Dynamic Taylor Cone Part II: Numeric Scheme

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CONTENTS

Numeric scheme

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1. Numerical approximation

1.1. *Quintic spline*

The accuracy of the numerical solution to boundary integral equation is directly affected by the precision and smoothness of the underlying geometry, in this case a planar curve. Numerically we compute a quintic spline curve that interpolates through a discrete set of marker points that is at least continuous up to fourth derivative (Mund *et al.* 1975).

Consider $N+1$ marker points $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_N\}$. A natural choice for the intrinsic coordinate that can be used for interpolation is the chord length,

$$h_j = \|\boldsymbol{x}_{j+1} - \boldsymbol{x}_j\| \quad \text{for} \quad j = 0, \ldots, N-1 \tag{1.1}$$

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where h_j is the the euclidean distance between adjacent marker points $\boldsymbol{x}j$ and $\boldsymbol{x}_{j+1}$. Then we introduce $N+1$ chord coordinates $\{l_0,\ldots,l_N\}$,

$$l_0 = 0, \quad l_j = \sum_{k=0}^{j-1} h_k \quad \text{for} \quad j = 1,\ldots,N \tag{1.2}$$

Global spline $\boldsymbol{s}(l)$ is a collection of local splines $\{\boldsymbol{s}^{(0)},\ldots,\boldsymbol{s}^{(N-1)}\}$ where each $\boldsymbol{s}^{(j)}$ is a fifth degree polynomial defined on the interval $l \in [l_j, l_{j+1}]$,

$$\boldsymbol{s}^{(j)}(l) = \boldsymbol{x}_j + \sum_{k=1}^{5} \boldsymbol{c}_k^{(j)} (l - l_j)^k, \quad \text{for} \quad j = 0,\ldots,N-1 \tag{1.3}$$

Continuity between neighboring splines is satisfied up to the fourth derivative,

$$\left.\frac{\mathrm{d}^p \boldsymbol{s}^{(j)}}{\mathrm{d}l^p}\right|_{l_{j+1}} = \left.\frac{\mathrm{d}^p \boldsymbol{s}^{(j+1)}}{\mathrm{d}l^p}\right|_{l_{j+1}}, \quad \text{for} \quad p = 0,\ldots,4 \tag{1.4}$$

These continuity conditions yield five equations

$$\begin{aligned}
\boldsymbol{c}_3^{(j-1)} h_{j-1}^3 + \boldsymbol{c}_4^{(j-1)} h_{j-1}^4 + \boldsymbol{c}_5^{(j-1)} h_{j-1}^5 &= \boldsymbol{x}_j - \boldsymbol{x}_{j-1} \\
&\quad - \boldsymbol{c}_1^{(j-1)} h_{j-1} - \boldsymbol{c}_2^{(j-1)} h_{j-1}^2 && (1.5a) \\
3\boldsymbol{c}_3^{(j-1)} h_{j-1}^2 + 4\boldsymbol{c}_4^{(j-1)} h_{j-1}^3 + 5\boldsymbol{c}_5^{(j-1)} h_{j-1}^4 &= -\boldsymbol{c}_1^{(j-1)} - 2\boldsymbol{c}_2^{(j-1)} h_{j-1} + \boldsymbol{c}_1^{(j)} && (1.5b) \\
6\boldsymbol{c}_3^{(j-1)} h_{j-1} + 12\boldsymbol{c}_4^{(j-1)} h_{j-1}^2 + 20\boldsymbol{c}_5^{(j-1)} h_{j-1}^3 &= -2\boldsymbol{c}_2^{(j-1)} + 2\boldsymbol{c}_2^{(j)} && (1.5c) \\
\boldsymbol{c}_3^{(j-1)} + 4h_{j-1}\boldsymbol{c}_4^{(j-1)} + 10h_{j-1}^2\boldsymbol{c}_5^{(j-1)} &= \boldsymbol{c}_3^{(j)} && (1.5d) \\
\boldsymbol{c}_4^{(j-1)} + 5h_{j-1}\boldsymbol{c}_5^{(j-1)} &= \boldsymbol{c}_4^{(j)} && (1.5e)
\end{aligned}$$

We express the entire spline only in terms of the first and second coefficients $\{\boldsymbol{c}_1^{(j)}\}$ and $\{\boldsymbol{c}_2^{(j)}\}$. We first solve equations (1.5 a, 1.5 c, 1.5 b) for $\boldsymbol{c}_3^{(j-1)}$, $\boldsymbol{c}_4^{(j-1)}$ and $\boldsymbol{c}_5^{(j-1)}$,

$$\begin{aligned}
h_{j-1}^3 \boldsymbol{c}_3^{(j-1)} &= +10(\boldsymbol{x}_j - \boldsymbol{x}_{j-1}) \\
&\quad - 6h_{j-1}\boldsymbol{c}_1^{(j-1)} - 3h_{j-1}^2\boldsymbol{c}_2^{(j-1)} - 4h_{j-1}\boldsymbol{c}_1^{(j)} + h_{j-1}^2\boldsymbol{c}_2^{(j)} && (1.6a) \\
h_{j-1}^4 \boldsymbol{c}_4^{(j-1)} &= -15(\boldsymbol{x}_j - \boldsymbol{x}_{j-1}) \\
&\quad + 8h_{j-1}\boldsymbol{c}_1^{(j-1)} + 3h_{j-1}^2\boldsymbol{c}_2^{(j-1)} + 7h_{j-1}\boldsymbol{c}_1^{(j)} - 2h_{j-1}^2\boldsymbol{c}_2^{(j)} && (1.6b) \\
h_{j-1}^5 \boldsymbol{c}_5^{(j-1)} &= +\ 6(\boldsymbol{x}_j - \boldsymbol{x}_{j-1}) \\
&\quad - 3h_{j-1}\boldsymbol{c}_1^{(j-1)} - \ h_{j-1}^2\boldsymbol{c}_2^{(j-1)} - 3h_{j-1}\boldsymbol{c}_1^{(j)} + \ h_{j-1}^2\boldsymbol{c}_2^{(j)} && (1.6c)
\end{aligned}$$

Once we have $\{\boldsymbol{c}_1^{(j)}\}$ and $\{\boldsymbol{c}_2^{(j)}\}$ computed, $\{\boldsymbol{c}_3^{(j)}\}$, $\{\boldsymbol{c}_4^{(j)}\}$ and $\{\boldsymbol{c}_5^{(j)}\}$ can be derived.

Define $\lambda = h_j/h_{j-1}$. We have $2(N-1)$ equations for $j = 1,\ldots,N-1$,

$$\left.\begin{aligned}
10\left[+\lambda^3\boldsymbol{x}_{j-1} - (1+\lambda^3)\boldsymbol{x}_j + \boldsymbol{x}_{j+1}\right] = \quad & 4h_j\lambda^2\boldsymbol{c}_1^{(j-1)} \quad +6h_j(\lambda^2-1)\boldsymbol{c}_1^{(j)} \quad -4h_j\boldsymbol{c}_1^{(j+1)} \\
& +h_j^2\lambda\boldsymbol{c}_2^{(j-1)} \quad -3h_j^2(1+\lambda)\boldsymbol{c}_2^{(j)} \quad +h_j^2\boldsymbol{c}_2^{(j+1)} \\
15\left[-\lambda^4\boldsymbol{x}_{j-1} + (\lambda^4-1)\boldsymbol{x}_j + \boldsymbol{x}_{j+1}\right] = \quad & 7h_j\lambda^3\boldsymbol{c}_1^{(j-1)} \quad +8h_j(1+\lambda^3)\boldsymbol{c}_1^{(j)} \quad +7h_j\boldsymbol{c}_1^{(j+1)} \\
& +2h_j^2\lambda^2\boldsymbol{c}_2^{(j-1)} \quad +3h_j^2(1-\lambda^2)\boldsymbol{c}_2^{(j)} \quad -2h_j^2\boldsymbol{c}_2^{(j+1)}
\end{aligned}\right\} \tag{1.7}$$

for $2(N+1)$ unknowns, $\{\boldsymbol{c}_1^{(0)},\ldots,\boldsymbol{c}_1^{(N)}\}$ and $\{\boldsymbol{c}_2^{(0)},\ldots,\boldsymbol{c}_2^{(N)}\}$. The imposed boundary conditions at the beginning and end of the global spline produce four additional equations for $\{\boldsymbol{c}_1^{(0)}, \boldsymbol{c}_2^{(0)}, \boldsymbol{c}_1^{(N)}, \boldsymbol{c}_2^{(N)}\}$. Note we have implicitly introduced a ghost spline $\boldsymbol{s}^{(N)}$ which satisfies all continuity conditions with $\boldsymbol{s}^{(N-1)}$. The purpose of $\boldsymbol{s}^{(N)}$ is to deal with boundary conditions at the end of spline.

In general there are three types of boundary conditions at each end. Let x, s and $c^{(j)}$ be one of the scalar components of vector $\boldsymbol{x}_j$, spline $\boldsymbol{s}^{(j)}$ and coefficient $\boldsymbol{c}^{(j)}$. We first consider boundary conditions at $l = l_0$,

$$\text{Even:} \quad 0 = \frac{\mathrm{d}s^{(0)}}{\mathrm{d}l} = \frac{\mathrm{d}^3 s^{(0)}}{\mathrm{d}l^3} \qquad \text{at} \quad l = l_0 \tag{1.8}$$

$$\text{Odd:} \quad 0 = s^{(0)} = \frac{\mathrm{d}^2 s^{(0)}}{\mathrm{d}l^2} = \frac{\mathrm{d}^4 s^{(0)}}{\mathrm{d}l^4} \quad \text{at} \quad l = l_0 \tag{1.9}$$

$$\text{Mix:} \quad \alpha = \frac{\mathrm{d}s^{(0)}}{\mathrm{d}l}, \quad \beta = \frac{\mathrm{d}^2 s^{(0)}}{\mathrm{d}l^2} \qquad \text{at} \quad l = l_0 \tag{1.10}$$

which lead to two equations,

$$\left.\begin{aligned} c_1^{(0)} &= 0 \\ 6h_0 c_1^{(0)} + 3h_0^2 c_2^{(0)} + 4h_0 c_1^{(1)} - h_0^2 c_2^{(1)} &= -10x_0 + 10x_1 \end{aligned}\right\} \quad \text{Even} \tag{1.11a}$$

$$\left.\begin{aligned} -8h_0 c_1^{(0)} - 3h_0^2 c_2^{(0)} - 7h_0 c_1^{(1)} + 2h_0^2 c_2^{(1)} &= +15x_0 - 15x_1 \\ c_2^{(0)} &= 0 \end{aligned}\right\} \quad \text{Odd} \tag{1.11b}$$

$$\left.\begin{aligned} c_1^{(0)} &= \alpha \\ c_2^{(0)} &= \beta/2 \end{aligned}\right\} \quad \text{Mix} \tag{1.11c}$$

We then consider boundary conditions at $l = l_N$,

$$\text{Even:} \quad 0 = \frac{\mathrm{d}s^{(N-1)}}{\mathrm{d}l} = \frac{\mathrm{d}^3 s^{(N-1)}}{\mathrm{d}l^3} \qquad \text{at} \quad l = l_N \tag{1.12}$$

$$\text{Odd:} \quad 0 = s^{(N-1)} = \frac{\mathrm{d}^2 s^{(N-1)}}{\mathrm{d}l^2} = \frac{\mathrm{d}^4 s^{(N-1)}}{\mathrm{d}l^4} \quad \text{at} \quad l = l_N \tag{1.13}$$

$$\text{Mix:} \quad \alpha = \frac{\mathrm{d}s^{(N-1)}}{\mathrm{d}l}, \quad \beta = \frac{\mathrm{d}^2 s^{(N-1)}}{\mathrm{d}l^2} \qquad \text{at} \quad l = l_N \tag{1.14}$$

which also lead to two equations,

$$\left.\begin{aligned} c_1^{(N)} &= 0 \\ -4h_{N-1} c_1^{(N-1)} - h_{N-1}^2 c_2^{(N-1)} - 6h_{N-1} c_1^{(N)} + 3h_{N-1}^2 c_2^{(N)} &= 10x_{N-1} - 10x_N \end{aligned}\right\} \quad \text{Even} \tag{1.15a}$$

$$\left.\begin{aligned} -7h_{N-1} c_1^{(N-1)} - 2h_{N-1}^2 c_2^{(N-1)} - 8h_{N-1} c_1^{(N)} + 3h_{N-1}^2 c_2^{(N)} &= 15x_{N-1} - 15x_N \\ c_2^{(N)} &= 0 \end{aligned}\right\} \quad \text{Odd} \tag{1.15b}$$

$$\left.\begin{aligned} c_1^{(N)} &= \alpha \\ c_2^{(N)} &= \beta/2 \end{aligned}\right\} \quad \text{Mix} \tag{1.15c}$$

If we arrange the unknowns into a vector,

$$\left[c_1^{(0)}, c_2^{(0)}, \ldots, c_1^{(j)}, c_2^{(j)}, \ldots, c_1^{(N)}, c_2^{(N)}\right] \tag{1.16}$$

equations (1.7) with one of (1.11) and one of (1.15) result in a $2(N+1)$-by-$2(N+1)$ system of linear equations, which corresponds to a banded diagonal sparse matrix with at most 6 non-zero elements in each row. The rest coefficients $\boldsymbol{c}_3^{(j)}$, $\boldsymbol{c}_4^{(j)}$ and $\boldsymbol{c}_5^{(j)}$ can be reconstruct using equation (1.6).

It is convenient to re-parametrize each local spline with a new variable $t \in [0,1]$,

$$\boldsymbol{s}^{(j)}(t) = \boldsymbol{x}_j + \sum_{k=1}^{5} \boldsymbol{c}_k^{(j)} t^k, \quad \text{for} \quad j = 0, \ldots, N-1 \tag{1.17}$$

by redefining $\boldsymbol{c}_k^{(j)} \to \boldsymbol{c}_k^{(j)} h_j^k$. We can construct Lagrange interpolations along the arc-length of the global spline by the local arc-length L_j and its local fraction ξ,

$$L_j = \int_0^1 \dot{\boldsymbol{s}}^{(j)}\,\mathrm{d}t', \quad \xi(t) = \frac{1}{L_j}\int_0^t \dot{\boldsymbol{s}}^{(j)}\,\mathrm{d}t', \tag{1.18}$$

Convention for normal vector $\boldsymbol{n}$,

$$\boldsymbol{n} = \frac{1}{\|\dot{\boldsymbol{s}}\|}(-\dot{z}, \dot{r}) \tag{1.19}$$

In addition the total curvature 2κ of the surface of revolution obtained by rotating curve $\boldsymbol{s}(t)$ about the z-axis is given by,

$$2\kappa = \frac{\dot{r}\ddot{z} - \dot{z}\ddot{r}}{(\dot{r}^2 + \dot{z}^2)^{\frac{3}{2}}} + \frac{1}{r}\frac{\dot{z}}{\sqrt{\dot{r}^2 + \dot{z}^2}} = 2\frac{\ddot{z}}{\dot{r}^2} \text{ if } r \to 0 \tag{1.20}$$

1.2. *Gaussian quadrature*

Standard m-point Gauss-Legendre quadrature,

$$\int_0^1 f(t)\,\mathrm{d}t \approx \sum_{k=1}^{m} w_k f(x_k), \quad \int_a^b f(t)\,\mathrm{d}t \approx \sum_{k=1}^{m}(b-a) w_k f(a + (b-a)x_k) \tag{1.21}$$

Logarithmic-weighted Gauss quadrature,

$$\int_0^1 f(t)\ln t\,\mathrm{d}t \approx -\sum_{k=1}^{m} w_k^{\log} f(x_k), \quad \int_0^1 f(t)\ln(1-t)\,\mathrm{d}t \approx -\sum_{k=1}^{m} w_k^{\log} f(1-x_k) \tag{1.22}$$

When logarithmic singularity $\tau \in (0,1)$,

$$\begin{aligned}
\int_0^1 f(t)\ln|t-\tau|\,\mathrm{d}t &= \int_0^\tau f(t)\ln(\tau - t)\,\mathrm{d}t + \int_\tau^1 f(t)\ln(t-\tau)\,\mathrm{d}t \\
= \tau \int_0^1 & f(\tau s)\ln(\tau - \tau s)\,\mathrm{d}s + (1-\tau)\int_0^1 f(\tau + (1-\tau)s)\ln((1-\tau)s)\,\mathrm{d}s \\
= \tau \int_0^1 & f(\tau s)\ln(1-s)\,\mathrm{d}s + (1-\tau)\int_0^1 f(\tau + (1-\tau)s)\ln s\,\mathrm{d}s \\
&+ \tau\ln\tau \int_0^1 f(\tau s)\,\mathrm{d}s + (1-\tau)\ln(1-\tau)\int_0^1 f(\tau + (1-\tau)s)\,\mathrm{d}s
\end{aligned}$$

Logarithmic-weighted Gauss quadrature for singularity $\tau \in (0,1)$,

$$\int_0^1 f(t)\ln|t-\tau|\,\mathrm{d}t \approx \begin{cases} \tau\ln\tau\sum_{k=1}^{m} w_k f\left[\tau x_k\right] \\ +(1-\tau)\ln(1-\tau)\sum_{k=1}^{m} w_k f\left[\tau+(1-\tau)x_k\right] \\ -\tau\sum_{k=1}^{m} w_k^{\log} f\left[\tau(1-x_k^{\log})\right] \\ -(1-\tau)\sum_{k=1}^{m} w_k^{\log} f\left[\tau+(1-\tau)x_k^{\log}\right] \end{cases} \tag{1.23}$$

1.3. *Complete elliptic integral*

The complete elliptic integral of the first and second kind,

$$K(m) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-m\sin^2\theta}}, \quad E(m) = \int_0^{\pi/2} \sqrt{1-m\sin^2\theta}\,\mathrm{d}\theta \tag{1.24}$$

We can approximate $K(m)$ and $E(m)$ with

$$K(m) \approx K_P(m) - \ln(1-m)K_Q(m), \quad E(m) \approx E_P(m) - \ln(1-m)E_Q(m) \tag{1.25}$$

where K_P, E_P, K_Q, E_Q are tenth-order polynomials. Associate Legendre polynomial of 1/2 order can be conveniently expressed as,

$$P_{1/2}(x) = \frac{2}{\pi}\left\{2E\left(\frac{1-x}{2}\right) - K\left(\frac{1-x}{2}\right)\right\} \tag{1.26}$$

From the recursion relations,

$$\begin{aligned} (1+x^2)\frac{\mathrm{d}P_\ell(x)}{\mathrm{d}x} &= (\ell+1)xP_\ell(x) - (\ell+1)P_{\ell+1}(x) \\ \frac{\mathrm{d}K}{\mathrm{d}m} &= \frac{E(m)}{2m(1-m)} - \frac{K(m)}{2m} \\ \frac{\mathrm{d}E}{\mathrm{d}m} &= \frac{1}{2m}\left[E(m)-K(m)\right] \end{aligned} \tag{1.27}$$

we can bootstrap higher-order associate Legendre polynomials of $1/2+1$, $1/2+2$, ...

$$P_\ell(x) = c_\ell\left\{P_{\ell,E}(x)E\left(\frac{1-x}{2}\right) + P_{\ell,K}(x)K\left(\frac{1-x}{2}\right)\right\} \tag{1.28}$$

where c_ℓ is some constant. $P_{\ell,E}(x)$ and $P_{\ell,E}(x)$ are polynomials of $(\ell-1/2)$ order whose coefficients are computed symbolically and given in Table 1.

2. Axisymmetric boundary integral

Consider a *field point* $\boldsymbol{\eta}'$ and a *source point* $\boldsymbol{\eta}$ in the axisymmetric coordinate. We introduce the axisymmetric Green's function $G(\boldsymbol{\eta}';\boldsymbol{\eta})$ by integrating out the azimuthal

ℓ	c_ℓ	$P_{\ell,E}(x)$	$P_{\ell,K}(x)$
$1/2$	$2/\pi$	2	-1
$3/2$	$1/3\pi$	$16x$	$-2(4x+1)$
$5/2$	$1/15\pi$	$4\left(32x^2-9\right)$	$-2\left(32x^2+8x-9\right)$
$7/2$	$1/105\pi$	$64x\left(24x^2-13\right)$	$-2\left(384x^3+96x^2-208x-25\right)$
$9/2$	$1/315\pi$	$8192x^4-6528x^2+588$	$-2\left(2048x^4+512x^3-1632x^2-264x+147\right)$

TABLE 1. Coefficients of associate Legendre polynomials of $\ell = 1/2$ family

dependence (Lennon *et al.* 1979),

$$G(\boldsymbol{\eta}';\boldsymbol{\eta}) = \frac{1}{\pi\sqrt{a+b}}K(m) \tag{2.1}$$

$$\frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}} = \frac{1}{\pi r\sqrt{a+b}}\left\{\left[\frac{n_r}{2} + \frac{\boldsymbol{n}\cdot(\boldsymbol{x}'-\boldsymbol{x})}{\|\boldsymbol{x}'-\boldsymbol{x}\|^2}r\right]E(m) - \frac{n_r}{2}K(m)\right\} \tag{2.2}$$

with auxiliary variables defined as,

$$a = r'^2 + r^2 + (z'-z)^2, \quad b = 2r'r, \quad m = \frac{2b}{a+b} \tag{2.3}$$

Note $\boldsymbol{\eta} \to \boldsymbol{\eta}'$ implies $m \to 1$ which leads to a logarithmic blow-up of the Green's function (recall elliptic integral $E(m) \sim \ln(1-m)$ as $m \to 1$).

Consider a closed, simply connected region Ω and its boundary γ. Green's third identity yields the axisymmetric boundary integral equation,

$$\int_\gamma \left\{G(\boldsymbol{\eta}';\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{n}}(\boldsymbol{\eta}) - \phi(\boldsymbol{\eta})\frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial\boldsymbol{n}}(\boldsymbol{\eta})\right\} r\,\mathrm{d}\gamma(\boldsymbol{\eta}) = \begin{cases} \dfrac{\beta(\boldsymbol{\eta}')}{2\pi}\phi(\boldsymbol{\eta}') & \text{if } \boldsymbol{\eta}' \in \gamma \\ \phi(\boldsymbol{\eta}') & \text{if } \boldsymbol{\eta}' \in \Omega \end{cases} \tag{2.4}$$

Given boundary conditions on γ, we first solve the boundary integral equation to completely determine the value of $\partial\phi/\partial\boldsymbol{n}$ and ϕ on γ. Then any field value interior to Ω can be integrated according to the classical form of Green's third identity.

Boundary integral equation (2.4) can be numerically solved by collocation method (Kress 2013). We seek an approximate solution from a finite dimensional subspace by requiring equation (2.4) is satisfied only at a finite number of *collocation points*. Lagrange interpolation along arc-length,

$$\phi(\boldsymbol{\eta}) \approx \sum_e\sum_k p_{e_k}\mathcal{N}_k^{(e)}(\xi), \quad \frac{\partial\phi}{\partial\boldsymbol{n}}(\boldsymbol{\eta}) \approx \sum_e\sum_k q_{e_k}\mathcal{N}_k^{(e)}(\xi) \tag{2.5}$$

where $\mathcal{N}_k$ is the Lagrange basis of the local fractional arc-length variable ξ. We are interested in the first and second order basis which provide sufficient accuracy for our purpose.

$$\text{1st order:} \quad \mathcal{N}_0 = 1-\xi, \qquad \mathcal{N}_1 = \xi \tag{2.6}$$

$$\text{2nd order:} \quad \mathcal{N}_0 = (1-\xi)(1-2\xi), \qquad \mathcal{N}_1 = 4\xi(1-\xi), \qquad \mathcal{N}_2 = \xi(2\xi-1) \tag{2.7}$$

2.1. *Single layer integral*

Single-layer integral reduces to a summation of local integrals over local arc γ_e of element e,

$$\int_\gamma G(\boldsymbol{\eta}';\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{n}}(\boldsymbol{\eta})r\,\mathrm{d}\gamma(\boldsymbol{\eta}) \approx \sum_e\sum_k q_{e_k}\int_{\gamma_e} G(\boldsymbol{\eta}';\boldsymbol{\eta})\mathcal{N}_k^{(e)}(\xi)r\,\mathrm{d}\gamma(\boldsymbol{\eta})$$
$$= \sum_e\sum_k q_{e_k}\int_0^1 \frac{rJ}{\pi\sqrt{a+b}}\mathcal{N}_k^{(e)}(\xi)K(m)\,\mathrm{d}t \tag{2.8}$$

When singularity $\boldsymbol{\eta}'\in\gamma_e$ is located at $t=\tau\in[0,1]$,

$$\int_0^1 \frac{rJ}{\pi\sqrt{a+b}}\mathcal{N}_k^{(e)}K(m)\,\mathrm{d}t = \int_0^1 \frac{rJ}{\pi\sqrt{a+b}}\mathcal{N}_k^{(e)}\left[P_K - Q_K\ln\frac{1-m}{(t-\tau)^2} - 2Q_K\ln|t-\tau|\right]\mathrm{d}t$$

Introduce helper functions,

$$f_K^{\text{single}}(t) = \frac{rJ}{\pi\sqrt{a+b}} \tag{2.9}$$
$$R_{K/E}(m,\tau) = P_{K/E}(m) - Q_{K/E}(m)\ln\frac{1-m}{(t-\tau)^2} \tag{2.10}$$

If we define integrand $I^{(o)}$ for source location $\boldsymbol{\eta}'$ outside of γ_e and $I^{(i)}$ for $\boldsymbol{\eta}'$ interior to γ_e,

$$I^{(o)}(t) = f_K^{\text{single}}(t)K(m)\mathcal{N}_k^{(e)}(\xi) \tag{2.11}$$
$$I^{(i)}(t,\tau) = f_K^{\text{single}}(t)R_K(m,\tau)\mathcal{N}_k^{(e)}(\xi) \tag{2.12}$$
$$I_{\log}^{(i)}(t) = -2f_K^{\text{single}}(t)Q_K(m)\mathcal{N}_k^{(e)}(\xi) \tag{2.13}$$

then the local single-layer integral can be compactly represented with a regular part and a logarithmic-singular one,

$$\boldsymbol{\eta}'\notin\gamma_e: \text{ compute} \int_0^1 I^{(o)}(t)\,\mathrm{d}t \tag{2.14}$$
$$\boldsymbol{\eta}' \text{ at } t=0: \text{ compute} \int_0^1 I^{(i)}(t,0)\,\mathrm{d}t + \int_0^1 I_{\log}^{(i)}(t)\ln t\,\mathrm{d}t \tag{2.15}$$
$$\boldsymbol{\eta}' \text{ at } t=\tau: \text{ compute} \int_0^1 I^{(i)}(t,\tau)\,\mathrm{d}t + \int_0^1 I_{\log}^{(i)}(t)\ln|\tau-t|\,\mathrm{d}t \tag{2.16}$$
$$\boldsymbol{\eta}' \text{ at } t=1: \text{ compute} \int_0^1 I^{(i)}(t,1)\,\mathrm{d}t + \int_0^1 I_{\log}^{(i)}(t)\ln(1-t)\,\mathrm{d}t \tag{2.17}$$

Element order	o	
Number of elements	n	
Number of nodes	$n \times o + 1$	
Nodal index of source point	i	$0 \leqslant i \leqslant n \times o$
Elemental index of receiving point	j	$0 \leqslant j \leqslant n - 1$
Local node index	k	$0 \leqslant k \leqslant o$
Indices of elements $\ni$ i-th node if i mod $o = 0$	$\lfloor i/o \rfloor - 1,\ \lfloor i/o \rfloor$	
Indices of elements $\ni$ i-th node if i mod $o \neq 0$	$\lfloor i/o \rfloor - 1$	
Nodal indices of j-th element	$j \times o + k$	$0 \leqslant k \leqslant o$

TABLE 2. Index involved

2.2. *Double layer integral*

Similar to singular-layer integral, double-layer integral reduces to a summation of local integrals over local arc γ_e of element e,

$$\int_\gamma \frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}} \phi(\boldsymbol{\eta}) r \,\mathrm{d}\gamma(\boldsymbol{\eta}) \approx \sum_e \sum_k p_{e_k} \int_{\gamma_e} \frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}} \mathcal{N}_k^{(e)}(\xi) r \,\mathrm{d}\gamma(\boldsymbol{\eta})$$
$$= \sum_e \sum_k p_{e_k} \int_0^1 \mathcal{N}_k^{(e)}(\xi) \frac{J}{\pi\sqrt{a+b}} \left\{ \left[\frac{n_r}{2} + \frac{\boldsymbol{n}\cdot(\boldsymbol{x}'-\boldsymbol{x})}{\|\boldsymbol{x}'-\boldsymbol{x}\|^2} r \right] E(m) - \frac{n_r}{2} K(m) \right\} \mathrm{d}t$$
$$= \sum_e \sum_k p_{e_k} \int_0^1 \frac{\mathcal{N}_k^{(e)}(\xi)}{\pi\sqrt{a+b}} \left\{ \left[\frac{\dot{r}(z'-z) - \dot{z}(r'-r)}{(r'-r)^2 + (z'-z)^2} r - \frac{\dot{z}}{2} \right] E(m) + \frac{\dot{z}}{2} K(m) \right\} \mathrm{d}t \quad (2.18)$$

Technically we don't need to treat integrand involving elliptic integral of the second kind $E(m)$. However, convergence rate of standard Gauss-Legendre quadrature depends magnitude of derivatives.

$$f_E^{\text{double}} = \frac{1}{\pi\sqrt{a+b}} \left[\frac{\dot{r}(z'-z) - \dot{z}(r'-r)}{a-b} r - \frac{\dot{z}}{2} \right], \quad f_K^{\text{double}} = \frac{1}{\pi\sqrt{a+b}} \frac{\dot{z}}{2} \quad (2.19)$$

If we define integrand $I^{(o)}$ for source location $\boldsymbol{\eta}'$ outside of γ_e and $I^{(i)}$ for $\boldsymbol{\eta}'$ interior to γ_e,

$$I^{(o)}(t) = \left[f_E^{\text{double}} E(m) + f_K^{\text{double}} K(m) \right] \mathcal{N}_k^{(e)}(\xi) \quad (2.20)$$
$$I^{(i)}(t,\tau) = \left[f_E^{\text{double}} R_E(m,\tau) + f_K^{\text{double}} R_K(m,\tau) \right] \mathcal{N}_k^{(e)}(\xi) \quad (2.21)$$
$$I_{\log}^{(i)}(t) = -2 f_E^{\text{double}} Q_E(m) - 2 f_K^{\text{double}} Q_K(m) \quad (2.22)$$

When $\boldsymbol{\eta}'$ is located the symmetry axis where $r' = 0$, we can simplify the elliptic integrals,

$$\int_\gamma G(\boldsymbol{\eta}';\boldsymbol{\eta}) \frac{\partial \phi}{\partial \boldsymbol{n}}(\boldsymbol{\eta}) r \,\mathrm{d}\gamma(\boldsymbol{\eta}) \approx \sum_e \sum_k q_{e_k} \int_0^1 \mathcal{N}_k^{(e)}(\xi) r \frac{J}{2\sqrt{r + (z-z')^2}} \,\mathrm{d}t \quad (2.23)$$
$$\int_\gamma \frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}} \phi(\boldsymbol{\eta}) r \,\mathrm{d}\gamma(\boldsymbol{\eta}) \approx \sum_e \sum_k p_{e_k} \int_0^1 \mathcal{N}_k^{(e)}(\xi) r \frac{\dot{z} r + \dot{r}(z-z')}{2\left[r^2 + (z'-z)^2\right]^{3/2}} \,\mathrm{d}t \quad (2.24)$$

Standard Gauss-Legendre quadrature is sufficient to evaluate the above integral since the integrands have well-defined limit as $\boldsymbol{\eta} \to \boldsymbol{\eta}'$ assuming $\ddot{z}/\dot{r}$ is finite at $r = 0$.

Algorithm 1 Counting mismatches between two packed strings

function DISTANCE(x, e)
  for $0 \leqslant i \leqslant N_x - 1$ **do** # We can parallelize this loop
    if $i \bmod o = 0$ **then**
      $I_{\text{lower}} = I_{\text{upper}}$
    else
      $I_{\text{lower}} = \lfloor i/o \rfloor - 1, \quad I_{\text{upper}} = \lfloor i/o \rfloor$
    end if
    for $0 \leqslant i \leqslant N_x - 1$ **do**
    end for
  end for
end function

2.3. *Matrix assembly*

The collocation procedure for a single C^1 continuous boundary γ leads to a matrix equation,

$$(\boldsymbol{B} + \boldsymbol{D})\boldsymbol{p} = \boldsymbol{H}\boldsymbol{p} = \boldsymbol{S}\boldsymbol{q} \tag{2.25}$$

Consider two adjacent boundaries γ_0 and γ_1. The assembly process can be decomposed into four parts: $\{\boldsymbol{\eta}' \in \gamma_0, \boldsymbol{\eta}' \in \gamma_1\} \otimes \{\boldsymbol{\eta} \in \gamma_0, \boldsymbol{\eta} \in \gamma_1\}$ which leads to a block matrix equation,

$$\begin{bmatrix} \boldsymbol{H}_{00} & \boldsymbol{H}_{01} \\ \boldsymbol{H}_{10} & \boldsymbol{H}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_{00} & \boldsymbol{S}_{01} \\ \boldsymbol{S}_{10} & \boldsymbol{S}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_0 \\ \boldsymbol{q}_1 \end{bmatrix} \tag{2.26}$$

We have two identical rows in $\boldsymbol{H}$ and $\boldsymbol{S}$ and extra equation for continuity of potential,

$$\left.\begin{aligned} \mathrm{row}_{-1}(\boldsymbol{H}_{00}) \cdot \boldsymbol{p}_0 + \mathrm{row}_{-1}(\boldsymbol{H}_{01}) \cdot \boldsymbol{p}_1 &= \mathrm{row}_{-1}(\boldsymbol{S}_{00}) \cdot \boldsymbol{q}_0 + \mathrm{row}_{-1}(\boldsymbol{S}_{01}) \cdot \boldsymbol{q}_1 \\ \mathrm{row}_0(\boldsymbol{H}_{10}) \cdot \boldsymbol{p}_0 + \mathrm{row}_0(\boldsymbol{H}_{11}) \cdot \boldsymbol{p}_1 &= \mathrm{row}_0(\boldsymbol{S}_{10}) \cdot \boldsymbol{q}_0 + \mathrm{row}_0(\boldsymbol{S}_{11}) \cdot \boldsymbol{q}_1 \\ (p_0)_{-1} - (p_1)_0 &= 0 \end{aligned}\right\} \tag{2.27}$$

We simply replace one of identical rows with the continuity condition,

$$\begin{aligned} \mathrm{row}_{-1}(\boldsymbol{H}_{00}) &= [0, \cdots, 0, 1], \quad \mathrm{row}_{-1}(\boldsymbol{H}_{01}) = [-1, 0, \cdots, 0] \\ \mathrm{row}_{-1}(\boldsymbol{S}_{00}) &= [0, \cdots, 0, 0], \quad \mathrm{row}_{-1}(\boldsymbol{S}_{01}) = [0, 0, \cdots, 0] \end{aligned} \tag{2.28}$$

For mixed-type boundary condition, i.e., Dirichlet and Neumann on different segments of the boundary, we simply rearrange the block matrices,

$$\begin{bmatrix} \boldsymbol{H}_{00} & -\boldsymbol{S}_{01} \\ \boldsymbol{H}_{10} & -\boldsymbol{S}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{q}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_{00} & -\boldsymbol{H}_{01} \\ \boldsymbol{S}_{10} & -\boldsymbol{H}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_0 \\ \boldsymbol{p}_1 \end{bmatrix} \tag{2.29}$$

Note the continuity between discrete potential vectors $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ is still implied. What cannot be achieved is to prescribe identical value of potential both before and after the transition point. In this case, continuity of potential can no longer provide an extra equation.

2.4. *Benchmark*

We validate our implementation of boundary integral solver for test problems posed on a smooth boundary γ and on a piecewise smooth boundary $\gamma_0 \cup \gamma_1$. The latter contains a geometric discontinuity of a 90° corner at the transition point between γ_0 and γ_1.

Consider the following parametrisation of the boundaries,

$$\gamma_0 = \left\{ 2\Big(1 + \frac{1}{4}\cos(8\theta - \pi)\Big) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \,\Big|\, \theta \in [0, \pi/2] \right\} \tag{2.30}$$

$$\gamma_1 = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} \,\Big|\, \theta \in [0, 3/2] \right\} \tag{2.31}$$

The general form of axisymmetric harmonic potential can be constructed from Legendre polynomial P_ℓ of order ℓ. We consider the interior problem for the potential ϕ,

$$\phi = (r^2 + z^2) P_2\Big(\frac{z}{\sqrt{r^2+z^2}}\Big), \quad \frac{\partial\phi}{\partial \boldsymbol{n}} = \boldsymbol{n}\cdot(-r, 2z) \tag{2.32}$$

If we prescribe the potential value of ϕ on γ_0 and its normal flux $\partial\phi/\partial\boldsymbol{n}$ on γ_1, our solver is expected to reproduce $\partial\phi/\partial\boldsymbol{n}$ on γ_0 and ϕ on γ_1,

$$\text{Test problem:} \quad \text{given} \left\{ \begin{array}{r} \phi \text{ on } \gamma_0 \\ \partial\phi/\partial\boldsymbol{n} \text{ on } \gamma_1 \end{array} \right., \ \text{find} \left\{ \begin{array}{r} \partial\phi/\partial\boldsymbol{n} \text{ on } \gamma_0 \\ \phi \text{ on } \gamma_1 \end{array} \right. \tag{2.33}$$

In addition for a plane curve given parametrically as $\gamma(t) = (r(t), z(t))$, the total curvature 2κ of the surface of revolution obtained by rotating curve γ about the z-axis is given by,

$$2\kappa = \frac{\dot{r}\ddot{z} - \dot{z}\ddot{r}}{(\dot{r}^2 + \dot{z}^2)^{\frac{3}{2}}} + \frac{1}{r}\frac{\dot{z}}{\sqrt{\dot{r}^2 + \dot{z}^2}} \tag{2.34}$$

We verify the accuracy of quintic spline interpolation against the analytic form derived from the parametrisation (2.32). All errors between numerical and analytical solutions are measured in the l^∞-norm,

$$\|\mathrm{err_d}\|_\infty = \max_{\boldsymbol{x}_i \in \gamma_1} |p_i^{\mathrm{num}} - \phi(\boldsymbol{x}_i)| \tag{2.35}$$

$$\|\mathrm{err_n}\|_\infty = \max_{\boldsymbol{x}_i \in \gamma_0} |q_i^{\mathrm{num}} - \partial\phi/\partial\boldsymbol{n}(\boldsymbol{x}_i)| \tag{2.36}$$

$$\|\mathrm{err_c}\|_\infty = \max_{\boldsymbol{x}_i \in \gamma_0} |2\kappa_i^{\mathrm{spline}} - 2\kappa(\boldsymbol{x}_i)| \tag{2.37}$$

In figure 1(a) and 1(b) we plot these errors against the total number of degree of freedom (DOF) used by the solver.

3. Self-similar Taylor cone

3.1. *Preparation for iteration*

In the last section we outline the numerical procedure to solve Laplace equation subject to mixed boundary conditions. Initial guess is a C^3 continuous function f_{guess},

$$f_{\mathrm{guess}}(r) = \begin{cases} f_0 r + f_2 r^2 + f_4 r^4 + f_6 r^6, & \text{if } r \leqslant r_c \\ c_0 r + \dfrac{c_1}{\sqrt{r}} + \dfrac{c_3}{r^{7/2}} + \dfrac{c_4}{r^5}, & \text{if } r > r_c \end{cases} \tag{3.1}$$

where r_c is the connection point where continuities up to the third derivative are enforced. Given a reasonable c_1, function $f_{\mathrm{guess}}(r)$ agrees with analytic prediction in the far-field. The effect of varying r_c and c_1 is illustrated in figure 2.

We clamp the spline function, its tangent direction and total curvature at truncation point r_*

$$\dot{z} = \dot{r}\frac{\mathrm{d}f}{\mathrm{d}r}, \quad \ddot{z} = \frac{\ddot{r}\dot{z} + 2\kappa(\dot{r}^2 + \dot{z}^2)^{3/2}}{\dot{r}} - \frac{\dot{z}\left(\dot{r}^2 + \dot{z}^2\right)}{\dot{r}r} \tag{3.2}$$

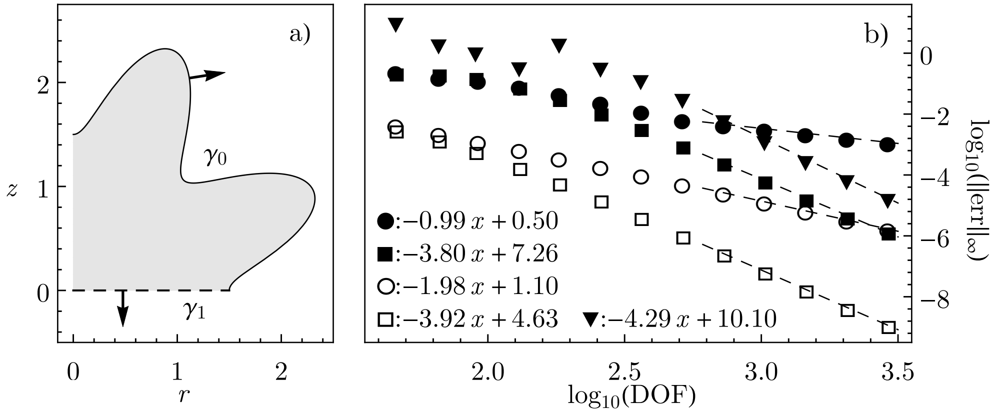

FIGURE 1. Error convergence between numerical and analytic solutions measured in l^∞-norm against degrees of freedom (DOF) used for the test problem (2.33): (a) Boundaries γ_0 (solid) and γ_1 (dashed) given by parametrisation (2.32). (b) Error convergence: curvature error (2.37) (▼) on γ_0; Neumann error (2.36) on γ_0 with linear (-) and quadratic (■) shape functions; Dirichlet error (2.35) on γ_1 with linear (#) and quadratic (2) shape functions. Inset: linear fit (dashed line) of the last five points of each error.

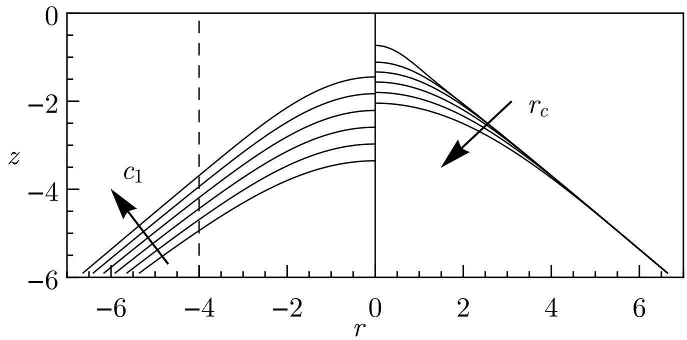

FIGURE 2. C^3 function $f_{\text{guess}}(r)$: increasing r_c (right) and c_1 (left)

Forward, central and backward finite difference approximation to first derivative with three data points $\{(x_i, f_i)\}_{i=0,1,2}$,

$$f'(x_0) \approx -f_0\left(\frac{1}{h_0+h_1}+\frac{1}{h_0}\right)+f_1\left(\frac{1}{h_0}+\frac{1}{h_1}\right)-f_2\left(\frac{1}{h_1}-\frac{1}{h_0+h_1}\right) \tag{3.3}$$

$$f'(x_1) \approx +f_0\left(\frac{1}{h_0+h_1}-\frac{1}{h_0}\right)+f_1\left(\frac{1}{h_0}-\frac{1}{h_1}\right)+f_2\left(\frac{1}{h_1}-\frac{1}{h_0+h_1}\right) \tag{3.4}$$

$$f'(x_2) \approx -f_0\left(\frac{1}{h_0+h_1}-\frac{1}{h_0}\right)-f_1\left(\frac{1}{h_0}+\frac{1}{h_1}\right)+f_2\left(\frac{1}{h_0+h_1}+\frac{1}{h_1}\right) \tag{3.5}$$

$$f''(x_0) \approx f''(x_1) \approx f''(x_2) \approx \frac{2}{h_0(h_0+h_1)}f_0-\frac{2}{h_0h_1}f_1+\frac{2}{h_1(h_0+h_1)}f_2 \tag{3.6}$$

where $h_i = x_{i+1} - x_i$. For curvature purpose, we only list the forward and backward finite difference approximation to the second derivative with four-point stencil,

$$
\begin{aligned}
f''(x_3) \approx \quad & -\frac{2(h_1+2h_2)}{h_0(h_0+h_1)(h_0+h_1+h_2)} f_0 \qquad + \frac{2(h_0+h_1+2h_2)}{h_0 h_1(h_1+h_2)} f_1 \\
& - \frac{2(h_0+2(h_1+h_2))}{h_1 h_2(h_0+h_1)} f_2 \quad + \frac{2(h_0+2h_1+3h_2)}{h_2(h_1+h_2)(h_0+h_1+h_2)} f_3
\end{aligned} \tag{3.7}
$$

Thus we use backward finite difference method to approximate the first and second derivatives of r-component of the spline at the truncation point r_*. With prescribed slope and curvature, we can compute the first and second derivatives of z-component.

Gradient of axisymmetric harmonic function,

$$\frac{\partial}{\partial r} R^\ell P_\ell(\cos\theta) = \frac{\ell r^2 - (\ell+1)z^2}{r} R^{\ell-2} P_\ell\left(\frac{z}{R}\right) + \frac{(\ell+1)z}{r} R^{\ell-1} P_{\ell+1}\left(\frac{z}{R}\right) \tag{3.8}$$

$$\frac{\partial}{\partial z} R^\ell P_\ell(\cos\theta) = (2\ell+1) z R^{\ell-2} P_\ell\left(\frac{z}{R}\right) - (\ell+1) R^{\ell-1} P_{\ell+1}\left(\frac{z}{R}\right) \tag{3.9}$$

$$\frac{\partial}{\partial r} R^{-\ell-1} P_\ell(\cos\theta) = \frac{(\ell+1)z}{r} R^{-\ell-2} P_{\ell+1}\left(\frac{z}{R}\right) - \frac{\ell+1}{r} R^{-\ell-1} P_\ell\left(\frac{z}{R}\right) \tag{3.10}$$

$$\frac{\partial}{\partial z} R^{-\ell-1} P_\ell(\cos\theta) = -(\ell+1) R^{-\ell-2} P_{\ell+1}\left(\frac{z}{R}\right) \tag{3.11}$$

Note $\partial/\partial r$ of these axisymmetric harmonic functions at $z = 0$ is well-defined and always zero due to symmetry.

3.2. *Newton-Raphson iteration*

We look for possible root to the self-similar Bernoulli's equation,

$$\mathcal{B}(\boldsymbol{\eta}) = \frac{2}{3}\boldsymbol{\eta}\cdot\nabla\phi - \frac{1}{3}\phi + \frac{1}{2}|\nabla\phi|^2 - 2\kappa - \frac{1}{2}|\nabla\psi|^2 \tag{3.12}$$

subject to the boundary conditions,

$$\frac{2}{3}\boldsymbol{n}\cdot\boldsymbol{\eta} + \frac{\partial\phi}{\partial\boldsymbol{n}} = 0, \quad \psi = 0 \quad \text{on} \quad \gamma_0 \tag{3.13}$$

We rewrite $\mathcal{B}$ in terms of position $\boldsymbol{\eta}$ and velocity potential ϕ on γ_0,

$$
\begin{aligned}
\mathcal{B}(\boldsymbol{\eta}) &= \frac{2}{3}(\boldsymbol{n}\cdot\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{n}} + \frac{2}{3}(\boldsymbol{s}\cdot\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{s}} - \frac{1}{3}\phi + \frac{1}{2}\left(\frac{\partial\phi}{\partial\boldsymbol{n}}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial\boldsymbol{s}}\right)^2 - 2\kappa \\
&= \frac{1}{2}\left(\frac{\partial\phi}{\partial\boldsymbol{s}}\right)^2 + \frac{2}{3}(\boldsymbol{s}\cdot\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{s}} - \frac{1}{3}\phi - \frac{2}{9}(\boldsymbol{n}\cdot\boldsymbol{\eta})^2 - 2\kappa - \frac{1}{2}\left(\frac{\partial\psi}{\partial\boldsymbol{n}}\right)^2 = 0
\end{aligned} \tag{3.14}
$$

This problem is nonlinear in velocity field which is normal for inertia-dominate flow but more importantly nonlinear in the geometry.

4. Results and discussion

4.1. *Burton & Taborek*

We first recover the self-similar cone formation at the end of an inviscid drop reported by Burton & Taborek (2011).

REFERENCES

Burton, J. C. & Taborek, P. 2011 Simulations of coulombic fission of charged inviscid drops. *Phys. Rev. Lett.* **106**, 144501.

Algorithm 2 Nonlinear iteration

function NEWTON ITERATION(c_1, a_0, r_*) ▷ truncation distance r_*
    compute $c_0, \ldots, c_4$ ▷ $c_2 = 0$
    initialize guess shape γ_0 ▷ fix 1st and 2nd derivatives at the end
    initialize fluid and vacuum patches, γ_1 and γ_2
    while error($\boldsymbol{\mathcal{B}}$) $<$ tolerance **do** ▷ l^∞ error norm
        for all knots $\boldsymbol{\eta}_i$ of spline γ_0 **do** ▷ exclude ending knot
            perturb $\boldsymbol{\eta}_i \to \boldsymbol{\eta}_i + \epsilon \boldsymbol{n}_i$
            compute ending derivatives $\dot{r}, \ddot{r}, \dot{z}, \ddot{z}$
            initialize perturbed spline $\underline{\gamma}_0$
            initialize perturbed Neumann condition on $\underline{\gamma}_0$
            compute solution $\underline{\phi}$ on $\underline{\gamma}_0$
            compute perturbed residue vector $\underline{\boldsymbol{\mathcal{B}}}$ ▷ for knots only
            $\mathrm{col}_i(\boldsymbol{J}) = (\underline{\boldsymbol{\mathcal{B}}} - \boldsymbol{\mathcal{B}})/\epsilon$
        end for
        solve $\boldsymbol{J}\,\delta\boldsymbol{n} = -\boldsymbol{\mathcal{B}}$ ▷ $\dim(\boldsymbol{J}) = \#_{\mathrm{node}} \times (\#_{\mathrm{knot}} - 1)$
        update $\boldsymbol{\eta}_i \to \boldsymbol{\eta}_i + \alpha\delta\boldsymbol{n}_i$ with damping rate α
        compute new residue vector $\boldsymbol{\mathcal{B}}$
    end while
end function

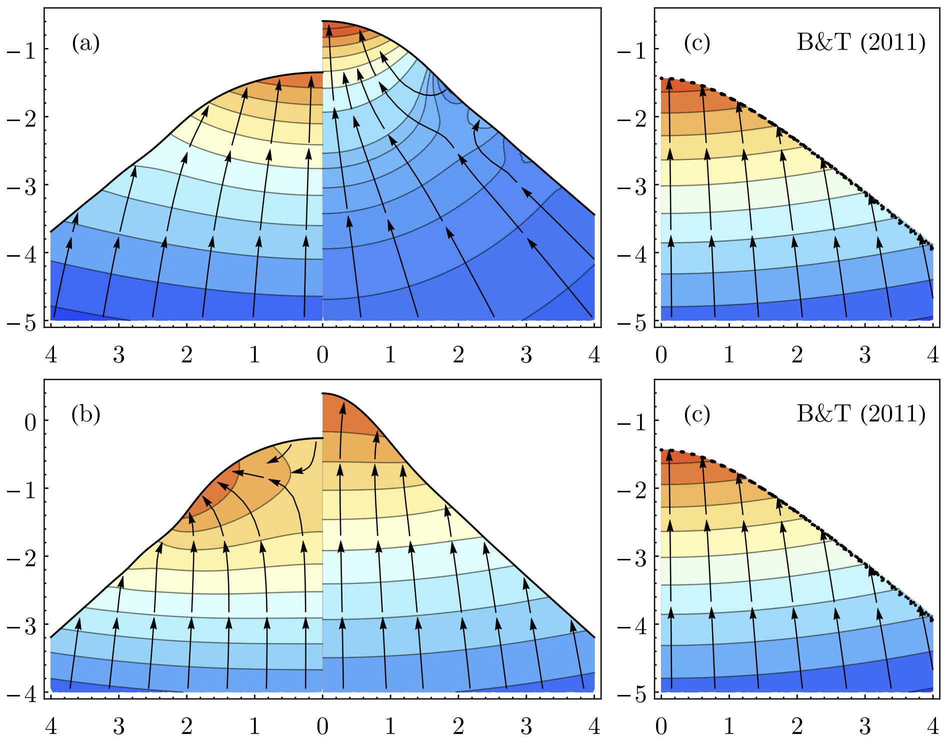

FIGURE 3. Burton & Taborek (2011) ??

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