Q.4 A person has three coins A, B and C in his pocket out of which A is a fair coin. The probability of B showing head is $\frac{2}{3}$ and that of C is $\frac{1}{3}$. He selected one of the coins at random and tossed it three times and observed 2 heads and 1 tail. The probability that the selected coin is A is

(A) $\frac{7}{25}$ (B) $\frac{18}{25}$ (C) $\frac{9}{25}$ (D) $\frac{16}{25}$

Q.5 A bag contains 6R and 4W balls. 4 balls are drawn one by one without replacement and were found to be at least two white. Find the probability that the next draw of a ball from this bag will give a white ball.

Q.6 Box A contains nine cards numbered 1 through 9, and box B contains five cards numbered 1 through 5. A box is chosen at random and a card drawn; if the card shows an even number, another card is drawn from the same box. If the card shows an odd number, a card is drawn from the other box ;

(i) What is the probability that both cards show even numbers?

(ii) If both cards show even numbers, what is the probability that they come from box A?

(iii) What is the probability that both cards show odd numbers?

Answer key

Q.1 $\frac{28}{45}$ Q.2 B Q.3 D Q.4 C Q.5 $\frac{34}{115}$

Q.6 (i) 2/15 ; (ii) 5/8 ; (iii) 1/3

MATHEMATICAL EXPECTATION (PRACTICAL USE OF PROBABILITY IN DAY TO DAY LIFE):

It is worthwhile indicating that if 'P' represents a person's chance of success in any venture and 'M 'the sum of money which he will receive in case of success, then the sum of money denoted by 'P·M' is called his expectation.

Illustration :

Two players of equal skill A and B are playing a game. They leave off playing (due to some force majeure conditions) when A wants 3 points and B wants 2 to win. If the prize money is Rs.16000/-. How can the referee divide the money in a fair way.

Sol. *Since, A wins if he scores 3 points before B scores 2.*

Probability of A's scoring a point = Probability of B's scoring at point = $\frac{1}{2}$

Hence, required probability that A succeeds

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{5}{16}$$

Probability that B succeeds $= 1 - \frac{5}{16} = \frac{11}{16}$

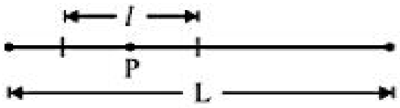
$\therefore$ $A\text{'s expectation} = \frac{5}{16} \times 16000 = 5000$

$B\text{'s expectation} = \frac{11}{16} \times 16000 = 11000$

GEOMETRICAL PROBABILITY (CONTINUOUS SAMPLE SPACE) :

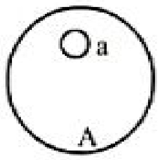
(1) One-dimensional Probability :

$$P = \frac{\text{favourable length}}{\text{total length}}$$



(2) Two-dimensional Probability :

$$P = \frac{\text{favourable area}}{\text{total area}}$$



(3) Three-dimensional Probability :

$$P = \frac{\text{favourable volume}}{\text{total volume}}$$

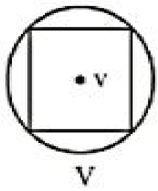
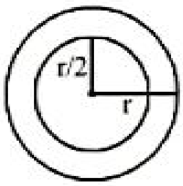


Illustration :

A point is taken inside a circle of radius find the probability that the point is closer to the centre as a circumference.

Sol. $n(s) = \pi r^2$



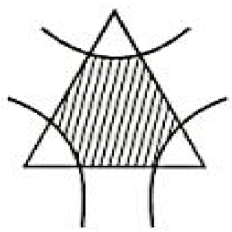
$$n(A) = \pi\left(\frac{r}{2}\right)^2$$

$$P = \frac{\pi\left(\frac{r}{2}\right)^2}{\pi r^2} = \frac{1}{4} \qquad \textit{Ans.}$$

Illustration :

A point is selected random inside a equilateral triangle whose length of side is 3. Find the probability that its distance from any corner is greater than 1.

Sol. $\text{Area of sector} = \frac{r^2\theta}{2}$



$$n(s) = \frac{\sqrt{3}}{4}\cdot 9$$

$$n(A) = \frac{\sqrt{3}}{4}\cdot 9 - 3\,\frac{1\cdot 1}{2}\cdot\frac{\pi}{3} \qquad \therefore\ P(A) = \frac{\frac{\sqrt{3}}{4}\cdot 9 - \frac{\pi}{2}}{\frac{\sqrt{3}}{4}\cdot 9} = 1 - \frac{2\pi}{9\sqrt{3}}$$

Illustration :

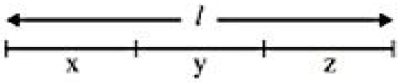
A sphere of radius r is circumscribed about a cube. Find the probability that a point lies in the sphere but out side the cube.

Sol. $$P = \frac{\text{fav. volume}}{\text{total volume}} = \frac{\frac{4\pi}{3}r^3 - \left(\frac{2r}{\sqrt{3}}\right)^3}{\frac{4}{3}\pi r^3} = 1 - \frac{2}{\sqrt{3}\pi}$$ ***Ans.***

Illustration :

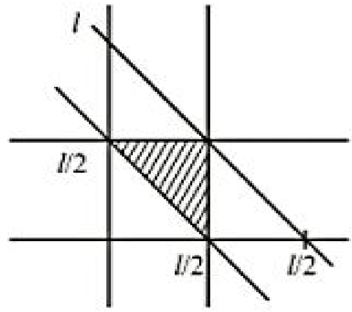
A stick of length l is broken into three parts find the probability that these three parts from a triangle.

Sol.



$x > 0 \quad y > 0, \; l-(x+y) > 0 \quad z = l-(x+y)$

For a triangle sum of two sides should be greater than third side



$x + y > l - (x+y) \qquad x + y > \frac{l}{2}$ $\qquad$ ......(i)

$x + l - (x+y) > y \qquad l > 2y \qquad y < \frac{l}{2}$ $\qquad$ ......(ii)

Similarly $\qquad x < \frac{l}{2}$ $\qquad$ ....(iii)

So required probability $$= \frac{\frac{l}{2}\times\frac{l}{2}\times\frac{l}{2}}{\frac{1}{2}\times l\times l} = \frac{1}{4}.$$ ***Ans.***

COINCIDENCE TESTIMONY :

If p_1 and p_2 are the probabilities of speaking the truth of two independent witnesses A and B who give the same statement

$$\text{P (their combined statement is true)} = P(H_1 / H_1 \cup H_2) = \frac{p_1 p_2}{p_1 p_2 + (1-p_1)(1-p_2)}.$$

where H_1 means both speaks the truth and H_2 means both speaks false.

In this case it has been assumed that we have no knowledge of the event except the statement made by A and B.

However if p is the probability of the happening of the event before their statement then

$$\text{P (their combined statement is true)} = \frac{p p_1 p_2}{p p_1 p_2 + (1-p)(1-p_1)(1-p_2)}.$$

Here it has been assumed that the statement given by all the independent witnesses can be given in two ways only, so that if all the witnesses tell falsehoods they agree in telling the same falsehood.
If this is not the case and c is the chance of their coincidence testimony then the

Probability that the statement is true = $P p_1 p_2$

Probability that the statement is false = $(1-p).c\,(1-p_1)(1-p_2)$

However chance of coincidence testimony is taken only if the joint statement is not contradicted by any witness.

Illustration :

A speaks the truth 3 out of 4 times, and B 5 out of 6 times. What is the probability that they will contradict each other in stating the same fact.

Sol. $P(A) = \frac{3}{4}$; $P(B) = \frac{5}{6}$

$$P(contradict) = \frac{3}{4}\cdot\frac{1}{6}+\frac{5}{6}\cdot\frac{1}{4}=\frac{8}{24}=\frac{1}{3}$$

Illustration :

A speaks truth 3 times out of 4, and B 7 times out of 10. They both assert that a white ball has been drawn from a bag containing 6 balls of different colours; find the probability of the truth of their assertion. P(A) = 3/4; P(B)=7/10

Sol. *There are 2 hypothesis*   *(i) their coincidence testimony is true*
(ii) it is false

H_1 *: white ball is actually drawn & both speaks the truth*

$$P(H_1) = \frac{1}{6}\cdot\frac{3}{4}\cdot\frac{7}{10}$$

H_2 *: (white has not been withdrawn) and (their statement coincides) and they both speaks false*

$$P(H_2) = \frac{5}{6}\times\frac{1}{5}\times\frac{1}{5}\times\frac{1}{4}\times\frac{3}{10}$$

Let E : their assertion is true

$$\therefore \quad P(E) = P\left(\frac{H_1}{H_1 \cup H_2}\right) = \frac{\frac{1}{6}\cdot\frac{3}{4}\cdot\frac{7}{10}}{\frac{1}{6}\cdot\frac{3}{4}\cdot\frac{7}{10}+\frac{5}{6}\left(\frac{1}{25}\right)\cdot\frac{1}{4}\cdot\frac{3}{10}} = \frac{35}{36}$$

Practice Problem

Q.1 10 witnesses each of whom the probability of speaking the truth as 5/6, assert that certain event took place. If the probability of this event before their statement is $\frac{1}{1+5^9}$, find the probability of the truth of their assertion.

Q.2 A man is know to speak the truth 3 out if 4 times. He throws a die and reports that it is a six. The probability that it is actually a six is

(A) $\frac{3}{8}$ (B) $\frac{1}{5}$ (C) $\frac{3}{4}$ (D) None of these

Q.3 2 hunters A and B shot at a bear simultaneously. The bear was shot dead with only one hole in its body. Probability of A shooting the bear is 0.8 and that of B shooting the bear is 0.4. The hide was sold for Rs 280/-. divide this money in a fair way.

Q.4 Two points are picked at random on the unit circle $x^2 + y^2 = 1$. The probability that the chord joining the two points has length at least 1, is

(A) 4/9 (B) 1/3 (C) 1/6 (D) 2/3

Q.5 A circle of radius 'a' is inscribed in a square of side 2a. Find the probability that a point chosen at random is inside the square but outside the circle.

Q.6 A cloth of length 10 meter is to be randomly distributed among three brothers. Find the probability that no one gets more than 4 meter of cloth.

Q.7 A starts from a town 'X' any time between 1PM to 4PM walks towards the town 'Y' on a straight road with a speed of 4km/hr and B starts from 'Y' at any time between 1 PM to 4PM and walks towards 'X' at 4 km/hr. Assuming all times of starting all equally likely, find the odds in favour of their meeting on the way.

Answer key

Q.1 5/6 Q.2 A

Q.3 A's share = $\frac{0.48}{0.56} \times 280 = 240/-$; B's share = $\frac{0.08}{0.56} \times 280 = 40/-$

Q.4 D Q.5 $P(E) = 1 - p/4$ Q.6 $\frac{1}{25}$ Q.7 5 : 4

RANDOM VARIABLE :

Random variables are of two types :

(i) Discreate random variable.
(ii) Continuous random variables.

(i) A random variable is called a discrete random variable if it can take only finitely many values. For example, in the experiment of drawing three cards from a pack of playing cards, the random variable "number of kings drawn" is a discrete random variable taking value either 0 or 1 or 2 and 3.

(ii) A random variable is called a continuous random variable if it can take any value between certain limits. For example, height, weight of students in a class are continuous random variables.

Probability Distribution of a discrete random variable :

Let x be a discrete random variable assuming values $x_1, x_2, x_3, \ldots\ldots, x_n$ corresponding to the various outcomes of a random experiment. If the probability of occurrence of $x = x_i$ is $P(x_i) = p_i$, $1 \le i \le n$ such that $p_1 + p_2 + p_3 + \ldots\ldots + p_n = 1$, then the function, $P(x_i) = P_i$, $1 \le i \le n$ is called the probability function of the random variable x and the set $\{P(x_1), P(x_2), P(x_3), \ldots\ldots, P(x_n)\}$ is called the probability distribution of x.

Illustration :

Three balls are drawn are by one without replacement from a bag containing 5 white and 4 red balls. Find the probability distribution of the number of red balls drawn.

5 White
4 Red

Sol. *Let x denote the discrete random variable "number of red balls".*

$\therefore$ *The possible values of x are 0, 1, 2, 3.*

Let R_i be the event of drawing a red ball from the bag in the ith draw, i = 1, 2, 3.

$$P(x=0) = P(\bar{R}_1\bar{R}_2\bar{R}_3) = P(\bar{R}_1)\; P\left(\frac{\bar{R}_2}{\bar{R}_1}\right) P\left(\frac{\bar{R}_3}{\bar{R}_1\bar{R}_2}\right) = \frac{5}{9}\times\frac{4}{8}\times\frac{3}{7} = \frac{60}{504} = \frac{5}{42}$$

$$P(x=1) = P\left(R_1\bar{R}_2\bar{R}_3 \quad \text{or} \quad \bar{R}_1R_2\bar{R}_3 \quad \text{or} \quad \bar{R}_1\bar{R}_2R_3\right)$$

$$= P(R_1)\; P\left(\frac{\bar{R}_2}{R_1}\right) P\left(\frac{\bar{R}_3}{R_1\bar{R}_2}\right) + \frac{P(\bar{R}_1)}{P\left(\frac{R_2}{\bar{R}_1}\right)P\left(\frac{\bar{R}_3}{\bar{R}_1R_2}\right)} + P(\bar{R}_1)\; P\left(\frac{\bar{R}_2}{\bar{R}_1}\right) P\left(\frac{R_3}{\bar{R}_1\bar{R}_2}\right)$$

$$= \frac{4}{9}\times\frac{5}{8}\times\frac{4}{7} + \frac{5}{9}\times\frac{4}{8}\times\frac{4}{7} + \frac{5}{9}\times\frac{4}{8}\times\frac{4}{7} = \frac{240}{504} = \frac{10}{21}$$

$$P(x = 2) = P\left(R_1 R_2 \overline{R}_3 \quad \text{or} \quad R_1 \overline{R}_2 R_3 \quad \text{or} \quad \overline{R}_1 R_2 R_3\right)$$

$$= P(R_1)\, P\left(\frac{R_2}{R_1}\right) P\left(\frac{\overline{R}_3}{R_1R_2}\right) + P(R_1)\, P\left(\frac{\overline{R}_2}{R_1}\right) P\left(\frac{R_3}{R_1\overline{R}_2}\right) + P(\overline{R}_1)P\left(\frac{R_2}{\overline{R}_1}\right) P\left(\frac{R_3}{\overline{R}_1R_2}\right)$$

$$= \frac{4}{9}\times\frac{3}{8}\times\frac{5}{7}+\frac{4}{9}\times\frac{5}{8}\times\frac{3}{7}+\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}=\frac{180}{504}=\frac{5}{14}$$

$$P(x = 3) = P(R_1R_2R_3) = R(R_1)\, P\left(\frac{R_2}{R_1}\right) P\left(\frac{R_3}{R_1R_2}\right) = \frac{4}{9}\times\frac{3}{8}\times\frac{2}{7}=\frac{24}{504}=\frac{1}{21}$$

The required Probability distribution is

x	0	1	2	3
$P(x)$	$\frac{5}{42}$	$\frac{10}{21}$	$\frac{5}{14}$	$\frac{1}{21}$

MEAN AND VARIANCE OF A PROBABILITY DISTRIBUTION :

(1) Mean :-

If a random variable X assumes the values $x_1, x_2, \ldots\ldots, x_n$ with probabilities $p_1, p_2, \ldots\ldots, p_n$ respectively then the mean of X is defined by

Rendom variable (x_i)	Probability (p_i)	p_ix_i
x_1	p_1	p_1x_1
x_2	p_2	p_2x_2
$\vdots$	$\vdots$	$\vdots$
x_n	p_n	p_nx_n

Then mean $(\mu) = \dfrac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i} = \sum_{i=1}^{n} x_i\, p_i$ $\qquad \left\{\sum_{i=1}^{n} p_i = 1\right\}$

(2) Variance :

$$\sigma^2 = \sum_{i=1}^{n} p_i(x_i-\mu)^2 = \sum_{i=1}^{n} p_i\left(x_i^2 - 2\mu x_i + \mu^2\right) = \sum_{i=1}^{n}\left(p_ix_i^2 - 2\mu p_ix_i + p_i\mu^2\right)$$

$$= \sum_{i=1}^{n} p_ix_i^2 - 2\mu\sum_{i=1}^{n} p_ix_i + \mu^2\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} p_ix_i^2 - \mu^2$$

(3) Standard Deviation :

$$SD = +\sqrt{\sigma^2}$$

Illustration :

Two bad eggs are accidently mixed with 10 good eggs 3 eggs are drawn simultaneously from the basket. Find the mean and variance of the number of bad eggs drawn.

Sol. $\dfrac{x=0}{P(0)} = \dfrac{^{10}C_3}{^{12}C_3} = \dfrac{10\cdot 9\cdot 8}{12\cdot 11\cdot 10} = \dfrac{6}{11}$

$x = 1$

$$P(1) = \frac{^{2}C_1 \times {^{10}C_2}}{^{12}C_3} = \frac{9}{22}$$

$x = 2$

$$P(2) = \frac{^{2}C_2 \cdot {^{10}C_1}}{^{12}C_3} = \frac{10\cdot 6}{12\cdot 11\cdot 10} = \frac{1}{22}$$

x_i	p_i	$p_i x_i$
0	$\frac{6}{11}$	0
1	$\frac{9}{22}$	$\frac{9}{22}$
2	$\frac{1}{22}$	$\frac{2}{22}$

$$\mu = \sum p_i x_i = \frac{11}{22} = \frac{1}{2}$$

$$\therefore \sum p_i x_i^2 = 0 + \frac{9}{22} + \frac{4}{22}$$

$$\sigma^2 = \sum p_i x_i^2 - \mu^2 = \frac{13}{22} - \frac{1}{4} = \frac{15}{44}$$ ***Ans.***

BINOMIAL PROBABILITY DISTRIBUTION :

Let an experiment has n independent trials and each of the trial has two possible out comes i.e. success or failure.

If getting number of successes in the experiment is a random variable then probability of getting exactly r-successes is -

$$P(x = r) = {}^nC_r\, p^r \cdot q^{n-r}$$

where p = probability of getting success

and q = probability of getting failure

x_i	p_i	$p_i x_i$	$p_i x_i^2$
0	${}^nC_0 p^0 q^n$	$0 \times {}^nC_0 p^0 q^n$	$0^2 \cdot {}^nC_0 p^0 q^n$
1	${}^nC_1 p^1 q^{n-1}$	$1 \times {}^nC_1 p^1 q^{n-1}$	$1^2 \cdot {}^nC_1 p^1 q^{n-1}$
2	${}^nC_2 p^2 q^{n-2}$	$\vdots$	$\vdots$
3	${}^nC_3 p^3 q^{n-3}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	${}^nC_r p^r q^{n-r}$	$r \times {}^nC_r p^r \cdot q^{n-r}$	$r^2 \cdot {}^nC_r p^r q^{n-r}$

Mean of BPD of a random variable

$$\mu = \sum p_i x_i = \sum_{r=0}^{n} r \cdot {}^nC_r \cdot p^r \cdot q^{n-r} = \sum_{r=0}^{n} r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r} = p \cdot n \sum_{r=1}^{n} {}^{n-1}C_{r-1} \cdot p^{r-1} q^{n-r}$$

$$= np\,[{}^{n-1}C_0 \cdot p^0\, q^{n-1} + {}^{n-1}C_1 \cdot p^1 q^{n-2} + \ldots\ldots + {}^{n-1}C_{n-1}\, p^{n-1}\, q^0]$$

$$= np\,(p+q)^{n-1} = np$$

Variance of BPD of a random variable :

$$\sigma^2 = \sum p_i x_i^2 - \mu^2$$

$$\sum p_i x_i^2 = \sum_{n=0}^{n} r^2 \cdot {}^nC_r \cdot p^r\, q^{n-r} = \sum_{r=0}^{n} r^r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r}$$

$$= \sum_{r=0}^{n} r^2 \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r} = \sum_{r=0}^{n} (r-1+1) \cdot {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r}$$

$$= \left[\sum_{r=0}^{n} (r-1) \cdot {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r} + \sum_{r=0}^{n} {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r}\right]$$

$$= p^2\, n \cdot (n-1) \sum_{n=r}^{n} {}^{n-2}C_{r-2} \cdot p^r \cdot q^{n-r} + pn \sum_{r=1}^{n} {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r}$$

$$= p^2 \cdot n(n-1)\,(p+q)^{n-2} + pn \cdot (p+q)^{n-1}$$

$$= p^2 \cdot n(n-1) + pn$$

$\therefore \quad \sigma^2 = p^2 n^2 - p^2 n + pn - n^2 p^2 \qquad \{\because\ \mu = np\}$

$\sigma^2 = pn(1-p) = npq$ **Ans.**

Standard deviation of BPD of a random variable :

Positive value of square root of variance is called standrard deviation.

$$SD = +\sqrt{\sigma^2} = \sqrt{npq}$$

Illustration :

A pair of dice is thrown 5 times if getting a doublet is considered as a success then find the mean & variance of the successes.

Sol. *Here* $n = 5$ *and favorable sample space are* $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)$

$\therefore\ p = 1/6$ *and* $q = 5/6$

Mean $= \mu = np = 5/6$

Variance $= \sigma^2 = npq = 25/36$

Illustration :

If difference between mean & variance of a BPD is 1 and difference between squares is 11 then find the probability of getting exactly 3 successes.

Sol. *Given*

$$np - npq = 1 \qquad \ldots\ldots(i)$$

$$n^2p^2 - n^2p^2q^2 = 11 \qquad \ldots\ldots(ii)$$

$$\therefore \quad np + npq = 11$$

$$npq = 5 \qquad q = 5/6 \qquad p = 1/6 \qquad n = 36$$

$$\therefore \text{Required probability} = {}^{36}C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{33}$$

Illustration :

If X follows a binomial distribution with mean 3 and variance $\left(\frac{3}{2}\right)$, *find*

(i) $P(X \geq 1)$ *(ii)* $P(X \leq 5)$

Sol. *We know that mean* $= np$ *and variance* $= npq$

$$\therefore \quad np = 3 \text{ and } npq = \frac{3}{2} \Rightarrow 3q = \frac{3}{2} \Rightarrow q = \frac{1}{2}$$

$$\therefore \quad p = (1 - q) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

Now, $np = 3$ *and* $p = \frac{1}{2} \Rightarrow n \times \frac{1}{2} = 3 \Rightarrow n = 6$

So, the binomial distribution is given by

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^6C_r \cdot \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{(6-r)} = {}^6C_r \left(\frac{1}{2}\right)^6$$

(i) $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^6C_0 \cdot \left(\frac{1}{2}\right)^6 = \left(1 - \frac{1}{64}\right) = \frac{63}{64}$$

(ii) $P(X \leq 5) = 1 - P(X = 6)$

$$= 1 - {}^6C_6 \left(\frac{1}{2}\right)^6 = \left(1 - \frac{1}{64}\right) = \frac{63}{64}$$

Illustration :

If the sum of the mean and variance of a binomial distribution for 5 trials is 1.8, find the distribution.

Sol. *We know that*

mean = up and variance = npq

It is being given that $n = 5$ *and mean + variance = 1.8*

$\therefore\ np + npq = 1.8,$ where $n = 5$

$\Leftrightarrow 5p + 5pq = 1.8$

$\Leftrightarrow p + p(1 - p) = 0.36 \qquad [\because\ q = (1 - p)]$

$\Leftrightarrow p^2 - 2p + 0.36 = 0$

$\Leftrightarrow 100\,p^2 - 200p + 36 = 0$

$\Leftrightarrow 25p^2 - 50p + 9 = 0$

$\Leftrightarrow 25p^2 - 45p - 5p + 9 = 0$

$\Leftrightarrow 5p\,(5p - 9) - (5p - 9) = 0$

$\Leftrightarrow (5p - 9)\,(5p - 1) = 0$

$\Leftrightarrow p = \frac{1}{5} = 0.2 \qquad [\because\ p \text{ cannot exceed } 1]$

Thus, $n = 5,\ p = 0.2,$ and $q = (1 - p) = (1 - 0.2) = 0.8$

Let X denote the binomial variate. Then, the required distribution is

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^5C_r \cdot (0.2)^r \cdot (0.8)^{(5-r)} \quad \text{where } r = 0, 1, 2, 3, 4, 5$$

Practice Problem

Q.1 A pair of fair dice is thrown. Let X be the random variable which denotes the minimum of the two numbers which appear. Find the probability distribution, mean and variance of X.

Q.2 If the mean and SD of a binomial variate X are 9 and 3/2 respectively. Find the probability that X takes a value greater than one.

Q3 A pair of dice is thrown 4 times. If getting a total of 9 in a single throw is considered as a success then find the mean and variance of the successes.

Q4 The sum of the product of the mean and variance of a binomial distribution are 24 and 128 respectively. Find the distribution.

Q.5 A fair coin is tossed four times. Let Y denotes the random variable denoting the "longest string of heads" occurring. Find the probability distribution, mean, variance and S.D. of Y.

Answer key

Q.1 $\mu = 2.5$; $\text{var}(X) = 2.1$

Q.2 $P(X>1) = 1 - P(X = 0 \text{ or } 1)$

Q3 Mean = 4/9 ; Variance = 32/81

Q4 ${}^{32}C_r \cdot \left(\frac{1}{2}\right)^{32}$

Q.5 $\mu = 1.7$; $\sigma^2 = 0.9$; SD = 0.95

Solved Examples

Q.1 If three coins are tossed randomly then the probability of getting

(i) all three tails (ii) atleast one head

(ii) one head and two tails (iv) exactly two tails

Sol. For a single coin,

$S_1 = \{H, T\}, n(S_1) = 2$

For two coins,

$S_2 = \{H, H\}, (H, T), (T, H), (T, T), n(S_2) = 2^2 = 4$

For three coins,

$S \equiv S_2 \times S_2 = \{H, T\} \times \{(H, H), (H, T), (T, H), (T, T)\}$

$\equiv \{(H, H, H), (H, H, T), (H, T, H), (T, H, H), (H, T, T), (T, H, T)\}, (T, T, H), (T, T, T),$

$\therefore \quad n(S_2) = 2^3 = 8$

(i) Let E_1 = Event of getting all three tails $\{(T, T, T)\}$

$n(E_1) = 1$

$$\therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{\text{Fav. case}}{\text{Total case}} = \frac{1}{8}$$

(ii) Let E_2 = Event of getting at least one head.

$E_2 \equiv \{(H, H, H), (H, H, T), (H, T, H), (T, H, H), (H, T, T), (T, H, T), (T, T, H)\}$

$n(E_2) = 7$

$$\therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{7}{8}$$

(iii) Let E_3 = Event of getting one head and two tails.

$E_3 \equiv \{(H, T, T), (T, H, T), (T, T, H)\}$

$n(E_3) = 3$

$$\therefore \quad P(E_3) = \frac{n(E_3)}{n(S)} = \frac{3}{8}$$

(iv) Let E_4 = Event of getting exactly two tails.

$E_2 \equiv \{(H, T, T), (T, H, T), (T, T, H)\}$

$n(E_4) = 3$

$$\therefore \quad P(E_4) = \frac{n(E_4)}{n(S)} = \frac{3}{8}.$$

Q.2 If three tair and unbaised dice are rolled the ludo board at once. Find the probability that

(i) numbers shown are equal. (ii) numbers shown are (totally) different

(ii) sum of numbers is 10 (iv) sum of numbers is 15

Sol. Here,

$n(S) = 6^3 = 216$

E_1 = Event to show equal number on each.

$E_1 = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6)\}$

$n(E_1) = 6$

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{216} = \frac{1}{36}$$

(ii) E_2 = Event to show different number on each.

$n(E_2) = {}^6P_3$ = to arrange any three different face out of six faces.

$$\therefore \qquad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{{}^6P_3}{216} = \frac{5}{9}$$

(iii) E_3 = Events to have sum of all three dice appeared as 10.

$$E_3 \to \left.\begin{array}{l} (1, 3, 6) \to 3! \;= 6 \\ (1, 4, 5) \to 3! \;= 6 \\ (2, 2, 6) \to 3!/2 = 3 \\ (2, 3, 5) \to 3! \;= 6 \\ (2, 4, 4) \to 3/2 = 3 \\ (3, 3, 4) \to 3/2! = 3 \end{array}\right\} \; n(E_3) = 27$$

$$P(E_3) = \frac{n(E_3)}{n(S)} = \frac{27}{216}$$

Q.3 A bag contains 20 identical balls of which 8 are black and 12 are blue. Three balls are taken out at random from the bag one after the other without replacement. Find the probability that all the three balls drawn are blue.

Sol. The probability that the first ball drawn is blue is $\frac{12}{20}$,

since there are 12 blue balls among 20 balls in the bag. If the first ball is blue, then the probability that the second ball drawn is blue is $\frac{11}{19}$, since 11 of the remaining 19 are blue.

Similarly, if the first two balls drawn are blue, then the probability that the third ball drawn is blue is $\frac{10}{18}$.

The required probability is $\frac{12}{20}\cdot\frac{11}{19}\cdot\frac{10}{18}=\frac{11}{57}$

Note : If the drawn ball is replaced every time, then the probability is $\left(\frac{12}{20}\right)^3=\left(\frac{3}{5}\right)^3$

Q.4 A and B are two events of a random experiment. If $P(A\cap B)=\frac{1}{4}$ and $P(\bar{A})=\frac{5}{8}$, then $P(A\cap\bar{B})$ is equal to

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{3}{8}$

Sol. We have $P(A\cap\bar{B})=P(A-B)=P(A)-P(A\cap B)$

$$=\left(1-\frac{5}{8}\right)-\frac{1}{4}=\frac{3}{8}-\frac{1}{4}=\frac{1}{8}$$

Q.5 A class contains 20 boys and 20 girls of which half the boys and half the girls have cat eyes. If one student is selected from the class, the probability that either the student is a boy or has cat eyes is

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$ (C) $\frac{3}{8}$ (D) $\frac{2}{3}$

Sol. Let A be the event of a boy and B the event of having cat eyes. So

$$P(A)=\frac{20}{40}=\frac{1}{2} \quad \text{and} \quad P(B)=\frac{20}{40}=\frac{1}{2}$$

Now $P(A\cap B)=\frac{10}{40}=\frac{1}{4}$

Therefore $P(A\cup B)=P(A)+P(B)-P(A\cup B)$

$$=\frac{1}{2}+\frac{1}{2}-\frac{1}{4}=\frac{3}{4}$$

Q.6 An urn contains 10 black and 5 white balls. Two balls are drwn from the urn one after the other without replacement. What is the probability that both drawn balls are black?

Sol. Let E and F denote respectively the events that first and second ball drawn are black. We have to find $P(E \cap F)$ or P(EF).

Now $\quad P(E) = P(\text{black ball in first draw}) = \dfrac{10}{15}$

Also given that the first ball drawn is black, i.e., event E has occurred, now there are 9 black balls and five white balls left in the urn. Therefore, the probability that the second ball drawn is black, given that the ball in the first draw is black, is nothing but the conditional probability of F given that E has occurred.

i.e. $\quad P(F|E) = \dfrac{9}{14}$

By multiplication cube fo probability, we have

$$P(E \cap F) = P(E) \cdot P\left(\frac{F}{E}\right) = \frac{10}{15} \cdot \frac{9}{14} = \frac{3}{7}$$

Q.7 Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that first two cards are kings and the third card drawn is an ace?

Sol. Let K denote the event that the card drawn is king and A be the event that the card drawn is an ace. Clearly, we have to find P(KKA)

Now $\quad P(K) = \dfrac{4}{52}$

Also, P(K|K) is the probability of second king with the condition that one king has already been drawn. Now there are three kings in $(52 - 1) = 51$ cards.

Therefore $\quad P(K|K) = \dfrac{3}{51}$

Lastly, P(A|KK) is the probability of third drawn card to be an ace, with the condition that two kings have already been drawn. Now there are four aces in left 50 cards.

Therefore $\quad P(A|KK) = \dfrac{4}{50}$

By multiplication law of probability, we have

$$P(KKA) = P(K) \quad P(K|K) \quad P(A|KK)$$

$$= \frac{4}{52} \times \frac{3}{51} \times \frac{4}{50} = \frac{2}{5525}$$

Q.8 The probability that a candidate securing admission in IIT through entrance test is $\frac{1}{10}$. Seven candidates are selected at random from a centre. The probability that two will get admission in IIT through entrance test is

(A) $20(0.1)^2(0.9)^5$ (B) $15(0.1)^2(0.9)^5$ (C) $21(0.1)^2(0.9)^5$ (D) $2(0.1)^2(0.9)^5$

Sol. Let $p = \text{Probability of success} = \frac{1}{10} = 0.1$

$q = \text{Probability of failure} = 1 - 0.1 = 0.9$

Therefore $P(X = 2) = \text{Probability of 2 success and 5 failures}$

$$= {}^7C_2\,(0.1)^2\,(0.9)^5$$

$$= 21\,(0.1)^2\,(0.9)^5$$

Q.9 A book writer writes a good block with probability $\frac{1}{2}$. If it is a good book, the probability that it will be published is $\frac{2}{3}$, otherwise it is $\frac{1}{4}$. If he writes 2 books, the probability that at least one book will be published is

(A) $\frac{407}{576}$ (B) $\frac{411}{576}$ (C) $\frac{405}{576}$ (D) $\frac{307}{576}$

Sol. Let G = Event of good book

G' = Event of not a good book

E = Event of publication

Then $E = (G \cup G') \cap E = (G \cap E) \cup (G' \cap E)$

Now $P\left(\frac{E}{G}\right) = \frac{2}{3}$, $P\left(\frac{E}{G'}\right) = \frac{1}{4}$, $P(G) == P(G')$

Therefore $P(E) = \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{4} = \frac{11}{24}$

Further, X denotes the number of books published. Then P(at least one book will be published)

$$= P(X = 1) + P(X = 2) = {}^2C_1\left(\frac{11}{24}\right)\left(\frac{13}{24}\right) + {}^2C_2\left(\frac{11}{24}\right)^2\left(\frac{13}{24}\right)^0 = 2 \times \frac{11}{24} \times \frac{13}{24} + \left(\frac{11}{24}\right)^2 = \frac{407}{576}$$

Q.10 $P_1, P_2 \ldots\ldots, P_8$ are equally strong players (i.e., for each of them the probability of win or lose is 1/2) are participating in tennis singles tournament. If the probability of P_1 losing to eventual winner of the tournament is m/n then $n - m$ is equal to ______

Sol. Let X be the eventual winner. P_1 may lose to X in I, II and III rounds.

$P(P_1 \text{ to lose in I}) = P(P_1 \text{ pairing with X and losing}) = \frac{1}{4} \times \frac{1}{2}$

since $P(P_1 \text{ pairing with X}) = \frac{1}{4}$ as there are four pairs.

$P(P_1 \text{ to lose in II}) = P(P_1 \text{ wins in I})\ P(P_1 \text{ paring with X in II})\ P(P_1 \text{ losing})$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$P(P_1 \text{ to lose in III}) = P(P_1 \text{ wining I and II})\ P(P_1 \text{ loosing}) = \left(\frac{1}{2} \times \frac{1}{2}\right) \times \frac{1}{2} = \frac{1}{8}$

Therefore probability of P_1 losing to X is $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

Hence $n - m = 8 - 3 = 5$

Q.11 The odds against an event A is 2 : 3 and odds in favour of another event B is 1 : 2. If A and B are independent and $P(A \cup B) = \frac{m}{n}$, then $|m - n|$ is _______ . Here m and n do not have proper common divisor.

Sol. We have

$$P(A \cup B) = P(A) + P(B) - (A \cap B) = \frac{3}{5} + \frac{1}{3} - \frac{3}{5} \times \frac{1}{3} = \frac{9 + 5 - 3}{15} = \frac{11}{15}$$

Therefore $m = 11$ and $n = 15$. So $|m - n| = 4$.

Q.12 A natural number is selected at random from the first 100 natural numbers. Let A, B and C denote the events of selection of even number, a multiple of 3 and a multiple of 5, respectively. Then

(A) $P(A \cap B) = \frac{4}{25}$

(B) $P(B \cap C) = \frac{3}{50}$

(C) $P(C \cap A) = \frac{1}{10}$

(D) $P(A \cup B \cup C) = \frac{37}{50}$

Sol. We have

Number of even numbers ≤ 100 is equal to 50.

Number of multiples of $3 \leq 100$ is 33.

Number of multiples of $5 \leq 100$ is 20.

Number of common multiples of 2 and 3 is 16.

Number of common multiples of 3 and 5 is 16.

Number of common multiples of 2 and 5 is 10.

Number of common multiples of 2, 3 and 5 is 3.

Now $\quad P(A) = \frac{50}{100}, \; P(B) = \frac{33}{100}, \; P(C) = \frac{20}{100}$

$$P(A \cap B) = \frac{16}{100}, \; P(B \cap C) = \frac{6}{100}, \; P(C \cap A) = \frac{10}{100}$$

Also

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{50}{100} + \frac{33}{100} + \frac{20}{100} - \frac{16}{100} - \frac{6}{100} - \frac{10}{100} + \frac{3}{100} = \frac{106-32}{100} = \frac{74}{100} = \frac{37}{50}$$

Hence all (A), (B), (C) and (D) are correct.

Q.13 Boxes B_1, B_2, B_3 contain different coloured balls as given in table. The prbabilities of selecting boxes are, respectively, $\frac{1}{6}$, $\frac{1}{2}$ and $\frac{1}{3}$. One of the boxes is chosen at random and a ball is drawn from it. If the probability of the drawn ball is black is $\frac{23}{90}$ then the value of n is equal to ______.

Table : Integer answer type question 5(n is a positive integer)

	White	Black	Red
B_1	2	n	2
B_2	3	2	4
B_3	4	3	2

Sol. Let B be denote the event of drawing a black ball. Then

$$B = (B_1 \cup B_2 \cup B_3) \cap B \; = (B_1 \cap B) \cup (B_2 \cap B) \cup (B_3 \cap B)$$

Therefore $\quad P(B_1) = P(B_1)\left(\frac{B}{B_1}\right) + P(B_2)\, P\left(\frac{B}{B_2}\right) + P(B_3)\, P\left(\frac{B}{B_3}\right)$

$$= \frac{1}{6} \times \frac{n}{n+4} + \frac{1}{2} \times \frac{2}{9} + \frac{1}{3} \times \frac{3}{9} = \frac{n}{6(n+4)} + \frac{2}{9} = \frac{3n + 4(n+4)}{18(n+4)}$$

By hypothesis $P(B) = \frac{23}{90}$

Therefore $\frac{3n + 4(n+4)}{18(n+4)} = \frac{23}{90}$

$35n + 80 = 23\,(n+4)$

$12n = 12$

$n = 1$ **Ans.**

Q.14 A and B are two independent events whose probabilities are, respectively, $\frac{1}{n}$ and $\frac{1}{(n+1)}$. If the probability of $A \cap B$ is $\frac{1}{12}$, then n equals _____ .

Sol. A and B are independent events. This implies $P(A \cap B) = P(A)\,P(B)$

Therefore $\frac{1}{12} = P(A)\,P(B) = \frac{1}{n(n+1)}$ which gives $n = 3$ **Ans.**

Q.15 A number x is selected from the set of first 9 natural numbers (i.e., $x = 1, 2, 3, \ldots\ldots, 9$). If the probability that $f(f(x)) = x$ where $f(x) = x^2 - 3x + 3$ is $\frac{m}{9}$, then m is equal to _____

Sol. Clearly all the solutions of $f(x) = x$ are also solutions of $f(f(x)) = x$. First, we solve $f(x) = x$

$f(x) = 0 \Rightarrow x^2 - 3x + 3 = x \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1, 3$

Therefore $x = 1, 3$ are also solutions of $f(f(x)) = x$. We want to seek if there are any more solutions of $f(f(x)) = x$ other than 1 and 3.

$f(f(x)) = x \Rightarrow f(x^2 - 3x + 3) = x \Rightarrow (x^2 - 3x + 3)^2 - 3(x^2 - 3x + 3) + 3 = 0$

$\Rightarrow x^4 - 6x^3 + 12x^2 - 9x + 3 = 0 \Rightarrow (x^2 - 4x + 3)(x^2 - 2x + 1) = 0$

$\Rightarrow (x-1)(x-3)(x-1)^2 = y \Rightarrow x = 1, 3$

In this case we have no additional solutions. Therefore the probability that x satisfies equation $f(f(x)) = x$ is $\frac{2}{9}$. Therefore $m = 2$. Ans.