

Dynamic Taylor Cone Part II: Numeric Scheme

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Numeric scheme ([Taylor 1964](#))

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1. Quintic spline

Given N marker points $\{\mathbf{x}_0, \dots, \mathbf{x}_N\}$, we first compute $(N - 1)$ chord lengths $\{h_0, \dots, h_{N-1}\}$ by euclidean norm between adjacent marker points,

$$h_j = \|\mathbf{x}_{j+1} - \mathbf{x}_j\| \quad \text{for } j = 0, \dots, N - 1 \quad (1.1)$$

Then we introduce N chord coordinates $\{l_0, \dots, l_N\}$ with $l_0 = 0$ and

$$l_j = \sum_{k=0}^{j-1} h_k \quad \text{for } j = 1, \dots, N \quad (1.2)$$

Global spline is a collection of local splines $\{\mathbf{s}^{(0)}, \dots, \mathbf{s}^{(N-1)}\}$ where each $\mathbf{s}^{(j)}$ is a fifth degree polynomial defined on the interval $l \in [l_j, l_{j+1}]$,

$$\mathbf{s}^{(j)}(l) = \mathbf{x}_j + \sum_{k=1}^5 \mathbf{c}_k^{(j)} (l - l_j)^k, \quad \text{for } j = 0, \dots, N - 1 \quad (1.3)$$

Continuity between neighboring splines is satisfied up to the fourth derivative,

$$\left. \frac{d^p \mathbf{s}^{(j)}}{dl^p} \right|_{l_{j+1}} = \left. \frac{d^p \mathbf{s}^{(j+1)}}{dl^p} \right|_{l_{j+1}}, \quad \text{for } p = 0, \dots, 4 \quad (1.4)$$

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The continuity conditions yield five equations

$$\left. \begin{aligned} \mathbf{c}_3^{(j-1)} h_{j-1}^3 + \mathbf{c}_4^{(j-1)} h_{j-1}^4 + \mathbf{c}_5^{(j-1)} h_{j-1}^5 &= \mathbf{x}_j - \mathbf{x}_{j-1} - \mathbf{c}_1^{(j-1)} h_{j-1} - \mathbf{c}_2^{(j-1)} h_{j-1}^2 \\ 3\mathbf{c}_3^{(j-1)} h_{j-1}^2 + 4\mathbf{c}_4^{(j-1)} h_{j-1}^3 + 5\mathbf{c}_5^{(j-1)} h_{j-1}^4 &= -\mathbf{c}_1^{(j-1)} - 2\mathbf{c}_2^{(j-1)} h_{j-1} + \mathbf{c}_1^{(j)} \\ 6\mathbf{c}_3^{(j-1)} h_{j-1} + 12\mathbf{c}_4^{(j-1)} h_{j-1}^2 + 20\mathbf{c}_5^{(j-1)} h_{j-1}^3 &= -2\mathbf{c}_2^{(j-1)} + 2\mathbf{c}_2^{(j)} \\ \mathbf{c}_3^{(j-1)} + 4h_{j-1}\mathbf{c}_4^{(j-1)} + 10h_{j-1}^2\mathbf{c}_5^{(j-1)} &= \mathbf{c}_3^{(j)} \\ \mathbf{c}_4^{(j-1)} + 5h_{j-1}\mathbf{c}_5^{(j-1)} &= \mathbf{c}_4^{(j)} \end{aligned} \right\} \quad (1.5)$$

We first solve for $\mathbf{c}_3^{(j-1)}$, $\mathbf{c}_4^{(j-1)}$ and $\mathbf{c}_5^{(j-1)}$,

$$\left. \begin{aligned} h_{j-1}^3 \mathbf{c}_3^{(j-1)} &= -6h_{j-1} \mathbf{c}_1^{(j-1)} - 3h_{j-1}^2 \mathbf{c}_2^{(j-1)} - 4h_{j-1} \mathbf{c}_1^{(j)} + h_{j-1}^2 \mathbf{c}_2^{(j)} + 10(\mathbf{x}_j - \mathbf{x}_{j-1}) \\ h_{j-1}^4 \mathbf{c}_4^{(j-1)} &= +8h_{j-1} \mathbf{c}_1^{(j-1)} + 3h_{j-1}^2 \mathbf{c}_2^{(j-1)} + 7h_{j-1} \mathbf{c}_1^{(j)} - 2h_{j-1}^2 \mathbf{c}_2^{(j)} - 15(\mathbf{x}_j - \mathbf{x}_{j-1}) \\ h_{j-1}^5 \mathbf{c}_5^{(j-1)} &= -3h_{j-1} \mathbf{c}_1^{(j-1)} - h_{j-1}^2 \mathbf{c}_2^{(j-1)} - 3h_{j-1} \mathbf{c}_1^{(j)} + h_{j-1}^2 \mathbf{c}_2^{(j)} + 6(\mathbf{x}_j - \mathbf{x}_{j-1}) \end{aligned} \right\} \quad (1.6)$$

Define $\lambda = h_j/h_{j-1}$. We have $2(N-1)$ equations for $j = 1, \dots, N-1$,

$$\left. \begin{aligned} 10 [\lambda^3 \mathbf{x}_{j-1} - (1 + \lambda^3) \mathbf{x}_j + \mathbf{x}_{j+1}] &= 4h_j \lambda^2 \mathbf{c}_1^{(j-1)} + 6h_j (\lambda^2 - 1) \mathbf{c}_1^{(j)} - 4h_j \mathbf{c}_1^{(j+1)} \\ &\quad + h_j^2 \lambda \mathbf{c}_2^{(j-1)} - 3h_j^2 (1 + \lambda) \mathbf{c}_2^{(j)} + h_j^2 \mathbf{c}_2^{(j+1)} \\ 15 [-\lambda^4 \mathbf{x}_{j-1} + (\lambda^4 - 1) \mathbf{x}_j + \mathbf{x}_{j+1}] &= 7h_j \lambda^3 \mathbf{c}_1^{(j-1)} + 8h_j (1 + \lambda^3) \mathbf{c}_1^{(j)} + 7h_j \mathbf{c}_1^{(j+1)} \\ &\quad + 2h_j^2 \lambda^2 \mathbf{c}_2^{(j-1)} + 3h_j^2 (1 - \lambda^2) \mathbf{c}_2^{(j)} - 2h_j^2 \mathbf{c}_2^{(j+1)} \end{aligned} \right\} \quad (1.7)$$

for $2(N+1)$ unknowns, $\{\mathbf{c}_1^{(0)}, \dots, \mathbf{c}_1^{(N)}\}$ and $\{\mathbf{c}_2^{(0)}, \dots, \mathbf{c}_2^{(N)}\}$. The imposed boundary conditions at the beginning and end of the global spline produce four additional equations for $\{\mathbf{c}_1^{(0)}, \mathbf{c}_2^{(0)}, \mathbf{c}_1^{(N)}, \mathbf{c}_2^{(N)}\}$. Note we have implicitly introduced a ghost spline $\mathbf{s}^{(N)}$ which satisfies all continuity conditions with $\mathbf{s}^{(N-1)}$. The purpose of $\mathbf{s}^{(N)}$ is to deal with boundary conditions at the end of spline.

In general there are three types of boundary conditions at each end. Let x , s and $c^{(j)}$ be one of the scalar components of vector \mathbf{x}_j , spline $\mathbf{s}^{(j)}$ and coefficient $\mathbf{c}^{(j)}$. We first consider boundary conditions at $l = l_0$,

$$\text{Even: } 0 = \frac{ds^{(0)}}{dl} = \frac{d^3 s^{(0)}}{dl^3} \quad \text{at } l = l_0 \quad (1.8)$$

$$\text{Odd: } 0 = s^{(0)} = \frac{d^2 s^{(0)}}{dl^2} = \frac{d^4 s^{(0)}}{dl^4} \quad \text{at } l = l_0 \quad (1.9)$$

$$\text{Mix: } \alpha = \frac{ds^{(0)}}{dl}, \quad \beta = \frac{d^2 s^{(0)}}{dl^2} \quad \text{at } l = l_0 \quad (1.10)$$

which lead to two equations,

$$\left. \begin{aligned} c_1^{(0)} &= 0 \\ 6h_0c_1^{(0)} + 3h_0^2c_2^{(0)} + 4h_0c_1^{(1)} - h_0^2c_2^{(1)} &= -10x_0 + 10x_1 \\ -8h_0c_1^{(0)} - 3h_0^2c_2^{(0)} - 7h_0c_1^{(1)} + 2h_0^2c_2^{(1)} &= +15x_0 - 15x_1 \end{aligned} \right\} \text{ Even} \quad (1.11a)$$

$$\left. \begin{aligned} c_2^{(0)} &= 0 \\ c_1^{(0)} &= \alpha \end{aligned} \right\} \text{ Odd} \quad (1.11b)$$

$$\left. \begin{aligned} c_1^{(0)} &= \alpha \\ c_2^{(0)} &= \beta/2 \end{aligned} \right\} \text{ Mix} \quad (1.11c)$$

We then consider boundary conditions at $l = l_N$,

$$\text{Even: } 0 = \frac{ds^{(N-1)}}{dl} = \frac{d^3s^{(N-1)}}{dl^3} \quad \text{at } l = l_N \quad (1.12)$$

$$\text{Odd: } 0 = s^{(N-1)} = \frac{d^2s^{(N-1)}}{dl^2} = \frac{d^4s^{(N-1)}}{dl^4} \quad \text{at } l = l_N \quad (1.13)$$

$$\text{Mix: } \alpha = \frac{ds^{(N-1)}}{dl}, \quad \beta = \frac{d^2s^{(N-1)}}{dl^2} \quad \text{at } l = l_N \quad (1.14)$$

which also lead to two equations,

$$\left. \begin{aligned} c_1^{(N)} &= 0 \\ -4h_{N-1}c_1^{(N-1)} - h_{N-1}^2c_2^{(N-1)} - 6h_{N-1}c_1^{(N)} + 3h_{N-1}^2c_2^{(N)} &= 10x_{N-1} - 10x_N \end{aligned} \right\} \text{ Even} \quad (1.15a)$$

$$\left. \begin{aligned} -7h_{N-1}c_1^{(N-1)} - 2h_{N-1}^2c_2^{(N-1)} - 8h_{N-1}c_1^{(N)} + 3h_{N-1}^2c_2^{(N)} &= 15x_{N-1} - 15x_N \\ c_2^{(N)} &= 0 \end{aligned} \right\} \text{ Odd} \quad (1.15b)$$

$$\left. \begin{aligned} c_1^{(N)} &= \alpha \\ c_2^{(N)} &= \beta/2 \end{aligned} \right\} \text{ Mix} \quad (1.15c)$$

If we arrange the unknowns into a vector,

$$[c_1^{(0)}, c_2^{(0)}, \dots, c_1^{(j)}, c_2^{(j)}, \dots, c_1^{(N)}, c_2^{(N)}] \quad (1.16)$$

equations (1.7) with one of (1.11) and one of (1.15) result in a $2(N+1)$ -by- $2(N+1)$ system of linear equations, which corresponds to a banded diagonal sparse matrix with at most 6 non-zero elements in each row. The rest coefficients $c_3^{(j)}$, $c_4^{(j)}$ and $c_5^{(j)}$ can be reconstruct using equation (1.6).

It is convenient to re-parametrize each local spline with a new variable $t \in [0, 1]$,

$$\mathbf{s}^{(j)}(t) = \mathbf{x}_j + \sum_{k=1}^5 \mathbf{c}_k^{(j)} t^k, \quad \text{for } j = 0, \dots, N-1 \quad (1.17)$$

by redefining $\mathbf{c}_k^{(j)} \rightarrow \mathbf{c}_k^{(j)} h_j^k$. We can construct Lagrange interpolations along the arc-length of the global spline by the local arc-length L_j and its local fraction ξ ,

$$L_j = \int_0^1 \dot{\mathbf{s}}^{(j)} dt', \quad \xi(t) = \frac{1}{L_j} \int_0^t \dot{\mathbf{s}}^{(j)} dt', \quad (1.18)$$

Convention for normal vector \mathbf{n} ,

$$\mathbf{n} = \frac{(-\dot{z}, \dot{r})}{\|\dot{\mathbf{s}}\|} \quad (1.19)$$

2. Numerical integration and approximation

Standard m -point Gauss-Legendre quadrature,

$$\int_0^1 f(t) dt \approx \sum_{k=1}^m w_k f(x_k), \quad \int_a^b f(t) dt \Rightarrow \begin{matrix} x_k \rightarrow a + (b-a)x_k \\ w_k \rightarrow (b-a)w_k \end{matrix} \quad (2.1)$$

Logarithmic-weighted Gauss quadrature,

$$\int_0^1 f(t) \ln t dt \approx - \sum_{k=1}^m w_k^{\log} f(x_k), \quad \int_0^1 f(t) \ln(1-t) dt \approx - \sum_{k=1}^m w_k^{\log} f(1-x_k) \quad (2.2)$$

When logarithmic singularity $\tau \in (0, 1)$,

$$\begin{aligned} \int_0^1 f(t) \ln |t - \tau| dt &= \int_0^\tau f(t) \ln(\tau - t) dt + \int_\tau^1 f(t) \ln(t - \tau) dt \\ &= \tau \int_0^1 f(\tau s) \ln(\tau - \tau s) ds + (1 - \tau) \int_0^1 f(\tau + (1 - \tau)s) \ln((1 - \tau)s) ds \\ &= \tau \int_0^1 f(\tau s) \ln(1 - s) ds + (1 - \tau) \int_0^1 f(\tau + (1 - \tau)s) \ln s ds \\ &\quad + \tau \ln \tau \int_0^1 f(\tau s) ds + (1 - \tau) \ln(1 - \tau) \int_0^1 f(\tau + (1 - \tau)s) ds \end{aligned}$$

Logarithmic-weighted Gauss quadrature for singularity $\tau \in (0, 1)$,

$$\int_0^1 f(t) \ln |t - \tau| dt \approx \left\{ \begin{aligned} &\tau \ln \tau \sum_{k=1}^m w_k f[x_k] \\ &+ (1 - \tau) \ln(1 - \tau) \sum_{k=1}^m w_k f[\tau + (1 - \tau)x_k] \\ &- \tau \sum_{k=1}^m w_k^{\log} f[\tau(1 - x_k^{\log})] \\ &- (1 - \tau) \sum_{k=1}^m w_k^{\log} f[\tau + (1 - \tau)x_k^{\log}] \end{aligned} \right. \quad (2.3)$$

Elliptic integral of the first and second kind,

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}, \quad E(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \theta} d\theta \quad (2.4)$$

Elliptic integral of the first and second kind,

$$K(m) \approx P_K(m) - \ln(1 - m)Q_K(m), \quad E(m) \approx P_E(m) - \ln(1 - m)Q_E(m) \quad (2.5)$$

where P_K, P_E, Q_K, Q_E are tenth order polynomials. Associate Legendre polynomial of $1/2$ order can be conveniently expressed as,

$$P_{1/2}(x) = \frac{2}{\pi} \left\{ 2E\left(\frac{1-x}{2}\right) - K\left(\frac{1-x}{2}\right) \right\} \quad (2.6)$$

3. Axisymmetric boundary integral

Axisymmetric Green's function $G(\boldsymbol{\eta}'; \boldsymbol{\eta})$,

$$G(\boldsymbol{\eta}'; \boldsymbol{\eta}) = \frac{1}{\pi\sqrt{a+b}} K(m) \quad (3.1)$$

$$\frac{\partial G(\boldsymbol{\eta}'; \boldsymbol{\eta})}{\partial \mathbf{n}} = \frac{1}{\pi r \sqrt{a+b}} \left\{ \left[\frac{n_r}{2} + \frac{\mathbf{n} \cdot (\mathbf{x}' - \mathbf{x})}{\|\mathbf{x}' - \mathbf{x}\|^2} r \right] E(m) - \frac{n_r}{2} K(m) \right\} \quad (3.2)$$

Auxiliary variables,

$$a = r'^2 + r^2 + (z' - z)^2, \quad b = 2r'r, \quad m = \frac{2b}{a+b} \quad (3.3)$$

Note $\boldsymbol{\eta} \rightarrow \boldsymbol{\eta}'$ implies $m \rightarrow 1$. Axisymmetric boundary integral equation,

$$\frac{\beta(\boldsymbol{\eta}')}{2\pi} \phi(\boldsymbol{\eta}') = \int_{\gamma} \left\{ G(\boldsymbol{\eta}'; \boldsymbol{\eta}) \frac{\partial \phi}{\partial \mathbf{n}}(\boldsymbol{\eta}) - \phi(\boldsymbol{\eta}) \frac{\partial G(\boldsymbol{\eta}'; \boldsymbol{\eta})}{\partial \mathbf{n}}(\boldsymbol{\eta}) \right\} r \, d\gamma(\boldsymbol{\eta}) \quad (3.4)$$

Lagrange interpolation along arc-length,

$$\phi(\boldsymbol{\eta}) \approx \sum_e \sum_k p_{ek} \mathcal{N}_k^{(e)}(\xi), \quad \frac{\partial \phi}{\partial \mathbf{n}}(\boldsymbol{\eta}) \approx \sum_e \sum_k q_{ek} \mathcal{N}_k^{(e)}(\xi) \quad (3.5)$$

where \mathcal{N}_k is the Lagrange basis of the local fractional arc-length variable ξ ,

$$\mathcal{N}_0 = (1 - \xi)(1 - 2\xi), \quad \mathcal{N}_1 = 4\xi(1 - \xi), \quad \mathcal{N}_2 = \xi(2\xi - 1) \quad (3.6)$$

3.1. Single-layer integral

Single-layer integral reduces to a summation of local integrals over local arc γ_e of element e ,

$$\begin{aligned} \int_{\gamma} G(\boldsymbol{\eta}'; \boldsymbol{\eta}) \frac{\partial \phi}{\partial \mathbf{n}}(\boldsymbol{\eta}) r \, d\gamma(\boldsymbol{\eta}) &\approx \sum_e \sum_k q_{ek} \int_{\gamma_e} G(\boldsymbol{\eta}'; \boldsymbol{\eta}) \mathcal{N}_k^{(e)}(\xi) r \, d\gamma(\boldsymbol{\eta}) \\ &= \sum_e \sum_k q_{ek} \int_0^1 \frac{rJ}{\pi\sqrt{a+b}} \mathcal{N}_k^{(e)}(\xi) K(m) \, dt \end{aligned} \quad (3.7)$$

When singularity $\boldsymbol{\eta}' \in \gamma_e$ is located at $t = \tau \in [0, 1]$,

$$\int_0^1 \frac{rJ}{\pi\sqrt{a+b}} \mathcal{N}_k^{(e)} K(m) \, dt = \int_0^1 \frac{rJ}{\pi\sqrt{a+b}} \mathcal{N}_k^{(e)} \left[P_K - Q_K \ln \frac{1-m}{(t-\tau)^2} - 2Q_K \ln |t-\tau| \right] \, dt$$

Introduce helper functions,

$$f_K^{\text{single}}(t) = \frac{rJ}{\pi\sqrt{a+b}} \quad (3.8)$$

$$R_{K/E}(m, \tau) = P_{K/E}(m) - Q_{K/E}(m) \ln \frac{1-m}{(t-\tau)^2} \quad (3.9)$$

If we define integrand $I^{(o)}$ for source location $\boldsymbol{\eta}'$ outside of γ_e and $I^{(i)}$ for $\boldsymbol{\eta}'$ interior to γ_e ,

$$I^{(o)}(t) = f_K^{\text{single}}(t) K(m) \mathcal{N}_k^{(e)}(\xi) \quad (3.10)$$

$$I^{(i)}(t, \tau) = f_K^{\text{single}}(t) R_{K/E}(m, \tau) \mathcal{N}_k^{(e)}(\xi) \quad (3.11)$$

$$I_{\log}^{(i)}(t) = -2f_K^{\text{single}}(t) Q_{K/E}(m) \mathcal{N}_k^{(e)}(\xi) \quad (3.12)$$

then the local single-layer integral can be compactly represented with a regular part and a logarithmic-singular one,

$$\boldsymbol{\eta}' \notin \gamma_e : \text{ compute } \int_0^1 I^{(o)}(t) dt \quad (3.13)$$

$$\boldsymbol{\eta}' \text{ at } t = 0 : \text{ compute } \int_0^1 I^{(i)}(t, 0) dt + \int_0^1 I_{\log}^{(i)}(t) \ln t dt \quad (3.14)$$

$$\boldsymbol{\eta}' \text{ at } t = \tau : \text{ compute } \int_0^1 I^{(i)}(t, \tau) dt + \int_0^1 I_{\log}^{(i)}(t) \ln |\tau - t| dt \quad (3.15)$$

$$\boldsymbol{\eta}' \text{ at } t = 1 : \text{ compute } \int_0^1 I^{(i)}(t, 1) dt + \int_0^1 I_{\log}^{(i)}(t) \ln(1 - t) dt \quad (3.16)$$

3.2. Double-layer integral

Similar to singular-layer integral, double-layer integral reduces to a summation of local integrals over local arc γ_e of element e ,

$$\begin{aligned} \int_{\gamma} \frac{\partial G(\boldsymbol{\eta}'; \boldsymbol{\eta})}{\partial \mathbf{n}} \phi(\boldsymbol{\eta}) r d\gamma(\boldsymbol{\eta}) &\approx \sum_e \sum_k p_{ek} \int_{\gamma_e} \frac{\partial G(\boldsymbol{\eta}'; \boldsymbol{\eta})}{\partial \mathbf{n}} \mathcal{N}_k^{(e)}(\xi) r d\gamma(\boldsymbol{\eta}) \\ &= \sum_e \sum_k p_{ek} \int_0^1 \mathcal{N}_k^{(e)}(\xi) \frac{J}{\pi \sqrt{a+b}} \left\{ \left[\frac{n_r}{2} + \frac{\mathbf{n} \cdot (\mathbf{x}' - \mathbf{x})}{\|\mathbf{x}' - \mathbf{x}\|^2} r \right] E(m) - \frac{n_r}{2} K(m) \right\} dt \\ &= \sum_e \sum_k p_{ek} \int_0^1 \frac{\mathcal{N}_k^{(e)}(\xi)}{\pi \sqrt{a+b}} \left\{ \left[\frac{\dot{r}(z' - z) - \dot{z}(r' - r)}{(r' - r)^2 + (z' - z)^2} r - \frac{\dot{z}}{2} \right] E(m) + \frac{\dot{z}}{2} K(m) \right\} dt \end{aligned} \quad (3.17)$$

Technically we don't need to treat integrand involving elliptic integral of the second kind $E(m)$. However, convergence rate of standard Gauss-Legendre quadrature depends magnitude of derivatives.

$$f_E^{\text{double}} = \frac{1}{\pi \sqrt{a+b}} \left[\frac{\dot{r}(z' - z) - \dot{z}(r' - r)}{a - b} r - \frac{\dot{z}}{2} \right], \quad f_K^{\text{double}} = \frac{1}{\pi \sqrt{a+b}} \frac{\dot{z}}{2} \quad (3.18)$$

If we define integrand $I^{(o)}$ for source location $\boldsymbol{\eta}'$ outside of γ_e and $I^{(i)}$ for $\boldsymbol{\eta}'$ interior to γ_e ,

$$I^{(o)}(t) = [f_E^{\text{double}} E(m) + f_K^{\text{double}} K(m)] \mathcal{N}_k^{(e)}(\xi) \quad (3.19)$$

$$I^{(i)}(t, \tau) = [f_E^{\text{double}} R_E(m, \tau) + f_K^{\text{double}} R_K(m, \tau)] \mathcal{N}_k^{(e)}(\xi) \quad (3.20)$$

$$I_{\log}^{(i)}(t) = -2f_E^{\text{double}} Q_E(m) - 2f_K^{\text{double}} Q_K(m) \quad (3.21)$$

When $\boldsymbol{\eta}'$ is located the symmetry axis where $r' = 0$, we can simplify the elliptic integrals,

$$\int_{\gamma} G(\boldsymbol{\eta}'; \boldsymbol{\eta}) \frac{\partial \phi}{\partial \mathbf{n}}(\boldsymbol{\eta}) r d\gamma(\boldsymbol{\eta}) \approx \sum_e \sum_k q_{ek} \int_0^1 \mathcal{N}_k^{(e)}(\xi) \frac{J}{2\sqrt{1 + [(z - z')/r]^2}} dt \quad (3.22)$$

$$\int_{\gamma} \frac{\partial G(\boldsymbol{\eta}'; \boldsymbol{\eta})}{\partial \mathbf{n}} \phi(\boldsymbol{\eta}) r d\gamma(\boldsymbol{\eta}) \approx \sum_e \sum_k p_{ek} \int_0^1 \mathcal{N}_k^{(e)}(\xi) \frac{\dot{z}/r + \dot{r}(z - z')/r^2}{2\{1 + [(z - z')/r]^2\}^{3/2}} dt \quad (3.23)$$

The above integrands have well-defined limit as $\boldsymbol{\eta} \rightarrow \boldsymbol{\eta}'$ assuming \dot{z}/\dot{r} is finite at $r = 0$.

3.3. Assembly

$$(\mathbf{B} + \mathbf{D})\mathbf{p} = \mathbf{S}\mathbf{q} \quad (3.24)$$

Element order	o	
Number of elements	n	
Number of nodes	$n \times o + 1$	
Nodal index of source point	i	
Elemental index of receiving point	j	
Local node index	k	$0 \leq k \leq o$
Indices of elements $\ni i$ -th node if $i \bmod o = 0$	$\lfloor i/o \rfloor - 1, \lfloor i/o \rfloor$	check $0 \leq \text{id} \leq n - 1$
Indices of elements $\ni i$ -th node if $i \bmod o \neq 0$	$\lfloor i/o \rfloor - 1$	check $0 \leq \text{id} \leq n - 1$
Nodal indices of j -th element	$j \times o + k$	$0 \leq k \leq o$

TABLE 1. Index involved

Algorithm 1 Counting mismatches between two packed strings

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1: function DISTANCE( $x, e$ )
2:   for  $0 \leq i \leq N_x - 1$  do                                     # We can parallelize this loop
3:     if  $i \bmod o = 0$  then
4:        $I_{\text{lower}} = I_{\text{upper}}$ 
5:     else
6:        $I_{\text{lower}} = \lfloor i/o \rfloor - 1, \quad I_{\text{upper}} = \lfloor i/o \rfloor$ 
7:     end if
8:     for  $0 \leq i \leq N_x - 1$  do
9:       end for
10:   end for
11: end function

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REFERENCES

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