LIN241
Winter 2021
Lecture 7 summary: Quantifiers

1. Quantifiers and variables

(1) The following sentences of English contain quantifiers, namely **every** and **some**:

a. **Some people like South Park.**

b. **Every philosopher knows Kant.**

In order to translate them into Predicate Logic, we need to use variables, such as x in the following formulas:

c. ∃x[PEOPLE(x) & LIKE(x, s)]
'There is some x such that x is people and x likes s.'

d. ∀x[PHILOSOPHER(x) → KNOW(x, k)]
'For every x, if x is a philosopher, then x knows Kant.'

In these formulas, we say that the quantifier *binds* a variable in the formula it is attached to. In (1c), ∃x binds the occurrences of the variable x in the formula PEOPLE(x) & LIKE(x, s).

A formula that contains a variable that is not bound is an open formula. This is an open formula:

PEOPLE(x) & LIKE(x, s)

A formula that contains no variables that are not bound is a closed formula. This is a closed formula:

∃x[PEOPLE(x) & LIKE(x, s)]

(2) In (1c) and (1d), x does not refer to a specific individual. It is a variable that can take several values, like the variables you may be used to from algebra:

$f(x) = 3x + 2$

The function f returns different outputs for different values of x that are given as inputs: $f(1) = 5$, $f(2) = 8$, $f(3) = 11$, etc.

What are the possible values of the variables in statements like (1c) and (1d) above? They are individuals from the universe U of the model in which we are interpreting the formulas. The quantifiers tell us that the formula that follow them must be true for a certain 'quantity' of possible values given to *x*.

$[\![\exists x[\mathrm{PEOPLE}(x) \mathbin{\&} \mathrm{LIKE}(x, s)]]\!]^M = T$ iff there is at least one entity d in U such that
$d \in [\![\mathrm{PEOPLE}]\!]^M$ and $\langle d, [\![s]\!]^M\rangle \in [\![\mathrm{LIKE}]\!]^M$

$[\![\forall x[\mathrm{PHILOSOPHER}(x) \rightarrow \mathrm{KNOW}(x, k)]]\!]^M = T$ iff for every entity d in U,
if $d \in [\![\mathrm{PHILOSOPHER}]\!]^M$ then $\langle d, [\![k]\!]^M\rangle \in [\![\mathrm{KNOW}]\!]^M$

It is important to remember that individual variables are not interpreted like individual constants.

In order not to confuse them with individual constants, we use different letters to write down variables in Predicate Logic: $x, x', x'', \ldots, y, y', y'', \ldots z, z', z''\ldots$

2. Translating English sentences with Quantifiers into Predicate Logic

(3) Quantifiers in English express relations between two expressions, known as a restriction and a nuclear scope.

The restriction is the noun phrase that is determined by the quantifier.

The nuclear scope is more difficult to define informally, but it corresponds to the part of the clause that the quantifier relates to its restriction.

a. **John met every student**.
Restriction: **student**
Nuclear scope: **John met __**

b. **Every student is happy**.
Restriction: **student**
Nuclear scope: **__ is happy**

(4) The translation of a sentence with a quantifier must contain these two parts. The essential points to remember are that:

1. The restriction of the quantifier is translated as a predicate or an open formula,

2. The nuclear scope is translated as a predicate or an open formula,

3. If the quantifier is existential, the restriction and nuclear scope are related by &

4. If the quantifier is universal, the restriction and nuclear scope are related by →

5. The grammatical function of the quantifier (subject, object, ….) dictates where the variable that it binds should be placed in the nuclear scope.

(5) Examples:

a. **Some student is happy**.
$\exists x[\text{STUDENT}(x) \,\&\, \text{HAPPY}(x)]$

b. **A student knows Pia**.
$\exists x[\text{STUDENT}(x) \,\&\, \text{KNOW}(x,p)]$

c. **Conan interviewed some politician**.
$\exists x[\text{POLITICIAN}(x) \,\&\, \text{INTERVIEW}(c,x)]$

d. **Every student is happy**.
$\forall x[\text{STUDENT}(x) \rightarrow \text{HAPPY}(x)]$

e. **Everyone knows Pia**.
$\forall x[\text{PERSON}(x) \rightarrow \text{KNOW}(x,p)]$

f. **Pia knows every student**.
$\forall x[\text{STUDENT}(x) \rightarrow \text{KNOW}(p,x)]$

Note how the position of the variable x changes in (5b) vs (5c) and (5e) vs (5f).

It is useful to paraphrase these translations in English:

b. $\exists x[\text{STUDENT}(x) \,\&\, \text{KNOW}(x,p)]$
There is an x such that x is a student and x knows Pia.

c. $\exists x[\text{POLITICIAN}(x) \,\&\, \text{INTERVIEW}(c,x)]$
There is an x such that x is a politician and Conan interviewed x.

e. $\forall x[\text{PERSON}(x) \rightarrow \text{KNOW}(x,p)]$
For all x, if x is a person, then x knows Pia.

f. $\forall x[\text{STUDENT}(x) \rightarrow \text{KNOW}(p,x)]$
For all x, if x is a student then Pia knows x.

(6) Negative quantifiers can be expressed using either existential quantification or universal quantification.

(7) **No customer complained**.

$\sim\exists x[\text{CUSTOMER}(x) \,\&\, \text{COMPLAINED}(x)]$
'It is not the case that there is an x such that x is a customer and x complained.'

$\forall x[\text{CUSTOMER}(x) \rightarrow \sim \text{COMPLAINED}(x)]$
'For every x, if x is a customer then x didn't complain.'

These two translations are equivalent: there does not exist a customer who complained just in case for every customer, it false that this customer complained.

(8) **Not every customer complained**.

$\exists x[\text{CUSTOMER}(x) \ \& \ {\sim}\text{COMPLAINED}(x)]$
'There is an x such that x is a customer and x didn't complain.'

${\sim}\forall x[\text{CUSTOMER}(x) \rightarrow \text{COMPLAINED}(x)]$
'It is not the case that for all x, if x is a customer then x complained.'

These two translations are equivalent: there is a customer who didn't complain just in case it is false that every customer complained.

3. Scope Ambiguities in Predicate Logic

(9) Sentences of English can be ambiguous due to the presence of several quantifiers.

A dog is chasing every cat.

(10) These sentences can be disambiguated in Predicate Logic:

(9a) $\exists x[\text{DOG}(x) \ \& \ \forall y[\text{CAT}(y) \rightarrow \text{CHASE}(x,y)]]$
There is an x that is a dog such that for all y, if y is a cat, x is chasing y.

(9b) $\forall y[\text{CAT}(y) \rightarrow \exists x[\text{DOG}(x) \ \& \ \text{CHASE}(x,y)]]$
For all y, if y is a cat then there is an x that is dog such that x is chasing y.

(11) You see that in order to check whether (9a) is true, we must check that there is at least one dog such that this dog is chasing every cat. The universal statement is interpreted as a part of the existential statement. We say that the universal quantifier is inside the scope of the existential quantifier, or that the existential quantifier takes scope over the universal quantifier.

By contrast, in order to check whether (9b) is true, we must check that every entity that is a cat is such that this cat is chased by some dog. The existential statement is interpreted as a part of the universal statement. We say that the existential quantifier is inside the scope of the universal quantifier, or that the universal quantifier takes scope over the existential quantifier.

The ambiguity observed in (9) is a scope ambiguity.

(12) As these examples illustrate, translating a sentence into Predicate Logic forces us to resolve its scope ambiguities: the syntax of Predicate Logic is defined in such a way that well-formed formulas never have scope ambiguities. In other words, the scope of an expression in Predicate Logic is always represented unambiguously in the syntactic structure of the formulas where this expression occurs.

(13) The definition of scope for quantifiers of Predicate Logic is very simple:

- In the expressions $\forall x[\varphi]$, the formula φ is called the scope of the quantifier $\forall x$
- In the expressions $\exists x[\varphi]$, the formula φ is called the scope of the quantifier $\exists x$

(14) Let us illustrate:

- the scope of $\exists x$ in (9a) is the formula: $DOG(x)\ \&\ \forall y[CAT(y) \rightarrow CHASE(x,y)]$
- the scope of $\forall y$ in (9a) is the formula: $CAT(y) \rightarrow CHASE(x,y)$
- the scope of $\exists x$ in (9b) is the formula: $DOG(x)\ \&\ CHASE(x,y)$
- the scope of $\forall y$ in (9b) is the formula: $CAT(y) \rightarrow \exists x[DOG(x)\ \&\ CHASE(x,y)]$

(15) When thinking about the semantic effect of quantifier scope in formulas of Predicate Logic, it is often useful to draw diagrams of situations in which the formulas are true.

More precisely, let us say that you are looking at a formula with two quantifiers, and you think that there is a scope ambiguity between the quantifiers. For instance:

a. **Every child read a book**.

b. $\forall y[CHILD(y) \rightarrow \exists x[BOOK(x)\ \&\ READ(x,y)]]$

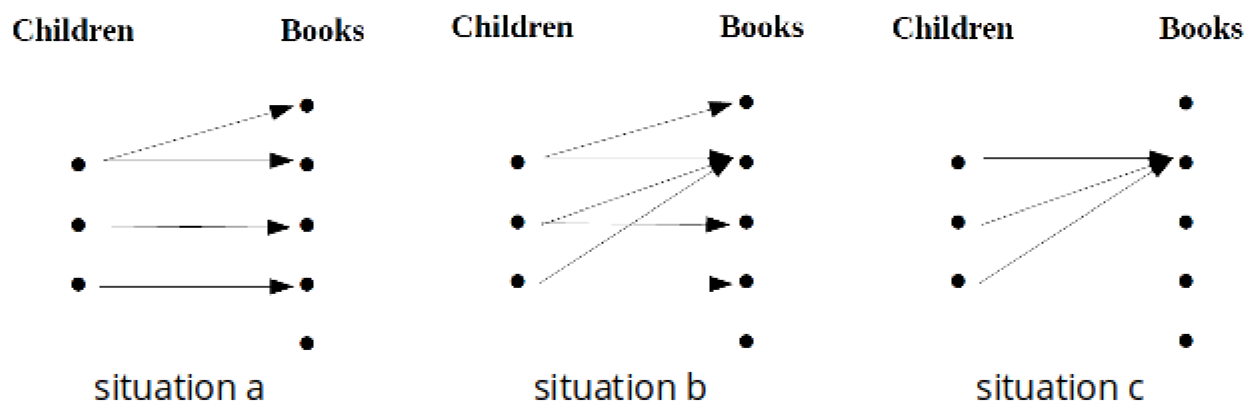
c. $\exists x[BOOK(x)\ \&\ \forall y[CHILD(y) \rightarrow READ(x,y)]]$

By translating (15a) as (15b) and (15c), you have assumed that the sentence is ambiguous between two scope options for the quantifiers **some** and **every**. At this point, you want to ask whether (15b) and (15c) have the same meaning, and if not, what is the difference in meaning between these two formulas.

In order to show that (15b) and (15c) have different meanings, you can show that there is a situation in which one of them is true and the other is false.

We can do this by representing situations with diagrams, where children and books are represented by dots, and lines between dots represent the reading relation. Three different situations of this sort are represented on the next page.

(16) Three situations in which the truth-value of (15b) and (15c) can be evaluated:



The truth value of (15b) and (15c) in these situations are:

Formula	situation a	situation b	situation c
(15b): $\forall y[\text{CHILD}(y) \rightarrow \exists x[\text{BOOK}(x) \,\&\, \text{READ}(x,y)]]$	T	T	T
(15c): $\exists x[\text{BOOK}(x) \,\&\, \forall y[\text{CHILD}(y) \rightarrow \text{READ}(x,y)]]$	F	T	T

Since (15c) is false in situation (a) but (15b) is true in that situation, the two formulas are not equivalent. They have different meanings.

(17) Note that (15c) entails (15b), but not the other way around.

Indeed, if there is a book that every child read (for instance, *The Wonderful Wizard of Oz*), then for every child, there is a book that this child read. However, it might be true that for every child, there is a book that this child read, while it is false that there is a book that every child read. This is the case for instance when every child read a different book, and no single book was read by all the children.

(18) **Surface Scope and Inverse scope**

Have a look again at (15a) and its translation as (15b) and (15c).

You see that in (15a), the universal quantifier is in subject position, and the existential quantifier is in object position. We know from syntax that subjects are ‘higher’ than objects in English. Syntacticians say that subjects c-command objects. In other words, the existential quantifier c-commands the universal quantifier in the *syntactic structure* of (15a).

The c-command relation between the quantifiers in (15a) is reflected in their respective scope in (15b), where $\exists x$ is in the scope of $\forall y$. In that case, we say that the quantifiers in (15a) are interpreted with “direct scope” or “surface scope.”

By contrast, the scope of the quantifiers in (15c) does not reflect the c-command relation between the quantifiers in (15a), since $\forall y$ is in the scope of $\exists x$. We say that in (15c), the quantifiers of (15a) are interpreted with “inverse scope.”

4. Generalized quantifiers

(19) Not all quantifiers of English and other natural languages can be translated into Predicate Logic. In order to analyze natural language quantifiers, we can think of quantifiers as expressing relations between sets.

(20) In the theory of Generalized Quantifiers (GQ), determiner quantifiers like **all** or **some** express relations between sets of entities.

In GQ theory[1], the universal quantifier **all** conveys that its restriction is included in its nuclear scope:[2]

$[\![\ \textbf{all students are smart}\ ]\!]^M = T$ iff $[\![\ \textbf{students}\ ]\!]^M \subseteq [\![\ \textbf{smart}\ ]\!]^M$

Remember that $[\![\ \textbf{students}\ ]\!]^M$ is a set. It is the set of all students in the model M.

Likewise, $[\![\ \textbf{smart}\ ]\!]^M$ is a set. It is the set of all smart individuals in the model M.

The sentence is true if and only if the set of students is included in the set of smart individuals.

Another simple example is **some**. In GQ theory, **some** conveys that the intersection of its restriction and its nuclear scope is not empty:[3]

$[\![\ \textbf{some cats are smart}\ ]\!]^M = T$ iff $[\![\ \textbf{cats}\ ]\!]^M \cap [\![\ \textbf{smart}\ ]\!]^M \neq \emptyset$

The sentence is true if and only if the intersection of the set of cats with the set of smart individuals is not empty: that is to say, when you can find at least one entity that is a member of both sets.

(21) We can define the semantics of different quantifiers in this way. We illustrate this with **most**:

(α) $[\![\ \textbf{most NP VP}\ ]\!]^M = T$ iff $|\ [\![\textbf{NP}]\!]^M \cap [\![\textbf{VP}]\!]^M\ |\ /\ |[\![\textbf{NP}]\!]^M| > 1/2$

Remember that if S is a set, |S| is the cardinality of S, i.e. the number of members of S.

Then (α) says that $[\![\ \textbf{most NP VP}\ ]\!]^M$ is true iff the cardinality of the intersection of $[\![\textbf{NP}]\!]^M$ and $[\![\textbf{VP}]\!]^M$ divided by the cardinality of $[\![\textbf{NP}]\!]^M$ is greater than 1/2.

That is to say: more than half of the individuals in $[\![\textbf{NP}]\!]^M$ are in $[\![\textbf{VP}]\!]^M$.

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[1] Note that in GQ theory, what is called a Generalized Quantifier is the combination of the determiner with its restriction. So **all students** and **some cats** are Generalized Quantifiers. In this course, we will keep referring to **all** and **some** as quantifiers regardless.

[2] Given to sets A and B, $A \subseteq B$ iff every member of A is a member of B.

[3] Given to sets A and B, $A \cap B$ is the set of every entity that is both a member of A and a member of B.

(22) The following notation allows us to extend Predicate Logic with quantifiers that express relations between sets:

[Q x : Φ] Ψ
Where Q is a quantifier, Φ is the restriction of Q and Ψ is the nuclear scope of Q

Examples:

[*all* x: STUDENT(x)] SMART(x)

[*some* x: CAT(x)] SMART

A more detailed discussion of this notation can be found in section 14.3 of your textbook.