Question 1

a) $$M^* \Phi^* = -\nabla \cdot D(r,E)\,\nabla \phi^*(r,E) + \Sigma_t(r,E)\cdot\phi^*(r,E) - \int_{E'} \Sigma_a(r,E\to E')\,\phi^*(r,E')\cdot dE'$$

$$F^* \Phi^* = \nu\Sigma_f(r,E)\int_{E'} \chi(E')\,\phi^*(r,E')\,dE'$$

b) $$[F\phi]_g = \chi_g \cdot \sum_{g'=1}^{6} \nu\Sigma_{fg'}\,\phi_{g'}$$

$$LHS = \phi_1^* \chi_1[\nu\Sigma_{f,1}\phi_1 + \nu\Sigma_{f,2}\phi_2] + \phi_2^*\chi_2[\nu\Sigma_{f,1}\phi_1 + \nu\Sigma_{f,2}\phi_2]$$

$$= \phi_1(\nu\Sigma_{f,1})\cdot[\chi_1\phi_1^* + \chi_2\phi_2^*] + \phi_2(\nu\Sigma_{f,2})[\chi_1\phi_1^* + \chi_2\phi_2^*]$$

If the RHS is also expanded as follows,

$$RHS = \phi_1\cdot(\nu\Sigma_{f,1})\cdot[\chi_1\cdot\phi_1^* + \chi_2\phi_2^*] + \phi_2\cdot(\nu\Sigma_{f,2})[\chi_1\phi_1^* + \chi_2\phi_2^*]$$

clearly LHS = RHS, so

$$\sum_{g=1}^{2}\phi_g^*[F\phi]_g = \sum_{g=1}^{2}\phi_g[F^*\phi^*]_g$$

c)

$$M\phi = \lambda F \phi \quad \text{—— (1)}$$

$$M^*\phi^* = \lambda^* F^* \phi^* \quad \text{—— (2)}$$

from (1)

$$\langle \phi^*, M\phi \rangle = \lambda \langle \phi^*, F\phi \rangle \quad \text{—— (3)}$$

from (2)

$$\langle \phi, M^*\phi \rangle = \lambda^* \langle \phi, F^*\phi^* \rangle$$

$$\Rightarrow \quad \langle \phi^*, M\phi \rangle = \lambda^* \langle \phi^*, F\phi \rangle \quad \text{—— (4)}$$

(3) − (4), we'll have

$$0 = (\lambda - \lambda^*) \langle \phi^*, F\phi \rangle$$

since this is true for all ϕ, ϕ^*, therefore,

$$\lambda = \lambda^*$$

d) $$I = \int_E \left[\int_{E'} \Sigma_s(E' \to E)\, \phi(E')\, dE' \right] \phi^*(E)\, dE$$

$$= \int_E \int_{E'} \Sigma_s(E' \to E)\, \phi(E')\, \phi^*(E)\, dE' \cdot dE$$

Similarily

$$I^* = \int_E \int_{E'} \Sigma_s^*(E' \to E)\, \phi^*(E')\, \phi(E) \cdot dE' dE$$

mathematically, we can switch the order of integration

$$I^* = \int_{E'} \int_E \Sigma_s^*(E \to E')\, \phi^*(E)\, \phi(E') \cdot dE' dE$$

let $I = I^*$ for all $\phi(E)$, we can see it holds only when

$$\Sigma_s(E' \to E) = \Sigma_s^*(E - E')$$

Question 2:

a)
$$M \Phi = \lambda F \Phi$$
$$M_0 \Phi_0 = \lambda_0 F_0 \Phi_0$$
$$M_0^* \Phi_0^* = \lambda_0 F_0^* \Phi_0^* \quad \text{(unperturbed)}$$

take the inner product

$$\langle \Phi, M_0^* \Phi_0^* \rangle = \lambda_0 \langle \Phi, F_0^* \Phi_0^* \rangle$$

by the definition of adjoint,

$$\langle \Phi_0^*, M_0 \Phi \rangle = \lambda_0 \langle \Phi_0^*, F_0 \Phi \rangle$$

Since $F = F_0 + \Delta F$, therefore

$$\langle \Phi_0^*, M_0 \Phi \rangle = \lambda_0 \langle \Phi_0^*, F \Phi \rangle - \lambda_0 \langle \Phi_0^*, \Delta F \Phi \rangle \quad -(1)$$

then, from the perturbed system,

$$\langle \Phi_0^*, M \Phi \rangle = \lambda \langle \Phi_0^*, F \Phi \rangle \quad -(2)$$

subtracting (1) (2), we have

$$\langle \Phi_0^*, M \Phi \rangle - \langle \Phi_0^*, M_0 \Phi \rangle = \lambda \langle \Phi_0^*, F \Phi \rangle - \lambda_0 \langle \Phi_0^*, F \Phi \rangle + \lambda_0 \langle \Phi_0^*, \Delta F \Phi \rangle$$

$$\Rightarrow \quad \langle \Phi_0^*, \Delta M \Phi \rangle - \langle \Phi_0^*, \lambda_0 \Delta F \Phi \rangle = (\lambda - \lambda_0) \langle \Phi_0^*, F \Phi \rangle$$

$$\Rightarrow \quad \langle \Phi_0^*, (\Delta M - \lambda_0 \Delta F)\Phi \rangle = \Delta\lambda \langle \Phi_0^*, F\Phi \rangle$$

$$\Rightarrow \quad \Delta\rho = -\Delta\lambda = \frac{\langle \Phi_0^*, (\lambda_0 \Delta F - \Delta M)\Phi \rangle}{\langle \Phi_0^*, F\Phi \rangle}$$

b)

$$\langle \Phi_0^*, M_0 \Phi \rangle = \lambda_0 \langle \Phi_0^*, F_0 \Phi \rangle \quad \text{— (1)}$$

$$\langle \Phi_0^*, M \Phi \rangle = \lambda \langle \Phi_0^*, F \Phi \rangle \quad \text{— (2)}$$

Since $F = F_0 + \Delta F$, eq (2) becomes,

$$\langle \Phi_0^*, M\Phi \rangle = \lambda \langle \Phi_0^*, F_0 \Phi \rangle + \lambda \langle \Phi_0^*, \Delta F \Phi \rangle \quad \text{— (3)}$$

subtract (1) (3), we have,

$$\langle \Phi_0^*, M_0 \Phi \rangle - \langle \Phi_0^*, M\Phi \rangle = \lambda_0 \langle \Phi_0^*, F_0\Phi \rangle - \lambda \langle \Phi_0^*, F_0\Phi \rangle - \lambda \langle \Phi_0^*, \Delta F \Phi \rangle$$

$$\Rightarrow \quad -\langle \Phi_0^*, \Delta M \Phi \rangle = -\Delta\lambda \langle \Phi_0^*, F_0 \Phi \rangle - \lambda \langle \Phi_0^*, \Delta F \Phi \rangle$$

$$\Rightarrow \quad \Delta\rho = -\Delta\lambda = \frac{\langle \Phi_0^*, (\lambda \Delta F - \Delta M)\Phi \rangle}{\langle \Phi_0^*, F_0 \Phi \rangle}$$

c)

$$\because \quad M_0 \Phi_0 = \lambda_0 F_0 \Phi_0$$

$$\therefore \quad \langle \Phi^*, M_0 \phi_0 \rangle = \lambda_0 \langle \phi^*, \Delta F \phi_0 \rangle$$

$$\Rightarrow \quad \langle \Phi^*, M_0 \Phi_0 \rangle = \lambda_0 \langle \Phi^*, F \phi_0 \rangle - \lambda_0 \langle \Phi^*, \Delta F \Phi_0 \rangle \quad — (1)$$

plus, since $M^* \phi^* = \lambda F^* \phi^*$ (adjoint)

we have $\langle \phi_0, M^* \phi^* \rangle = \lambda \langle \phi_0, F^* \phi^* \rangle$

$$\Rightarrow \quad \langle \phi^*, M \phi_0 \rangle = \lambda \langle \phi^*, F \phi_0 \rangle \quad — (2)$$

substracting (1) (2), we got

$$\langle \phi^*, M_0 \phi_0 \rangle - \langle \phi^*, M \phi_0 \rangle = \lambda_0 \langle \phi^*, F \phi_0 \rangle - \lambda_0 \langle \phi^*, \Delta F \phi_0 \rangle$$
$$- \lambda \langle \phi^*, F \phi_0 \rangle$$

$$\Rightarrow \quad - \langle \phi^*, \Delta M \phi_0 \rangle + \lambda_0 \langle \phi^*, \Delta F \phi_0 \rangle = - \Delta \lambda \langle \phi^*, F \phi_0 \rangle$$

$$\Rightarrow \quad \Delta \rho = - \Delta \lambda = \frac{\langle \phi^*, (\lambda_0 \Delta F - \Delta M) \phi \rangle}{\langle \phi^*, F \phi_0 \rangle}$$

d> $M_0 \phi_0 = \lambda_0 F_0 \phi_0 \Rightarrow \langle \phi^*, M_0 \phi_0 \rangle = \lambda_0 \langle \phi^*, F_0 \phi_0 \rangle$ — (1)

with the adjoint $M^* \phi^* = \lambda F^* \phi^*$

$\Rightarrow \langle \phi_0, M^* \phi^* \rangle = \lambda \langle \phi_0, F^* \phi^* \rangle$

$\Rightarrow \langle \phi^*, M^* \phi_0 \rangle = \lambda \langle \phi^*, F \phi_0 \rangle$

$= \lambda \langle \phi^*, F_0 \phi_0 \rangle$

$+ \lambda \langle \phi^*, \Delta F \phi_0 \rangle$ — (2)

subtracting (1) (2), we have,

$\langle \phi^*, M_0 \phi_0 \rangle - \langle \phi^*, M \phi_0 \rangle = \lambda_0 \langle \phi^*, F_0 \phi_0 \rangle - \lambda \langle \phi^*, F_0 \phi_0 \rangle$

$- \lambda \langle \phi^*, \Delta F \phi_0 \rangle$

$\Rightarrow$

$- \langle \phi^*, \Delta M \phi_0 \rangle + \lambda \langle \phi^*, \Delta F \phi_0 \rangle = - \Delta\lambda \langle \phi^*, F_0 \phi_0 \rangle$

$$\Rightarrow \Delta \rho = - \Delta \lambda = \frac{\langle \phi^*, (\lambda \Delta F - \Delta M) \phi_0 \rangle}{\langle \phi^*, F_0 \phi_0 \rangle}$$

Question 3:

insert the values into the equation, we get

$$\begin{pmatrix} 9 & 0 \\ -6 & 2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \lambda \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

in which $M = \begin{pmatrix} 9 & 0 \\ -6 & 2 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$

the system can also be written as,

$$(M^{-1} F)\phi = \frac{1}{\lambda} \phi$$

$$\because \quad M^{-1} F = \begin{pmatrix} 0 & \frac{1}{3} \\ 0 & 1 \end{pmatrix}$$

and its eigenvalues are 0 and 1, so

$\lambda_0 = 1$ (critical condition)

the eigen vector when $\lambda = 1$ is $\phi_0 = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}$

then we look into the adjoint problem,

$$M_0^* \phi_0^* = \lambda_0 F_0^* \phi_0^*$$

$$M_0^* = \begin{pmatrix} 9 & -6 \\ 0 & 2 \end{pmatrix} \qquad F_0^* = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$$

$$\Rightarrow \quad M_0^{*-1} \cdot F_0^* = \begin{pmatrix} \frac{1}{9} & \frac{1}{3} \\ 0 & \frac{1}{2} \end{pmatrix}$$

solve for $\lambda_0^* = 1$, and $\phi_0^* = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$

and $\Delta\rho = \dfrac{\langle \phi_0^*, (\Delta M - \lambda_0 \Delta F)\phi_0 \rangle}{\langle \phi_0^*, F_0 \phi_0 \rangle}$

$\Delta M = M - M_0$

$\delta F = F - F_0$

FOPT

$$\Delta\rho = \frac{\langle \phi_0^*, (\lambda_0 \Delta F - \Delta M)\phi \rangle}{\langle \phi_0^*, F\phi \rangle} \quad \text{(EPT)}$$

Inserting all known numbers into the previous two equations, we have the $\Delta\rho$ results as shown below

Case	Direct Substraction	FOPT[1]	EPT[2]	Relative Error of FOPT
a) $\delta N^2 = 0.01E24$	-5×10^{-2}	-5×10^{-2}	-5×10^{-2}	—
b) $\delta N^2 = 0.10E24$	-5×10^{-1}	-5×10^{-1}	-5×10^{-1}	—
c) $\delta\sigma^1_{f2} = 0.01b$	4.9505×10^{-3}	5×10^{-3}	4.9505×10^{-3}	0.0099
d) $\delta\sigma^1_{f2} = 0.50b$	0.1666⋯	0.25	0.1666⋯	0.5
e) $\delta\sigma^1_{12} = 1.00b$	0.047619	0.055⋯	0.047619	0.166⋯

Discussion,

1. $\Delta\rho$ in a) and b) is negative since poison is added to the system.
2. $\Delta\rho$ in c) and d) is positive since fission cross section is increased.

3. $\Delta\rho$ in e) is positive since the scattering into the thermal group increased.
4. The EPT method gives the "exact" results in all five cases.
5. FOPT gives the exact results in the first two cases, but not in the other three cases. This is because σ_f and $\theta_{1,2}$ affect the system quadratically. FOPT doesn't model this so there's a discrepency.

Question 4:

$$\phi(r) = \frac{\phi \sin\left(\frac{\pi r}{B}\right)}{r} \quad \& \quad B = \frac{\pi}{R}$$

$$\Delta\rho = \frac{-\int \Delta\Sigma_a \, \phi(r) \cdot dV}{\int \nu \Sigma_{f_0} \, \phi(r) \, dV}$$

$$dV = d\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 \cdot dr$$

$$\Rightarrow \Delta\rho = \frac{-\int_0 \Delta\Sigma_a \, (4\pi r^2) \cdot \frac{\phi \cdot \sin\left(\frac{\pi r}{R}\right)}{r} \cdot dr}{\int_0^R \nu \Sigma_{f_0} \, (4\pi r^2) \, \frac{\phi_0 \sin\left(\frac{\pi r}{R}\right)}{r} \cdot dr}$$

$$= \frac{-\Delta\Sigma_a \int_0^{r_0} r \sin\left(\frac{\pi r}{R}\right) \cdot dr}{\nu \Sigma_{f_0} \int_0^R r \sin\left(\frac{\pi r}{R}\right) \cdot dr}$$

$$= \left(-\frac{\Delta\Sigma_a}{\nu \Sigma_{f_0}}\right) \frac{-\frac{r \cdot R}{\pi} \cos\left(\frac{\pi r}{R}\right)\Big|_0^{r_0} + \int_0^{r_0} \frac{R}{\pi} \cos\left(\frac{\pi r}{R}\right) dr}{-\frac{rR}{\pi} \cos\left(\frac{\pi r}{R}\right)\Big|_0^R + \int_0^R \frac{R}{\pi} \cos\left(\frac{\pi r}{R}\right) \cdot dr}$$

$$= \frac{\Delta\Sigma_a}{\nu \Sigma_{f_0}} \left(\frac{r}{R} \cos\left(\frac{\pi r}{R}\right) - \frac{1}{\pi} \sin\left(\frac{\pi r}{R}\right) \right)$$

this is from the reaction rate,

Then try from FOPT.

$$\Delta\rho = \frac{-\int \Delta\Sigma_a \phi(r) \cdot dV}{\int \nu \Sigma_{f_0} \phi(r)\, dV}$$

$$= \frac{-\int_0^{r_a} \Delta\Sigma_a (4\pi r^2) \frac{\phi_0^2 \sin^2(\frac{\pi r}{R})}{r^2} dr}{\int_0^R \nu \Sigma_{f_0} (4\pi r^2) \frac{\phi_0^2 \sin^2(\frac{\pi r}{R})}{r^2} dr}$$

$$= \frac{-\Delta\Sigma_a \int_0^{r_a} \sin^2(\frac{\pi r}{R})\, dr}{\nu \Sigma_{f_0} \int_0^R \sin^2(\frac{\pi r}{R}) \cdot dr}$$

$$= \frac{-\Delta\Sigma_a \int_0^{r_a} \frac{1-\cos(\frac{2\pi r}{R})}{2} \cdot dr}{\nu \Sigma_{f_0} \int_0^R \frac{1-\cos(\frac{2\pi r}{R})}{2} \cdot dr}$$

$$= \frac{-\Delta\Sigma_a}{\nu \Sigma_{f_0}} \frac{\left(\frac{r}{2} - \frac{R}{4\pi} \sin(\frac{2\pi r}{R})\right)\Big|_0^{r_a}}{\left(\frac{r}{2} - \frac{R}{4\pi} \sin(\frac{2\pi r}{R})\right)\Big|_0^{R}}$$

$$= \left(-\frac{\Delta\Sigma_a}{\nu \Sigma_f}\right) \frac{\frac{r}{2} - \frac{R}{4\pi} \sin(\frac{2\pi r_a}{R})}{\frac{R}{2}}$$

$$\Rightarrow \Delta\rho = \frac{\Delta\Sigma_a}{\nu \Sigma_{f_0}} \left(\frac{1}{2\pi} \sin(\frac{2\pi r_0}{R}) - \frac{r_a}{R}\right)$$

(a) from Eq. (3.15b)

$$k = \frac{\nu \Sigma_f}{\Sigma_a} \frac{1}{1 + L^2 B^2}$$

in this case $B = \frac{\pi}{R}$ and $L = \sqrt{\frac{D}{\Sigma_a}}$

therefore, $$k = \frac{\nu \Sigma_f}{\Sigma_a + D\left(\frac{\pi}{R}\right)^2} = \frac{0.126}{0.125 + 0.25 \cdot \left(\frac{\pi}{50}\right)^2}$$

$$k = 1.0001035$$

(b) $$\Delta \Sigma_a = \Delta \Sigma_c = (c\%) \cdot \Sigma_c$$

EPT: $$\Delta \rho = \frac{\Delta \Sigma_a}{\nu \Sigma_{f_0}} \left(\frac{r_a}{R} \cos\left(\frac{\pi r_a}{R}\right) - \frac{1}{\pi} \sin\left(\frac{\pi r_a}{R}\right) \right)$$

FOPT: $$\Delta \rho = \frac{\Delta \Sigma_a}{\nu \Sigma_{f_0}} \left(\frac{1}{2\pi} \sin\left(\frac{2\pi r_a}{R}\right) - \frac{r_a}{R} \right)$$

Inserting all known values, we'll have

$$\text{EPT} \Rightarrow \Delta \rho = \frac{(c\%)\ 0.073}{0.126} \left(\frac{1}{10} \cos\left(\frac{\pi}{10}\right) - \frac{1}{\pi} \sin \frac{\pi}{10} \right)$$

$$= -0.00188728908 \cdot c\%$$

$$\text{FOPT} \Rightarrow \Delta \rho = \frac{(c\%) \cdot 0.073}{0.126} \cdot \left(\frac{1}{2\pi} \cdot \sin\left(\frac{\pi}{5}\right) - \frac{1}{10} \right)$$

$$= -0.0037875562 \cdot c\%$$

It can be seen that FOPT predicts more reactivity change.