

Problem Set II.3

$$1. \quad \kappa(ATA) = \|ATA\| \|A^T A^{-1}\| \\ = \|ATA\| \|A^{-1} (A^{-1})^T\|$$

$$= \|A\|^2 \|A^{-1}\|^2 \\ = (\kappa(A))^2$$

$$2. \quad A = \begin{pmatrix} 2000 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{1}{2000} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\kappa(A) = \|A\| \cdot \|A^{-1}\| = 2000 \times 1 = 2000$$

$$\kappa(A^{-1}) = \kappa(A) = 2000 > 1000$$

$$3. \quad A \Delta x = \Delta b$$

$$\Delta x = A^{-1} \Delta b$$

$$\|\Delta x\| = \|A^{-1} \Delta b\| \leq \|A^{-1}\| \|\Delta b\|$$

$$Ax = b$$

$$\|\Delta b\| = \|Ax\| \leq \|A\| \|x\|$$

$$\|\Delta x\| \|\Delta b\| \leq \|A\| \|A^{-1}\| \|\Delta b\| \|\Delta x\|$$

$$\frac{\|\Delta x\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|\Delta b\|}{\|b\|}$$

4. positive definite A .

$$\|A\| = \lambda_{\max}$$

$$\|A^{-1}\| = 1 / \lambda_{\min}$$

$$\kappa(A) = \frac{\lambda_{\max}}{\lambda_{\min}}$$

$$5. \quad \kappa(Q) = \|Q\| \|Q^T\| \\ \geq \|Q\| \|Q^T\| = 1 \\ \Leftrightarrow Q \text{ is a constant multiple of } B.$$

$$8. \quad g_1 = (\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3})$$

$$g_2 = (\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6})$$

$$9. \quad A = QRP^T + \varepsilon$$
$$Q = \begin{pmatrix} \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{6} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \frac{\sqrt{6}}{3} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{3} \end{pmatrix}, \quad P^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(b) \quad \text{rank}(A) = 3.$$

Can not factor $A = YB\text{rank } Z$.