# Dynamic Taylor Cone Part II: Numeric Scheme

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Numeric scheme (Taylor 1964)

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## 1. Quintic spline

Given $N$ marker points $\{\boldsymbol{x}_0, \dots, \boldsymbol{x}_N\}$, we first compute $(N-1)$ chord lengths $\{h_0, \dots, h_{N-1}\}$ by euclidean distance between adjacent marker points,

$$h_j = \|\boldsymbol{x}_{j+1} - \boldsymbol{x}_j\|_2 \quad \text{for} \quad j = 0, \dots, N-1 \tag{1.1}$$

Then we introduce $N$ chord coordinates $\{l_0, \dots, l_N\}$ with $l_0 = 0$ and

$$l_j = \sum_{k=0}^{j-1} h_k \quad \text{for} \quad j = 1, \dots, N \tag{1.2}$$

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Global spline is a collection of local splines $\{\boldsymbol{s}^{(0)},\ldots,\boldsymbol{s}^{(N-1)}\}$ where each $\boldsymbol{s}^{(j)}$ is a fifth degree polynomial defined on the interval $l\in[l_j,l_{j+1}]$,

$$\boldsymbol{s}^{(j)}(l)=\boldsymbol{x}_j+\sum_{k=1}^{5}\boldsymbol{c}_k^{(j)}(l-l_j)^k,\quad \text{for}\quad j=0,\ldots,N-1 \tag{1.3}$$

Continuity between neighboring splines is satisfied up to the fourth derivative,

$$\left.\frac{\mathrm{d}^p\boldsymbol{s}^{(j)}}{\mathrm{d}l^p}\right|_{l_{j+1}}=\left.\frac{\mathrm{d}^p\boldsymbol{s}^{(j+1)}}{\mathrm{d}l^p}\right|_{l_{j+1}},\quad \text{for}\quad p=0,\ldots,4 \tag{1.4}$$

The continuity conditions yield five equations

$$\left.\begin{aligned}
\boldsymbol{c}_3^{(j-1)}h_{j-1}^3+\boldsymbol{c}_4^{(j-1)}h_{j-1}^4+\boldsymbol{c}_5^{(j-1)}h_{j-1}^5&=\boldsymbol{x}_j-\boldsymbol{x}_{j-1}-\boldsymbol{c}_1^{(j-1)}h_{j-1}-\boldsymbol{c}_2^{(j-1)}h_{j-1}^2\\
3\boldsymbol{c}_3^{(j-1)}h_{j-1}^2+4\boldsymbol{c}_4^{(j-1)}h_{j-1}^3+5\boldsymbol{c}_5^{(j-1)}h_{j-1}^4&=-\boldsymbol{c}_1^{(j-1)}-2\boldsymbol{c}_2^{(j-1)}h_{j-1}+\boldsymbol{c}_1^{(j)}\\
6\boldsymbol{c}_3^{(j-1)}h_{j-1}+12\boldsymbol{c}_4^{(j-1)}h_{j-1}^2+20\boldsymbol{c}_5^{(j-1)}h_{j-1}^3&=-2\boldsymbol{c}_2^{(j-1)}+2\boldsymbol{c}_2^{(j)}\\
\boldsymbol{c}_3^{(j-1)}+4h_{j-1}\boldsymbol{c}_4^{(j-1)}+10h_{j-1}^2\boldsymbol{c}_5^{(j-1)}&=\boldsymbol{c}_3^{(j)}\\
\boldsymbol{c}_4^{(j-1)}+5h_{j-1}\boldsymbol{c}_5^{(j-1)}&=\boldsymbol{c}_4^{(j)}
\end{aligned}\right\} \tag{1.5}$$

We first solve for $\boldsymbol{c}_3^{(j-1)}$, $\boldsymbol{c}_4^{(j-1)}$ and $\boldsymbol{c}_5^{(j-1)}$,

$$\left.\begin{aligned}
h_{j-1}^3\boldsymbol{c}_3^{(j-1)}&=\\
&-6h_{j-1}\boldsymbol{c}_1^{(j-1)}-3h_{j-1}^2\boldsymbol{c}_2^{(j-1)}-4h_{j-1}\boldsymbol{c}_1^{(j)}+h_{j-1}^2\boldsymbol{c}_2^{(j)}+10(\boldsymbol{x}_j-\boldsymbol{x}_{j-1})\\
h_{j-1}^4\boldsymbol{c}_4^{(j-1)}&=\\
&+8h_{j-1}\boldsymbol{c}_1^{(j-1)}+3h_{j-1}^2\boldsymbol{c}_2^{(j-1)}+7h_{j-1}\boldsymbol{c}_1^{(j)}-2h_{j-1}^2\boldsymbol{c}_2^{(j)}-15(\boldsymbol{x}_j-\boldsymbol{x}_{j-1})\\
h_{j-1}^5\boldsymbol{c}_5^{(j-1)}&=\\
&-3h_{j-1}\boldsymbol{c}_1^{(j-1)}-h_{j-1}^2\boldsymbol{c}_2^{(j-1)}-3h_{j-1}\boldsymbol{c}_1^{(j)}+h_{j-1}^2\boldsymbol{c}_2^{(j)}+6(\boldsymbol{x}_j-\boldsymbol{x}_{j-1})
\end{aligned}\right\} \tag{1.6}$$

Define $\lambda=h_j/h_{j-1}$. We have $2(N-1)$ equations for $j=1,\ldots,N-1$,

$$\left.\begin{aligned}
10\left[+\lambda^3\boldsymbol{x}_{j-1}-(1+\lambda^3)\boldsymbol{x}_j+\boldsymbol{x}_{j+1}\right]=&\;4h_j\lambda^2\boldsymbol{c}_1^{(j-1)}+6h_j(\lambda^2-1)\boldsymbol{c}_1^{(j)}-4h_j\boldsymbol{c}_1^{(j+1)}\\
&+h_j^2\lambda\boldsymbol{c}_2^{(j-1)}-3h_j^2(1+\lambda)\boldsymbol{c}_2^{(j)}+h_j^2\boldsymbol{c}_2^{(j+1)}\\
15\left[-\lambda^4\boldsymbol{x}_{j-1}+(\lambda^4-1)\boldsymbol{x}_j+\boldsymbol{x}_{j+1}\right]=&\;7h_j\lambda^3\boldsymbol{c}_1^{(j-1)}+8h_j(1+\lambda^3)\boldsymbol{c}_1^{(j)}+7h_j\boldsymbol{c}_1^{(j+1)}\\
&+2h_j^2\lambda^2\boldsymbol{c}_2^{(j-1)}+3h_j^2(1-\lambda^2)\boldsymbol{c}_2^{(j)}-2h_j^2\boldsymbol{c}_2^{(j+1)}
\end{aligned}\right\} \tag{1.7}$$

for $2(N+1)$ unknowns, $\{\boldsymbol{c}_1^{(0)},\ldots,\boldsymbol{c}_1^{(N)}\}$ and $\{\boldsymbol{c}_2^{(0)},\ldots,\boldsymbol{c}_2^{(N)}\}$. The imposed boundary conditions at the beginning and end of the global spline produce four additional equations for $\{\boldsymbol{c}_1^{(0)},\boldsymbol{c}_2^{(0)},\boldsymbol{c}_1^{(N)},\boldsymbol{c}_2^{(N)}\}$. Note we have implicitly introduced a ghost spline $\boldsymbol{s}^{(N)}$ which satisfies all continuity conditions with $\boldsymbol{s}^{(N-1)}$. The purpose of $\boldsymbol{s}^{(N)}$ is to deal with boundary conditions at the end of spline.

In general there are three types of boundary conditions at each end. Let $x$, $s$ and $c^{(j)}$ be one of the scalar components of vector $\boldsymbol{x}_j$, spline $\boldsymbol{s}^{(j)}$ and coefficient $\boldsymbol{c}^{(j)}$. We first consider boundary conditions at $l = l_0$,

$$\text{Even:} \quad 0 = \frac{\mathrm{d}s^{(0)}}{\mathrm{d}l} = \frac{\mathrm{d}^3 s^{(0)}}{\mathrm{d}l^3} \qquad \text{at} \quad l = l_0 \tag{1.8}$$

$$\text{Odd:} \quad 0 = s^{(0)} = \frac{\mathrm{d}^2 s^{(0)}}{\mathrm{d}l^2} = \frac{\mathrm{d}^4 s^{(0)}}{\mathrm{d}l^4} \quad \text{at} \quad l = l_0 \tag{1.9}$$

$$\text{Mix:} \quad \alpha = \frac{\mathrm{d}s^{(0)}}{\mathrm{d}l}, \quad \beta = \frac{\mathrm{d}^2 s^{(0)}}{\mathrm{d}l^2} \qquad \text{at} \quad l = l_0 \tag{1.10}$$

which lead to two equations,

$$\left.\begin{aligned} c_1^{(0)} &= 0 \\ 6h_0 c_1^{(0)} + 3h_0^2 c_2^{(0)} + 4h_0 c_1^{(1)} - h_0^2 c_2^{(1)} &= -10x_0 + 10x_1 \end{aligned}\right\} \quad \text{Even} \tag{1.11$a$}$$

$$\left.\begin{aligned} -8h_0 c_1^{(0)} - 3h_0^2 c_2^{(0)} - 7h_0 c_1^{(1)} + 2h_0^2 c_2^{(1)} &= +15x_0 - 15x_1 \\ c_2^{(0)} &= 0 \end{aligned}\right\} \quad \text{Odd} \tag{1.11$b$}$$

$$\left.\begin{aligned} c_1^{(0)} &= \alpha \\ c_2^{(0)} &= \beta/2 \end{aligned}\right\} \quad \text{Mix} \tag{1.11$c$}$$

We then consider boundary conditions at $l = l_N$,

$$\text{Even:} \quad 0 = \frac{\mathrm{d}s^{(N-1)}}{\mathrm{d}l} = \frac{\mathrm{d}^3 s^{(N-1)}}{\mathrm{d}l^3} \qquad \text{at} \quad l = l_N \tag{1.12}$$

$$\text{Odd:} \quad 0 = s^{(N-1)} = \frac{\mathrm{d}^2 s^{(N-1)}}{\mathrm{d}l^2} = \frac{\mathrm{d}^4 s^{(N-1)}}{\mathrm{d}l^4} \quad \text{at} \quad l = l_N \tag{1.13}$$

$$\text{Mix:} \quad \alpha = \frac{\mathrm{d}s^{(N-1)}}{\mathrm{d}l}, \quad \beta = \frac{\mathrm{d}^2 s^{(N-1)}}{\mathrm{d}l^2} \qquad \text{at} \quad l = l_N \tag{1.14}$$

which also lead to two equations,

$$\left.\begin{aligned} c_1^{(N)} &= 0 \\ -4h_{N-1} c_1^{(N-1)} - h_{N-1}^2 c_2^{(N-1)} - 6h_{N-1} c_1^{(N)} + 3h_{N-1}^2 c_2^{(N)} &= 10x_{N-1} - 10x_N \end{aligned}\right\} \quad \text{Even} \tag{1.15$a$}$$

$$\left.\begin{aligned} -7h_{N-1} c_1^{(N-1)} - 2h_{N-1}^2 c_2^{(N-1)} - 8h_{N-1} c_1^{(N)} + 3h_{N-1}^2 c_2^{(N)} &= 15x_{N-1} - 15x_N \\ c_2^{(N)} &= 0 \end{aligned}\right\} \quad \text{Odd} \tag{1.15$b$}$$

$$\left.\begin{aligned} c_1^{(N)} &= \alpha \\ c_2^{(N)} &= \beta/2 \end{aligned}\right\} \quad \text{Mix} \tag{1.15$c$}$$

If we arrange the unknowns into a vector,

$$\left[c_1^{(0)}, c_2^{(0)}, \ldots, c_1^{(j)}, c_2^{(j)}, \ldots, c_1^{(N)}, c_2^{(N)}\right] \tag{1.16}$$

equations (1.7) with one of (1.11) and one of (1.15) result in a $2(N+1)$-by-$2(N+1)$ system of linear equations, which corresponds to a banded diagonal sparse matrix with at most 6 non-zero elements in each row. The rest coefficients $\boldsymbol{c}_3^{(j)}$, $\boldsymbol{c}_4^{(j)}$ and $\boldsymbol{c}_5^{(j)}$ can be reconstruct using equation (1.6).

It is convenient to re-parametrize each local spline with a new variable $t \in [0,1]$,

$$\boldsymbol{s}^{(j)}(t) = \boldsymbol{x}_j + \sum_{k=1}^{5} \boldsymbol{c}_k^{(j)} t^k, \quad \text{for} \quad j = 0, \dots, N-1 \tag{1.17}$$

by redefining $\boldsymbol{c}_k^{(j)} \to \boldsymbol{c}_k^{(j)} h_j^k$. We can construct Lagrange interpolations along the arc-length of the global spline by the local arc-length $L_j$ and its local fraction $\xi$,

$$L_j = \int_0^1 \dot{\boldsymbol{s}}^{(j)}\, \mathrm{d}t', \quad \xi(t) = \frac{1}{L_j} \int_0^t \dot{\boldsymbol{s}}^{(j)}\, \mathrm{d}t', \tag{1.18}$$

Convention for normal vector $\boldsymbol{n}$,

$$\boldsymbol{n} = \frac{1}{\|\dot{\boldsymbol{s}}\|}(-\dot{z}, \dot{r}) \tag{1.19}$$

In addition the total curvature $2\kappa$ of the surface of revolution obtained by rotating curve $\boldsymbol{s}(t)$ about the $z$-axis is given by,

$$2\kappa = \frac{\dot{r}\ddot{z} - \dot{z}\ddot{r}}{(\dot{r}^2 + \dot{z}^2)^{\frac{3}{2}}} + \frac{1}{r} \frac{\dot{z}}{\sqrt{\dot{r}^2 + \dot{z}^2}} = 2\frac{\ddot{z}}{\dot{r}^2} \text{ if on symmetry axis} \tag{1.20}$$

## 2. Numerical approximation

### 2.1. *Gaussian quadrature*

Standard $m$-point Gauss-Legendre quadrature,

$$\int_0^1 f(t)\, \mathrm{d}t \approx \sum_{k=1}^{m} w_k f(x_k), \quad \int_a^b f(t)\, \mathrm{d}t \approx \sum_{k=1}^{m} (b-a) w_k f(a + (b-a)x_k) \tag{2.1}$$

Logarithmic-weighted Gauss quadrature,

$$\int_0^1 f(t) \ln t\, \mathrm{d}t \approx -\sum_{k=1}^{m} w_k^{\log} f(x_k), \quad \int_0^1 f(t) \ln(1-t)\, \mathrm{d}t \approx -\sum_{k=1}^{m} w_k^{\log} f(1 - x_k) \tag{2.2}$$

When logarithmic singularity $\tau \in (0,1)$,

$$\begin{aligned}
&\int_0^1 f(t) \ln|t-\tau|\, \mathrm{d}t = \int_0^\tau f(t) \ln(\tau - t)\, \mathrm{d}t + \int_\tau^1 f(t) \ln(t-\tau)\, \mathrm{d}t \\
&= \tau \int_0^1 f(\tau s) \ln(\tau - \tau s)\, \mathrm{d}s + (1-\tau) \int_0^1 f(\tau + (1-\tau)s) \ln((1-\tau)s)\, \mathrm{d}s \\
&= \tau \int_0^1 f(\tau s) \ln(1-s)\, \mathrm{d}s + (1-\tau) \int_0^1 f(\tau + (1-\tau)s) \ln s\, \mathrm{d}s \\
&\quad + \tau \ln\tau \int_0^1 f(\tau s)\, \mathrm{d}s + (1-\tau)\ln(1-\tau) \int_0^1 f(\tau + (1-\tau)s)\, \mathrm{d}s
\end{aligned}$$

Logarithmic-weighted Gauss quadrature for singularity $\tau \in (0,1)$,

$$\int_0^1 f(t)\ln|t-\tau|\,\mathrm{d}t \approx \begin{cases} \tau\ln\tau\sum_{k=1}^{m} w_k f\left[\tau x_k\right] \\ +(1-\tau)\ln(1-\tau)\sum_{k=1}^{m} w_k f\left[\tau+(1-\tau)x_k\right] \\ -\tau\sum_{k=1}^{m} w_k^{\log} f\left[\tau(1-x_k^{\log})\right] \\ -(1-\tau)\sum_{k=1}^{m} w_k^{\log} f\left[\tau+(1-\tau)x_k^{\log}\right] \end{cases} \tag{2.3}$$

### 2.2. *Complete elliptic integral*

The complete elliptic integral of the first and second kind,

$$K(m) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-m\sin^2\theta}}, \quad E(m) = \int_0^{\pi/2} \sqrt{1-m\sin^2\theta}\,\mathrm{d}\theta \tag{2.4}$$

We can approximate $K(m)$ and $E(m)$ with

$$K(m) \approx P_K(m) - \ln(1-m)Q_K(m), \quad E(m) \approx P_E(m) - \ln(1-m)Q_E(m) \tag{2.5}$$

where $P_K$, $P_E$, $Q_K$, $Q_E$ are tenth order polynomials. Associate Legendre polynomial of $1/2$ order can be conveniently expressed as,

$$P_{1/2}(x) = \frac{2}{\pi}\left\{2E\left(\frac{1-x}{2}\right) - K\left(\frac{1-x}{2}\right)\right\} \tag{2.6}$$

From the recursion relations,

$$\begin{aligned} (1+x^2)\frac{\mathrm{d}P_\ell(x)}{\mathrm{d}x} &= (\ell+1)xP_\ell(x) - (\ell+1)P_{\ell+1}(x) \\ \frac{\mathrm{d}K}{\mathrm{d}m} &= \frac{E(m)}{2m(1-m)} - \frac{K(m)}{2m} \\ \frac{\mathrm{d}E}{\mathrm{d}m} &= \frac{1}{2m}\left[E(m)-K(m)\right] \end{aligned} \tag{2.7}$$

we can bootstrap associate Legendre polynomials of higher orders, $3/2$ , $5/2$ , ..., The form is again

$$P_\ell(x) = c_\ell\left\{P_{\ell,E}(x)E\left(\frac{1-x}{2}\right) + P_{\ell,K}(x)K\left(\frac{1-x}{2}\right)\right\} \tag{2.8}$$

where $c_\ell$ is some constant. $P_{\ell,E}(x)$ and $P_{\ell,E}(x)$ are polynomials of $(\ell-1/2)$ order whose coefficients are computed symbolically and given in Table 1.

## 3. Axisymmetric boundary integral

Axisymmetric Green's function $G(\boldsymbol{\eta}';\boldsymbol{\eta})$,

$$G(\boldsymbol{\eta}';\boldsymbol{\eta}) = \frac{1}{\pi\sqrt{a+b}}K(m) \tag{3.1}$$

$$\frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}} = \frac{1}{\pi r\sqrt{a+b}}\left\{\left[\frac{n_r}{2} + \frac{\boldsymbol{n}\cdot(\boldsymbol{x}'-\boldsymbol{x})}{\|\boldsymbol{x}'-\boldsymbol{x}\|^2}r\right]E(m) - \frac{n_r}{2}K(m)\right\} \tag{3.2}$$

| $\ell$ | $c_\ell$ | $P_{\ell,E}(x)$ | $P_{\ell,K}(x)$ |
|---|---|---|---|
| 1/2 | $2/\pi$ | $2$ | $-1$ |
| 3/2 | $1/3\pi$ | $16x$ | $-2(4x+1)$ |
| 5/2 | $1/15\pi$ | $4\left(32x^2-9\right)$ | $-2\left(32x^2+8x-9\right)$ |
| 7/2 | $1/105\pi$ | $64x\left(24x^2-13\right)$ | $2\left(-384x^3-96x^2+208x+25\right)$ |
| 9/2 | $1/315\pi$ | $8192x^4-6528x^2+588$ | $-2\left(2048x^4+512x^3-1632x^2-264x+147\right)$ |

TABLE 1. Coefficients of associate Legendre polynomials of $\ell = 1/2$ family

Auxiliary variables,

$$a = r'^2 + r^2 + (z'-z)^2, \quad b = 2r'r, \quad m = \frac{2b}{a+b} \tag{3.3}$$

Note $\eta \to \eta'$ implies $m \to 1$. Axisymmetric boundary integral equation,

$$\frac{\beta(\boldsymbol{\eta}')}{2\pi}\phi(\boldsymbol{\eta}') = \int_\gamma \left\{ G(\boldsymbol{\eta}';\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{n}}(\boldsymbol{\eta}) - \phi(\boldsymbol{\eta})\frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial\boldsymbol{n}}(\boldsymbol{\eta})\right\} r\,\mathrm{d}\gamma(\boldsymbol{\eta}) \tag{3.4}$$

Lagrange interpolation along arc-length,

$$\phi(\boldsymbol{\eta}) \approx \sum_e\sum_k p_{e_k}\mathcal{N}_k^{(e)}(\xi), \quad \frac{\partial\phi}{\partial\boldsymbol{n}}(\boldsymbol{\eta}) \approx \sum_e\sum_k q_{e_k}\mathcal{N}_k^{(e)}(\xi) \tag{3.5}$$

where $\mathcal{N}_k$ is the Lagrange basis of the local fractional arc-length variable $\xi$,

$$\mathcal{N}_0 = (1-\xi)(1-2\xi), \quad \mathcal{N}_1 = 4\xi(1-\xi), \quad \mathcal{N}_2 = \xi(2\xi-1) \tag{3.6}$$

### 3.1. *Single layer integral*

Single-layer integral reduces to a summation of local integrals over local arc $\gamma_e$ of element $e$,

$$\begin{aligned}\int_\gamma G(\boldsymbol{\eta}';\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{n}}(\boldsymbol{\eta})r\,\mathrm{d}\gamma(\boldsymbol{\eta}) &\approx \sum_e\sum_k q_{e_k}\int_{\gamma_e} G(\boldsymbol{\eta}';\boldsymbol{\eta})\mathcal{N}_k^{(e)}(\xi)r\,\mathrm{d}\gamma(\boldsymbol{\eta}) \\ &= \sum_e\sum_k q_{e_k}\int_0^1 \frac{rJ}{\pi\sqrt{a+b}}\mathcal{N}_k^{(e)}(\xi)K(m)\,\mathrm{d}t \end{aligned} \tag{3.7}$$

When singularity $\boldsymbol{\eta}' \in \gamma_e$ is located at $t = \tau \in [0,1]$,

$$\int_0^1 \frac{rJ}{\pi\sqrt{a+b}}\mathcal{N}_k^{(e)}K(m)\,\mathrm{d}t = \int_0^1 \frac{rJ}{\pi\sqrt{a+b}}\mathcal{N}_k^{(e)}\left[P_K - Q_K\ln\frac{1-m}{(t-\tau)^2} - 2Q_K\ln|t-\tau|\right]\mathrm{d}t$$

Introduce helper functions,

$$f_K^{\mathrm{single}}(t) = \frac{rJ}{\pi\sqrt{a+b}} \tag{3.8}$$

$$R_{K/E}(m,\tau) = P_{K/E}(m) - Q_{K/E}(m)\ln\frac{1-m}{(t-\tau)^2} \tag{3.9}$$

If we define integrand $I^{(o)}$ for source location $\boldsymbol{\eta}'$ outside of $\gamma_e$ and $I^{(i)}$ for $\boldsymbol{\eta}'$ interior to $\gamma_e$,

$$I^{(o)}(t) = f_K^{\text{single}}(t) K(m) \mathcal{N}_k^{(e)}(\xi) \tag{3.10}$$

$$I^{(i)}(t,\tau) = f_K^{\text{single}}(t) R_K(m,\tau) \mathcal{N}_k^{(e)}(\xi) \tag{3.11}$$

$$I_{\log}^{(i)}(t) = -2 f_K^{\text{single}}(t) Q_K(m) \mathcal{N}_k^{(e)}(\xi) \tag{3.12}$$

then the local single-layer integral can be compactly represented with a regular part and a logarithmic-singular one,

$$\boldsymbol{\eta}' \notin \gamma_e : \ \text{compute} \int_0^1 I^{(o)}(t)\,\mathrm{d}t \tag{3.13}$$

$$\boldsymbol{\eta}' \text{ at } t=0 : \ \text{compute} \int_0^1 I^{(i)}(t,0)\,\mathrm{d}t + \int_0^1 I_{\log}^{(i)}(t) \ln t \,\mathrm{d}t \tag{3.14}$$

$$\boldsymbol{\eta}' \text{ at } t=\tau : \ \text{compute} \int_0^1 I^{(i)}(t,\tau)\,\mathrm{d}t + \int_0^1 I_{\log}^{(i)}(t) \ln|\tau - t| \,\mathrm{d}t \tag{3.15}$$

$$\boldsymbol{\eta}' \text{ at } t=1 : \ \text{compute} \int_0^1 I^{(i)}(t,1)\,\mathrm{d}t + \int_0^1 I_{\log}^{(i)}(t) \ln(1-t) \,\mathrm{d}t \tag{3.16}$$

### 3.2. *Double layer integral*

Similar to singular-layer integral, double-layer integral reduces to a summation of local integrals over local arc $\gamma_e$ of element $e$,

$$\begin{aligned}
&\int_\gamma \frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}} \phi(\boldsymbol{\eta}) r \,\mathrm{d}\gamma(\boldsymbol{\eta}) \approx \sum_e \sum_k p_{e_k} \int_{\gamma_e} \frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}} \mathcal{N}_k^{(e)}(\xi) r \,\mathrm{d}\gamma(\boldsymbol{\eta}) \\
&= \sum_e \sum_k p_{e_k} \int_0^1 \mathcal{N}_k^{(e)}(\xi) \frac{J}{\pi\sqrt{a+b}} \left\{ \left[ \frac{n_r}{2} + \frac{\boldsymbol{n}\cdot(\boldsymbol{x}'-\boldsymbol{x})}{\|\boldsymbol{x}'-\boldsymbol{x}\|^2} r \right] E(m) - \frac{n_r}{2} K(m) \right\} \mathrm{d}t \\
&= \sum_e \sum_k p_{e_k} \int_0^1 \frac{\mathcal{N}_k^{(e)}(\xi)}{\pi\sqrt{a+b}} \left\{ \left[ \frac{\dot{r}(z'-z) - \dot{z}(r'-r)}{(r'-r)^2 + (z'-z)^2} r - \frac{\dot{z}}{2} \right] E(m) + \frac{\dot{z}}{2} K(m) \right\} \mathrm{d}t
\end{aligned} \tag{3.17}$$

Technically we don't need to treat integrand involving elliptic integral of the second kind $E(m)$. However, convergence rate of standard Gauss-Legendre quadrature depends magnitude of derivatives.

$$f_E^{\text{double}} = \frac{1}{\pi\sqrt{a+b}} \left[ \frac{\dot{r}(z'-z) - \dot{z}(r'-r)}{a-b} r - \frac{\dot{z}}{2} \right], \quad f_K^{\text{double}} = \frac{1}{\pi\sqrt{a+b}} \frac{\dot{z}}{2} \tag{3.18}$$

If we define integrand $I^{(o)}$ for source location $\boldsymbol{\eta}'$ outside of $\gamma_e$ and $I^{(i)}$ for $\boldsymbol{\eta}'$ interior to $\gamma_e$,

$$I^{(o)}(t) = \left[ f_E^{\text{double}} E(m) + f_K^{\text{double}} K(m) \right] \mathcal{N}_k^{(e)}(\xi) \tag{3.19}$$

$$I^{(i)}(t,\tau) = \left[ f_E^{\text{double}} R_E(m,\tau) + f_K^{\text{double}} R_K(m,\tau) \right] \mathcal{N}_k^{(e)}(\xi) \tag{3.20}$$

$$I_{\log}^{(i)}(t) = -2 f_E^{\text{double}} Q_E(m) - 2 f_K^{\text{double}} Q_K(m) \tag{3.21}$$

| | | |
|---|---|---|
| Element order | $o$ | |
| Number of elements | $n$ | |
| Number of nodes | $n \times o + 1$ | |
| Nodal index of source point | $i$ | |
| Elemental index of receiving point | $j$ | |
| Local node index | $k$ | $0 \leqslant k \leqslant o$ |
| Indices of elements $\ni$ $i$-th node if $i \bmod o = 0$ | $\lfloor i/o \rfloor - 1,\ \lfloor i/o \rfloor$ | check $0 \leqslant \mathrm{id} \leqslant n-1$ |
| Indices of elements $\ni$ $i$-th node if $i \bmod o \neq 0$ | $\lfloor i/o \rfloor - 1$ | check $0 \leqslant \mathrm{id} \leqslant n-1$ |
| Nodal indices of $j$-th element | $j \times o + k$ | $0 \leqslant k \leqslant o$ |

TABLE 2. Index involved

When $\boldsymbol{\eta'}$ is located the symmetry axis where $r' = 0$, we can simplify the elliptic integrals,

$$\int_\gamma G(\boldsymbol{\eta}';\boldsymbol{\eta})\frac{\partial \phi}{\partial \boldsymbol{n}}(\boldsymbol{\eta}) r\,\mathrm{d}\gamma(\boldsymbol{\eta}) \approx \sum_e \sum_k q_{e_k} \int_0^1 \mathcal{N}_k^{(e)}(\xi) r \frac{J}{2\sqrt{r + (z-z')^2}}\,\mathrm{d}t \tag{3.22}$$

$$\int_\gamma \frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}} \phi(\boldsymbol{\eta}) r\,\mathrm{d}\gamma(\boldsymbol{\eta}) \approx \sum_e \sum_k p_{e_k} \int_0^1 \mathcal{N}_k^{(e)}(\xi) r \frac{\dot{z} r + \dot{r}(z-z')}{2\big[r^2 + (z'-z)^2\big]^{3/2}}\,\mathrm{d}t \tag{3.23}$$

The above integrands have well-defined limit as $\boldsymbol{\eta} \to \boldsymbol{\eta}'$ assuming $\ddot{z}/\dot{r}$ is finite at $r = 0$.

### 3.3. *Assembly*

**Algorithm 1** Counting mismatches between two packed strings

**function** DISTANCE($x$, $e$)
  **for** $0 \leqslant i \leqslant N_x - 1$ **do** # We can parallelize this loop
    **if** $i \bmod o = 0$ **then**
      $I_{\text{lower}} = I_{\text{upper}}$
    **else**
      $I_{\text{lower}} = \lfloor i/o \rfloor - 1, \quad I_{\text{upper}} = \lfloor i/o \rfloor$
    **end if**
    **for** $0 \leqslant i \leqslant N_x - 1$ **do**
    **end for**
  **end for**
**end function**

$$(\boldsymbol{B} + \boldsymbol{D})\boldsymbol{p} = \boldsymbol{H}\boldsymbol{p} = \boldsymbol{S}\boldsymbol{q} \tag{3.24}$$

Consider two adjacent boundaries $\gamma_0$ and $\gamma_1$. We have a block matrix equation,

$$\begin{bmatrix} \boldsymbol{H}_{00} & \boldsymbol{H}_{01} \\ \boldsymbol{H}_{10} & \boldsymbol{H}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_{00} & \boldsymbol{S}_{01} \\ \boldsymbol{S}_{10} & \boldsymbol{S}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_0 \\ \boldsymbol{q}_1 \end{bmatrix} \tag{3.25}$$

We have two identical rows in $\boldsymbol{H}$ and $\boldsymbol{S}$ and extra equation for continuity of potential,

$$\left.\begin{aligned} \mathrm{row}_{-1}(\boldsymbol{H}_{00}) \cdot \boldsymbol{p}_0 + \mathrm{row}_{-1}(\boldsymbol{H}_{01}) \cdot \boldsymbol{p}_1 &= \mathrm{row}_{-1}(\boldsymbol{S}_{00}) \cdot \boldsymbol{q}_0 + \mathrm{row}_{-1}(\boldsymbol{S}_{01}) \cdot \boldsymbol{q}_1 \\ \mathrm{row}_{0}(\boldsymbol{H}_{10}) \cdot \boldsymbol{p}_0 + \mathrm{row}_{0}(\boldsymbol{H}_{11}) \cdot \boldsymbol{p}_1 &= \mathrm{row}_{0}(\boldsymbol{S}_{10}) \cdot \boldsymbol{q}_0 + \mathrm{row}_{0}(\boldsymbol{S}_{11}) \cdot \boldsymbol{q}_1 \\ (p_0)_{-1} - (p_1)_0 &= 0 \end{aligned}\right\} \tag{3.26}$$

We simply replace one of identical rows with the continuity condition,

$$\begin{aligned} \mathrm{row}_{-1}(\boldsymbol{H}_{00}) &= [0, \cdots, 0, 1], \quad \mathrm{row}_{-1}(\boldsymbol{H}_{01}) = [-1, 0, \cdots, 0] \\ \mathrm{row}_{-1}(\boldsymbol{S}_{00}) &= [0, \cdots, 0, 0], \quad \mathrm{row}_{-1}(\boldsymbol{S}_{01}) = [0, 0, \cdots, 0] \end{aligned} \tag{3.27}$$

For mixed-type boundary condition, i.e., Dirichlet and Neumann on different segments of the boundary, we simply rearrange the block matrices,

$$\begin{bmatrix} \boldsymbol{H}_{00} & -\boldsymbol{S}_{01} \\ \boldsymbol{H}_{10} & -\boldsymbol{S}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{q}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_{00} & -\boldsymbol{H}_{01} \\ \boldsymbol{S}_{10} & -\boldsymbol{H}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_0 \\ \boldsymbol{p}_1 \end{bmatrix} \tag{3.28}$$

Note the continuity between discrete potential vectors $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ is still implied.

### 3.4. *Benchmark*

We validate our implementation of boundary integral solver for test problems posed on a smooth boundary $\gamma$ and on a piecewise smooth boundary $\gamma_0 \cup \gamma_1$. The latter contains a geometric discontinuity of a 90° corner at the transition point between $\gamma_0$ and $\gamma_1$. Consider the following parametrisation of the boundaries,

$$\gamma_0 = \left\{ 2\Big(1 + \frac{1}{4}\cos(8\theta - \pi)\Big) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\Big|\; \theta \in [0, \pi/2] \right\} \tag{3.29}$$

$$\gamma_1 = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} \;\Big|\; \theta \in [0, 3/2] \right\} \tag{3.30}$$

The general form of axisymmetric harmonic potential can be constructed from Legendre polynomial $P_\ell$ of order $\ell$. We consider the interior problem for the potential $\phi$,

$$\phi = (r^2 + z^2) P_2\Big(\frac{z}{\sqrt{r^2 + z^2}}\Big), \quad \frac{\partial \phi}{\partial \boldsymbol{n}} = \boldsymbol{n} \cdot (-r, 2z) \tag{3.31}$$

If we prescribe the potential value of $\phi$ on $\gamma_0$ and its normal flux $\partial\phi/\partial\boldsymbol{n}$ on $\gamma_1$, our solver is expected to reproduce $\partial\phi/\partial\boldsymbol{n}$ on $\gamma_0$ and $\phi$ on $\gamma_1$,

$$\text{Test problem:} \quad \text{given} \begin{cases} \phi \text{ on } \gamma_0 \\ \partial\phi/\partial\boldsymbol{n} \text{ on } \gamma_1 \end{cases}, \ \text{find} \begin{cases} \partial\phi/\partial\boldsymbol{n} \text{ on } \gamma_0 \\ \phi \text{ on } \gamma_1 \end{cases} \tag{3.32}$$

In addition for a plane curve given parametrically as $\gamma(t) = (r(t), z(t))$, the total curvature $2\kappa$ of the surface of revolution obtained by rotating curve $\gamma$ about the $z$-axis is given by,

$$2\kappa = \frac{\dot{r}\ddot{z} - \dot{z}\ddot{r}}{(\dot{r}^2 + \dot{z}^2)^{\frac{3}{2}}} + \frac{1}{r}\frac{\dot{z}}{\sqrt{\dot{r}^2 + \dot{z}^2}} \tag{3.33}$$

We verify the accuracy of quintic spline interpolation against the analytic form derived from the parametrisation (3.31). All errors between numerical and analytical solutions are measured in the $l^\infty$-norm,

$$\|\mathrm{err_d}\|_\infty = \max_{\boldsymbol{x}_i \in \gamma_1} |p_i^{\mathrm{num}} - \phi(\boldsymbol{x}_i)| \tag{3.34}$$

$$\|\mathrm{err_n}\|_\infty = \max_{\boldsymbol{x}_i \in \gamma_0} |q_i^{\mathrm{num}} - \partial\phi/\partial\boldsymbol{n}(\boldsymbol{x}_i)| \tag{3.35}$$

$$\|\mathrm{err_c}\|_\infty = \max_{\boldsymbol{x}_i \in \gamma_0} |\kappa_i^{\mathrm{spline}} - \kappa(\boldsymbol{x}_i)| \tag{3.36}$$

In figure 1(a) and 1(b) we plot these errors against the total number of degree of freedom (DOF) used by the solver.

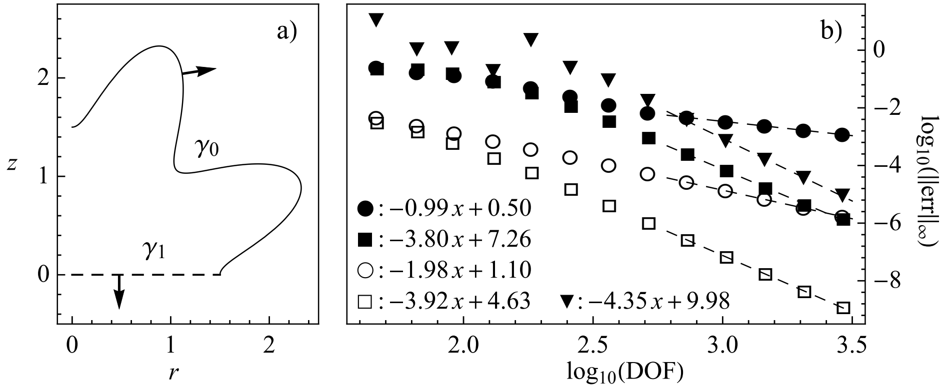


FIGURE 1. Error convergence between numerical and analytic solutions measured in $l^\infty$-norm against degrees of freedom (DOF) used for the test problem (3.32): (a) Boundaries $\gamma_0$ (solid) and $\gamma_1$ (dashed) given by parametrisation (3.31). (b) Convergence of total curvature (▼) on $\gamma_0$; convergence of Neumann data on $\gamma_0$ with linear (●) and quadratic (■) shape functions; convergence of Dirichlet data on $\gamma_1$ with linear (○) and quadratic (□) shape functions; linear fit of the last five points of each error convergence (inset).

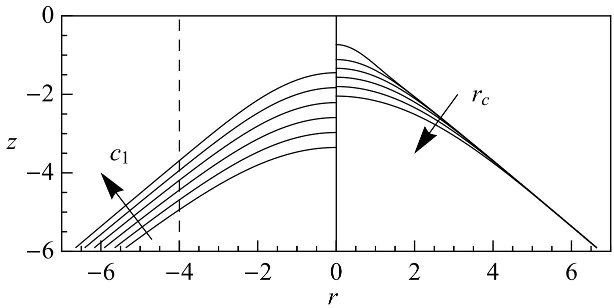


FIGURE 2. $C^3$ function $f_{\text{guess}}(r)$: increasing $r_c$ (right) and $c_1$ (left)

## 4. Self-similar Taylor cone

### 4.1. *Prepare for iteration*

In the last section we outline the numerical procedure to solve Laplace equation subject to mixed boundary conditions. Initial guess is a $C^3$ continuous function $f_{\text{guess}}$,

$$f_{\text{guess}}(r) = \begin{cases} f_0 r + f_2 r^2 + f_4 r^4 + f_6 r^6, & \text{if } r \leqslant r_c \\ c_0 r + \dfrac{c_1}{\sqrt{r}} + \dfrac{c_3}{r^{7/2}} + \dfrac{c_4}{r^5}, & \text{if } r > r_c \end{cases} \tag{4.1}$$

where $r_c$ is the connection point where continuities up to the third derivative are enforced. Given a reasonable $c_1$, function $f_{\text{guess}}(r)$ agrees with analytic prediction in the far-field. The effect of varying $r_c$ and $c_1$ is illustrated in figure 2.

We clamp the spline function, its tangent direction and total curvature at truncation point $r_*$

$$\dot{z} = \dot{r}\frac{\mathrm{d}f}{\mathrm{d}r}, \quad \ddot{z} = \frac{\ddot{r}\dot{z} + \kappa(\dot{r}^2 + \dot{z}^2)^{3/2}}{\dot{r}} - \frac{\dot{z}\left(\dot{r}^2 + \dot{z}^2\right)}{\dot{r}r} \tag{4.2}$$

Forward, central and backward finite difference approximation to first derivative with three data points $\{(x_i, f_i)\}_{i=0,1,2}$,

$$f'(x_0) \approx -f_0\left(\frac{1}{h_0+h_1}+\frac{1}{h_0}\right)+f_1\left(\frac{1}{h_0}+\frac{1}{h_1}\right)-f_2\left(\frac{1}{h_1}-\frac{1}{h_0+h_1}\right) \tag{4.3}$$

$$f'(x_1) \approx +f_0\left(\frac{1}{h_0+h_1}-\frac{1}{h_0}\right)+f_1\left(\frac{1}{h_0}-\frac{1}{h_1}\right)+f_2\left(\frac{1}{h_1}-\frac{1}{h_0+h_1}\right) \tag{4.4}$$

$$f'(x_2) \approx -f_0\left(\frac{1}{h_0+h_1}-\frac{1}{h_0}\right)-f_1\left(\frac{1}{h_0}+\frac{1}{h_1}\right)+f_2\left(\frac{1}{h_0+h_1}+\frac{1}{h_1}\right) \tag{4.5}$$

where $h_i = x_{i+1} - x_i$. For curvature purpose, we only list the backward finite difference approximation to the second derivative with three- and four-point stencil,

$$f''(x_2) \approx \frac{2}{h_0}\frac{1}{h_0+h_1}f_0 - \frac{2}{h_0h_1}f_1 + \frac{2}{h_0}\left(\frac{1}{h_1}-\frac{1}{h_0+h_1}\right)f_2 \tag{4.6}$$

$$\begin{aligned} f''(x_3) \approx & -\frac{2(h_1+2h_2)}{h_0(h_0+h_1)(h_0+h_1+h_2)}f_0 + \frac{2(h_0+h_1+2h_2)}{h_0h_1(h_1+h_2)}f_1 \\ & -\frac{2(h_0+2(h_1+h_2))}{h_1h_2(h_0+h_1)}f_2 + \frac{2(h_0+2h_1+3h_2)}{h_2(h_1+h_2)(h_0+h_1+h_2)}f_3 \end{aligned} \tag{4.7}$$

Thus we use backward finite difference method to approximate the first and second derivatives of $r$-component of the spline at the truncation point $r_*$. With prescribed slope and curvature, we can compute the first and second derivatives of $z$-component.

Axisymmetric harmonic function,

$$\frac{\partial}{\partial r}R^\ell P_\ell(\cos\theta) = \frac{\ell r^2-(\ell+1)z^2}{r}R^{\ell-2}P_\ell\left(\frac{z}{R}\right)+\frac{(\ell+1)z}{r}R^{\ell-1}P_{\ell+1}\left(\frac{z}{R}\right) \tag{4.8}$$

$$\frac{\partial}{\partial z}R^\ell P_\ell(\cos\theta) = (2\ell+1)zR^{\ell-2}P_\ell\left(\frac{z}{R}\right)-(\ell+1)R^{\ell-1}P_{\ell+1}\left(\frac{z}{R}\right) \tag{4.9}$$

$$\frac{\partial}{\partial r}R^{-\ell-1}P_\ell(\cos\theta) = \frac{(\ell+1)z}{r}R^{-\ell-2}P_{\ell+1}\left(\frac{z}{R}\right)-\frac{\ell+1}{r}R^{-\ell-1}P_\ell\left(\frac{z}{R}\right) \tag{4.10}$$

$$\frac{\partial}{\partial z}R^{-\ell-1}P_\ell(\cos\theta) = -(\ell+1)R^{-\ell-2}P_{\ell+1}\left(\frac{z}{R}\right) \tag{4.11}$$

Note $\partial/\partial r$ of these axisymmetric harmonic functions at $z=0$ is well-defined and always zero due to symmetry.

### 4.2. *Newton iteration*

We aim to find the root

$$\mathcal{B}(\boldsymbol{\eta}) = \frac{2}{3}\boldsymbol{\eta}\cdot\nabla\phi - \frac{1}{3}\phi + \frac{1}{2}|\nabla\phi|^2 - 2\kappa \tag{4.12}$$

with

$$\frac{2}{3}\boldsymbol{n}\cdot\boldsymbol{\eta} + \frac{\partial\phi}{\partial\boldsymbol{n}} = 0 \tag{4.13}$$

Rewrite $\mathcal{B}$,

$$\begin{aligned} \mathcal{B}(\boldsymbol{\eta}) &= \frac{2}{3}(\boldsymbol{n}\cdot\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{n}} + \frac{2}{3}(\boldsymbol{s}\cdot\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{s}} - \frac{1}{3}\phi + \frac{1}{2}\left(\frac{\partial\phi}{\partial\boldsymbol{n}}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial\boldsymbol{s}}\right)^2 - 2\kappa \\ &= \frac{1}{2}\left(\frac{\partial\phi}{\partial\boldsymbol{s}}\right)^2 + \frac{2}{3}(\boldsymbol{s}\cdot\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{s}} - \frac{1}{3}\phi - \frac{2}{9}(\boldsymbol{n}\cdot\boldsymbol{\eta})^2 - 2\kappa \end{aligned} \tag{4.14}$$

**Algorithm 2** Nonlinear iteration

**function** NEWTON ITERATION($c_1$, $a_0$, $r_*$) ▷ truncation distance $r_*$<br>
  compute $c_0, \ldots, c_4$ ▷ $c_2 = 0$<br>
  initialize guess shape $\gamma_0$ ▷ fix 1st and 2nd derivatives at the end<br>
  initialize patch $\gamma_1$<br>
  **while** error($\boldsymbol{\mathcal{B}}$) $<$ tolerance **do** ▷ $l^\infty$ error norm<br>
    **for all** knots $\boldsymbol{\eta}_i$ of spline $\gamma_0$ **do**<br>
      perturb $\boldsymbol{\eta}_i \to \boldsymbol{\eta}_i + \epsilon \boldsymbol{n}_i$<br>
      compute ending derivatives $\dot{r}, \ddot{r}, \dot{z}, \ddot{z}$<br>
      initialize perturbed spline $\underline{\gamma}_0$<br>
      initialize perturbed Neumann condition on $\underline{\gamma}_0$<br>
      compute solution $\underline{\phi}$ on $\underline{\gamma}_0$<br>
      compute perturbed residue vector $\underline{\boldsymbol{\mathcal{B}}}$ on knots only<br>
      $\mathrm{col}_i(\boldsymbol{J}) = (\underline{\boldsymbol{\mathcal{B}}} - \boldsymbol{\mathcal{B}})/\epsilon$<br>
    **end for**<br>
    solve $\boldsymbol{J}\,\delta\boldsymbol{n} = -\boldsymbol{\mathcal{B}}$<br>
    update $\boldsymbol{\eta}_i \to \boldsymbol{\eta}_i + \alpha\delta\boldsymbol{n}_i$ with damping rate $\alpha$<br>
    compute new residue vector $\boldsymbol{\mathcal{B}}$<br>
  **end while**<br>
**end function**

## REFERENCES

TAYLOR, G. I. 1964 Disintegration of water drops in an electric field. *Proceedings of the Royal Society of London Series A: Mathematical and Physical Sciences* **280** (1382), 383–397.