PERMUTATION AND COMBINATION

FUNDAMENTAL PRINCIPAL OF COUNTING :

If an event can occur in m different ways following which another event can occur in n different ways, then total number of simultaneous occurrence of both the events in a **definite order** is $(m \times n)$. This can be extended to any number of events

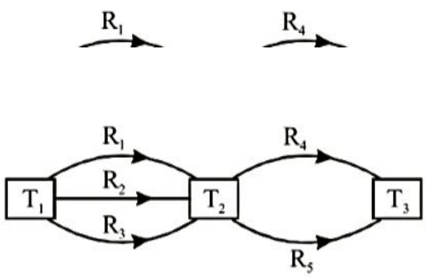
Eg : For an event to be occur in $m_1, m_2, \ldots\ldots, m_n$ ways then number of ways is $m_1 \times m_2 \ldots\ldots m_n$

Important Point :

FPC (Fundamental Principal of Counting) is used to count some event without actually counting them.

Let us take help of some model.

Model- I : Find number of ways of in which one can travel from T_1 (town 1) to T_3 (town 3) via T_2 (town 2).



Total ways : –

T_1 — R_1 — T_2 — R_4 — T_3
T_1 — R_1 — T_2 — R_5 — T_3
T_1 — R_2 — T_2 — R_4 — T_3
T_1 — R_2 — T_2 — R_5 — T_3
T_1 — R_3 — T_2 — R_4 — T_3
T_1 — R_3 — T_2 — R_5 — T_3 = 6 ways

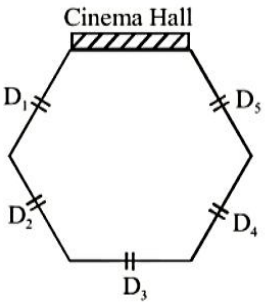
It is easy to proceed by FPC T_1 to $T_2 \longrightarrow 3$

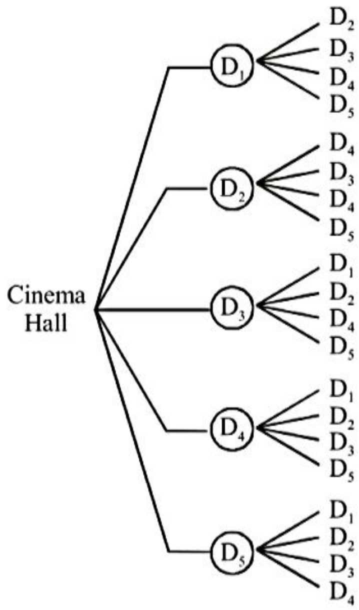
T_2 to $T_3 \longrightarrow 2$

Total ways = $3 \times 2 = 6$

Model- II :

(i) To find the number of ways by which a person can enter and leave cinema hall by a different door.





$4 + 4 + 4 + 4 + 4 = 20$

By F.P.C.
A person can enter in cinema hall by 5 ways & leave by 4 ways $= 5 \times 4 = 20$.

(ii) If he can enter and leave by any door then number of ways $= 5 \times 5 = 25$.

(ii) If he can enter and leave by any door then number of ways $= 5 \times 5 = 25$.

Basic Steps to Remember :

Step-I : Identify the independent events involved in a given problem.
Step-II : Find the number of ways performing/occuring each event
Step-III : Multiply these numbers to get the total number of ways of performing/occuring all the events

Illustration :

How many 3 digit numbers can be formed by the digit 1, 2, 3, 4, 5 without repetition.

Sol. *Hundred's place digit can be selected in 5 ways.*
Ten's place digit can be selected in 4 ways.
Unit's place digit can be selected in 3 ways.
So, $5 \times 4 \times 3 = 60$

Illustration :

In an examination of 10 T/F question, How many sequence of answers are possible.

Sol. *Any question can be answered in two ways , i.e. true or false.*
So total task of anwering tan question can be done in
$2 \times 2 \times 2 \times \ldots\ldots$ *10 times* $= 2^{10}$ *ways*

Illustration :

10 students complete in a swiming race. In how many ways can they occupy the first 3 positions.

Sol. *1^{st} place can be occupied in 10 ways*
2^{nd} place can be occupied in 9 ways
3^{rd} place can be occupied in 8 ways.
So total number of ways = $10 \times 9 \times 8 = 720$

Illustration :

There are 7 flags of diffrent colour. Find the number of diffrent signals that can be transmitted by the use of 2 flags one above the other.

Sol. *1^{st} place can be occupied in 7 ways*
2^{nd} place can be occupied in 6 ways
So total number of ways = $7 \cdot 6 = 42$

Illustration :

A letter lock consists of 3 rings each marked with 10 different letters. In how many ways, its is possible to make an unsuccessful attempt to open the lock?

possible to make an unsuccessful attempt to open the lock?

Sol. *2 ring may have same letters at a time. One ring can have any one of 10 different letters in 10 ways.*
Similarly 2^{nd} and 3^{rd} ring can have any one of 10 different letters in 10 ways respectively.
Total number of attempts = $10 \times 10 \times 10 = 10^3 = 1000$
But out of these 1000 attempt, only one attempt is successful.
Required number of unsuccessful attempt = $1000 - 1 \quad = 999$

Illustration :

How many 6 digits odd number greater than 6,00,000 can be formed from the digits 5,6,7, 8, 9, 0 if repetition of digit is allowed ?

Sol. *Numbers greater than 6,00,000 and formed with the digit 5, 6, 7, 8, 9, 0 are of 6 digit but begin with 6, 7, 8 or 9.*
Also, the numbers which end with 5, 7, 9 are odd.
Hence, first place can be filled by 4 ways (out of 6, 7, 8 or 9). Last place can be filled by 3 ways.
Hence, first and last place can be filled by 4×3 ways.
Also 2^{nd} place can be filled by 6 ways.
3^{rd} place can be filled by 6 ways
4^{th} place can be filled by 6 ways.
5^{th} place can be filled by 6 ways
Hence, all the 6 places can be filled by
$4 \times 3 \times 6 \times 6 \times 6 \times 6 = 15552$ ways.

Illustration :

How many integers greater than 5000 can be formed with the digit 7, 6, 5, 4 & 3 using each digit at most once.

Sol. *For 4 digits number $\Rightarrow$ First position can be filled by 7, 6 or 5 (that is in three ways). Hence*

4 *digit number* $= 3 \times 4 \times 3 \times 2 = 72$

5 *digit number* $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Total number $= 192$

Illustration :

How many natural number less than 30000 can be made from the digits 0, 1, 2, 3, 4, 5, 6.

Sol. *Let a five digit number be denoted by*

a	b	c	d	e

Each of the places can be filled by either of 0, 1, 2, 3, 4, 5, 6 in 7 ways.

The place marked by "a" can be filled by the digits 0, 1 or 2 (since number is to be less than 30000)

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Hence number of intergers $= 3 \times 7 \times 7 \times 7 \times 7 = 3 \times 7^4$

In these numbers one case includes "00000" which is not a natural numbers

Hence number of natural numbers $= 3 \times 7^4 - 1$

Illustration :

Consider the word DAUGHTER. How many 4 letter word can be formed from the letter of above word so that each word contain letter G.

Sol. *4 possible positions for G.*

Remaining three by $\Rightarrow 4 \times 7 \times 6 \times 5 = 28 \times 30 = 840.$

Alternative method :

Total number of ways by which 4 letter word can be formd $= 8 \times 7 \times 6 \times 5$

Number of four letter word without G $= 7 \times 6 \times 5 \times 4$

$= 8 \times 7 \times 6 \times 5 - 7 \times 6 \times 5 \times 4.$

Illustration :

How many diffrent words can be formed using all the letters in the word "MIRACLE".

(a) If vowels may accupy the even position.

(b) If vowels may occupy odd position.

Sol.

(a) Even position $\Rightarrow$

1	2	3	4	5	6	7
	×		×		×	

vowels $\rightarrow$ *I, E, A*

consonants $\rightarrow$ *M, R, C, L*

Three vowels at three position $\Rightarrow 3 \times 2 \times 1 = 6$

Four consonants at four position $\Rightarrow 4 \times 3 \times 2 \times 1 = 24$

Total number of ways $= 6 \times 24 = 144$.

(b)

1	2	3	4	5	6	7
×		×		×		×

1^{st} *position can be filled by any one of the four vowel.*

2^{nd} *position can be filled by any one of the three vowel.*

3^{rd} *position can be filled by any one of the two vowel.*

Thus total ways $= (4 \times 3 \times 2) \times (4 \times 3 \times 2 \times 1) = 576$

Illustration :

There are m men and n monkey. Number of ways in which every monkey has a master, if a man can have any number of monkey.

Sol. *Monkey is distributed among in masters, like 1 monkey can go to* $\rightarrow$ *m masters*

Total number of ways $= m \times m \times \ldots\ldots\ldots m = m^n$

Illustration :

Number of ways in which m diffrent toys can be distributed in "n" children if every child may receive any number of toys

Sol. *Object of distribution toys*

One toy $\rightarrow$ *n children*

Total number of ways $= \quad n \times n \ldots\ldots\ldots\ldots n = n^m$

Illustration :

Find the number of ways in which we can post 5 letters in 10 letter boxes.

Sol.

1^{st} *letter* $\Rightarrow$ *10 boxes*

2^{nd} *letter* $\Rightarrow$ *10 boxes*

3^{rd} *letter* $\Rightarrow$ *10 boxes*

4^{th} *letter* $\Rightarrow$ *10 boxes*

5^{th} *letter* $\Rightarrow$ *10 boxes*

$= 10^5$.

Illustration :

In a car plate number containing only 3 or 4 digits not containing the digit 0. What is the maximum numbers of cars that can be numbered?

Sol. *Here repetition of digits is allowed.*
Also, numbers are formed with the digit1, 2, 3, …… 9

Case-I : *When car plate numbers contain 3 digit number of places to be filled up $r = 3$.*
Out of the 9 digit first place can be filled by 9 ways.
Similarly, 2^{nd} and 3^{rd} place can be filled in 9 ways respectively.
So, when car plate number contais 3 digit, maximum number of cars = 9^3.

Case-II : *When car plate number contains 4 digit, in this case number of cars to be filled up $r = 4$.*
1^{st} place can be filled in 9 ways.
2^{nd} place can be filled by 9 ways and so on.
Maximum number of cars that can be numbered.

$$= 9^3 + 9^4 = 7290$$

Lexicography illustration (Lexicography is called science of making words) :

(a) Find total number of 5 letter word that can be formed from letters of word "TOUGH"

(a) Find total number of 5 letter word that can be formed from letters of word "TOUGH".

$5 \times 4 \times 3 \times 2 \times 1 = 120$

(b) Find the rank of "TOUGH" if all the letters of the word are arranged in all possible orders & written out as in a dictionary.

The number of letters in the word "TOUGH" is 5 & all the five letters are different.
Alphabatical order of all the letters is G, H, O, T, U

Number of words begining with	G	=	$4 \times 3 \times 2 \times 1$
Number of words begining with	H	=	$4 \times 3 \times 2 \times 1$
Number of words begining with	O	=	$4 \times 3 \times 2 \times 1$
Number of words begining with	TG	=	$3 \times 2 \times 1$
Number of words begining with	TH	=	$3 \times 2 \times 1$
Number of words begining with	TOG	=	2×1
Number of words begining with	TOH	=	2×1

Next words begining with "TOU" and it is "TOUGH" = 1.
Rank = 24 + 24 + 24 + 6 + 6 + 2 + 2 + 1 = 89

Try Yourself

Find the rank of "SACHIN" if all the letters of the word are arranged in all possible order & written out as in a dicitonary. [Ans. 601]

Practice Problem

Q.1 If the letters of the word "VARUN" are written in all possible ways and then are arranged as in a dictionary, then the rank of the word VARUN is :
(A) 98 (B) 99 (C) 100 (D) 101

Q.2 How many natural numbers are their from 1 to 1000 which have none of their digits repeated.

Q.3 A man has 3 jackets, 10 shirts, and 5 pairs of slacks. If an outfit consists of a jacket, a shirt, and a pair of slacks, how many different outfits can the man make?

Q.4 There are 6 roads between A & B and 4 roads between B & C.
(i) In how many ways can one drive from A to C by way of B?
(ii) In how many ways can one drive from A to C and back to A, passing through B on both trips ?
(iii) In how many ways can one drive the circular trip described in (ii) without using the same road more than once.

Q.5 (i) Find the number of four letter word that can be formed from the letters of the word HISTORY. (each letter to be used at most once)
(ii) How many of them contain only consonants?
(iii) How many of them begin & end in a consonant?
(iv) How many of them begin with a vowel?
(v) How many contain the letters Y?

(iv) How many of them begin with a vowel?
(v) How many contain the letters Y?
(vi) How many begin with T & end in a vowel?
(vii) How many begin with T & also contain S?
(viii) How many contain both vowels?

Q.6 If repetitions are not permitted
(i) How many 3 digit numbers can be formed from the six digits 2, 3, 5, 6, 7 & 9 ?
(ii) How many of these are less than 400 ? (iii) How many are even ?
(iv) How many are odd ? (v) How many are multiples of 5?

Q.7 A letter lock consists of three rings each marked with 10 different letters. Find the number of ways in which it is possible to make an unsuccessful attempts to open the lock.

Q.8 It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

Answer key

Q.1 C Q.2 738 Q.3 150 Q.4 (i) 24; (ii) 576; (iii) 360
Q.5 (i) 840 ; (ii) 120 ; (iii) 400; (iv) 240 ; (v) 480; (vi) 40; (vii) 60; (vii) 240
Q.6 (i) 120; (ii) 40 ; (iii) 40; (iv) 80; (v) 20
Q.7 999
Q.8 2880

NOTATION OF FACTORIAL :

Notation of factorial & its Algebra :

The continued product of first n, natural number is called as " n factorial" and denoted by n! or $\underline{|n}$

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots\ldots (n-1)n!$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
$3! = 1 \cdot 2 \cdot 3 = 6$
$5! = 120; \qquad 61 = 720; \qquad 7! = 5040$

Special Results :

❖ $0! = 1$ i.e. factorial of zero is 1

Proof: $n! = (n-1)! \cdot n$

Putting $n = 1$

$1! = (1-1)!\ 1 \quad \Rightarrow \quad 0! = 1$

❖ Factorial of negative number is undefined

$(n-1)! = \dfrac{n!}{n}$ if $n = 0$ then $(-1)! = \dfrac{0!}{0} = \dfrac{1}{0}$ Not defined.

Asking:

(i) Find n if $(n+1)! = 12 \times (n-1)!$
(ii) $(n+2)! = 2550\ n!$

Sol.

Sol.

(i) $(n+1)\, n\, (n-1)! = 12 \times (n-1)!$
$n^2 + n - 12 = 0;$
$(n+4)(n-3) = 0 \therefore n = 3$

(ii) $(n+2)(n+1) = 2550;$
$(n+52)(n-49) = 0 \therefore n = 49$

❖ $(2n!) = 2^n \cdot n!\ [1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)]$

Proof:

$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \ldots\ldots (2n-1)(2n-2)$

Take 2 common each n even terms

$= 2^n (1 \cdot 3 \cdot 5 \ldots\ldots (2n+1))(1 \cdot 2 \cdot 3 \ldots\ldots n)$
$= 2^n (n!)(1 \cdot 3 \cdot 5 \ldots\ldots (2n+1))$

Illustration :

Using above find exponent of prime 2 in (100 !)

Sol. $(2 \cdot 50)! = 2^{(50)} \cdot 50!\ (1 \cdot 3 \cdot 5 \ldots\ldots 99) \to 50 \quad 2's$
$(50!) = (2 \cdot 25)! = 2^{(25)} \cdot 25!\ (1 \cdot 3 \cdot 5 \ldots\ldots 49) \to 25 \quad 2's$
$(24!) = (2 \cdot 12)! = 2^{(12)} \times 12!\ (1 \cdot 3 \cdot 5 \ldots\ldots 23) \to 12 \quad 2's$
$(12!) = (2 \cdot 6)! = 2^{(6)} \cdot 6!\ (1 \cdot 3 \cdot 5 \ldots\ldots 11) \to 6 \quad 2's$
$(6!) = (2 \cdot 3)! = 2^{(3)} \cdot 3!\ (1 \cdot 3 \cdot 5 \ldots\ldots 5)$
$= 2^{(3)} \cdot 3!\ (1 \cdot 3 \cdot 5) \to 3 \quad 2's$
$(3!) = 3 \cdot 2 \cdot 1 = 2^{(1)} \cdot (1 \cdot 3) \to 1 \quad 2's$
$50 + 25 + 12 + 6 + 3 + 1 = 97$

Alternative Method :

Let p be a prime number and n be a positive integer. Then, the last integer amongst 1, 2, 3, ……, (n – 1), n which is divisible by p is $\left[\frac{n}{p}\right]p$*, where* $\left[\frac{n}{p}\right]$ *denotes the greatest integer less than or equal to* $\frac{n}{p}$.

For example, $\left[\frac{10}{3}\right] = 3, \left[\frac{12}{5}\right] = 2, \left[\frac{15}{3}\right] = 5$ etc

Let $E_p(n)$ *denote the exponent of the prime p in the positive integer n. Then,*

$$E_p(n!) = E_p(1\cdot2\cdot3 \ldots\ldots (n-1)n)$$

$$= E_p\left(p\cdot2p\cdot3p\ldots\ldots\left[\frac{n}{p}\right]p\right) \qquad \left[\because \begin{array}{l}\text{Remaining integers between}\\ \text{1 and n are not divisible by p}\end{array}\right]$$

$$= \left[\frac{n}{p}\right] + E_p\left(1\cdot2\cdot3\ldots\left[\frac{n}{p}\right]\right)$$

Now, the last integer amongst 1, 2, 3, ……, $\left[\frac{n}{p}\right]$ *which is divisible by p is*

$\left[\frac{n/p}{\;}\right]p = \left[\frac{n}{\;}\right]p$

Now, the last integer amongst 1, 2, 3, ……, $\left[\frac{\;}{p}\right]$ *which is divisible by p is*

$$\left[\frac{n/p}{p}\right]p = \left[\frac{n}{p^2}\right]p$$

$$\therefore \qquad E_p(n!) = \left[\frac{n}{p}\right] + E_p\left(p\cdot2p\cdot3p\ldots\ldots\left[\frac{n}{p^2}\right]p\right) \qquad \left[\because \begin{array}{l}\text{Remaining integers between}\\ \text{1 and}\left[\frac{n}{p}\right]\text{are not divisible by p}\end{array}\right]$$

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots\ldots + \left[\frac{n}{p^s}\right]$$

where s is the largest positive integer such that $p^s \le n < p^{s+1}$

Alternative Method :

$$E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$

$$= 50 + 25 + 12 + 6 + 3 + 1$$

Illustration :

Find number of zeros of the end of (1000)!

Sol. *In any usual factorial of a natural number of 2s are more than number of 5s. Hence number of 10s are same as number of 5s.*

Special Note :

Such kind of counting is possible only on the basis of primes.

Basic Method :

Now we decide number of 5s in (1000)! (1000!)

5, 10, 15 .... 1000 : 200	$\Rightarrow$	200 number containing at least one 5
25, 50 .... 1000 : 40	$\Rightarrow$	40 number containing at least two 5
125, 250 .... 1000 8	$\Rightarrow$	8 number containing at least three 5
625 1	$\Rightarrow$	1 number containing at least four 5

Total = 200 + 40 + 8 + 1 = 249

Objective approach

$$E_5\,(1000!) = \left[\frac{1000}{5}\right] + \left[\frac{1000}{5^2}\right] + \left[\frac{1000}{5^3}\right] + \left[\frac{1000}{5^4}\right]$$

$$= 200 + 40 + 8 + 1 = 249$$

Illustration :

Find number of zeros at the end of 2007!

Find number of zeros at the end of 2007!

Sol. $$E_5\,(2007!) = \left[\frac{2007}{5}\right] + \left[\frac{2007}{5^2}\right] + \left[\frac{2007}{5^3}\right] + \left[\frac{2007}{5^4}\right]$$

$$= 401 + 80 + 16 + 3 = 500$$

Illustration :

Find exponent of 3 in 100!

Sol. *Basic method :*

	Term		
One 3	*3, 6, 9, 12, ... 99 = 33*	$\Rightarrow$	*33 number containing at least one 3*
Two 3	*9, 18, 27 .... 99 = 11*	$\Rightarrow$	*11 number containing at least two 3*
Third 3s	*27 + 54 + 81 = 3*	$\Rightarrow$	*3 number containing at least three 3*
4s	*81 = 1*	$\Rightarrow$	*1 number containing at least four 3*

Sum = 33 + 11 + 3 + 1 = 48

Objective approach

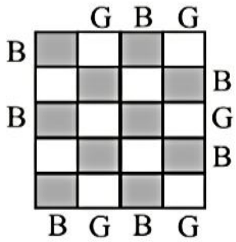
$$E_3\,(100!) = \left[\frac{100}{3}\right] + \left[\frac{100}{3^2}\right] + \left[\frac{100}{3^3}\right] + \left[\frac{100}{3^4}\right]$$

$$= 33 + 11 + 3 + 1 = 48$$

Illustration :

In a class of 20 students having 20 chairs in 5 raw's of 4 each. If the class has 10 boys and 10 girls, in how many ways, can the student's be placed in the chair's such that no boy is sitting in front of, behind, or next to another boy and no girl is sitting in front of, behind or next to another girl.
(A) $^{20}C_{10}\,(10!)^2$ *(B)* $2x^{20}\,(10!)^2$ *(C)* $2x\,(10!)^2$ *(D) not possible*

Sol. *Let us colour the desk of chair's like chessboard pattern.*
The given arrangement is possible if boys lies on black and girl's on white or girl's on black and boys white can be done $\underbrace{{}^2C_1}_{\text{Black or White}}(10!)^2$.



DIVISIBILITY OF NUMBERS :

The following chart shows the conditions of divisibility of numbers by 2,3,4,5,6,8,9,25

Divisible by	Condition	
2	whose last digit is even	(2, 4, 6, 8, 0)
3	sum of whose digits is divisible by 3	
4	whose last two digits number is divisible by 4	
[illegible]	[illegible]	
4	whose last two digits number is divisible by 4	
5	whose last digit is either 0 or 5	
6	which is divisible by both 2 and 3	
8	whose last three digits number is divisible by 8	
9	sum of whose digits is divisible by 9	
25	whose last two digits are divisible by 25	

PERMUTATION & COMBINATION :

Permutation :

Permutation means arrangement in a definte order of things which may be alike or different taken some or all at a time. Hence permutation refers to the situation where order of occurrence of the events is important.

Combination :

Combination/selection/collection/committee refers to the situation where order of occurrence of the event is not important. Combination is sellection of one or more things out of n things which may be alike or different taken some or all at a time.

Note :
Things which are alike and which are different. All god made things in general are treated to be different and all man made things are to be spelled weather like or different.
Hence we say that permutation is arrangement of things in definite order.

Example :

(i) Out of A, B, C, D take 3 letters & form number plate of car. [Pemutation]

(ii) Out of four letters A, B, C, D take any 3 letters & form triangle (possible). [Combination]

In 1^{st} case arrangement of letters are there, in 2^{nd} case only selection will form the triangle, arrangment is not required.

Theorm related to application of Permutation and Combination :

Theorm-1 :

Number of permutations of n distinct things taken all at a time symbolised as :

$$^nP_n = P(n, n) = A_n^n = n!$$

Proof :

Let these are n things arranged at n places

$$n \cdot (n-1) \cdot (n-2) \ldots\ldots 3 \cdot 2 \cdot 1 = n!$$

We also say that number of ways in which n distinct objects can be arranged amongst themselves in $^nP_n = n!$

i.e. Find total number of words that of 10 letters that can be formed from all the letters of word GANESHPURI.

$$A = {^nP_n} = 10! = n!$$

$$A = {^nP_n} = 10! = n!$$

Theorm-2 :

Number of permutations of n distinct things taken r at a time

$$0 \le r \le n$$

$$^nP_r = P(n, r) = A_r^n = \frac{n!}{(n-r)!}$$

Things $T_1, T_2 \ldots\ldots T_n$

Places $1, 2, 3, \ldots\ldots r$

Choice $n \cdot (n-1) \cdot (n-2) \ldots\ldots [n-(r-1)]$

$$\text{Total way} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

Hence we can say that

$$^nP_r = \frac{n!}{(n-r)!}$$

Eg : In how many ways can 5 person be made to occupy

(a) Five different chairs

$$^5P_5 = 5! = 120$$

(b) Three different chairs

$$^5P_3 = 5 \times 4 \times 3 = 60$$

Theorm-3 :

Number of combination / selection of n distinct things taken r at a time

$$^{n}C_{r} = C(n, r) = \binom{n}{r} = \frac{n!}{(r)!(n-r)!}$$

Proof :

Let 10 different objects are given as A, B, C, D, E, F, G, H, I, J

Let combinations taking 3 at a time = x

Arrangement = (x) × (3!)

$$x \cdot 3! = {}^{10}P_{3}$$

$$x = \frac{^{10}P_{3}}{3!} = \frac{10!}{(10-3)3!}$$

Theorm-4 :

Number of combination of n different things taken r at a time when p particular things are always included.

$= {}^{n-p}C_{r-p}$

i.e. Find total number of ways of selecting 11 player out of 15 player when Mahendra Sing Dhoni and Yuvraj Singh are always included = $^{15-2}C_{11-2} = {}^{13}C_{9}$

Theorm-5 :

Number of combination of n different things taken r at a time when p particular thing are always excluded.

$^{n-p}C_{r}$

Eg : How many different selections of 6 books can be made from 11 different books if two particular

$^{n-p}C_{r}$

Eg : How many different selections of 6 books can be made from 11 different books if two particular

books are never selected = $^{11-2}C_{6} = {}^{9}C_{6}$

Important Results :

(i) $^{n}C_{0} = 1$

(ii) $^{n}C_{n} = 1$

(iii) $^{n}C_{n-r} = {}^{n}C_{r}$ If $^{n}C_{x} = {}^{n}C_{y}$ $\Rightarrow$ $x + y = n$ or $x = y$

(iv) $^{n}P_{r} = {}^{n}C_{r}\, r!$

i.e. permutation is defined total number of combination of object then arragement of objects.

(v) $^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

Examples on $^{n}C_{r}$ and $^{n}P_{r}$:

Illustration :

There are n points in a plane, no three of which collinear, find

(a) Number of straight lines *(b) Number of triangles*

(c) Number of digonals in a polygon of n sides.

Sol.

(a) Number of ways by which we can select any two points gives total number of straight lines = $^{n}C_{2}$.

(b) Number of ways by which we can select any 3 non collinear points, gives total number of triangle = $^{n}C_{3}$.

(c) In any polygon number of vertices = number of sides

$^{n}C_{2}$ *= number of sides + number of diagonals*

$^{n}C_{2} - n$ *= Number of diagonals.*

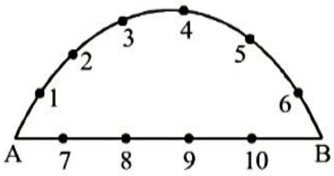
Illustration :

There are 10 points in a plane of which 4 are collinear and rest are non-collinear. Find

(i) *Number of lines* ***(ii)*** *Number of triangles*

Sol.

(i) *Number of lines*



Method-I :

Given 4 points are collinear so total number of ways of selecting any two points = $^{10}C_2$.

If 4 collinear points give only one line (AB). So over counted number of lines formed by collinear

points = $^{4}C_2 - 1$.

Thus total lines = $^{10}C_2 - {}^{4}C_2 + 1$

Method-II :

Take any two points from upper arc = $^{6}C_2$ *ways.*

Take one point from upper arc and one point from line AB = $^{6}C_1 \times {}^{4}C_1$ *ways.*

Take both the points from line AB then number of lines = 1 (Line AB).

Total number of lines = $^{6}C_1 \times {}^{4}C_1 + {}^{6}C_2 + 1$

Take both the points from line AB then number of lines = 1 (Line AB).

Total number of lines = $^{6}C_1 \times {}^{4}C_1 + {}^{6}C_2 + 1$

(ii) *Method-I :*

Number of ways of taking any three points = $^{10}C_3$

Number of ways of taking any three points from four collinear points = $^{4}C_3$

Number of triangles formed = $^{10}C_3 - {}^{4}C_3$

Method-II :

Two points from upper arc and one point from line AB = $^{6}C_2 \times {}^{4}C_1$

Two points from line AB and one point from upper arc = $^{6}C_1 \times {}^{4}C_2$

All the three points from upper arc = $^{6}C_3$

Total number of triangle = $^{6}C_2 \times {}^{4}C_1 + {}^{6}C_1 \times {}^{4}C_2 + {}^{6}C_3$

Illustration :

The sides AB, BC, CA of a triangle ABC have 3, 4, 5 points respectively on them. Find the number of triangle that can be constructed using these points as vertices.

Sol. *Total ways* $= {}^{12}C_3 - {}^{3}C_3 - {}^{4}C_3 - {}^{5}C_3 = 205$

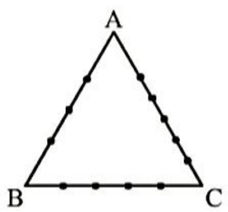


Illustration :

10 diffrent letters of English alphabet are given. Words of 5 letters are formed from these given letters. How many words are formed when at least one letter is repeated.

Sol. $10^5 - 10 \cdot 9 \cdot 7 \cdot 6 = 6976$

Illustration :

Everybody in a room shakes hand with every body else. The total number of handshakes is 66. Find total number of person in the room.

Sol. *Let total number of persons are n. For every two persons possible handshake is 1.*
So, total hand shake = ${}^nC_2 = 66 \Rightarrow n^2 - n - 132 = 0$
$n = 12,\ -11 \Rightarrow n = 12,\ n \neq -11.$

Illustration :

The number of ways in which a team of 11 players can be selected from 22, players including two of them and excluding four of them

Sol. *As two players are already included so only 9 players to be selected from* $22 - 4 - 2 = 16$
Number of ways = ${}^{16}C_9$

Illustration :

Illustration :

How many differnt words can be formed from the letters of the word GANESHPURI, when the letters E, H, P are never together.

Sol.

	EHP						

$\boxed{\text{EHP}} \longrightarrow$ String

(Consider EHP as a letter or string)

So, we have to arrange one string & 7 letters
Hence total number of ways = $10! - 8! \times 3!$

$\hookrightarrow$ *EHP can be arranged amongst themselves in 3! ways.*

Illustration :

How many ways can the seven diffrent colour of a rainbow be arranged so that the blue and green never come together

Sol. *Total number of arrangement without constraint* $= 7!$

	BG				

$\boxed{\text{BG}} \longrightarrow$ String *(Consider BG as a letter or string)*

If BG always come together then number of ways = $(\underset{\text{letters}}{1} + \overset{\text{string}}{5})!\ \underbrace{2!}_{\text{Arrangement of B \& G}}$

Required number of ways = Total possible (without restriction) – (ways when BG together)
$= 7! - 6! \times 2!$

Illustration :

In a morse telegraphy there are 4 arms and each arm is capable of taking 5 distinct position including the position of rest. Find total no. of different signals.

Sol. *One arm is capable of taking 5 positions*
Hence total number of ways = 5^4
Out all above these arise a unique condition in which no signal is transfer. It is R R R R
Hence number of different signal = $5^4 - 1$

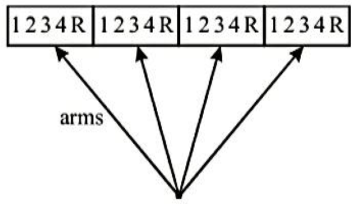
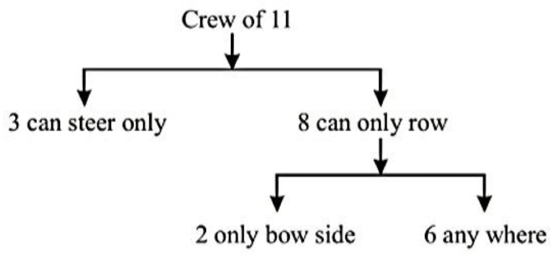
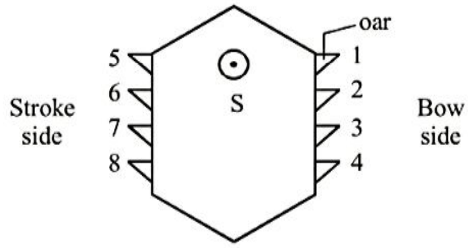


Illustration :

An 8 oared boat to be manned from a crew of 11 of which 3 can only steer. In how many ways the staff can be arranged if 2 of the men can row only on the bow side.





Sol. *For steering position 3 person's can be selected in ${}^3C_1$ ways.*
Out of 4 places of bow sides two are selected for 2 men by ${}^4C_2$ ways and arranged by ${}^4C_2 \times 2!$ ways

Sol. *For steering position 3 person's can be selected in ${}^3C_1$ ways.*
Out of 4 places of bow sides two are selected for 2 men by ${}^4C_2$ ways and arranged by ${}^4C_2 \times 2!$ ways
Remaining six person's can be arranged by 6! ways at six places (As number of seats and number of person's are equal)

Thus required ways = $\underbrace{{}^3C_1}_{\text{Steer}} \times \underbrace{({}^4C_2\,(2!))}_{\text{Row side}} \times \underbrace{(6!)}_{\text{Any side}}$

Illustration :

4 Boys & 4 Girls are to be seated in a line find number of ways

(i) *They can be seated so that "No two girls are together"*

(ii) *If not all the girls are together*
or
If at least one girl is separated from rest of girls.

(iii) *Boys and girls are alternate*

(iv) *If there are 4 married couples then number of ways in which they can be seated so theat each couple is together.*

Sol.

(i) *Out of four standing boys five gaps are there*

$$\left|\, B_1 \,\right|\, B_2 \,\left|\, B_3 \,\right|\, B_4 \,\left|\right.$$

Out of five select any four gaps by ${}^5C_4$ ways in which girls are arranged by = ${}^5C_4 \times 4!$ ways.
Also standing boys are arranged by 4! ways
Required ways = $(4!) \times ({}^5C_4 \cdot 4!)$

Note :- Arrangement by this method is called as gap method.

(ii) *Consider* $G_1\,G_2\,G_3\,G_4 \rightarrow$ *as one string.*

If all girls are together then total ways = $\underbrace{(1}_{\text{String}} + \overbrace{4)!}^{\text{Four boys}}\ \underbrace{(4!)}_{\text{Arrangement of four girls in string}} = (5! \times 4!)$

Total number of arrangment without any restriction = 8!
Total number of ways by which not all girls are together = $8\,! - (5\,! \times 4\,!)$

(iii) *If boys and girls are alternate then two ways of arranging respective position of boys and girls are shown below*

OR

Number of ways of arranging boys $\Rightarrow 4!$
Number of ways of arranging girls $\Rightarrow 4!$
Required ways = $2 \times 4! \times 4!$

(iv) *Let the couples are as below*

B G	B G	B G	B G

(iv) *Let the couples are as below*

B_1G_1	B_2G_2	B_3G_3	B_4G_4

There are four string and each string has husband and wife which can be arranged in 2! ways.
Ways of arranging strings = 4!
Required ways = $(2!)^4 \times 4!$

Illustration :

Let C be the set of 6 consonants (b, c, d, f, g, h) and V be a set of 5 vowels (a, e, i, o, u) and W be the set of seven letter words that can be formed with these 11 letters using both the following rules.

(a) The vowels and consonant in the word must alternate.
(b) No letter can be used more than once in a single word.

If the number of words in the set W are 10 K. Find K.

Sol. *Consonant* $\rightarrow$ *b, c, d, f, g, h (6)*
Vowels $\rightarrow$ *a, e, i, o, u (5)*
W : | × | | × | | × | | × |

Case I : If word begins with consonants
then $({}^6C_4 \times 4!) \times ({}^5C_3 \times 3!) = 360 \times 60 = 21600$
Case II : If word begins with vowels
$({}^5C_4 \times 4!) \times ({}^6C_3 \times 3!) = 120 \times 120 = 14400$
Total $= 36000 \Rightarrow 10K = 36000 \Rightarrow K = 3600$ *Ans.*

Illustration :

Find total number of permutations of n different things taken not more than m at a time when each thing may be repeated any number of times?

Sol. *Number of permutation of one thing* $= n$
Number of permutation of two things $= n \times n = n^2$
Number of permutation of three things $= n \times n \times n = n^3$
and similarly for m things
So, number of permutation of n things taken not more then m at a time when repetition is allowed

$$= n + n^2 + n^3 \ldots\ldots + n^m = \frac{n(n^m - 1)}{n - 1}$$

Illustration :

In how many ways 10 examination papers be arranged so that the best and the worst papers never come together?

Sol. *For the number of arrangements of 10 examination papers without restriction*
$n = 10 \qquad r = 10$
total number of arrangement of 10 examination papers with any restriction $= {}^{10}P_{10} = 10!$
Keeping best and worst together, number of different papers = 9.
(i.e. in this case $n = 9,\ r = 9$*)*
Number of arrangements when best and worst are kept together $= {}^{9}P_{9} \cdot 2!$
Hence, the required number of arrangements of 10 examination papers so that best and worst papers are not together is $10! - {}^{9}P_{9} \cdot 2! = 8 \times 9!$

Hence, the required number of arrangements of 10 examination papers so that best and worst papers are not together is $10! - {}^{9}P_{9} \cdot 2! = 8 \times 9!$

Illustration :

On a new year day, every students of a class sends a card to every other student, the postman delivers 600 cards. How many students are there in class?

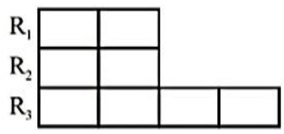
Sol. *Total number of students* $= n$
Number of pair of students $= {}^{n}C_2$
Two students out of n can be selected in ${}^{n}C_2$ *ways.*
Here for each pair of students, number of cards sent is 2.
If P sends card to Q, the Q also sends a card to P.
Number of cards sent $= 2 \times {}^{n}C_2$
According to the problem $2 \times {}^{n}C_2 = 600$

$$\Rightarrow \quad 2 \times \frac{n(n-1)}{2!} = 600 \quad n^2 - n - 600 = 0$$

$$\Rightarrow \quad n = 25 \qquad [\because\ n \neq -24]$$

Illustration :

In how many ways the letters of the word 'PERSON' can be placed in the squares of the given figure shown so that no row remain empty?



Sol. *There are 6 different letters in the word 'PERSON'*
Total required way = Total possible way – (1st *row empty + 2*nd *row empty)*
Total possible ways $= {}^{8}P_{6}$ *(arrangement of 6 different letters in 8 boxes)*

ways when 1st row empty = 6! (arrangement of 6 different letters in remaining 6 boxes)
way when 2nd row empty = 6! (arrangement of 6 different letters in remaining 6 boxes)
Total required ways = $^8P_6 - (6! + 6!)$
$= 18720$

Illustration :

Consider all the six digit numbers that can be formed using the digits 1, 2, 3, 4, 5 and 6, each digit being used exactly once. Each of such six digit numbers have the property that for each digit, not more than two digits smaller than that digit appear to the right of that digit. Find the number of such six digit numbers having the desired property.

Sol. *Begin with six open positions.*
To get an arrangements of digits having the desired property, choose a position to place the digit 6. The 6 can be placed in any of the 3 rightmost positions.
Once the position of the 6 has been chosen, choose a position to place the digit 5.
The 5 can be placed in any of the 3 rightmost positions not occupied by the 6.
Continuing in this way, the digits 6, 5, 4 and 3 can all be placed in one of 3 positions, the 3 rightmost positions which are left open.
Finally, there will be 2 available positions to place the 2, and only one position left for the 1.
Thus, the number of ways to choose an arrangement with the desired property is $3^4 \cdot 2 = 162$.

Practice Problem

Q.1 In how many ways can 5 persons be made to occupy

Practice Problem

Q.1 In how many ways can 5 persons be made to occupy
(i) five different chairs (ii) three different chairs

Q.2 A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. What is the total number of signals that canbe made.

Q.3 A gentleman has 6 friends to invite. In how many ways can invitation cards be sent to them if he has three servants.

Q.4 An English school and a Vernacular school are both under one superintendent. Suppose that the superintendentship, the four teachership of English and Vernacular school each, are vacant, if there be altogether 11 candidates for the appointments, 3 of whom apply exclusively for the superintendentship and 2 exclusively for the appointment in the English school, the number of ways in which the different appointments can be disposed of is :
(A) 4320 (B) 268 (C) 1080 (D) 25920

Q.5 If m denotes the number of 5 digit numbers if each successive digits are in their descending order of magnitude and n is the corresponding figure, when the digits are in their ascending order of magnitude then $(m - n)$ has the value
(A) $^{10}C_4$ (B) $^9C_5$ (C) $^{10}C_3$ (D) $^9C_3$

Answer key

Q.1 (i) 120, (ii) 60 Q.2 $4^5 - 1 = 1023$ Q.3 3^6
Q.4 D [Hint :-Similar to boat problem] Q.5 B

FORMATION OF GROUPS :

(I) No. of ways in which $(m+n)$ different things can be divided in two groups, one containing m things and other contains 'n' things is

$$^{m+n}C_n \text{ or } \frac{(m+n)!}{m!\,n!}.$$

Illustration :

Out of four players P_1, P_2, P_3, P_4 form two teams one contain 3 players and other one player.

Sol. *Number of groups* $= \frac{4!}{1!\,3!} = 4$

Note :

$\Rightarrow$ Acutual explanation of above 4 groups

These an 4 answers so obtains

Select - 3	Rejected - 1
P_1, P_2, P_3	P_4
P_2, P_3, P_4	P_1
P_3, P_4, P_1	P_2
P_4, P_1, P_2	P_3

(II) If groups are equal size i.e. $m = n$

P_4, P_1, P_2 | P_3

(II) If groups are equal size i.e. $m = n$

Total number of ways in which 2n different things can be divided into two equal groups $= \frac{^{2n}C_n}{2!}$

$$= \frac{(2n)!}{(n!)(n!)(2!)}$$

We divide by 2! to avoid false counting.

Proof :

Divide P_1, P_2, P_3, P_4 in two groups

Team – A	Team - B
P_1P_2	P_3P_4
P_1P_3	P_2P_4
P_1P_4	P_2P_3
P_2P_3	P_1P_4
P_2P_4	P_1P_3
P_3P_4	P_1P_2

We see that half of the case are repeated.

Thus $\frac{4!}{2!\,2!}$ gives us wrong answer.

Correct answer $= \frac{4!}{(2!)(2!)(2!)}$

Actually counting all such cases we observe that regrouping appear's when equal size group's are required. To avoid false counting we devide by factorial of number of equal size group's.

(III) Total numbers of ways in which (m + n + p) different things can be divided into three unequal groups m, n, p is $\dfrac{(m+n+p)!}{m!\,n!\,p!}$

Illustration :

If we divide 12 different things in three unequal groups (2, 3, 7) then total number of ways are

***Sol.** *$\text{12 digit things} \begin{array}{l} \nearrow 2 \\ \rightarrow 3 \\ \searrow 4 \end{array}$ $\qquad \dfrac{12!}{2!\,3!\,7!}$

Explanation :

$$12 \begin{array}{l} \nearrow 2 \\ \searrow 10 \end{array} \qquad \frac{12!}{2!\,10!} \qquad\qquad 12 \begin{array}{l} \nearrow 2 \\ \searrow 10 \begin{array}{l} \nearrow 3 \\ \searrow 7 \end{array} \end{array} \qquad \frac{10!}{3!\,7!}$$

$$\text{Total ways} = {}^{12}C_2 \times {}^{10}C_7 \times {}^{3}C_3 = \frac{12!}{2!\,10!} \times \frac{10!}{3!\,7!} = \frac{12!}{2!\,3!\,7!}$$

If groups are equal (m = n = p) then number of ways

$$= \frac{(3n)!}{(n!)^3\,3!}$$

Important Points :

The number of ways in which 'r' group's of n different object's can be formed in such a way that 'p' groups of n_1 object, q group of n_2 object each is ……

$$\text{Required ways} = \frac{n!}{(n_1!)^p (n_2!)^q (p!)(q!)}$$

$n = (n_1 + n_1$ .... p times) + $(n_2 + n_2$ .... q times)

Divide by factorial of number of equal size group.

Illustration :

In how many ways 3 team's of 11 player's each, 4 team's 6 player's each, 2 team's of 15 player's each can be formed out of 87 player's

Sol. $\therefore$ $\quad \text{Required ways} = \dfrac{87!}{(11!)^3 (6!)^4 (15!)^2} \times \dfrac{1}{(3!)(4!)(2!)}$

Illustration :

In how many ways 6 bundles of 12 different toys be made such that 2 bundles are of 3 toys each, 2 bundles are 2 toys each & 2 bundle of 1 toy each

Sol. $\text{Required ways} = \dfrac{(12!)}{(3!)^2 (2!)^2 (1!)^2} \times \dfrac{1}{(2!)(2!)(2!)}$

Illustration :

Total number of ways in which 200 person's can be divided into 100 equal group's.

Sol. Required ways $= \dfrac{200!}{(2!)^{100}(100)!}$

Corollary : Grouping and then arrangement.
If $(m+n+p)$ different thing's can be divided in 3 group's & can be distributed to three person's.

Required ways $= \dfrac{(m+n+p)!}{m!\,n!\,p!} \times 3!$

Illustration :

Find number of ways by which 30 Jawan's can be devided into three group's of 12, 10 & 8 and send to three different boarder's.

Sol. Total ways $= \dfrac{(30!)\times 3!}{(8!)(10!)(12!)}$

In above case if group's are equal size (i.e. group of 10 each)

→ Send to three boarder's

In above case if group's are equal size (i.e. group of 10 each)

$$= \dfrac{(30!)\times \overbrace{(3!)}^{\text{Send to three boarder's}}}{(10!)^3 \underbrace{(3)!}_{\text{Three equal size groups}}}$$

Illustration :

In how many ways six diff. books can be distributed between four persons, so that each person gets atleast one book.

Sol. *Two cases possible {1, 1, 1, 3}, {1, 1, 2, 2}*

$$\text{Groups}\left[\frac{6!}{(1!)^3\,3!\,3!}+\frac{6!}{(1!)^2\,(2!)^2\,2!\,2!}\right]4!$$

Illustration :

Find number of ways by which five different objects given to three students.

Sol. *Two cases possible {1, 1, 3} {1, 2, 2}*

$$\left[\frac{5!}{(1!)^2\,3!\times 2!}+\frac{5!}{1!\,(2!)^2\times 2!}\right]3!$$

Try yourself :

In how many ways eight different computers can be distributed in 5 institution so that each institute gets atleast one computer.

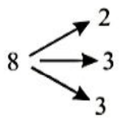
Illustration :

Number of ways in whcih 8 persons can be seated in three diff. taxies each reaching 3 seats for passengers and duly numbered is

(a) *If internal arrangement of persons inside the taxi is immaterial.*

(b) *If internal arrangement also matters*

Sol.

(a)  $\left[\dfrac{8!}{2!\,3!\,3!} \times \dfrac{1}{2!}\right] 3!$

×	×
×	D

(b) *Using grouping* $\left[\left(\dfrac{8\,!}{2!\ \ 3!\ \ 3!} \times \dfrac{1}{2!}\right) 3!\right] 3!\,3!\,3! = 9!$

or arrange 8 people in 9 seat ${}^{9}C_{8} \times 8! = 9!$

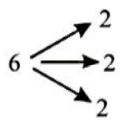
Illustration :

During election's 3 districts are to be convassed by 20, 15, 10 people respectively. If 45 volunter's there then number of ways in which they can be sent.

Sol. *Required ways* $= \dfrac{45\,!}{20!(5\,!)(10\,!)}$

Illustration :

In a jeep there are 3 seat in front and three in the back, number of different ways in which six persons of different height can be seated so that every one in front is shorter than the person directly behind him,

Sol.  $\dfrac{6! \times 3!}{(2\,!)^3\ 3!} = \dfrac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 6} \times 6 = 15 \times 6 = \mathbf{90\ Ans.}$

×	×	×
×	×	×

Practice Problem

Q.1 In how many ways can a pack of 52 cards be divided
(i) equally in four sets. (ii) equally among four players

Q.2 In how many ways can a pack of 52 cards be
(i) distributed among four players having 10, 12, 14 and 16 cards.
(ii) divided into four sets of 7, 15, 15 and 15 cards.

Q.3 (i) In how many ways can five people be divided into three groups.
(ii) In how many ways can five people be distributed in thre different rooms if no room must be empty.

Q.4 (i) In how many ways can 12 different balls be divided between 2 boys, any one receiving 5 and the other 7 balls?
(ii) In how many ways can these 12 balls be divided into groups of 5, 4 and 3 balls respectively?

Answer key

Q.1 (i) $\frac{52}{(13!)^4}\times\frac{1}{4!}$ (ii) $\left(\frac{52!}{(13!)^4}\times\frac{1}{4!}\right)\times 4!$ Q.2 (i) $\frac{52!}{10!12!14!16!}\times 4!$ (ii) $\frac{52!}{7!(15!)^3}\times\frac{1}{3!}$

Q.3 (i) 25, (ii) 150 Q.4 (i) $\frac{12!}{5!7!}\times 2!$ (ii) $\frac{12!}{5!4!3!}$

Q.3 (i) 25, (ii) 150 Q.4 (i) $\frac{12!}{5!7!}\times 2!$ (ii) $\frac{12!}{5!4!3!}$

PERMUTATION OF ALIKE OBJECTS :

Case – I: taken all at a time
Case – II: taken some at a time

Case-I: ***Permutation of a like objects taken all at a time***

Number of permuation of n things — p of one kind, q of another kind and r are all different — taken all at a time $= \frac{n!}{p!q!r!}$

Proof :
D A D D Y
Let three D's are different

$D_1 A D_2 D_3 Y$
$D_1 A D_3 D_2 Y$
$D_2 A D_3 D_1 Y$
$D_2 A D_1 D_3 Y$
$D_3 A D_2 D_1 Y$
$D_3 A D_1 D_2 Y$

One D A D D Y is counted as six different, word's

Let x is required ways

$x \times (6) = 6!$

$x = \frac{6!}{3!}$

Illustration :

Find total number of word's formed by using all letters of the word "IITJEE".

Sol. *ways are* $= \dfrac{6!}{2!(2!)}$

Illustration :

Consider word ASSASSINATION, find number of ways of arranging the letters.

(i) *Number of words using all.*
(ii) *If no two vowels are together.*
(iii) *If all S are seperated.*
(iv) *Atleast one S is seperated from rest of the S's*
(v) *vowels are in the same order.*
(vi) *Relative position of vowels and consonant remain same.*

Sol.

(i) *ASSASSINATION contains four S, three A, two N and two I.*

$13!$

(i) *ASSASSINATION contains four S, three A, two N and two I.*

Total ways $= \dfrac{13!}{(4!)(3!)(2!)(2!)}$

(ii) *We have six vowels as A, A, A, I, I, O and seven consonants as S, S, S, S, N, T, N*

$| S | S | S | S | N | T | N |$

Six vowels in 8 gap's

Total ways $= {}^{8}C_{6} \times \dfrac{6!}{(3!)(2!)} \times \dfrac{7!}{(4!)(2!)}$

(iii) $| A | A | I | N | A | T | I | O | N |$

Out of 10 gaps select 4

Total ways $= {}^{10}C_{4} \times \dfrac{9!}{(3!)(2!)(2!)}$

(iv) *Total – all four S together*

$$\frac{13!}{(4!)(3!)(2!)(2!)} - \frac{10!}{(3!)(2!)(2!)}$$

$\Rightarrow$ *Consider* $\boxed{\text{SSSS}}$ *as one string.*

(v) Total ways = $\underbrace{{}^{13}C_6 \times 1}_{\substack{\text{arrangement}\\\text{of vowels}}} \times \underbrace{\dfrac{7!}{4!\ 2!\ 1!}}_{\substack{\text{arrangement}\\\text{of consonants}}}$

(vi)

v			v		v		v		v	v	

Total ways = $\underbrace{\dfrac{6!}{3!\ 2!}}_{\substack{\text{arrangement}\\\text{of vowels}}} \times \underbrace{\dfrac{7!}{4!\ 2!}}_{\substack{\text{arrangement}\\\text{of consonants}}}$

Illustration :

How many words can be formed with the letters of the word 'PATALIPUTRA' without changing the relative positions of vowels and consonents?

Sol. The word 'PATALIPUTRA' has eleven letters, in which two P's, three A's, two T's, one L, one U, one R, one I, Vowels are AAIUA

one R, one I, Vowels are AAIUA

These vowels can be arranged themselves in $\frac{5!}{3!} = 20$ ways.

The consonents are PTLPTR these consonents can be arranged themselves in $\frac{6!}{2!\,2!} = 180$ ways.

$\therefore$ Required number of words = 20 × 180 = 3600 ways.

Illustration :

How many words can be formed using all the letters of the word HONOLULU if no two alike letters are together.

Sol. Let A represent's ways when OO together, B when LL together, C when UU together.

Required ways = Total ways – [When all three alike letters together + when 2 alike letters together + when one alike letter together]

$$= \text{Total ways} - [n(E_3) + n(E_2) + n(E_1)] \qquad ...(i)$$

$$\text{Total ways} = \frac{8!}{(2!)(2!)(2!)} = 5040$$

(a) $n(E_3) = A \cap B \cap C$ (Region 7)

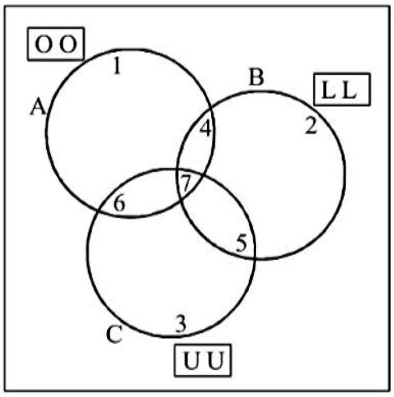
i.e.,

H	N	OO	LL	UU

$n(E_3) = 5! = 120$

(b) $n(E_2) = 3\,[(A \cap B) - (A \cap B \cap C)]$ or [Region 4 + 5 + 6]

$$\underbrace{= 3\left(\frac{6!}{2!} - 5!\right)}_{\to\, n(E_3)} = 720$$



i.e., | H | OO | N | LL | UU |

(c) $n(E_1) = 3\,[\,A - \{(A \cap B) + (A \cap C)\} + (A \cap B \cap C)]$

$$= \left[\frac{7!}{(2!)(2!)} - \left(\frac{6!}{2!} \times 2\right) + 5!\right] = 1980$$

Put in 1^{st}

Required ways = 5040 – [120 + 720 + 1980] = 2220 *Ans.*

Try Yourself :

How many 8 digit numbers can be formed using two 1's, two 2's, two 3's, 4 and one 5. So that no two consecutive digit is identical. [Ans. 2220]

Illustration :

consecutive digit is identical. [Ans. 2220]

Illustration :

Four faces of a tetrahedral dice are marked with 2, 3, 4, 5. The lowest face being considered as the outcome. In how many way a total of 30 can occur in 7 throws.

Sol. *7 throws outcome whose sum is equal to 30 can be obtained in following way.*

Category	**Numebr of ways**
5,5,5,5,5,2,3	$\frac{7!}{5!} = 42$
5,5,5,5,4,4,2	$\frac{7!}{(4!)(2!)} = 105$
5,5,5,5,4,3,3	$\frac{7!}{(4!)(2!)} = 105$
5,5,5,4,4,4,3	$\frac{7!}{(3!)(3!)} = 140$
5,5,4,4,4,4,4	$\frac{7!}{(2!)(5!)} = 21$

Total ways = 42 + 105 + 105 + 140 + 21 = 413 *Ans.*

Illustration :

Total number of ways of forming 6 letter word from the letters of word "PROPORTION".

Sol. *There are*

$P \to 2:$ $R \to 2,$ $O \to 3, T \to 1,$ $I \to 1,$ $N \to 1$

	Category	Process of Selection	Selection	Arrangement
(1)	2 alike, 4 different	(a) 2 alike PP, RR, OOO $\to {}^3C_1$ (b) 4 different from 5 possible option $\to {}^5C_4$	${}^3C_1 \times {}^5C_4 = 15$	$15 \times \frac{6!}{2!}$
(2)	3 alike, 3 different	(a) 3 alike OOO $\to {}^1C_1$ (b) 3 different out of 5 different option $\to {}^5C_3$	${}^1C_1 \times {}^5C_3 = 10$	$10 \times \frac{6!}{3!}$
(3)	2 alike of one kind	(a) 2 alike of one and 2 alike of		
		(b) 3 different out of 5 different option $\to {}^5C_3$	${}^1C_1 \times {}^5C_3 = 10$	$10 \times \frac{6!}{3!}$
(3)	2 alike of one kind 2 alike other kind & 2 different	(a) 2 alike of one and 2 alike of other PP, RR, OOO $\to {}^3C_2$ (b) 2 out of 4 different $\to {}^4C_2$	${}^3C_2 \times {}^4C_2 = 18$	$18 \times \frac{6!}{2!\,2!}$
(4)	3 alike of one kind + 2 a like of another kind + 1 different	(a) OOO, PP, RR $\to {}^1C_1 \times {}^2C_1$ (b) 2 out of 4 different $\to {}^4C_2$	${}^2C_1 \times {}^4C_1 = 8$	$8 \times \frac{6!}{3!\,2!}$
(5)	2 alike, 2 alike + 2 alike	PP, RR, OOO $\to {}^1C_1 \times {}^1C_1 \times {}^1C_1$	${}^1C_1 = 1$	$1 \times \frac{6!}{2!\,2!\,2!}$
(6)	All 6 different	6 possible option P, R, O, T, I, N $\to {}^6C_6$	${}^6C_6 = 1$	$6!$
	Total		Number of selection = 53	Number of words = 11130

Then total number of words formed = 11130

PERMUTATION OF THE OBJECTS TAKEN SOME AT A TIME :

Illustration :

How many 5 lettered words can be formed using the letters of the words "INDEPENDENCE".

Sol. *There are* $E \rightarrow 4,\ N \rightarrow 3,\ D \rightarrow 2,\ P \rightarrow 1,\ I \rightarrow 1,\ C \rightarrow 1$

Category	Selection	Arrangement
4 alike & diff.	$1 \times {}^5C_1 = 5$ eg: EEEED	$\frac{5!}{4!} \times 5 = 25$
3 alike & 2 diff.	${}^2C_1 \times {}^5C_2 = 20$ EEEDD	$\frac{5!}{3!} \times 20 = 400$
3 alike & 2 alike of diff. kind	${}^2C_1 \times {}^2C_1 = 4$ EEEDD	$\frac{5!}{3! \times 2!} \times 4 = 40$
2 alike & 3 diff.	${}^3C_1 \times {}^5C_3 = 30$ EENDI	$\frac{5!}{2!} \times 30 = 1800$
2 alike + 2 other alike and 1 diff.	${}^3C_2 \times 4 = 12$ EENND	$\frac{5!}{2! \times 2!} \times 12 = 360$
all five diff.	${}^6C_5 = 6$ EDIPC	$6 \times 5! = 720$

EDIPC

Illustration :

Find the number of words each consisting 5 letters from the letters of the word "MISSSISSIPPI"

Sol. $S \rightarrow 4,\ I \rightarrow 4,\ P \rightarrow 2,\ M \rightarrow 1$

	Category	Selection	Arrangement
(1)	*4 alike, 1 different*	${}^2C_1 \times {}^3C_1 = 6$	$6 \times \frac{5!}{4!}$
(2)	*3 alike, 2 alike*	${}^2C_1 \times {}^2C_1 = 4$	$4 \times \frac{6!}{2!\,3!}$
(3)	*3 alike, 2 different*	${}^2C_1 \times {}^3C_1 = 6$	$6 \times \frac{6!}{3!}$
(4)	*2 alike, 2 alike 1 different*	${}^3C_2 \times {}^2C_1 = 6$	$6 \times \frac{5!}{2!\,2!}$
(5)	*2 alike, 3 different*	${}^3C_1 \times {}^3C_3 = 3$	$3 \times \frac{5!}{2!}$
	Total	*Number of selection = 25*	*Number of arrangement = 1350*

Then total number of words formed = 1350

Practice Problem

Q.1 Determine the number of permutations of the letters of the word MATHEMATICS.

Q.2 Find the number of numbers greater than 10^6 that can be formed using the digits of the number 2334203 if all the digits of the given number must be used.

Q.3 In how many ways can be letters of the word RESTRICTION be arranged so that the vowels never occur together.

Q.4 How many ways can be formed with the letters of the word RESTRICTIONwithout changing the relative order of the vowels and consonants.

Q.5 In how many ways can the letters AAABBCD be arranged so that
(i) the two B's are together but no two A's are together.
(ii) no two B's and no two A's are together.

Answer key

Q.1 $\dfrac{11!}{2!\,2!\,2!}$ Q.2 360 Q.3 $\dfrac{8!}{2!\,2!} \times \dfrac{4!}{2!}$ Q.4 $\dfrac{7!}{(2!)^2} \times \dfrac{4!}{2!}$ Q.5 (i) 24, (ii) 96

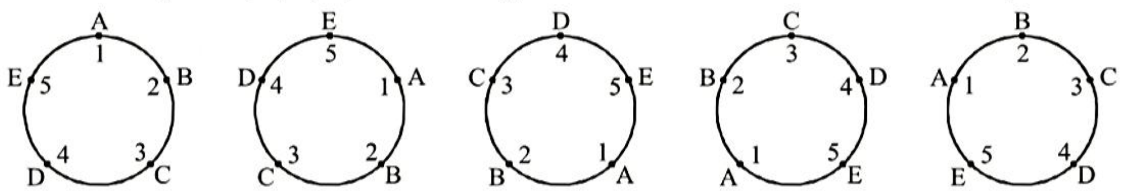
CIRCULAR PERMUTATION :

When object are different

Permutation of objects in a row is called as linear permutation. If we arrange the objects along a closed curve it is called as circular permutation.
Thus in, circular permutation, we consider one object fixed and the remaining objects are arranged as in the case of a linear arrangements.

Theorm-I :

The number of circular permutation of **n** distinct objects is $\mathbf{(n-1)\,!}$
Proof :-Consider 5objects A, B, C, D, E to be arranged around a closed curve is called circular permutation.



All are Same

Let the total number of circular permutation be x. Above circular permutation is equivalent to 5 linear permutations given by ABCDEF, EABCD, DEABC, CDEAB, BCDEA

that is one circular permutation is equivalent to 5x linear permutation given by

$$x \cdot 5 = 5!$$

$$x = \frac{5!}{5} = \frac{5.(5-1)!}{5} = (5-1)!$$

Similarly for n objects nx = n !

$$\boxed{x = \frac{n!}{n} = (n-1)!}$$

(i) n distinct things taken all at a time and arranged along circle in $(n-1)!$ ways

(ii) Taken r things out of n distinct things at a time and arranged along circle in ${}^nC_r \cdot (r-1)!$ ways.

Note :-

In the above theorem anti–clockwise and clockwise order of arrangements are considered as distinct permutations.

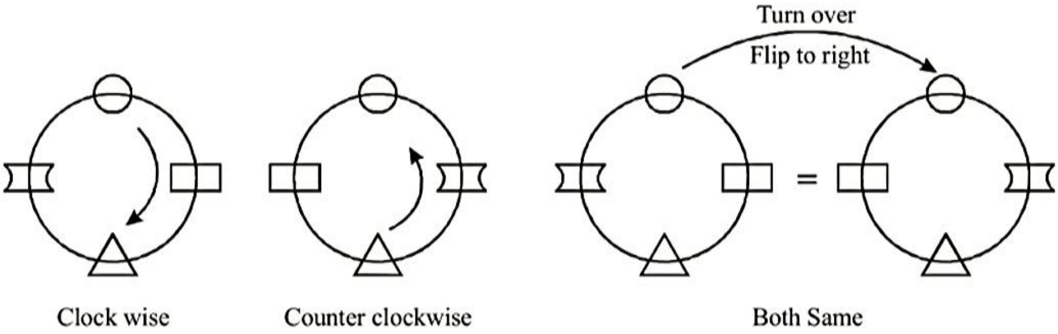
Theorm-II :

If aniclockwise and clockwise are considered to be same total number of circular permutation given by

If aniclockwise and clockwise are considered to be same total number of circular permutation given by

$$\frac{(n-1)!}{2}.$$

If we arrange flowers or garland beads in a neckless then there is no distinction between clockwise & anticlockwise direction.



Note:-

(i) If we have n diffrent things taken r at a time in form of a garland or neckless

Required number of arrangements $= \frac{{}^nC_r \cdot (r-1)!}{2}$.

(ii) The distinction between clockwise and anticlockwise is ignored when a number of people have to be seated around a table so as not to have the same neighours.

Illustration :

Find the number of ways in which 9 people can be seated on a round table so that all shall not have the same neighbours in any 2 arrangements.

Sol. *For same neighbour, clockwise and anticlockwise arrangements are same.*
So total number of ways will be arrangement of 9 people taken clockwise and anticlockwise same

and equal to $\frac{(9-1)!}{2} = \frac{8!}{2}$

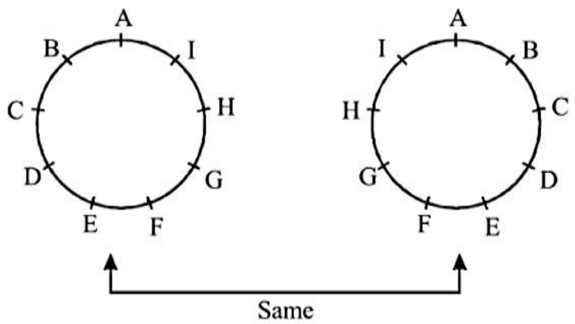


Illustration :

Illustration :

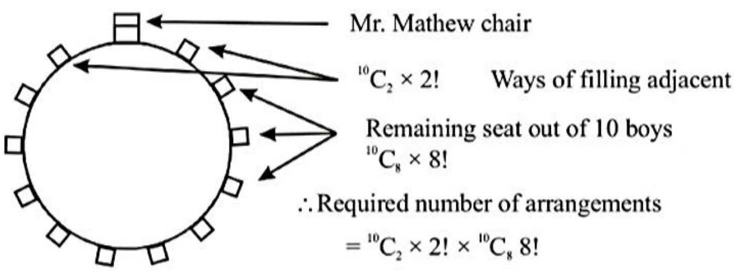
Find the number of ways in which 10 children can sit in a mary go round relative to one another.

Sol. *Here clockwise and anticlockwise arrangements are diffrent.*
Thus required ways = (10 – 1)! = 9!

Illustration :

The 10 students of batch B feel they have some cenceptual doubt on "Circular permutation". Mr. Mathew called them in disussion room and asked them to sit down around a circular table which is surrounded by 13 chairs. Mr mathew told that his adjacent seat should not remain empty. The number of ways, in which the students can sit around a round table if Mr. Mathew also sit around a chair.

Sol.



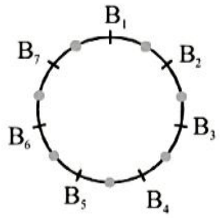
$\therefore$ Required number of arrangements

$= {}^{10}C_2 \times 2! \times {}^{10}C_8\ 8!$

Illustration :

Find number of ways in which 7 American and 7 British people can be seated on a round table so that no two Americans are consecutive.

Sol. *Circular arrangement of 7 British = (7 – 1) !*
There are 7 gap among 7 British.



Out of 7 gap's 7 American can be filled by (7!) ways.
Total ways = (7!) (7 – 1)!

Illustration :

There are n intermediate stations on a railway line from one terminus to another. In how many ways can the train stop at 3 of these intermediate stations if
(a) all the three stations are consecutive
(b) at least two of the stations are consecutive
(c) no two of these stations are consecutive.

Sol.
(a) The number of triples of consecutive stations, viz.
$S_1S_2S_3,\ S_2S_3S_4,\ S_3S_4S_5,\ \ldots\ldots,\ S_{n-2}S_{n-1}S_n$
is $(n-2)$.

(b) The total number of consecutive pair of stations, viz.
$S_1S_2,\ S_2S_3,\ \ldots\ldots\ S_{n-1}S_n$
is $(n-1)$.

Each of the above pair can be associated with a third station in $(n-2)$ ways. Thus, choosing a pair of stations and any third station can be done in $(n-1)(n-2)$ ways. The above count also includes the case of three consecutive stations. However, we can see that each such case has counted twice. For example, the pair S_4S_5 combined with S_6 and the pair S_5S_6 combined with S_4 are identical.

Hence, subtracting the excess counting, the number of ways which three stations can be chosen so that at least two of them are consecutive

$$= (n-1)(n-2) - (n-2) = (n-2)^2.$$

(c) Without restriction, the train can stop at any three stations in ${}^nC_3$ ways.
Hence, the number of ways the train can stop so that no two stations are consecutive

$$= {}^nC_3 - (n-2)^2 = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} - (n-2)^2$$

$$= (n-2)\left(\frac{n^2-n-6n+12}{6}\right) = \frac{(n-2)(n-3)(n-4)}{6} = {}^{n-2}C_3$$

Illustration :

n different things are arranged in a circle. In how many ways can three objects be selected if

(a) all the three objects are consecutive

(b) at least two of the objects are consecutive

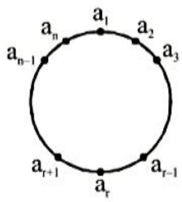
(c) no two objects are consecutive

Sol.

(a) The number of triaples of consecutive objects, viz.

$a_1a_2a_3,\ a_2a_3a_4,\ \ldots\ldots\ a_na_1a_2$

is n



(b) The total number of consecutive pair of objects, viz.

$a_1a_2,\ a_2a_3,\ \ldots\ldots\ a_na_1$

is n

Each of the above pair can be associated with a third objects as

$a_1a_2a_3,\ a_1a_2a_4,\ a_1a_2a_5,\ \ldots\ldots\ a_1a_2a_{n-1}$

in (n – 3) ways.

Note that $a_1a_2a_n$ *has not been counted since it will be counted when we write* $a_na_1a_2$*. Thus, choosing a pair of objects and any third object can be done in n(n – 3) ways.*

Hence, the number of ways of selecting three objects so that at least two of them are consecutive.

Note that $a_1a_2a_n$ *has not been counted since it will be counted when we write* $a_na_1a_2$*. Thus, choosing a pair of objects and any third object can be done in n(n – 3) ways.*

Hence, the number of ways of selecting three objects so that at least two of them are consecutive.

$= n\ (n - 3)$

(c) Without restriction, three objects can be selected in ${}^nC_3$ *ways*

Hence, the number of ways of selecting three objects so that no two of them are consecutive

$= {}^nC_3 - n\ (n - 3)$

Illustration :

Find number of circular permutation of n persons if two specific people are never together.

Sol. *Required ways = Total – when A & B are always together*

$= (n - 1) - (n - 2)! \times 2 = (n - 2)!\ [n - 1 - 2]$

$= (n - 2)\ !\ (n - 3).$

Illustration :

In how many ways 7 diffrent flowers can be formed into a garland.

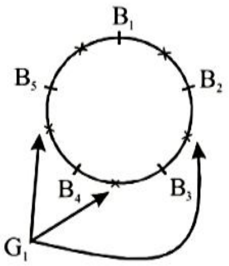
Sol. *Here clockwise and anticlockwise permutations are same*

Hence total ways $= \dfrac{6\,!}{2}.$

Illustration :

Find number of ways in which 5B and 5G can be seated on a circle alternately if a particular B_1 and G_1 are never adjacent to each other in any arrangement.

Sol.



For B_1, G_1 not to be together. G_1 must select from 3 possible gaps in ${}^3C_1$ ways

$$\underset{\text{Boy}}{\underset{\downarrow}{4!}} \times \underset{G_1}{\underset{\downarrow}{{}^3C_1}} \times 4! \quad = 1728 \qquad \textit{Ans.}$$

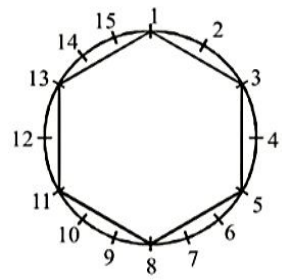
Try Yourself :

Out of 10 flowers of diffrent colours, how many diffrent garlands can be made if each garland condsist of 6 flowers of diffrent colurs. [Ans. ${}^{10}C_6 \dfrac{5!}{2}$]

Illustration :

How many hexagon's can be constructed by joining vertices of a quindecagon (15 side's polygon)

How many hexagon's can be constructed by joining vertices of a quindecagon (15 side's polygon) if none of the sides of Hexagon is also the sides of quindecagon .



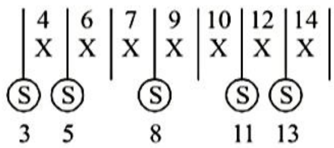
Sol. *Step*

(i) *Select the initial vertex say '1' (in ${}^{15}C_1$ ways)*

(ii) *Now 2 and 15 cannot be selected. From the remaining vertices 3 to 14 (twelve) we have to select 5 more for our hexagon.*

(iii) *Symbolise the vertices to be taken by S S S S S and the vertices not be taken (7 in this case) by X X X X X X X.*

(iv) *Identify the gaps between these crosses (8 in this case) and select*



any five out of these gaps in ${}^8C_5$ ways.

(v) *Serial number are to be alloted either to S or to 'X' whichever comes earlier. In the present problem the vertices corresponding to one selection are 3, 5, 8, 11, 13 and the hexagon as shown*

(vi) *For each selection we therefore have a haxagon with two non consecutive vertices.*

(vii) *Number of hexagons = ${}^{15}C_1 \times {}^8C_5$. However this particular hexagon 1, 3, 5, 8, 11, 13 will occur 6 times when we select the initial vertex as 3 or 5 or 8 or 11 or 13. Hence our answer is 6 times more.*

(viii) *Required number of hexagons* $= \dfrac{15 \times {}^8C_5}{6} = \dfrac{15 \times 56}{6} = \mathbf{140}$ ***Ans.***

Alternative Solution :

As in linear let us open this chain to have 1, 2, 3, …… 13, 14, 15   O O O O O O to be selected ;
| X | X | X | X | X | X | X | X | X | not to be selected number of ways = $^{10}C_6$
Required number of ways = $^{10}C_6$ *– number of ways when 1 and 15 are included, since in circular these become consecutive*
Now if 1 and 15 are laready selected , 2 and 14 cannot be taken. Remaining vertices are 3, 4, 5, ……, 11, 12, 13   (11);   O O O O (4); | X | X | X | X | X | X | (7)
Cases to be rejected = $^8C_4$
Required number of ways = $^{10}C_6 - {}^8C_4 \Rightarrow {}^{10}C_4 - {}^8C_4 \Rightarrow 210 - 70 = \mathbf{140}$ ***Ans.***

Illustration :

In how many ways triangle can be constructed by joining vertices of a queendecagen if name of the sides of triangle can be sides of quindecagon.

Sol. *By method in above question*

$$= \frac{^{15}C_1 \times {}^{11}C_2}{3} = 275$$

Case-II : Circular permutation which objects are alike.

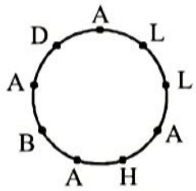
Illustration :

In how many ways letter's word ALLAHABAD can be arranged in a circle.

In how many ways letter's word ALLAHABAD can be arranged in a circle.

Sol. *There are four A's & two L's*

Required ways = $\frac{(9-1)!}{4!\,2!} = \frac{8!}{4!\,2!}$



Practice Problem

Q.1 In how many ways can 5 men and 5 women be seated at a round table if
(i) there is no restriction
(ii) all the five women sit together
(iii) no two women sit together
(iv) not more than four women sit together

Q.2 In how many ways can 5 men and 3 women be seated at a round table if
(i) no two women sit together
(ii) two particular women must sit together, while the third one must not sit beside those two.

Q.3 Find the number of ways in which n different beads can be arranged to form a necklace.

Answer key

Q.1 (i) 9!, (ii) 5! × 5! (iii) 4! × 5!, (iv) 9! – 5!

Q.2 (i) $4! \times {}^5P_3$ (ii) 5700

Q.3 $\frac{1}{2}(n-1)!$

TOTAL NUMBER OF COMBINATIONS OR SELECTION OR COLLECTION :

We know that

$$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \text{........} + {}^nC_rx^r + \text{.....} + {}^nC_n\, y^n.$$

Now replace x by 1

then $\quad 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n$

Case-I : Selection of one or more things out of n things. When all the things are different total number of selections.

One thing can be selected in ${}^nC_1$ ways

Two things can be selected in ${}^nC_2$ ways

Three things can be selected in ${}^nC_3$ ways

$\vdots$

$\vdots$

n things can be selected in ${}^nC_n$ ways

n things can be selected in ${}^nC_n$ ways

Total ways = ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1$

Illustration :

Sanjeev has 7 friend's. In how many ways can be invite one or more of them to dinner.

Sol. $\quad {}^7C_1 + {}^7C_2 + {}^7C_3 \ldots\ldots {}^7C_7 = 2^7 - 1$

Case-II : The number of selecting zero or more things out of n identical things is n + 1

Proof :

Selecting none thing = 1 way

Selecting 1 thing = 1 ways

Selecting 2 things = 1 way

$\vdots$

$\vdots$

Selecting n things = 1 way

Total number of ways = 1 + 1 + 1 ......(n + 1) times = (n + 1) Ans.

Case-III : Number of ways selecting one or more things out of which p are alike of one kind, q are alike of second kind, r alike of third kind, while s are different is $(p + 1)(q + 1)(r + 1)2^S - 1$

Proof :

Selecting none thing (out of p alike things) = 1 way
Selecting 1 thing (out of p alike things) = 1 ways
Selecting 2 things (out of p alike things) = 1 way
⋮
⋮
Selecting p things (out of p alike things) = 1 way

Total number of ways = 1 + 1 + 1 ......(p + 1) times = (p + 1) Ans.

Similarly for q alike, total ways = q + 1
Similarly for r alike, total ways = r + 1

For s different things total ways of selection will be 2^S, i.e. any item is selected or not.
So total number of required ways = $(p + 1)(q + 1)(r + 1)2^S - 1$
(1 is subtracted when no item is selected)

Illustration :

Find the number of ways in which one or more letter be selected from the letters "AAAABBCCCDEF"

Sol. Total number of ways = $\underset{A}{(4 + 1)}\,\underset{B}{(2 + 1)}\,\underset{C}{(3 + 1)}\,\underset{DEF}{2^3} - 1 = 479$

Sol. Total number of ways = $\underset{A}{(4 + 1)}\,\underset{B}{(2 + 1)}\,\underset{C}{(3 + 1)}\,\underset{DEF}{2^3} - 1 = 479$

Illustration :

It is given that 4 Apples, 3 Mangoes, 2 Bananas, 2 Oranges, consider the following cases.
Case-I: Fruits of same species are alike and rests are different ,then
(i) Find the number of ways, atleast one fruit is selected.
(ii) Find the number of ways, atleast one fruit of each kind is selected.
Case-II: Fruits of same species are different and rests are also different , then
(i) Find the number of ways, atleast one fruit is selected.
(ii) Find the number of ways, atleast one fruit of each kind is selected.

Sol. *Case-I :*

(i) *Apples can be selected in (4 + 1) ways*
Total number of ways = $(4 + 1)(3 + 1)(2 + 1)(2 + 1) - 1 = 179$

(ii) *Since we need atleast one fruit of each kind, Apples can be selected in 4 ways.*
So total number of ways = $4 \times 3 \times 2 \times 2 = 48$

Case-II :

(i) *Since fruits of same kind are different.*
Apples can be selected in 2^4 *ways*
Total number of ways = $2^4 \times 2^3 \times 2^2 \times 2^2 - 1 = 2047$

(ii) *Since we need atleast one fruit of each kind, Apples can be selected in* $2^4 - 1$ *ways.*
So total number of ways = $(2^4 - 1) \times (2^3 - 1) \times (2^2 - 1) \times (2^2 - 1)$

Total number of combinations in different cases :

(a) The number of combinations of n **different things** taking some or all (or atleast one) at a time
$= {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = \mathbf{2^n - 1}$

(b) The number of ways to select some or all out of $(p + q + r)$ things where p are alike of first kind, q are alike of second kind and r are alike of third kind is $= \mathbf{(p+1)(q+1)(r+1) - 1}$

(c) The number of ways to select some or all out of $(p + q + t)$ things where p are alike of first kind, q are alike of second kind and remaining t are different is $= \mathbf{(p+1)(q+1)2^t - 1}$.

PROMBLEMS BASED ON NUMBER THEORY :

Note that every natural number except 1 has atleast 2 divisors. If it has exactly two divisors then it is called a prime. System of prime numbers begins with 2. All primes except 2 are odd. A number having more than 2 divisors is called a composite. 2 is the only even number which is not composite. A pair of natural numbers are said to be relatively prime or coprime if their HCF is one. For two natural numbers to be relatively it is not necessay that one or both should be prime. It is possible that they both are composite but still coprime. eg. 4 and 25. Note that 1 is neither prime nor composite however it is coprime with every other natural number.A pair of primes are said to be twin if their non-negative difference is 2 e.g. 3 & 5 ; 5 & 7 e.t.c.

Number of divisors and their sum :

Number of divisors and their sum :

(a) Every natural number N can always be put in the form $N = p_1^{\alpha_1}, p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1, p_2, \ldots p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are non-negative integers.

(b) If $N = p_1^{\alpha_1}, p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ then number of divisor of N is equivalent of number of ways of selecting zero or more objects from the groups of identical objects, $(p_1, p_1, \ldots, \alpha_1$ times$)$ $(p_2, p_2, \ldots \alpha_2$ times$)$, $(p_k, p_k, \ldots, \alpha_k$ times$) = (\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1)$ which includes 1 and N also.

(c) All the divisors excluding 1 and N are called proper divisors.
Also number of divisiors of N can be seen as number of different terms in the expansion of

$$(p_1^0 + p_1^1 + p_1^2 + \ldots + p_1^{\alpha_1}) \times (p_2^0 + p_2^1 + p_2^2 + \ldots + p_2^{\alpha_2}) \times \ldots\ldots \times (1 + p_k + p_k^2 + \ldots + p_k^{\alpha_k})$$

Hence, sum of the divisors of N is

$$(1 + p_1 + p_1^2 + \ldots + p_1^{\alpha_1}) \times (1 + p_2 + p_2^2 + \ldots + p_2^{\alpha_2}) \ \ldots\ (1 + p_k + p_k^2 + \ldots + p_k^{\alpha_k})$$

$$= \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \ \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \ \ldots \ \frac{p_k^{\alpha_k+1} - 1}{p_k - 1}$$

(d) Number of ways of putting N as a product of two natural numbers is $\frac{1}{2}(a_1 + 1)(a_2 + 1)\ldots(a_k + 1)$ if N is not a perfect square.

If N is a perfect square, then this is $\frac{1}{2}[(a_1 + 1)(a_2 + 1)\ldots\ldots(a_k + 1) + 1]$.

(e) If $N = p^a \times q^b \times r^c$ ...... p, q are prime number & a, b are natural number.

$$I = \frac{(Total\ number\ of\ divisors)}{2}$$

where I is the number of required ways if N is not a perfect square.

$$I = \frac{(Total\ number\ of\ divisors\ +\ 1)}{2}$$

where I is the number of required ways if N is a perfect square.

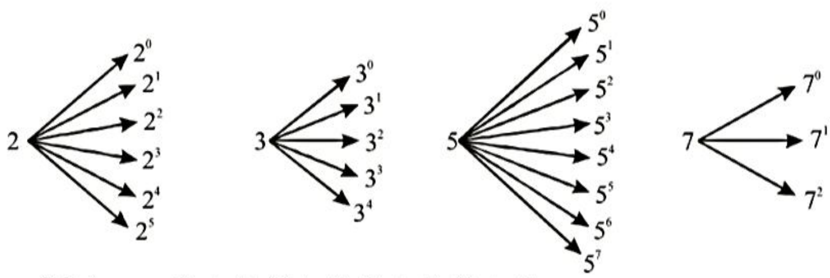
Illustration :

Consider the number $N = 2^5 \times 3^4 \times 5^7 \times 7^2$,
Now answer the following question.

(i) *Total number of divisor,*
(ii) *Number of proper divisor*
(iii) *Number of odd divisor*
(iv) *Number of even divisor.*
(v) *Number of divisors divisible by 5.*
(vi) *Number of divisors divisible by 10.*
(vii) *Number of divisors divisible by 2 but not by 4.*
(viii) *Sum of all divisors*
(ix) *Sum of even divisors*
(vii) *Number of divisors divisible by 2 but not by 4.*
(viii) *Sum of all divisors*
(ix) *Sum of even divisors*
(x) *Sum of odd divisors*
(xi) *Sum of divisor of the form* $(4n + 2)$, $n \in N$ $(4n + 2) \Rightarrow$ *Even number but not divisible by 4, so exactly one 2.*
(xii) *Number of ways in which N can be resolved as product of two divisor.*

Sol.

(i)



Number of divisor $= (5 + 1)(4 + 1)(7 + 1)(2 + 1)$
$= 6 \times 5 \times 8 \times 3 = 720$

(ii) *Proper divisor will be other than 1 and number itself.*
So proper divisor $= 720 - 2 = 718$.

(iii) *Number of odd divisor can be obtained by choosing* $3^4 \times 5^7 \times 7^2$
Which can be formed by $(4 + 1)(7 + 1) \times (2 + 1) = 120$ *ways*

(iv) It can be obtain by selecting atleast one '2' in $2^5 \cdot 3^4 \cdot 5^7 \cdot 7^2$
which can be done in $5 \times (4 + 1) \times (7 + 1) \times (2 + 1) = 600$

(v) Atleast one 5 must be there in $2^5 \times 3^4 \times 5^7 \times 7^2$
$(5 + 1)(4 + 1)(7)(2 + 1) = 630$

(vi) For divisibility by 10 number most be divisible by 2 & 5.
So total divisor must contain atleast one 2 and atleast one 5 in $2^5 \times 3^4 \times 5^7 \times 7^2$
So that ways $= 5 \times (4 + 1) \times (7) \times (2 + 1) = 525$

(vii) Exact one 2 should be there
Total ways $= 1 \times 5 \times 8 \times 3 = 120$

(viii) It can be obtained from
$(2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5)(3^0 + 3^1 + 3^2 + 3^3 + 3^4)$
$(5^0 + 5^1 + 5^2 + 5^3 + 5^4 + \ldots\ldots + 5^7)(7^0 + 7^1 + 7^2)$

$$= (2^6 - 1) \times \frac{(3^5 - 1)}{2} \times \frac{(5^8 - 1)}{4} \times \frac{(7^3 - 1)}{6}$$

(ix) At least one 2 must be there
Sum $= (2^1 + 2^2 + \ldots\ldots + 2^5)(3^0 + 3^1 + \ldots\ldots + 3^4)(5^0 + 5^1 + \ldots\ldots + 5^7)(7^0 + 7^1 + 7^2)$

Sum $= 2 \cdot (2^5 - 1) \times \frac{(3^5 - 1)}{} \times \frac{(5^8 - 1)}{} \times \frac{(7^3 - 1)}{}$

Sum $= 2 \cdot (2^5 - 1) \times \dfrac{(3^5 - 1)}{2} \times \dfrac{(5^8 - 1)}{4} \times \dfrac{(7^3 - 1)}{6}$

(x) No 2 can be selected
Sum $= (3^0 + 3^1 + 3^2 + 3^3 + 3^4)(5^0 + 5^1 + \ldots\ldots + 5^7)(7^0 + 7^1 + 7^2)$

(xi) Sum $= (2)(3^0 + 3^1 + 3^2 + 3^3 + 3^4)(5^0 + 5^1 + \ldots\ldots + 5^7)(7^0 + 7^1 + 7^2)$

$$= 2 \times \frac{(3^5 - 1)}{2} \times \frac{(5^8 - 1)}{4} \times \frac{(7^3 - 1)}{6} = \frac{(3^5 - 1)(5^8 - 1)(7^3 - 1)}{24}$$

(xii) When N is perfect square $N = 2^5 \times 3^4 \times 5^7 \times 7^2$ is not a perfect square

So $\quad I = \dfrac{(\text{Total number of divisors})}{2} = \dfrac{720}{2} = 360 \qquad$ Ans.

Illustration :

For $N = 75600$, find number of ways by which N can be resolved as product of 2 devisor.

Sol. $N = 2^4 \times 3^3 \times 5^2 \times 7^1$
Total number of divisor $(N_T) = (4 + 1) \times (3 + 1) \times (2 + 1) \times (1 + 1) = 120$

If (N_T) is even then required number of ways $= \dfrac{N_T}{2} = 60$

Illustration :

Prove that number of ways in which N can be resolved as a product of 2 divisors which are relatively prime = 2^{n-1}. Where n is the number of prime involed in the prime factorisation of

$$N = 2^6 \times 3^5 \times 5^4 \times 7^3 \times 11^2 \times 13^1$$

Sol. $N = \underset{a}{2^6} \times \underset{b}{3^5} \times \underset{c}{5^4} \times \underset{d}{7^3} \times \underset{e}{11^2} \times \underset{f}{13^1}$ *a, b, c, d, e, f are relatively prime*

$$\text{Total ways} = {}^6C_0 + {}^6C_1 + {}^6C_2 + \frac{{}^6C_3}{2}$$

${}^6C_0 \longrightarrow 1 \times abcdef$

${}^6C_1 \longrightarrow a \times bcdef,\ b \times acdef,\ c \times abdef,\ d \times abcef,\ e \times abcdf,\ f \times abcde,$

${}^6C_2 \longrightarrow ab \times cdef,\ ae \times bcdf,\ \ldots\ldots$

$\dfrac{{}^6C_3}{2} \longrightarrow$ *2 for equal size group*

$$\text{Total ways} = {}^6C_0 + {}^6C_1 + {}^6C_2 + \frac{{}^6C_3}{2} = 2^{6-1} = 32$$

Similarly for n prime factorisation total ways = 2^{n-1}

Illustration :

Illustration :

*Find the number of permutation of 6 digits from the set {1, 2, 3, 4, 5, 6} where each digit is to be used exactly once, so that the chosen permutation changes from increasing to decreasing or decreasing to increasing at most once **e.g.** the strings like 1 2 3 4 5 6, 6 5 4 3 2 1, 1 2 6 5 4 3 and 6 3 2 1 4 5 are acceptable but strings like 1 3 2 4 5 6 or 6 5 3 2 4 1 are not.*

Sol. *{1, 2, 3, 4, 5, 6}*

Case-1:	up	up	up	up	up	6	${}^5C_0$
Case-2:	up	up	up	up	6	down	${}^5C_1$
Case-3:	up	up	up	6	down	down	${}^5C_2$
Case-4:	up	up	6	down	down	down	${}^5C_3$
Case-5:	up	6	down	down	down	down	${}^5C_4$
Case-6:	6	down	down	down	down	down	${}^5C_5$
Case-7:	down	down	down	down	1	up	${}^5C_1$
Case-8:	down	down	down	1	up	up	${}^5C_2$
Case-9:	down	down	1	up	up	up	${}^5C_3$
Case-10:	down	1	up	up	up	up	${}^5C_4$

Total = 62 **Ans.**

Aliter: ${}^6C_1 \times 2 + {}^6C_2 \times 2 + {}^6C_3 = 12 + 30 + 20 = 62$ **Ans.**

Illustration :

Let N be the number of ordered pairs of non empty sets A and B that have the following properties:

(a) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(b) $A \cap B = \phi$

(c) *The number of elements of A is not the element of A.*

(d) *The number of elements of B is not an element of B.*

Find N.

Sol. ***Explanation:***

(a) and (b) $\Rightarrow n(A) + n(B) = 10$

(c) and (d) $\Rightarrow$ *if A is the two elements set then B is an eight elements set, therefore 2 is not in A and must be in B and 8 is not in B and must be in A.*

Also note that both set can not have equal number of elements because if A is a 5 element set then B will also be five element set and the elements can not be both in A and B.

No. of elements in A i.e. n(A)	n(B)	number of ways
1 element set (say) {9}	{1..........} 9 element set	$\underbrace{{}^8C_0}_{\text{group of 0 and 8}}$
2 element set {8............}	{2..........} 8 element set	$\underbrace{{}^8C_1}_{\text{group of 1 and 7}}$
3 element set {7............}	{3..........} 7 element set	$\underbrace{{}^8C_2}_{\text{group of 2 and 6}}$
3 element set {7............}	{3..........} 7 element set	$\underbrace{{}^8C_2}_{\text{group of 2 and 6}}$
4 element set {6............}	{4..........} 6 element set	${}^8C_3$
5 element set	Not possible	
6 element set	4 element	${}^8C_5$
7 element set	3 element	${}^8C_6$
8 element set	2 element	${}^8C_7$
9 element set	1 element	${}^8C_8$

Total $= ({}^8C_0 + {}^8C_1 + {}^8C_2 + \ldots\ldots\ldots + {}^8C_8) - {}^8C_4 = 2^8 - 70 = 256 - 70 = 186$ ***Ans.***

Illustration :

Find the number of non-empty subsets S of {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} such that, no two consecutive integers belong to S and if, S contains k elements, then S contains no number less than k.

Sol. $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

single element subset = 12

two element subset (1 cannot be taken – think! why?)

$$\begin{array}{cc} S \quad S & 2 \\ \underbrace{|\times|\times|\times|\times|\times|\times|\times|\times|\times|}_{\text{nine to be taken}} & 9 \end{array}$$

gaps 10 $\rightarrow {}^{10}C_2 = 45$

$|||^{ly}$ 3 element subset (1 and 2 cannot be taken)

3, 4, ... 11, 12

0 0 0

$|\times|\times|\times|\times|\times|\times|\times|$

$^8C_3 = 56$

4 element set (1, 2, 3, rejected)

0 0 0 0

$|\times|\times|\times|\times|\times|$

$^6C_4 = 15$

5 element (1, 2, 3, 4 rejected)

0 0 0 0 0

$|\times|\times|\times|$

$^4C_5 = 0$ *Not possible*

Total = 12 + 45 + 56 + 15 = 68 + 60 = 128 ***Ans.***

Practice Problem

Q.1 Consider the number $N = 75600\ (2^4.3^3.5^2.7^1)$ Find

(i) Number of divisors (ii) Number of proper divisors

(iii) Number of odd divisors (iv) Number of even divisors

(v) Number of divisors divisible by 5 (vi) Number of divisors divisible by 10

(vii) sum of all the divisors

(iii) Number of odd divisors (iv) Number of even divisors

(v) Number of divisors divisible by 5 (vi) Number of divisors divisible by 10

(vii) sum of all the divisors

Answer key

Q.1 (i) 120, (ii) 118, (iii) 24, (iv) 96, (v) 80, (vi) 64,

(vii) $(2^0 + 2^1 + 2^2 + 2^3 + 2^4)\ (3^0 + 3^1 + 3^2 + 3^3)(5^0 + 5^1 + 5^2)(7^0 + 7^1)$

SUMMATION OF NUMBERS (3 DIFFERENT WAYS) :

(a) Sum of all the numbers greater then 10000 formed by the digits 1,3,5,7,9 if no digit being repeated.

Method - 1 : All possible numbers = 5! = 120

If one occupies the units place then total numbers = 24.

Hence 1 enjoys units place 24 times $\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }\boxed{1}$

$|||^{ly}$ 1 enjoys each place 24 times

Sum due to $1 = 1 \times 24\ (1 + 10 + 10^2 + 10^3 + 10^4)$

$|||^{ly}$ Sum due to the

digit $3 = 3 \times 24\ (1 + 10 + 10^2 + 10^3 + 10^4)$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

Required total sum $= 24\ (1 + 10 + 10^2 + 10^3 + 10^4)\ (1 + 3 + 5 + 7 + 9)$

Method –2 : In 1^{st} column there are twenty four 1's , Twenty four 3's & so on and their sum is

$= 24 \times 25 = 600$

Hence add. in vertical column normally we get = 6666600

$$
\begin{array}{r|ccccc}
 & 5^{th} & & & 2^{nd} & 1^{st} \\
 & X & X & X & X & X \\
120 & X & X & X & X & X \\
\text{Number} & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & X & X & X & X & X \\
\hline
 & 666 & 6 & 6 & 0 & 0 & = 6666600
\end{array}
$$

Method–3 : Applicable only if the digits used are such that they have the same common difference. (valid even if the digits are repeating)

Writing all the numbers in ascending order of magnitude

$S = (13579 + 13597 + ... + 97513 + 97531)$

$S = (13579 + 99531) + (13597 + 97513) + ...$

$= (111110)\ 60$ time $= 6666600$ Ans

$S = \frac{n}{2}(l + L)$ where n = number of numbers, l = smallest, L = Largest

---

Illustration :

Illustration :

Find sum of all the numbers greater than 10000 formed by the digit 0, 1, 2, 4, 5, no digit being repeated.

Sol. *Using all the given digits we can form a five digit number except when zero is at first place.*
So to find the sum of all the possible five digit number
= (Sum of all possible arrangement) – (Sum of all the arrangements when zero is at first place)

$\Rightarrow$ *5 different digits can be arranged in 5! ways so each digit will appear at every place* $= \frac{5!}{5}$ *times*

i.e. 24 times

Sum of all digits at unit place $= 24\,(0 + 1 + 2 + 4 + 5)$

Sum of all digits at ten's place $= 24\,(0 + 1 + 2 + 4 + 5)$

………………………………………………………

Sum of all digits at 10000^{th} *place* $= 24\,(0 + 1 + 2 + 4 + 5)$

In this way sum of all possible arrangement $= 24\,(0 + 1 + 2 + 4 + 5)\,[1 + 10 + 10^2 + 10^3 + 10^4]$

when zero is at first place 4 digit number willbe formed.

Each number will appear 6 times at every place.

Sum of all 4 digit number at unit place $= 6\,(1 + 2 + 4 + 5)$

Sum of all 4 digit number at ten's place $= 6\,(1 + 2 + 4 + 5)$

………………………………………………………

Hence sum of all four digit numbers $= 6\,(1 + 2 + 4 + 5)\,(1 + 10 + 10^2 + 10^3)$

Required sum $= 24\,[0 + 1 + 2 + 3 + 4 + 5]\,[1 + 10 + 10^2 + 10^3 + 10^4]$

$\qquad - 6\,(1 + 2 + 4 + 5)\,(1 + 10 + 10^2 + 10^3)$

Illustration :

Find the sum of all the four digit numbers that can be formed with the digits 3, 2, 3, 4.

Sol. *The number of numbers having 2 in units place*

$= \frac{3!}{2!} = 3$ [$\because$ the other three places are to be filled by 3, 3 and 4]

Similarly the number of numbers having 4 in units places

$= \frac{3!}{2!} = 3$ [$\because$ the other three places are to be filled by 3, 3 and 2]

and the number of numbers having 3 in units places

$= 3! = 6$ [$\because$ the other three places are to be filled by 2, 3 and 4]

Thus, sum of the digits occurring in the units place

$= 2 \times 3 + 3 \times 6 + 4 \times 3 = 36$

We can see that the given digits (3, 2, 3, 4) occur at the tens, hundreds and thousands place, the same number of times as they occur at the units place.

Hence, the required sum of the numbers formed

$= 36\,(1 + 10 + 100 + 1000) = 39996$

Illustration :

Find the sum of the five digit numbers that can be formed using the digits 3, 4, 5, 6, 7 not using any digit more than once in any number.

Sol. *If 3 is placed at units place, the remaining 4 places can be filled in 4! = 24 ways*

Sol. *If 3 is placed at units place, the remaining 4 places can be filled in 4! = 24 ways*

Thus, 3 occurs at unit place 24 times.

The other digits similarly, each occurs at the unit places 24 times.

Similarly, each of the digit occurs at the other places tens, hundreds and so on, 24 times.

Hence, the required sum, is

$= 24\,(3 + 4 + 5 + 6 + 7)\,(10^0 + 10^1 + 10^2 + 10^3 + 10^4)$

$= 24 \times 25 \times 11111 = 6666600$

Illustration :

Find the number of permutations of the digits 1, 2, 3, 4 and 5 taken all at a time so that the sum of the digits at the first two places is smaller than the sum of the digits at the last two places.

Sol. *Total = 120;* $a_1\ a_2\ a_3\ a_4\ a_5$

Number of ways in which $a_1 + a_2 > a_4 + a_5$ *= number of ways in which* $a_1 + a_2 < a_4 + a_5$

$\therefore$ required number $= \frac{(120) - (a_1 + a_2 = a_4 + a_5)^*}{2}$

(* denotes the number of ways when $a_1 + a_2 = a_4 + a_5$)

Now $a_1 + a_2 = a_4 + a_5$ (1, 2, 3, 4, 5)

If $a_3 = 1$, 4 3 1 2 5 8 ways

$a_3 = 3$, 1 5 3 2 4 8 ways

$a_3 = 5$, 4 1 5 2 3 8 ways

If $a_3 = 2 / 4 / 6$ *not possible (think !)*

$\therefore$ required number $= \frac{(120) - 24}{2} = 48$ ***Ans.***

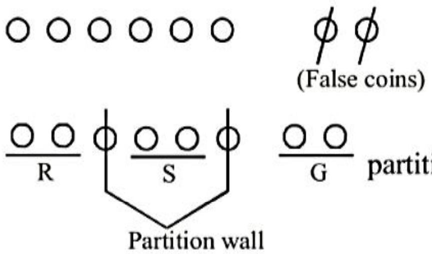
DESTRIBUTION OF ALIKE OBJECTS :

TYPE-1:

Total number of ways in which n identical coins can be distributed among p persons so that each person may get any number of coin is $^{n+p-1}C_{p-1} = \dfrac{(n+p-1)!}{(p-1)!\,(n)!}$

Proof:-

Let 6 identical coins can be distributed among 3 persons R|S|G



Out of eight places if we select any two 3 partitions are formed by $^{8}C_2$ ways

Illustration :

Find number of ways in which 30 mangos can be distributed among 5 persons.

Sol. $^{30+5-1}C_{5-1} = {}^{34}C_4$

Find number of ways in which 30 mangos can be distributed among 5 persons.

Sol. $^{30+5-1}C_{5-1} = {}^{34}C_4$

TYPE-2:

Total number of ways in which n identical items can be distributed among p persons such that each of them receive at least one item $^{n-1}C_{p-1}$.

Illustration :

Find total number of ways of distributing 7 identical computers to R|S|G. So that each receive atleast one computer

Sol. $^{7-1}C_{3-1} = {}^{6}C_2$.

Illustration :

Find number of natural solutions of equation $x + y + z = 102$, where $x,\ y,\ z \in N$.

Sol. $x + y + z = 102$ consider x, y, z as 3 beggers and 102 as coins, give 1 coin to each equation becomes
$X + Y + Z = 99$. $\quad X = x + 1, Y = y + 1, Z = z + 1$.
Now distributed 99 coins without constrants total ways $^{99+2}C_2 = {}^{101}C_2$.

Imp. Point :

Number of different terms in a complete homogeneous expression of degree m in n variables is equivalent to distribution of m identical coins among n beggers.
If expression is $(x_1 + x_2 + x_3 \ldots\ldots + x_n)^m$ number of terms $^{m+n-1}C_{n-1}$.

Illustration :

Find number of different terms in expansion of $(x+y+z)^{10}$.

Sol. $^{10+3-1}C_{3-1} = {}^{12}C_2$.

Illustration :

A man has to buy 25 mangoes in four different variety buying at least 4 of each variety. In how many ways can he plan his purchases; if mangoes of each variety are identical and available in abundance.

Sol. V_1, V_2, V_3, V_4 are considered as beggers give 4 to each

O O O O O O O O O $\quad \phi\ \phi\ \phi$

Now 9 mangoes to 4 verieties without any ristriction given as $^{9+4-1}C_{4-1} = {}^{12}C_3$.

Illustration :

Number of non-negative integral solution of the inequality $x+y+z+t \le 30$.

Sol. $(x+y+z+t) + \underset{\substack{\downarrow \\ \text{(False begger)}}}{w} = 30$

Required condition is equivalent to giving 30 coin's to 5 begger's $= {}^{30+5-1}C_{5-1} = {}^{34}C_4$.

Illustration :

Required condition is equivalent to giving 30 coin's to 5 begger's $= {}^{30+5-1}C_{5-1} = {}^{34}C_4$.

Illustration :

Number of ways in which 16 identical toys are to be distributed among 3 children such that each child does not receive less than 3 toys will be

(A) 96 (B) 16 (C) 36 (D) 46

Sol. Let x_1, x_2, x_3 be the number of toys received by the three children

Then, $x_1, x_2, x_3, \ge 3$ and $x_1, x_2, x_3 = 16$

Let $u_1 = x_1 - 3$, $u_2 = x_2 - 3$ and $u_3 = x_3 - 3$

Then, $u_1, u_2, u_3 \ge 0$ and $u_1 + u_2 + u_3 = 7$

Here, $n = 7$ and $r = 3$

$\therefore$ Number of ways $= {}^{n+r-1}C_{r-1} = {}^{9}C_2 = 36$

Illustration :

Number of non-negative integral solutions of $x_1 + x_2 + x_3 + 4x_4 = 20$ will be

(A) 496 (B) 516 (C) 536 (D) 546

Sol. Here, clearly $0 \le x_4 \le 5$, $x_1, x_2, x_3, \ge 0$ and $x_1 + x_2 + x_3 = 20 - 4x_4$

$\Rightarrow$ $r = 3$ and $n = 20 - 4x_1$

If $x_4 = 0$ number of ways $^{20+3-1}C_{3-1} = {}^{22}C_2$

If $x_4 = 1$ number of ways $^{16+3-1}C_{3-1} = {}^{18}C_2$ Similarly, if $x_4 = 2, 3, 4, 5$, number of ways $= {}^{14}C_2, {}^{10}C_2, {}^{6}C_2, {}^{2}C_2$ respectively

$\therefore$ Total number of ways

$= {}^{22}C_2 + {}^{18}C_2 + {}^{14}C_2 + {}^{10}C_2 + {}^{6}C_2 + {}^{2}C_2 = 536$

Illustration :

Number of ways 5 identical balls can be distributed into 3 different boxes so that no box remains empty will be

(A) 1 (B) 3 (C) 6 (D) 15

Sol. The required number of ways = ${}^{5-1}C_{3-1}$

$$= {}^{4}C_{2} = \frac{4.3}{1.2} = 6$$

Try yourself :

A shelf contain 6 separate compartment's. Find number of ways in which 12 identical marbles can be placed in the compartment so that no compartment is empty. [Ans. ${}^{11}C_5$]

MAXIMISE ${}^{n}C_{r}$:

$${}^{n}C_{r} \text{ is maximum at } \begin{cases} r = \dfrac{n}{2} & \text{if n is even} \\[2ex] r = \dfrac{n-1}{2} \text{ or } \dfrac{n+1}{2} & \text{if n is odd} \end{cases}$$

e.g. ${}^{15}C_r$ is maximum when r = 7 or 8, ${}^{12}C_r$ is maximum when r = 6

GRID PROBLEM :

Complete cartesian plane is partitioned by drawing line | | to x and y-axis equidistant apart like the lines

GRID PROBLEM :

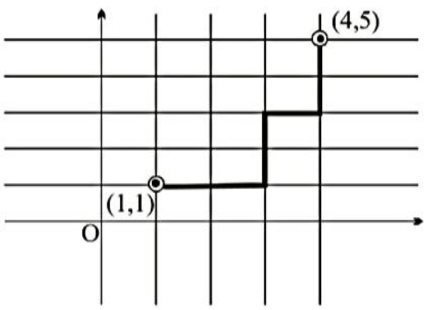
Complete cartesian plane is partitioned by drawing line | | to x and y-axis equidistant apart like the lines
on a chess board. Then the

Illustration :

Number of ways in which an ant can reach from (1, 1) to (4, 5) via shortest path.

Sol. Whatever may be the mode of travel of the ant; it has to traverse 3H (Horizontal) and 4V (Vertical) paths.

Hence required number of ways = $\dfrac{7!}{4!\,3!} = {}^{7}C_{3}$



Note : If there are n vertical and m horizontal lines then there will be (n – 1) horizontal and (m – 1) vertical paths

DERANGEMENT :

If n things are arranged in a row, the number of a ways they can be deranged so that r things occupy wrong places while (n – r) things occupy their original places, is

$$= {}^{n}C_{n-r}\, D_r$$

where $D_r = r\,!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+......+(-1)^r\frac{1}{r!}\right)$

If n things are arranged in a row, the number of ways they can be deranged so that none of them occupies its original place, is

$$= {}^{n}C_0\, D_n = n\,!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+......+(-1)^n\frac{1}{n!}\right)$$

Aliter :

D_n = ways in which n things are arranged so that all n things occupy wrong places

= arranged without restriction

– ways in which 1 thing is in correct position while (n – 1) things are deranged

– ways in which 2 things are in correct position while (n – 2) things are deranged

....

....

– ways in which all n things are in correct position and there is no derangement

$$= n! - {}^{n}C_1\, D_{n-1} - {}^{n}C_2 D_{n-2} - \ldots\ldots\ldots\ldots - {}^{n}C_n D_0$$

$$= n! - \sum^{n} {}^{n}C_r D_{n-r}$$

$$= n! - \sum_{r=1}^{n} {}^{n}C_r D_{n-r}$$

Thus, we have

$$D_0 = 0! = 1$$

$$D_1 = 1! - {}^{1}C_1 D_0 = 0$$

$$D_2 = 2! - {}^{2}C_1 D_1 - {}^{2}C_2 D_0 = 1$$

$$D_3 = 3! - {}^{3}C_1 D_2 - {}^{3}C_2 D_1 - {}^{3}C_3 D_0 = 2$$

$$D_4 = 4! - {}^{4}C_1 D_3 - {}^{4}C_2 D_2 - {}^{4}C_3 D_1 - {}^{4}C_4 D_0 = 9$$

$$D_5 = 5! - {}^{5}C_1 D_4 - {}^{5}C_2 D_3 - {}^{5}C_3 D_2 - {}^{5}C_4 D_1 - {}^{5}C_5 D_0 = 44$$

and so on.

Illustration :

A person writes letters to five friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelops so that

(a) all letters are in the wrong envelopes.

(b) at least three of them are in the wrong envelopes.

Sol.

(a) Required number of ways

$$= 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right) = \frac{5!}{2!}-\frac{5!}{3!}+\frac{5!}{4!}-\frac{5!}{5!}$$

(b) *Required number of ways*

$$= n!\sum_{n=1}^{n} {}^{n}C_r D_{n-r} \qquad \textit{where } n = 5$$

$$= {}^{5}C_2D_3 + {}^{5}C_1D_4 + {}^{5}C_0D_5$$

$$= 10 \times 3!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right) + 5 \times 4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right) + 1 \times 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)$$

$$= 10\,(3-1) + 5\,(12-4+1) + (60-20+5-1) = 20 + 45 + 44 = 109$$

Practice Problem

Q.1 The number of ways to put 4 letters in 4 addressed envelops so that all in wrong envelopes.

Q.2 The number of ways to put 5 letters in 5 addressed envelops so that all are in wrong envelopes.

Answer key

Q.1 9 Q.2 44

Q.1 9 Q.2 44

SOME IMPORTANT RESULTS ABOUT POINTS :

If there are n points in a plane of which m (< n) are collinear, then

(a) Total number of different straight lines obtained by joining these n points is

$${}^{n}C_2 - {}^{m}C_2 + 1$$

(b) Total number of different triangles formed by joining these n points is

$${}^{n}C_3 - {}^{m}C_3$$

(c) Number of diagonals in polygon of n sides is

$${}^{n}C_2 - n \qquad \text{i.e.} \quad \frac{n(n-3)}{2}$$

(d) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then total number of parallelograms so formed is

$${}^{m}C_2 \times {}^{n}C_2 \quad \text{i.e.} \quad \frac{mn(m-1)(n-1)}{4}$$

(e) Number of triangles formed by joining vertices of convex polygon of n sides is ${}^{n}C_3$ of which

(i) Number of triangles having exactly two sides common to the polygon = n

(ii) Number of triangles having exactly one side common to the polygon = n(n – 4)

(iii) Number of triangles having no side common to the polygon = $\dfrac{n(n-4)(n-5)}{6}$

Solved Examples

Single correct question

Q.1 In a bag there is a minimum of six old indian coins of every denominations (i.e. Athanni, Chavanni, Duanni, Ekanni). Number of ways in which one can take 6 coins from the bag is

(A) 120 (B) 90 (C) 84 (D) 60

Sol. Let Athanni $\rightarrow$ A Duanni $\rightarrow$ C
Chavanni $\rightarrow$ B Ekanni $\rightarrow$ D

$A + B + C + D = 6$ where $A \geq 0,\ B \geq 0, C \geq 0, D \geq 0$

Total number of arrangement $= {}^{n+r-1}C_{r-1} = {}^{9}C_{3} = 84$

Q.2 3 Indian and 3 American men and their wives are to be seated round to circular table. Let m denotes the number of ways when the indian couples are together and n denotes the number of ways when all the six couples are together. If $m = kn$ then k equals.

(A) 36 (B) 42 (C) 45 (D) 48

Sol. Indian couples can be seated together and rest 6 person are siting in a round table in $(6 + 3 - 1)! \times 2^3$
$= 8! \times 2^3$

where 2^3 are arrangement among themselves

$m = 8! \times 2^3$

Similarly 6 couples in a round table can be seated in

$n = (6 - 1)! \times 2^6$ ways $= 5! \times 2^6$

$$m = kn \quad \Rightarrow \quad \frac{m}{n} = k \quad \Rightarrow \quad \frac{8! \times 2^3}{5! \times 2^6} = k \quad \Rightarrow \quad k = 42$$

Similarly 6 couples in a round table can be seated in

$n = (6 - 1)! \times 2^6$ ways $= 5! \times 2^6$

$$m = kn \quad \Rightarrow \quad \frac{m}{n} = k \quad \Rightarrow \quad \frac{8! \times 2^3}{5! \times 2^6} = k \quad \Rightarrow \quad k = 42$$

Q.3 Golden temple express going from Amritsar to mumbai stops at 5 intermediate stations. 10 passengers enter the train during the journey with 10 different tickets of 'k' classes. If number of different sets of tickets they have is ${}^{45}C_{35}$ then k equals.

(A) 1 (B) 2 (C) 3 (D) 4

Sol. Let $I_1\ I_2\ I_3\ I_4\ I_5$ are the intermediate stations.
$A \quad I_1\ I_2\ I_3\ I_4\ I_5\ M$

Total number of tickets of one class are $= {}^{6}C_2$ (Selecting two station as origin & destination station)

Total tickets for k classes $= {}^{6}C_2 \times k$
$= 15k$

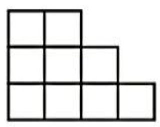
10 different tickets for 10 person can be choosen in ${}^{15k}C_{10}$

${}^{15k}C_{10} = {}^{45}C_{35} = {}^{45}C_{10}$

$15k = 45 \quad \Rightarrow \quad k = 3$

Q.4 The number of ways in which 5 X's can be placed in the squares of the figure so that no horizontal row remains empty is

(A) 97 (B) 98 (C) 100 (D) 126



Sol. Total number of x's are 5 so two horizontal rows can not be empty at a time that one horizontal row could be empty.

Total number of required ways $= {}^{9}C_5$ (total possible)
$-$ [top row empty + middle row empty
+ bottom row empty]

$= {}^{9}C_5 - [{}^{7}C_5 + {}^{6}C_5 + {}^{5}C_5] = 98$ Ans.

Q.5 Six boys and six girls sit along a line alternativelyin x ways and along a circle (again alternatively) in y ways then :

(A) $x = y$ (B) $y = 12\,x$ (C) $x = 10\,y$ (D) $x = 12\,y$

Sol. Lencar arrangement of 6 boy = 6! ways

$| B_1 | B_2 | B_3 | B_4 | B_5 | B_6$

Arrangement girls can be done in $6! \times 2$ ways

$x = 6! \times 6! \times 2$

For circular arrangement boys can be arranged in 5!

6 places can be filled by 6 girls in 6! ways

total ways $y = 5! \times 6!$

$x = 12\,y$ Ans.

Q.6 A forecast is to be made of the result of five cricket matches, each of which can be a win or a draw or a loss for indian team.

Let p = number of forecast with exactly 1 error

q = number of forecast with exactly 3 errors and

r = number of forecast with all five errors, then incorrect statement is :

(A) $8p = 5r$ (B) $2q = 5r$ (C) $8p = q$ (D) $2\,(p + q) > q$

Sol. Selection of one wrong forecast = ${}^5C_1$

Wrong forecast of 1 match can be done in 2 ways

$p = {}^5C_1 \times 2 = 10$

$q = {}^5C_3 \times 2 \times 2 \times 2 = 80$

$r = {}^5C \times 2 \times 2 \times 2 \times 2 \times 2 = 32$

Hence (A) Ans.

Q.7 The number of ten digit numbers that contain only 2 and 3 as its digit, but no any pairwise 3's joins together, is

(A) 145 (B) 143 (C) 129 (D) None

Sol.

One	3'	and	Nine	2's	→	$\lvert 2 \lvert 2 \lvert 2 \lvert 2 \lvert 2 \lvert 2 \lvert 2 \lvert 2 \lvert 2 \lvert \longrightarrow {}^{10}C_1$
Two	3's	and	8	2's	→	${}^9C_2$
Three	3's	and	7	2's	→	${}^8C_3$
Four	3's	and	6	2's	→	${}^7C_4$
Five	3's	and	5	2's	→	${}^6C_5$
Six	3's	and	6	2's	→	Not possible as repetition of 3's will come

Total number of ways = ${}^{10}C_1 + {}^9C_2 + {}^8C_3 + {}^7C_4 + {}^6C_5$

$= 143$ Ans.

Q.8 If the sum of all even positive dirisors of 100000 can be expressed in the form $k\,(5^2 + 5 + 1)\,(5^3 + 1)$ then the value of k is

(A) 31 (B) 62 (C) 64 (D) 93

Sol. $100000 = 2^5 \times 5^5$

Sum of all even divisor = $(2 + 2^2 + 2^3 + 2^4 + 2^5)\,(5^0 + 5^1 + \ldots\ldots + 5^5)$

$$= 2 \cdot \frac{2^5 - 1}{2 - 1} \cdot \frac{5^6 - 1}{5 - 1} = 2 \cdot \frac{31}{4}(5^3 - 1)\,(5^3 + 1)$$

$$= 62\,(5^2 + 5 + 1)\,(5^3 + 1) \quad \Rightarrow \quad k = 62$$

Q.9 Number of ways in which 15 indistinguishable oranges can be distributed in 3 different boxes so that every box R as atmost 8 oranges, are

(A) 52 (B) 108 (C) 76 (D) 28

Sol. Requried ways = (Total possible ways without restriction) – (ways when any box can ≥ 9 oranges)

Total possible ways are

$x + y + z = 15 \quad \Rightarrow \quad {}^{15+3-1}C_{3+1} = {}^{17}C_2$

Ways when any box can have 9 orarges

$x + y + z = 15$

either one of x, y, z can have more then 9 orages.

$x + y + z = 15 - 9 = 6$ with $x \ge 0,\ y \ge 0,\ z \ge 0$

Number of ways are ${}^3C_1 \times {}^{6+3-1}C_{3-1} = {}^3C_1 \times {}^8C_2$

Required ways are $= {}^{17}C_2 - {}^3C_1 \times {}^8C_2$

$= 52$ Ans.

Paragraph type

Paragraph for question nos. 10 to 12

Consider a polygon of sides 'n' which satisfies the equation $3 \cdot {}^np_4 = {}^{n-1}p_5$.

Q.10 Rajdhani express travelling from Delhi to Mumbai has n station enroute. Number of ways in which a train can be stopped at 3 stations if no two of the stopping station are consecutive, is

(A) 20 (B) 35 (C) 56 (D) 84

Q.11 Number of quadrilaterals that can be made using the vertices of the polygon of sides 'n' if exactly two adjacent sides of the quadrilateral are common to the sides of n - gon is

(A) 50 (B) 60 (C) 70 (D) None

Q.11 Number of quadrilaterals that can be made using the vertices of the polygon of sides 'n' if exactly two adjacent sides of the quadrilateral are common to the sides of n - gon is

(A) 50 (B) 60 (C) 70 (D) None

Q.12 Number of quadrilterals that can be formed using the vertices of a polygon of sides 'n' if exactly 1 side of the quadrilateral is common with the side of the n- gon is

(A) 150 (B) 100 (C) 96 (D) None

Sol. $3 \cdot \dfrac{n!}{(n-4)!} = \dfrac{(n-1)!}{(n-6)!} \qquad 3n = (n-4)(n-5)$

$\Rightarrow \quad 3n = n^2 - 9n + 20 \quad \Rightarrow \quad n^2 - 12n + 20 = 0$

$n = 10, \quad n = 2$ not possible

so $n = 10$

(i) Delhi $I_1\ I_2\ I_3\ I_4 \ldots\ldots I_{10}$ Mumbai

3 intermediate station such that no two are consecutive

Train is stopping at 3 station, for 7 remaining station there are 8 gaps.

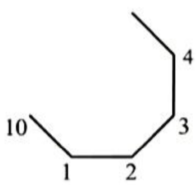
Filling these 3 station out of 8 gaps can be done in ${}^8C_3 = 56$ ways

(ii) $n = 10$

Contecutive two sides can be formed by (1, 2, 3) point so 4, 10 can't be

Seleted but one point out of remain 5 points can be selected in ${}^5C_1$ ways

So total quadrilaterals $= 10 \times {}^5C_1 = 50$



(iii) One side common means any two consecutive vertices can be selected in 10 ways. Now (1, 2) is selected 3, 10 can't be selected rest two non consecutive vertices can be selected from ${}^5C_2$ ways

Total number of quadrilateral $= 10 \cdot {}^5C_2 = 100$ ways.

Paragraph for question nos. 13 to 15

Consider the word "w" = C O M M I S I O N E R containing 12 letter of which five vowels and 7 consonants

Q.13 Number of 5 lettered word each comprising of 2 vowels and 3 consonants is
(A) 5120 (B) 6720 (C) 4960 (D) None

Q.14 Number of ways in which the letters of word 'w' can be arranged if alike letters are together but separated from the other alike letters is

(A) 2880 (B) 1120 (C) $\frac{12!-8!}{16}$ (D) None

Q.15 Number of ways is which the letters of the word 'w' can be arranged without changing the order of alike letters is

(A) $\frac{12!}{(2!)^4}$ (B) $^{12}C_8$ (C) $^{12}P_8$ (D) $^{12}P_4$

Sol. Vowels are O O I I E
Consonants are M M S S R C N

(i)	2 vowel &			3 consonant
(a)	vowels are alike		(a')	2 alike , 1 different
	$^2C_1 = 2$			$^2C_1 \times {}^4C_1 = 8$
(b)	vowels are different		(b')	All 3 different
	$^3C_2 = 3$			$^5C_3 = 10$

(i)	2 vowel &			3 consonant
(a)	vowels are alike		(a')	2 alike , 1 different
	$^2C_1 = 2$			$^2C_1 \times {}^4C_1 = 8$
(b)	vowels are different		(b')	All 3 different
	$^3C_2 = 3$			$^5C_3 = 10$

Total arrangement

$$aa' \times \frac{5!}{2! \times 2!} \times ab' \times \frac{5!}{2!} + ba' + \frac{5!}{2!} + bb' \times 5!$$

$$2 \times 8 \times \frac{5!}{4} + 2 \times 10 \times \frac{5!}{2!} + 3 \times 8 \times \frac{5!}{2!} + 3 \times 10 \times 5!$$

$$= 480 + 1200 + 1440 + 3600 = 6720 \text{ ways}$$

(ii) Alike word are N M, O O, I I, S S are together both seperated from other alike i.e. M M is seperated from OO.
so C N E R are arranged in 4! ways
5 placed are filled & arranged by $^5C_4 \times 4!$
total ways are $^5C_4 \times 4! \times 4! = 2880$

(iii) Total possible ways = 12! (considering all defferent)
ways of arranging 8 distinct letters are 8!

total required ways = $\frac{12!}{8!}$

Match the Column

Q.16 **Column-I** **Column-II**

(A) Number of all six digit natural numbers such that sum of their digits is 10 and each of the digit 0, 1, 2, 3 occurs atleast once in them is (P) 350

(B) A question paper consists of 2 parts A and B. Part A has 4 questions with 1 alternative each and Part-B has 3 question without any alternative. Number of ways in which one can select the question when atleast one question must be attempted from each part, is (Q) 405 (R) 490

(C) Number of ordered pairs of positive integers (a, b) such that the least common multiple of 'a' and 'b' is $2^2 \cdot 5^4 \cdot 11^4$ (S) 560

Sol.(A) 6 digit number containg 0, 1, 2, 3 atleast once are

S.No.	Fixed digit	Required digit	Total ways
(i)	0, 1, 2, 3	(4, 0)	$4 \times \frac{5!}{2!} = 240$
(ii)	0, 1, 2, 3	(3, 1)	$\frac{6!}{2! \times 2!} - \frac{5!}{2! \times 2!} = 150$
(iii)	0, 1, 2, 3	(2, 2)	$\frac{6!}{3!} - \frac{5!}{3!} = 100$

Total required ways = 240 + 150 + 100 = 490

(B) 3 option for question of paper A (i.e. 1st alternative 2nd alternative no solution)

2 option for question of paper B

Total ways = $(3^4 - 1)(2^3 - 1)$

$= (80)(7) = 560$

(C) $a = 2^r \cdot 5^s \cdot 11^t$ $\quad b = 2^{r'} \cdot 5^{s'} \cdot 11^{t'}$

If $r = 0, 1, 2$ then $r' = 2$

If $r = 2$ then $r' = 0, 1, 2$

Total ways $= 3 \times 2 - 1$ (for the repetition of (2, 2))

If $s = 0, 1, 2, 3, 4$ then $s^1 = 4$

If $s = 4$ then $s' = 0, 1, 2, 3, 4$

$= 4 \times 2 - 1$ (for repetition (4, 4))

$= 9$ ways

Similar for t & t' number of ways = 9 ways

total ways $5 \times 9 \times 9 = 405$ Ans.

Q.17 **Column-I** **Column-II**

(A) If the number of ways in which n different toys can be distributed in n children if exactly one child doesn't get any toy is 1200 then n equals (P) 3

(B) There are 2n white and 2n red counters. Counters are all alike exept for the colour. If the number of ways in which they can be arranged in a line so that they are symmetric w.r.t. a central mark is 70 then n equals to (Q) 4 (R) 5

(C) Total number of divisors of the number N = 360 which are of the form 4n + 2, n ≥ 0 is (S) 6

Sol.(A) Child can be rejected in $^nC_1$ ways

Now dividing n toys in (n – 1) boys.

$$= \frac{n!(n-1)!}{2!(n-2)!}$$

$$\text{Total ways} = {}^nC_1 \times \frac{n! \times (n-1)!}{2! \times (n-2)!} = n! \times {}^nC_2 = 1200$$

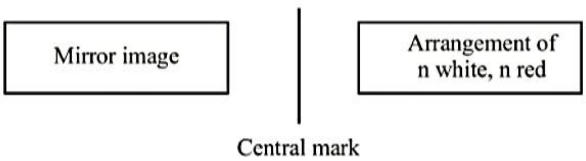
$$n = 5$$

(B) 2n red counters, 2n white counters

$$\text{Total ways} = {}^nC_1 \times \frac{n! \times (n-1)!}{2! \times (n-2)!} = n! \times {}^nC_2 = 1200$$

$$n = 5$$

(B) 2n red counters, 2n white counters



n white and n red can be arranged in $\frac{2n!}{n!\,n!}$ ways (Rest n white and n red are the mirror image)

$$\frac{2n!}{n!\,n!} = 70 \quad \Rightarrow \quad n = 4$$

(C) $360 = 2^3 \cdot 3^2 \cdot 5$

$4n + 2 \Rightarrow$ even number which is not divisible by 4 hence exacthy one '2' must be taken.

number of divisor $(1) \times (3) \times (2) = 6$

Q.18

	Column-I		Column-II
(A)	Number of five digit numbers of the form $d_1\, d_2\, d_3\, d_4\, d_5$ where d_{is} $i = 1, 2, 3, 4, 5$ are digit and satisfying $d_1 < d_2 \le d_3 < d_4 \le d_5$, is	(P)	${}^{10}C_5$
(B)	Number of five digit numbers of the form $d_1\, d_2\, d_3\, d_4\, d_5$ where $d_{is} = 1, 2, 3, 4, 5$ are digits satisfying $d_1 > d_2 \ge d_3 > d_4 > d_5$, is	(Q)	${}^{11}C_4$
(C)	Boby fischer and Boris Spassky play a unique game series in a chess tournament. They decide to play on till one of them won 5 matches. If each match end only in win or loss. Number of ways in which series can be won by either of them, is	(R)	${}^{11}C_6$
(D)	A badminton team has to be selected comprising of 5 students out of 10 students for inter shcool tournament. Number of ways this can be done if a perticular player is to be always included or always excluded from the team, is	(S)	$2 \cdot {}^9C_5$

Sol.(A) $d_1 < d_2 < d_3 < d_4 < d_5 \Rightarrow$ Number of numbers ${}^9C_5$ (0 not included)

If $d_1 < d_2 = d_3 < d_4 < d_5 \Rightarrow$ Number of numbers ${}^9C_4$ (0 not included and two same digit)

If $d_1 < d_2 < d_3 < d_4 = d_5 \Rightarrow$ Number of numbers ${}^9C_4$ (0 not included and two same digit)

If $d_1 < d_2 = d_3 < d_4 = d_5 \Rightarrow$ Number of number ${}^9C_3$ (0 not included & two pair of same digit)

Total = ${}^9C_5 + {}^9C_4 + {}^9C_4 + {}^9C_3 = {}^{10}C_5 + {}^{10}C_4 = {}^{11}C_6$

(B) $d_1 > d_2 \ge d_3 > d_4 > d_5$ ${}^{10}C_5$ (If all different)
${}^{10}C_4$ (If $d_2 = d_3$)

digit)

Total = ${}^9C_5 + {}^9C_4 + {}^9C_4 + {}^9C_3 = {}^{10}C_5 + {}^{10}C_4 = {}^{11}C_6$

(B) $d_1 > d_2 \ge d_3 > d_4 > d_5$ ${}^{10}C_5$ (If all different)
${}^{10}C_4$ (If $d_2 = d_3$)

$= {}^{10}C_5 + {}^{10}C_4 = {}^{11}C_5 = {}^{11}C_6$ Ans.

(C) Total number of ways in which the tournament can be won by either player = ${}^{10}C_5 = 2 \times {}^9C_5$

(D) Always excluded = ${}^9C_5$
Always included = ${}^9C_4$
Total ways = ${}^9C_5 + {}^9C_4 = {}^{10}C_5 = 2 \cdot {}^9C_5$

Reasoning type question

Q.19 Consider the 10 digits 0, 1, 2, 3, ... 9

Statement-1: Number of four digit even numbers that can be formed if each digit is to used only once in the number is 2268.

because

Statement-2: Total 4 digit numbers that can be formed if eah digit is used only once is 4536.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

Sol. Four digit even number

Case-I : If zero as least digit

$9 \times 8 \times 7\ \boxed{0} = 9 \times 8 \times 7$

Case-II : Zero is not last digit

$8 \times 8 \times 7\ \boxed{\uparrow} = 8 \times 8 \times 7$

4 ways (2, 4, 6, 8)

Four digit even numbers = $17 \times 56 = 952$
Total four digit numbers

$$\square \; \square \; \square \; \square$$
$$9 \times 9 \times 8 \times 7 = 4536$$

Q.20 **Statement-1:** Number of different terms in the expansive of $(a+b+c+d)^{12}$ is ${}^{15}C_3$
because
Statement-2: Number of ways in which n distinguishable objects can be distributed in p persons if each reciving none or one or more is ${}^{n+p-1}C_n$.
(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

Sol. Statement-1 :
Power of a $\rightarrow x_1$
Power of b $\rightarrow x_2$ $\quad x_1 + x_2 + x_3 + x_4 = 12$
Power of c $\rightarrow x_3$ $\quad$ total ways = ${}^{15}C_3$
Power of d $\rightarrow x_4$
Statement-2 :
n different object can be distributed to person in p^n ways.

Q.21 With usual notation
Statement-1: $c(n, r) \cdot p(r, r) = p(n, r)$ where $r, n \in N$
because
Statement-2: Every permutation of n distinct objects taken r at a time can be uniquely determine by

n different object can be distributed to person in p^n ways.

Q.21 With usual notation
Statement-1: $c(n, r) \cdot p(r, r) = p(n, r)$ where $r, n \in N$
because
Statement-2: Every permutation of n distinct objects taken r at a time can be uniquely determine by
first choosing r object out of these n object and then arranging thdese r objects.
(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

Sol. $c(n, r) = {}^nc_r$

$$p(r, r) = {}^rp_r = \frac{r!}{(r-r)!} = r!$$

$c(n, r) \cdot p(r, r) = p(n, r)$

Q.22 **Statement-1:** Number of rectangles on chessboard (which may be overlapped also) is ${}^8C_2 \times {}^8C_2$
because
Statement-2: To form a rectangles we have to select any two of the horizontal lines and any two from the vertical lines.
(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

Sol. Rectangle can be formed by selecting any two horizontal line and any two vertical lines.
Chess board has 9 horizontal and 9 vertical lines so total rectangles from a chessboard
$= {}^9C_2 \times {}^9C_2$