# Dynamic Taylor Cone Part II: Numeric Scheme

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Numeric scheme (Taylor 1964)

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## 1. Quintic spline

Given $N$ marker points $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_N\}$, we first compute $(N-1)$ chord lengths $\{h_0, \ldots, h_{N-1}\}$ by euclidean distance between adjacent marker points,

$$h_j = \|\boldsymbol{x}_{j+1} - \boldsymbol{x}_j\|_2 \quad \text{for} \quad j = 0, \ldots, N-1 \tag{1.1}$$

Then we introduce $N$ chord coordinates $\{l_0, \ldots, l_N\}$ with $l_0 = 0$ and

$$l_j = \sum_{k=0}^{j-1} h_k \quad \text{for} \quad j = 1, \ldots, N \tag{1.2}$$

Global spline is a collection of local splines $\{\boldsymbol{s}^{(0)}, \ldots, \boldsymbol{s}^{(N-1)}\}$ where each $\boldsymbol{s}^{(j)}$ is a fifth degree polynomial defined on the interval $l \in [l_j, l_{j+1}]$,

$$\boldsymbol{s}^{(j)}(l) = \boldsymbol{x}_j + \sum_{k=1}^{5} \boldsymbol{c}_k^{(j)} (l - l_j)^k, \quad \text{for} \quad j = 0, \ldots, N-1 \tag{1.3}$$

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Continuity between neighboring splines is satisfied up to the fourth derivative,

$$\left.\frac{\mathrm{d}^p \boldsymbol{s}^{(j)}}{\mathrm{d}l^p}\right|_{l_{j+1}} = \left.\frac{\mathrm{d}^p \boldsymbol{s}^{(j+1)}}{\mathrm{d}l^p}\right|_{l_{j+1}}, \quad \text{for} \quad p = 0, \ldots, 4 \tag{1.4}$$

The continuity conditions yield five equations

$$\left.\begin{aligned}
\boldsymbol{c}_3^{(j-1)} h_{j-1}^3 + \boldsymbol{c}_4^{(j-1)} h_{j-1}^4 + \boldsymbol{c}_5^{(j-1)} h_{j-1}^5 &= \boldsymbol{x}_j - \boldsymbol{x}_{j-1} - \boldsymbol{c}_1^{(j-1)} h_{j-1} - \boldsymbol{c}_2^{(j-1)} h_{j-1}^2 \\
3\boldsymbol{c}_3^{(j-1)} h_{j-1}^2 + 4\boldsymbol{c}_4^{(j-1)} h_{j-1}^3 + 5\boldsymbol{c}_5^{(j-1)} h_{j-1}^4 &= -\boldsymbol{c}_1^{(j-1)} - 2\boldsymbol{c}_2^{(j-1)} h_{j-1} + \boldsymbol{c}_1^{(j)} \\
6\boldsymbol{c}_3^{(j-1)} h_{j-1} + 12\boldsymbol{c}_4^{(j-1)} h_{j-1}^2 + 20\boldsymbol{c}_5^{(j-1)} h_{j-1}^3 &= -2\boldsymbol{c}_2^{(j-1)} + 2\boldsymbol{c}_2^{(j)} \\
\boldsymbol{c}_3^{(j-1)} + 4h_{j-1}\boldsymbol{c}_4^{(j-1)} + 10h_{j-1}^2\boldsymbol{c}_5^{(j-1)} &= \boldsymbol{c}_3^{(j)} \\
\boldsymbol{c}_4^{(j-1)} + 5h_{j-1}\boldsymbol{c}_5^{(j-1)} &= \boldsymbol{c}_4^{(j)}
\end{aligned}\right\} \tag{1.5}$$

We first solve for $\boldsymbol{c}_3^{(j-1)}$, $\boldsymbol{c}_4^{(j-1)}$ and $\boldsymbol{c}_5^{(j-1)}$,

$$\left.\begin{aligned}
h_{j-1}^3\boldsymbol{c}_3^{(j-1)} &= \\
&- 6h_{j-1}\boldsymbol{c}_1^{(j-1)} - 3h_{j-1}^2\boldsymbol{c}_2^{(j-1)} - 4h_{j-1}\boldsymbol{c}_1^{(j)} + h_{j-1}^2\boldsymbol{c}_2^{(j)} + 10(\boldsymbol{x}_j - \boldsymbol{x}_{j-1}) \\
h_{j-1}^4\boldsymbol{c}_4^{(j-1)} &= \\
&+ 8h_{j-1}\boldsymbol{c}_1^{(j-1)} + 3h_{j-1}^2\boldsymbol{c}_2^{(j-1)} + 7h_{j-1}\boldsymbol{c}_1^{(j)} - 2h_{j-1}^2\boldsymbol{c}_2^{(j)} - 15(\boldsymbol{x}_j - \boldsymbol{x}_{j-1}) \\
h_{j-1}^5\boldsymbol{c}_5^{(j-1)} &= \\
&- 3h_{j-1}\boldsymbol{c}_1^{(j-1)} - h_{j-1}^2\boldsymbol{c}_2^{(j-1)} - 3h_{j-1}\boldsymbol{c}_1^{(j)} + h_{j-1}^2\boldsymbol{c}_2^{(j)} + 6(\boldsymbol{x}_j - \boldsymbol{x}_{j-1})
\end{aligned}\right\} \tag{1.6}$$

Define $\lambda = h_j/h_{j-1}$. We have $2(N-1)$ equations for $j = 1, \ldots, N-1$,

$$\left.\begin{aligned}
10\left[+\lambda^3\boldsymbol{x}_{j-1} - (1+\lambda^3)\boldsymbol{x}_j + \boldsymbol{x}_{j+1}\right] = {} & 4h_j\lambda^2\boldsymbol{c}_1^{(j-1)} + 6h_j(\lambda^2-1)\boldsymbol{c}_1^{(j)} - 4h_j\boldsymbol{c}_1^{(j+1)} \\
& + h_j^2\lambda\boldsymbol{c}_2^{(j-1)} - 3h_j^2(1+\lambda)\boldsymbol{c}_2^{(j)} + h_j^2\boldsymbol{c}_2^{(j+1)} \\
15\left[-\lambda^4\boldsymbol{x}_{j-1} + (\lambda^4-1)\boldsymbol{x}_j + \boldsymbol{x}_{j+1}\right] = {} & 7h_j\lambda^3\boldsymbol{c}_1^{(j-1)} + 8h_j(1+\lambda^3)\boldsymbol{c}_1^{(j)} + 7h_j\boldsymbol{c}_1^{(j+1)} \\
& + 2h_j^2\lambda^2\boldsymbol{c}_2^{(j-1)} + 3h_j^2(1-\lambda^2)\boldsymbol{c}_2^{(j)} - 2h_j^2\boldsymbol{c}_2^{(j+1)}
\end{aligned}\right\} \tag{1.7}$$

for $2(N+1)$ unknowns, $\{\boldsymbol{c}_1^{(0)}, \ldots, \boldsymbol{c}_1^{(N)}\}$ and $\{\boldsymbol{c}_2^{(0)}, \ldots, \boldsymbol{c}_2^{(N)}\}$. The imposed boundary conditions at the beginning and end of the global spline produce four additional equations for $\{\boldsymbol{c}_1^{(0)}, \boldsymbol{c}_2^{(0)}, \boldsymbol{c}_1^{(N)}, \boldsymbol{c}_2^{(N)}\}$. Note we have implicitly introduced a ghost spline $\boldsymbol{s}^{(N)}$ which satisfies all continuity conditions with $\boldsymbol{s}^{(N-1)}$. The purpose of $\boldsymbol{s}^{(N)}$ is to deal with boundary conditions at the end of spline.

In general there are three types of boundary conditions at each end. Let $x$, $s$ and $c^{(j)}$ be one of the scalar components of vector $\boldsymbol{x}_j$, spline $\boldsymbol{s}^{(j)}$ and coefficient $\boldsymbol{c}^{(j)}$. We first consider boundary conditions at $l = l_0$,

$$\text{Even:} \quad 0 = \frac{\mathrm{d}s^{(0)}}{\mathrm{d}l} = \frac{\mathrm{d}^3 s^{(0)}}{\mathrm{d}l^3} \qquad \text{at} \quad l = l_0 \tag{1.8}$$

$$\text{Odd:} \quad 0 = s^{(0)} = \frac{\mathrm{d}^2 s^{(0)}}{\mathrm{d}l^2} = \frac{\mathrm{d}^4 s^{(0)}}{\mathrm{d}l^4} \quad \text{at} \quad l = l_0 \tag{1.9}$$

$$\text{Mix:} \quad \alpha = \frac{\mathrm{d}s^{(0)}}{\mathrm{d}l}, \quad \beta = \frac{\mathrm{d}^2 s^{(0)}}{\mathrm{d}l^2} \qquad \text{at} \quad l = l_0 \tag{1.10}$$

which lead to two equations,

$$\left.\begin{aligned} c_1^{(0)} &= 0 \\ 6h_0c_1^{(0)} + 3h_0^2c_2^{(0)} + 4h_0c_1^{(1)} - h_0^2c_2^{(1)} &= -10x_0 + 10x_1 \end{aligned}\right\} \quad \text{Even} \tag{1.11$a$}$$

$$\left.\begin{aligned} -8h_0c_1^{(0)} - 3h_0^2c_2^{(0)} - 7h_0c_1^{(1)} + 2h_0^2c_2^{(1)} &= +15x_0 - 15x_1 \\ c_2^{(0)} &= 0 \end{aligned}\right\} \quad \text{Odd} \tag{1.11$b$}$$

$$\left.\begin{aligned} c_1^{(0)} &= \alpha \\ c_2^{(0)} &= \beta/2 \end{aligned}\right\} \quad \text{Mix} \tag{1.11$c$}$$

We then consider boundary conditions at $l = l_N$,

$$\text{Even:} \quad 0 = \frac{\mathrm{d}s^{(N-1)}}{\mathrm{d}l} = \frac{\mathrm{d}^3s^{(N-1)}}{\mathrm{d}l^3} \qquad \text{at} \quad l = l_N \tag{1.12}$$

$$\text{Odd:} \quad 0 = s^{(N-1)} = \frac{\mathrm{d}^2s^{(N-1)}}{\mathrm{d}l^2} = \frac{\mathrm{d}^4s^{(N-1)}}{\mathrm{d}l^4} \quad \text{at} \quad l = l_N \tag{1.13}$$

$$\text{Mix:} \quad \alpha = \frac{\mathrm{d}s^{(N-1)}}{\mathrm{d}l}, \quad \beta = \frac{\mathrm{d}^2s^{(N-1)}}{\mathrm{d}l^2} \qquad \text{at} \quad l = l_N \tag{1.14}$$

which also lead to two equations,

$$\left.\begin{aligned} c_1^{(N)} &= 0 \\ -4h_{N-1}c_1^{(N-1)} - h_{N-1}^2c_2^{(N-1)} - 6h_{N-1}c_1^{(N)} + 3h_{N-1}^2c_2^{(N)} &= 10x_{N-1} - 10x_N \end{aligned}\right\} \quad \text{Even} \tag{1.15$a$}$$

$$\left.\begin{aligned} -7h_{N-1}c_1^{(N-1)} - 2h_{N-1}^2c_2^{(N-1)} - 8h_{N-1}c_1^{(N)} + 3h_{N-1}^2c_2^{(N)} &= 15x_{N-1} - 15x_N \\ c_2^{(N)} &= 0 \end{aligned}\right\} \quad \text{Odd} \tag{1.15$b$}$$

$$\left.\begin{aligned} c_1^{(N)} &= \alpha \\ c_2^{(N)} &= \beta/2 \end{aligned}\right\} \quad \text{Mix} \tag{1.15$c$}$$

If we arrange the unknowns into a vector,

$$\left[c_1^{(0)}, c_2^{(0)}, \ldots, c_1^{(j)}, c_2^{(j)}, \ldots, c_1^{(N)}, c_2^{(N)}\right] \tag{1.16}$$

equations (1.7) with one of (1.11) and one of (1.15) result in a $2(N+1)$-by-$2(N+1)$ system of linear equations, which corresponds to a banded diagonal sparse matrix with at most 6 non-zero elements in each row. The rest coefficients $\boldsymbol{c}_3^{(j)}$, $\boldsymbol{c}_4^{(j)}$ and $\boldsymbol{c}_5^{(j)}$ can be reconstruct using equation (1.6).

It is convenient to re-parametrize each local spline with a new variable $t \in [0, 1]$,

$$\boldsymbol{s}^{(j)}(t) = \boldsymbol{x}_j + \sum_{k=1}^{5} \boldsymbol{c}_k^{(j)} t^k, \quad \text{for} \quad j = 0, \ldots, N-1 \tag{1.17}$$

by redefining $\boldsymbol{c}_k^{(j)} \to \boldsymbol{c}_k^{(j)} h_j^k$. We can construct Lagrange interpolations along the arc-length of the global spline by the local arc-length $L_j$ and its local fraction $\xi$,

$$L_j = \int_0^1 \dot{\boldsymbol{s}}^{(j)}\, \mathrm{d}t', \quad \xi(t) = \frac{1}{L_j} \int_0^t \dot{\boldsymbol{s}}^{(j)}\, \mathrm{d}t', \tag{1.18}$$

Convention for normal vector $\boldsymbol{n}$,

$$\boldsymbol{n} = \frac{(-\dot{z}, \dot{r})}{\|\dot{\boldsymbol{s}}\|} \tag{1.19}$$

## 2. Numerical approximation

### 2.1. *Gaussian quadrature*

Standard $m$-point Gauss-Legendre quadrature,

$$\int_0^1 f(t)\,\mathrm{d}t \approx \sum_{k=1}^{m} w_k f(x_k), \quad \int_a^b f(t)\,\mathrm{d}t \Rightarrow \begin{array}{l} x_k \to a + (b-a)x_k \\ w_k \to (b-a)w_k \end{array} \tag{2.1}$$

Logarithmic-weighted Gauss quadrature,

$$\int_0^1 f(t)\ln t\,\mathrm{d}t \approx -\sum_{k=1}^{m} w_k^{\log} f(x_k), \quad \int_0^1 f(t)\ln(1-t)\,\mathrm{d}t \approx -\sum_{k=1}^{m} w_k^{\log} f(1-x_k) \tag{2.2}$$

When logarithmic singularity $\tau \in (0,1)$,

$$\begin{aligned}
\int_0^1 f(t)\ln|t-\tau|\,\mathrm{d}t &= \int_0^\tau f(t)\ln(\tau-t)\,\mathrm{d}t + \int_\tau^1 f(t)\ln(t-\tau)\,\mathrm{d}t \\
&= \tau\int_0^1 f(\tau s)\ln(\tau-\tau s)\,\mathrm{d}s + (1-\tau)\int_0^1 f(\tau+(1-\tau)s)\ln((1-\tau)s)\,\mathrm{d}s \\
&= \tau\int_0^1 f(\tau s)\ln(1-s)\,\mathrm{d}s + (1-\tau)\int_0^1 f(\tau+(1-\tau)s)\ln s\,\mathrm{d}s \\
&\quad + \tau\ln\tau\int_0^1 f(\tau s)\,\mathrm{d}s + (1-\tau)\ln(1-\tau)\int_0^1 f(\tau+(1-\tau)s)\,\mathrm{d}s
\end{aligned}$$

Logarithmic-weighted Gauss quadrature for singularity $\tau \in (0,1)$,

$$\int_0^1 f(t)\ln|t-\tau|\,\mathrm{d}t \approx \left\{ \begin{array}{c}
\tau\ln\tau\sum_{k=1}^{m} w_k f\left[\tau x_k\right] \\
+(1-\tau)\ln(1-\tau)\sum_{k=1}^{m} w_k f\left[\tau+(1-\tau)x_k\right] \\
-\tau\sum_{k=1}^{m} w_k^{\log} f\left[\tau(1-x_k^{\log})\right] \\
-(1-\tau)\sum_{k=1}^{m} w_k^{\log} f\left[\tau+(1-\tau)x_k^{\log}\right]
\end{array} \right. \tag{2.3}$$

### 2.2. *Complete elliptic integral*

The complete elliptic integral of the first and second kind,

$$K(m) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-m\sin^2\theta}}, \quad E(m) = \int_0^{\pi/2} \sqrt{1-m\sin^2\theta}\,\mathrm{d}\theta \tag{2.4}$$

We can approximate $K(m)$ and $E(m)$ with

$$K(m) \approx P_K(m) - \ln(1-m)Q_K(m), \quad E(m) \approx P_E(m) - \ln(1-m)Q_E(m) \tag{2.5}$$

| $\ell$ | $c_\ell$ | $P_{\ell,E}(x)$ | $P_{\ell,K}(x)$ |
|---|---|---|---|
| $1/2$ | $2/\pi$ | $2$ | $-1$ |
| $3/2$ | $1/3\pi$ | $16x$ | $-2(4x+1)$ |
| $5/2$ | $1/15\pi$ | $4\left(32x^2-9\right)$ | $-2\left(32x^2+8x-9\right)$ |
| $7/2$ | $1/105\pi$ | $64x\left(24x^2-13\right)$ | $2\left(-384x^3-96x^2+208x+25\right)$ |
| $9/2$ | $1/315\pi$ | $8192x^4-6528x^2+588$ | $-2\left(2048x^4+512x^3-1632x^2-264x+147\right)$ |

TABLE 1. Coefficients of associate Legendre polynomials of $\ell = 1/2$ family

where $P_K$, $P_E$, $Q_K$, $Q_E$ are tenth order polynomials. Associate Legendre polynomial of 1/2 order can be conveniently expressed as,

$$P_{1/2}(x) = \frac{2}{\pi}\left\{2E\left(\frac{1-x}{2}\right) - K\left(\frac{1-x}{2}\right)\right\} \tag{2.6}$$

From the recursion relations,

$$\begin{aligned}(1+x^2)\frac{\mathrm{d}P_\ell(x)}{\mathrm{d}x} &= (\ell+1)xP_\ell(x) - (\ell+1)P_{\ell+1}(x)\\ \frac{\mathrm{d}K}{\mathrm{d}m} &= \frac{E(m)}{2m(1-m)} - \frac{K(m)}{2m}\\ \frac{\mathrm{d}E}{\mathrm{d}m} &= \frac{1}{2m}\left[E(m)-K(m)\right]\end{aligned} \tag{2.7}$$

we can bootstrap associate Legendre polynomials of higher orders, 3/2 , 5/2 , ..., The form is again

$$P_\ell(x) = c_\ell\left\{P_{\ell,E}(x)E\left(\frac{1-x}{2}\right) + P_{\ell,K}(x)K\left(\frac{1-x}{2}\right)\right\} \tag{2.8}$$

where $c_\ell$ is some constant. $P_{\ell,E}(x)$ and $P_{\ell,E}(x)$ are polynomials of $(\ell-1/2)$ order whose coefficients are computed symbolically and given in Table 1.

## 3. Axisymmetric boundary integral

Axisymmetric Green's function $G(\boldsymbol{\eta}';\boldsymbol{\eta})$,

$$G(\boldsymbol{\eta}';\boldsymbol{\eta}) = \frac{1}{\pi\sqrt{a+b}}K(m) \tag{3.1}$$

$$\frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}} = \frac{1}{\pi r\sqrt{a+b}}\left\{\left[\frac{n_r}{2} + \frac{\boldsymbol{n}\cdot(\boldsymbol{x}'-\boldsymbol{x})}{\|\boldsymbol{x}'-\boldsymbol{x}\|^2}r\right]E(m) - \frac{n_r}{2}K(m)\right\} \tag{3.2}$$

Auxiliary variables,

$$a = r'^2 + r^2 + (z'-z)^2, \quad b = 2r'r, \quad m = \frac{2b}{a+b} \tag{3.3}$$

Note $\eta \to \eta'$ implies $m \to 1$. Axisymmetric boundary integral equation,

$$\frac{\beta(\boldsymbol{\eta}')}{2\pi}\phi(\boldsymbol{\eta}') = \int_\gamma \left\{G(\boldsymbol{\eta}';\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{n}}(\boldsymbol{\eta}) - \phi(\boldsymbol{\eta})\frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial\boldsymbol{n}}(\boldsymbol{\eta})\right\} r\,\mathrm{d}\gamma(\boldsymbol{\eta}) \tag{3.4}$$

Lagrange interpolation along arc-length,

$$\phi(\boldsymbol{\eta}) \approx \sum_e \sum_k p_{e_k} \mathcal{N}_k^{(e)}(\xi), \quad \frac{\partial \phi}{\partial \boldsymbol{n}}(\boldsymbol{\eta}) \approx \sum_e \sum_k q_{e_k} \mathcal{N}_k^{(e)}(\xi) \tag{3.5}$$

where $\mathcal{N}_k$ is the Lagrange basis of the local fractional arc-length variable $\xi$,

$$\mathcal{N}_0 = (1-\xi)(1-2\xi), \quad \mathcal{N}_1 = 4\xi(1-\xi), \quad \mathcal{N}_2 = \xi(2\xi-1) \tag{3.6}$$

### 3.1. *Single layer integral*

Single-layer integral reduces to a summation of local integrals over local arc $\gamma_e$ of element $e$,

$$\begin{aligned}\int_\gamma G(\boldsymbol{\eta}';\boldsymbol{\eta}) \frac{\partial \phi}{\partial \boldsymbol{n}}(\boldsymbol{\eta}) r \,\mathrm{d}\gamma(\boldsymbol{\eta}) &\approx \sum_e \sum_k q_{e_k} \int_{\gamma_e} G(\boldsymbol{\eta}';\boldsymbol{\eta}) \mathcal{N}_k^{(e)}(\xi) r \,\mathrm{d}\gamma(\boldsymbol{\eta}) \\ &= \sum_e \sum_k q_{e_k} \int_0^1 \frac{rJ}{\pi\sqrt{a+b}} \mathcal{N}_k^{(e)}(\xi) K(m) \,\mathrm{d}t \end{aligned} \tag{3.7}$$

When singularity $\boldsymbol{\eta}' \in \gamma_e$ is located at $t = \tau \in [0,1]$,

$$\int_0^1 \frac{rJ}{\pi\sqrt{a+b}} \mathcal{N}_k^{(e)} K(m) \,\mathrm{d}t = \int_0^1 \frac{rJ}{\pi\sqrt{a+b}} \mathcal{N}_k^{(e)} \left[ P_K - Q_K \ln \frac{1-m}{(t-\tau)^2} - 2Q_K \ln|t-\tau| \right] \mathrm{d}t$$

Introduce helper functions,

$$f_K^{\text{single}}(t) = \frac{rJ}{\pi\sqrt{a+b}} \tag{3.8}$$

$$R_{K/E}(m,\tau) = P_{K/E}(m) - Q_{K/E}(m) \ln \frac{1-m}{(t-\tau)^2} \tag{3.9}$$

If we define integrand $I^{(o)}$ for source location $\boldsymbol{\eta}'$ outside of $\gamma_e$ and $I^{(i)}$ for $\boldsymbol{\eta}'$ interior to $\gamma_e$,

$$I^{(o)}(t) = f_K^{\text{single}}(t) K(m) \mathcal{N}_k^{(e)}(\xi) \tag{3.10}$$

$$I^{(i)}(t,\tau) = f_K^{\text{single}}(t) R_K(m,\tau) \mathcal{N}_k^{(e)}(\xi) \tag{3.11}$$

$$I_{\log}^{(i)}(t) = -2 f_K^{\text{single}}(t) Q_K(m) \mathcal{N}_k^{(e)}(\xi) \tag{3.12}$$

then the local single-layer integral can be compactly represented with a regular part and a logarithmic-singular one,

$$\boldsymbol{\eta}' \notin \gamma_e: \ \text{compute} \int_0^1 I^{(o)}(t) \,\mathrm{d}t \tag{3.13}$$

$$\boldsymbol{\eta}' \text{ at } t=0: \ \text{compute} \int_0^1 I^{(i)}(t,0) \,\mathrm{d}t + \int_0^1 I_{\log}^{(i)}(t) \ln t \,\mathrm{d}t \tag{3.14}$$

$$\boldsymbol{\eta}' \text{ at } t=\tau: \ \text{compute} \int_0^1 I^{(i)}(t,\tau) \,\mathrm{d}t + \int_0^1 I_{\log}^{(i)}(t) \ln|\tau - t| \,\mathrm{d}t \tag{3.15}$$

$$\boldsymbol{\eta}' \text{ at } t=1: \ \text{compute} \int_0^1 I^{(i)}(t,1) \,\mathrm{d}t + \int_0^1 I_{\log}^{(i)}(t) \ln(1-t) \,\mathrm{d}t \tag{3.16}$$

| | | |
|---|---|---|
| Element order | $o$ | |
| Number of elements | $n$ | |
| Number of nodes | $n \times o + 1$ | |
| Nodal index of source point | $i$ | |
| Elemental index of receiving point | $j$ | |
| Local node index | $k$ | $0 \leqslant k \leqslant o$ |
| Indices of elements $\ni$ $i$-th node if $i \bmod o = 0$ | $\lfloor i/o \rfloor - 1,\ \lfloor i/o \rfloor$ | check $0 \leqslant \text{id} \leqslant n-1$ |
| Indices of elements $\ni$ $i$-th node if $i \bmod o \neq 0$ | $\lfloor i/o \rfloor - 1$ | check $0 \leqslant \text{id} \leqslant n-1$ |
| Nodal indices of $j$-th element | $j \times o + k$ | $0 \leqslant k \leqslant o$ |

TABLE 2. Index involved

### 3.2. *Double layer integral*

Similar to singular-layer integral, double-layer integral reduces to a summation of local integrals over local arc $\gamma_e$ of element $e$,

$$\begin{aligned}\int_\gamma \frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}}\phi(\boldsymbol{\eta}) r\,\mathrm{d}\gamma(\boldsymbol{\eta}) &\approx \sum_e\sum_k p_{e_k}\int_{\gamma_e}\frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}}\mathcal{N}_k^{(e)}(\xi) r\,\mathrm{d}\gamma(\boldsymbol{\eta})\\
&= \sum_e\sum_k p_{e_k}\int_0^1 \mathcal{N}_k^{(e)}(\xi)\frac{J}{\pi\sqrt{a+b}}\left\{\left[\frac{n_r}{2}+\frac{\boldsymbol{n}\cdot(\boldsymbol{x}'-\boldsymbol{x})}{\|\boldsymbol{x}'-\boldsymbol{x}\|^2}r\right]E(m)-\frac{n_r}{2}K(m)\right\}\mathrm{d}t\\
&= \sum_e\sum_k p_{e_k}\int_0^1 \frac{\mathcal{N}_k^{(e)}(\xi)}{\pi\sqrt{a+b}}\left\{\left[\frac{\dot r(z'-z)-\dot z(r'-r)}{(r'-r)^2+(z'-z)^2}r-\frac{\dot z}{2}\right]E(m)+\frac{\dot z}{2}K(m)\right\}\mathrm{d}t \end{aligned} \tag{3.17}$$

Technically we don't need to treat integrand involving elliptic integral of the second kind $E(m)$. However, convergence rate of standard Gauss-Legendre quadrature depends magnitude of derivatives.

$$f_E^{\text{double}} = \frac{1}{\pi\sqrt{a+b}}\left[\frac{\dot r(z'-z)-\dot z(r'-r)}{a-b}r-\frac{\dot z}{2}\right],\quad f_K^{\text{double}} = \frac{1}{\pi\sqrt{a+b}}\frac{\dot z}{2} \tag{3.18}$$

If we define integrand $I^{(o)}$ for source location $\boldsymbol{\eta}'$ outside of $\gamma_e$ and $I^{(i)}$ for $\boldsymbol{\eta}'$ interior to $\gamma_e$,

$$I^{(o)}(t) = \left[f_E^{\text{double}}E(m)+f_K^{\text{double}}K(m)\right]\mathcal{N}_k^{(e)}(\xi) \tag{3.19}$$

$$I^{(i)}(t,\tau) = \left[f_E^{\text{double}}R_E(m,\tau)+f_K^{\text{double}}R_K(m,\tau)\right]\mathcal{N}_k^{(e)}(\xi) \tag{3.20}$$

$$I^{(i)}_{\log}(t) = -2f_E^{\text{double}}Q_E(m)-2f_K^{\text{double}}Q_K(m) \tag{3.21}$$

When $\boldsymbol{\eta}'$ is located the symmetry axis where $r'=0$, we can simplify the elliptic integrals,

$$\int_\gamma G(\boldsymbol{\eta}';\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{n}}(\boldsymbol{\eta}) r\,\mathrm{d}\gamma(\boldsymbol{\eta}) \approx \sum_e\sum_k q_{e_k}\int_0^1 \mathcal{N}_k^{(e)}(\xi) r\frac{J}{2\sqrt{r+(z-z')^2}}\mathrm{d}t \tag{3.22}$$

$$\int_\gamma \frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}}\phi(\boldsymbol{\eta}) r\,\mathrm{d}\gamma(\boldsymbol{\eta}) \approx \sum_e\sum_k p_{e_k}\int_0^1 \mathcal{N}_k^{(e)}(\xi) r\frac{\dot z r+\dot r(z-z')}{2\left[r^2+(z'-z)^2\right]^{3/2}}\mathrm{d}t \tag{3.23}$$

The above integrands have well-defined limit as $\boldsymbol{\eta}\to\boldsymbol{\eta}'$ assuming $\ddot z/\dot r$ is finite at $r=0$.

### 3.3. *Assembly*

$$(\boldsymbol{B}+\boldsymbol{D})\boldsymbol{p} = \boldsymbol{H}\boldsymbol{p} = \boldsymbol{S}\boldsymbol{q} \tag{3.24}$$

**Algorithm 1** Counting mismatches between two packed strings

**function** DISTANCE($x$, $e$)
  **for** $0 \leqslant i \leqslant N_x - 1$ **do** # We can parallelize this loop
    **if** $i \bmod o = 0$ **then**
      $I_{\text{lower}} = I_{\text{upper}}$
    **else**
      $I_{\text{lower}} = \lfloor i/o \rfloor - 1, \quad I_{\text{upper}} = \lfloor i/o \rfloor$
    **end if**
    **for** $0 \leqslant i \leqslant N_x - 1$ **do**
    **end for**
  **end for**
**end function**

Consider two adjacent boundaries $\gamma_0$ and $\gamma_1$. We have a block matrix equation,

$$\begin{bmatrix} \boldsymbol{H}_{00} & \boldsymbol{H}_{01} \\ \boldsymbol{H}_{10} & \boldsymbol{H}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_{00} & \boldsymbol{S}_{01} \\ \boldsymbol{S}_{10} & \boldsymbol{S}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_0 \\ \boldsymbol{q}_1 \end{bmatrix} \tag{3.25}$$

We have two identical rows in $\boldsymbol{H}$ and $\boldsymbol{S}$ and extra equation for continuity of potential,

$$\left. \begin{aligned} \operatorname{row}_{-1}(\boldsymbol{H}_{00}) \cdot \boldsymbol{p}_0 + \operatorname{row}_{-1}(\boldsymbol{H}_{01}) \cdot \boldsymbol{p}_1 &= \operatorname{row}_{-1}(\boldsymbol{S}_{00}) \cdot \boldsymbol{q}_0 + \operatorname{row}_{-1}(\boldsymbol{S}_{01}) \cdot \boldsymbol{q}_1 \\ \operatorname{row}_{0}(\boldsymbol{H}_{10}) \cdot \boldsymbol{p}_0 + \operatorname{row}_{0}(\boldsymbol{H}_{11}) \cdot \boldsymbol{p}_1 &= \operatorname{row}_{0}(\boldsymbol{S}_{10}) \cdot \boldsymbol{q}_0 + \operatorname{row}_{0}(\boldsymbol{S}_{11}) \cdot \boldsymbol{q}_1 \\ (p_0)_{-1} - (p_1)_0 &= 0 \end{aligned} \right\} \tag{3.26}$$

We simply replace one of identical rows with the continuity condition,

$$\begin{aligned} \operatorname{row}_{-1}(\boldsymbol{H}_{00}) &= [0, \cdots, 0, 1], \quad \operatorname{row}_{-1}(\boldsymbol{H}_{01}) = [-1, 0, \cdots, 0] \\ \operatorname{row}_{-1}(\boldsymbol{S}_{00}) &= [0, \cdots, 0, 0], \quad \operatorname{row}_{-1}(\boldsymbol{S}_{01}) = [0, 0, \cdots, 0] \end{aligned} \tag{3.27}$$

For mixed-type boundary condition, i.e., Dirichlet and Neumann on different segments of the boundary, we simply rearrange the block matrices,

$$\begin{bmatrix} \boldsymbol{H}_{00} & -\boldsymbol{S}_{01} \\ \boldsymbol{H}_{10} & -\boldsymbol{S}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{q}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_{00} & -\boldsymbol{H}_{01} \\ \boldsymbol{S}_{10} & -\boldsymbol{H}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_0 \\ \boldsymbol{p}_1 \end{bmatrix} \tag{3.28}$$

Note the continuity between discrete potential vectors $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ is still implied.

### 3.4. *Benchmark*

We validate our implementation of boundary integral solver for test problems posed on a smooth boundary $\gamma$ and on a piecewise smooth boundary $\gamma_0 \cup \gamma_1$. The latter contains a geometric discontinuity of a 90° corner at the transition point between $\gamma_0$ and $\gamma_1$. Consider the following parametrisation of the boundaries,

$$\gamma_0 = \left\{ 2\Big(1 + \frac{1}{4}\cos(8\theta - \pi)\Big) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\Big|\; \theta \in [0, \pi/2] \right\} \tag{3.29}$$

$$\gamma_1 = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} \;\Big|\; \theta \in [0, 3/2] \right\} \tag{3.30}$$

The general form of axisymmetric harmonic potential can be constructed from Legendre polynomial $P_\ell$ of order $\ell$. We consider the interior problem for the potential $\phi$,

$$\phi = (r^2 + z^2) P_2\Big(\frac{z}{\sqrt{r^2 + z^2}}\Big), \quad \frac{\partial \phi}{\partial \boldsymbol{n}} = \boldsymbol{n} \cdot (-r, 2z) \tag{3.31}$$

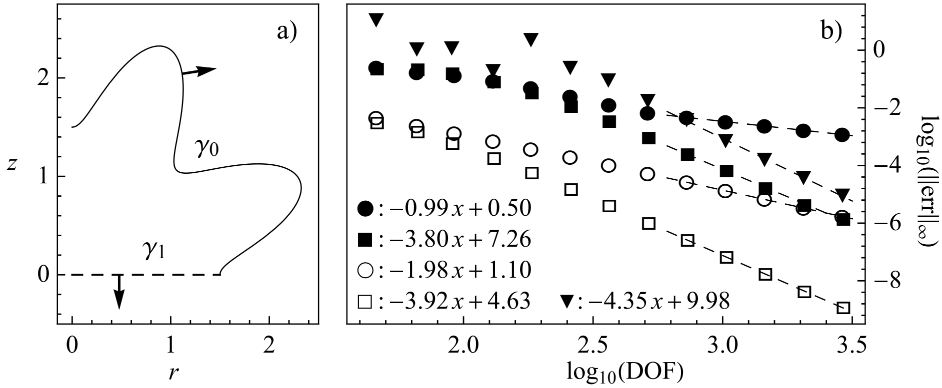


FIGURE 1. Error convergence between numerical and analytic solutions measured in $l^\infty$-norm against degrees of freedom (DOF) used for the test problem (3.32): (a) Boundaries $\gamma_0$ (solid) and $\gamma_1$ (dashed) given by parametrisation (3.31). (b) Convergence of total curvature (▼) on $\gamma_0$; convergence of Neumann data on $\gamma_0$ with linear (●) and quadratic (■) shape functions; convergence of Dirichlet data on $\gamma_1$ with linear (○) and quadratic (□) shape functions; linear fit of the last five points of each error convergence (inset).

If we prescribe the potential value of $\phi$ on $\gamma_0$ and its normal flux $\partial\phi/\partial\boldsymbol{n}$ on $\gamma_1$, our solver is expected to reproduce $\partial\phi/\partial\boldsymbol{n}$ on $\gamma_0$ and $\phi$ on $\gamma_1$,

$$\text{Test problem:}\quad \text{given}\begin{cases}\phi \text{ on } \gamma_0\\ \partial\phi/\partial\boldsymbol{n} \text{ on } \gamma_1\end{cases},\ \text{find}\begin{cases}\partial\phi/\partial\boldsymbol{n} \text{ on } \gamma_0\\ \phi \text{ on } \gamma_1\end{cases} \tag{3.32}$$

In addition for a plane curve given parametrically as $\gamma(t) = (r(t), z(t))$, the total curvature $2\kappa$ of the surface of revolution obtained by rotating curve $\gamma$ about the $z$-axis is given by,

$$2\kappa = \frac{\dot{r}\ddot{z}-\dot{z}\ddot{r}}{(\dot{r}^2+\dot{z}^2)^{\frac{3}{2}}} + \frac{1}{r}\frac{\dot{z}}{\sqrt{\dot{r}^2+\dot{z}^2}} \tag{3.33}$$

We verify the accuracy of quintic spline interpolation against the analytic form derived from the parametrisation (3.31). All errors between numerical and analytical solutions are measured in the $l^\infty$-norm,

$$\|\mathrm{err_d}\|_\infty = \max_{\boldsymbol{x}_i\in\gamma_1} |p_i^{\mathrm{num}} - \phi(\boldsymbol{x}_i)| \tag{3.34}$$

$$\|\mathrm{err_n}\|_\infty = \max_{\boldsymbol{x}_i\in\gamma_0} |q_i^{\mathrm{num}} - \partial\phi/\partial\boldsymbol{n}(\boldsymbol{x}_i)| \tag{3.35}$$

$$\|\mathrm{err_c}\|_\infty = \max_{\boldsymbol{x}_i\in\gamma_0} |\kappa_i^{\mathrm{spline}} - \kappa(\boldsymbol{x}_i)| \tag{3.36}$$

In figure 1(a) and 1(b) we plot these errors against the total number of degree of freedom (DOF) used by the solver.

## 4. Self-similar Taylor cone

In the last section we outline the numerical procedure to solve Laplace equation subject to mixed boundary conditions. Initial guess is a $C^3$ continuous function $f_{\mathrm{guess}}$,

$$f_{\mathrm{guess}}(r) = \begin{cases} f_0 r + f_2 r^2 + f_4 r^4 + f_6 r^6, & \text{if } r \leqslant r_c\\ c_0 r + \dfrac{c_1}{\sqrt{r}} + \dfrac{c_3}{r^{7/2}} + \dfrac{c_4}{r^5}, & \text{if } r > r_c\end{cases} \tag{4.1}$$

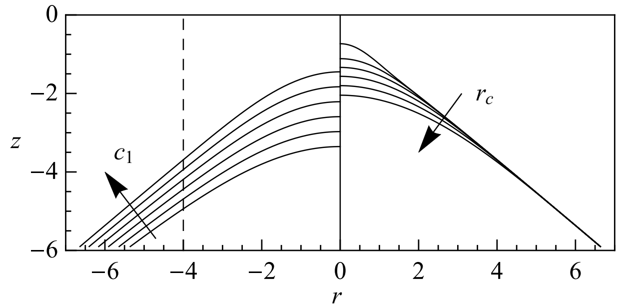


FIGURE 2. $C^3$ function $f_{\text{guess}}(r)$: increasing $r_c$ (right) and $c_1$ (left)

where $r_c$ is the connection point where continuities up to the third derivative are enforced. Given a reasonable $c_1$, function $f_{\text{guess}}(r)$ agrees with analytic prediction in the far-field. The effect of varying $r_c$ and $c_1$ is illustrated in figure 2.

## REFERENCES

TAYLOR, G. I. 1964 Disintegration of water drops in an electric field. *Proceedings of the Royal Society of London Series A: Mathematical and Physical Sciences* **280** (1382), 383–397.