Dynamic Taylor Cone Part II: Numeric Scheme

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Numeric scheme (Taylor 1964)

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1. Quintic spline

Given N marker points $\{\boldsymbol{x}_0, \dots, \boldsymbol{x}_N\}$, we first compute $(N-1)$ chord lengths $\{h_0, \dots, h_{N-1}\}$ by euclidean norm between adjacent marker points,

$$h_j = \|\boldsymbol{x}_{j+1} - \boldsymbol{x}_j\| \quad \text{for} \quad j = 0, \dots, N-1 \tag{1.1}$$

Then we introduce N chord coordinates $\{l_0, \dots, l_N\}$ with $l_0 = 0$ and

$$l_j = \sum_{k=0}^{j-1} h_k \quad \text{forb} \quad j = 1, \dots, N \tag{1.2}$$

Global spline is a collection of local splines $\{\boldsymbol{s}^{(0)}, \dots, \boldsymbol{s}^{(N-1)}\}$ where each $\boldsymbol{s}^{(j)}$ is a fifth degree polynomial defined on the interval $l \in [l_j, l_{j+1}]$,

$$\boldsymbol{s}^{(j)}(l) = \boldsymbol{x}_j + \sum_{k=1}^{5} \boldsymbol{c}_k^{(j)} (l - l_j)^k, \quad \text{for} \quad j = 0, \dots, N-1 \tag{1.3}$$

Continuity between neighboring splines is satisfied up to the fourth derivative,

$$\left.\frac{\mathrm{d}^p \boldsymbol{s}^{(j)}}{\mathrm{d}l^p}\right|_{l_{j+1}} = \left.\frac{\mathrm{d}^p \boldsymbol{s}^{(j+1)}}{\mathrm{d}l^p}\right|_{l_{j+1}}, \quad \text{for} \quad p = 0, \dots, 4 \tag{1.4}$$

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The continuity conditions yield five equations

$$\left.\begin{aligned}
\boldsymbol{c}_3^{(j-1)}h_{j-1}^3+\boldsymbol{c}_4^{(j-1)}h_{j-1}^4+\boldsymbol{c}_5^{(j-1)}h_{j-1}^5 &= \boldsymbol{x}_j-\boldsymbol{x}_{j-1}-\boldsymbol{c}_1^{(j-1)}h_{j-1}-\boldsymbol{c}_2^{(j-1)}h_{j-1}^2\\
3\boldsymbol{c}_3^{(j-1)}h_{j-1}^2+4\boldsymbol{c}_4^{(j-1)}h_{j-1}^3+5\boldsymbol{c}_5^{(j-1)}h_{j-1}^4 &= -\boldsymbol{c}_1^{(j-1)}-2\boldsymbol{c}_2^{(j-1)}h_{j-1}+\boldsymbol{c}_1^{(j)}\\
6\boldsymbol{c}_3^{(j-1)}h_{j-1}+12\boldsymbol{c}_4^{(j-1)}h_{j-1}^2+20\boldsymbol{c}_5^{(j-1)}h_{j-1}^3 &= -2\boldsymbol{c}_2^{(j-1)}+2\boldsymbol{c}_2^{(j)}\\
\boldsymbol{c}_3^{(j-1)}+4h_{j-1}\boldsymbol{c}_4^{(j-1)}+10h_{j-1}^2\boldsymbol{c}_5^{(j-1)} &= \boldsymbol{c}_3^{(j)}\\
\boldsymbol{c}_4^{(j-1)}+5h_{j-1}\boldsymbol{c}_5^{(j-1)} &= \boldsymbol{c}_4^{(j)}
\end{aligned}\right\} \tag{1.5}$$

We first solve for $\boldsymbol{c}_3^{(j-1)}$, $\boldsymbol{c}_4^{(j-1)}$ and $\boldsymbol{c}_5^{(j-1)}$,

$$\left.\begin{array}{lcccccc}
h_{j-1}^3\boldsymbol{c}_3^{(j-1)} &=& -6h_{j-1}\boldsymbol{c}_1^{(j-1)} & -3h_{j-1}^2\boldsymbol{c}_2^{(j-1)} & -4h_{j-1}\boldsymbol{c}_1^{(j)} & +h_{j-1}^2\boldsymbol{c}_2^{(j)} & +10(\boldsymbol{x}_j-\boldsymbol{x}_{j-1})\\
h_{j-1}^4\boldsymbol{c}_4^{(j-1)} &=& +8h_{j-1}\boldsymbol{c}_1^{(j-1)} & +3h_{j-1}^2\boldsymbol{c}_2^{(j-1)} & +7h_{j-1}\boldsymbol{c}_1^{(j)} & -2h_{j-1}^2\boldsymbol{c}_2^{(j)} & -15(\boldsymbol{x}_j-\boldsymbol{x}_{j-1})\\
h_{j-1}^5\boldsymbol{c}_5^{(j-1)} &=& -3h_{j-1}\boldsymbol{c}_1^{(j-1)} & -h_{j-1}^2\boldsymbol{c}_2^{(j-1)} & -3h_{j-1}\boldsymbol{c}_1^{(j)} & +h_{j-1}^2\boldsymbol{c}_2^{(j)} & +6(\boldsymbol{x}_j-\boldsymbol{x}_{j-1})
\end{array}\right\} \tag{1.6}$$

Define $\lambda = h_j/h_{j-1}$. We have $2(N-1)$ equations for $j=1,\dots,N-1$,

$$\left.\begin{array}{rcccc}
10\left[+\lambda^3\boldsymbol{x}_{j-1}-(1+\lambda^3)\boldsymbol{x}_j+\boldsymbol{x}_{j+1}\right]= & 4h_j\lambda^2\boldsymbol{c}_1^{(j-1)} & +6h_j(\lambda^2-1)\boldsymbol{c}_1^{(j)} & -4h_j\boldsymbol{c}_1^{(j+1)}\\
 & +h_j^2\lambda\boldsymbol{c}_2^{(j-1)} & -3h_j^2(1+\lambda)\boldsymbol{c}_2^{(j)} & +h_j^2\boldsymbol{c}_2^{(j+1)}\\
15\left[-\lambda^4\boldsymbol{x}_{j-1}+(\lambda^4-1)\boldsymbol{x}_j+\boldsymbol{x}_{j+1}\right]= & 7h_j\lambda^3\boldsymbol{c}_1^{(j-1)} & +8h_j(1+\lambda^3)\boldsymbol{c}_1^{(j)} & +7h_j\boldsymbol{c}_1^{(j+1)}\\
 & +2h_j^2\lambda^2\boldsymbol{c}_2^{(j-1)} & +3h_j^2(1-\lambda^2)\boldsymbol{c}_2^{(j)} & -2h_j^2\boldsymbol{c}_2^{(j+1)}
\end{array}\right\} \tag{1.7}$$

for $2(N+1)$ unknowns, $\{\boldsymbol{c}_1^{(0)},\dots,\boldsymbol{c}_1^{(N)}\}$ and $\{\boldsymbol{c}_2^{(0)},\dots,\boldsymbol{c}_2^{(N)}\}$. The imposed boundary conditions at the beginning and end of the global spline produce four additional equations for $\{\boldsymbol{c}_1^{(0)},\boldsymbol{c}_2^{(0)},\boldsymbol{c}_1^{(N)},\boldsymbol{c}_2^{(N)}\}$. Note we have implicitly introduced a ghost spline $\boldsymbol{s}^{(N)}$ which satisfies all continuity conditions with $\boldsymbol{s}^{(N-1)}$. The purpose of $\boldsymbol{s}^{(N)}$ is to deal with boundary conditions at the end of spline.

In general there are three types of boundary conditions at each end. Let x, s and $c^{(j)}$ be one of the scalar components of vector $\boldsymbol{x}_j$, spline $\boldsymbol{s}^{(j)}$ and coefficient $\boldsymbol{c}^{(j)}$. We first consider boundary conditions at $l=l_0$,

$$\text{Even:}\quad 0=\frac{\mathrm{d}s^{(0)}}{\mathrm{d}l}=\frac{\mathrm{d}^3s^{(0)}}{\mathrm{d}l^3}\qquad \text{at}\quad l=l_0 \tag{1.8}$$

$$\text{Odd:}\quad 0=s^{(0)}=\frac{\mathrm{d}^2s^{(0)}}{\mathrm{d}l^2}=\frac{\mathrm{d}^4s^{(0)}}{\mathrm{d}l^4}\qquad \text{at}\quad l=l_0 \tag{1.9}$$

$$\text{Mix:}\quad \alpha=\frac{\mathrm{d}s^{(0)}}{\mathrm{d}l},\quad \beta=\frac{\mathrm{d}^2s^{(0)}}{\mathrm{d}l^2}\qquad \text{at}\quad l=l_0 \tag{1.10}$$

which lead to two equations,

$$\left.\begin{aligned} c_1^{(0)} &= 0 \\ 6h_0c_1^{(0)} + 3h_0^2c_2^{(0)} + 4h_0c_1^{(1)} - h_0^2c_2^{(1)} &= -10x_0 + 10x_1 \end{aligned}\right\} \quad \text{Even} \tag{1.11a}$$

$$\left.\begin{aligned} -8h_0c_1^{(0)} - 3h_0^2c_2^{(0)} - 7h_0c_1^{(1)} + 2h_0^2c_2^{(1)} &= +15x_0 - 15x_1 \\ c_2^{(0)} &= 0 \end{aligned}\right\} \quad \text{Odd} \tag{1.11b}$$

$$\left.\begin{aligned} c_1^{(0)} &= \alpha \\ c_2^{(0)} &= \beta/2 \end{aligned}\right\} \quad \text{Mix} \tag{1.11c}$$

We then consider boundary conditions at $l = l_N$,

$$\text{Even:} \quad 0 = \frac{\mathrm{d}s^{(N-1)}}{\mathrm{d}l} = \frac{\mathrm{d}^3s^{(N-1)}}{\mathrm{d}l^3} \qquad \text{at} \quad l = l_N \tag{1.12}$$

$$\text{Odd:} \quad 0 = s^{(N-1)} = \frac{\mathrm{d}^2s^{(N-1)}}{\mathrm{d}l^2} = \frac{\mathrm{d}^4s^{(N-1)}}{\mathrm{d}l^4} \quad \text{at} \quad l = l_N \tag{1.13}$$

$$\text{Mix:} \quad \alpha = \frac{\mathrm{d}s^{(N-1)}}{\mathrm{d}l}, \quad \beta = \frac{\mathrm{d}^2s^{(N-1)}}{\mathrm{d}l^2} \qquad \text{at} \quad l = l_N \tag{1.14}$$

which also lead to two equations,

$$\left.\begin{aligned} c_1^{(N)} &= 0 \\ -4h_{N-1}c_1^{(N-1)} - h_{N-1}^2c_2^{(N-1)} - 6h_{N-1}c_1^{(N)} + 3h_{N-1}^2c_2^{(N)} &= 10x_{N-1} - 10x_N \end{aligned}\right\} \quad \text{Even} \tag{1.15a}$$

$$\left.\begin{aligned} -7h_{N-1}c_1^{(N-1)} - 2h_{N-1}^2c_2^{(N-1)} - 8h_{N-1}c_1^{(N)} + 3h_{N-1}^2c_2^{(N)} &= 15x_{N-1} - 15x_N \\ c_2^{(N)} &= 0 \end{aligned}\right\} \quad \text{Odd} \tag{1.15b}$$

$$\left.\begin{aligned} c_1^{(N)} &= \alpha \\ c_2^{(N)} &= \beta/2 \end{aligned}\right\} \quad \text{Mix} \tag{1.15c}$$

If we arrange the unknowns into a vector,

$$\left[c_1^{(0)}, c_2^{(0)}, \ldots, c_1^{(j)}, c_2^{(j)}, \ldots, c_1^{(N)}, c_2^{(N)}\right] \tag{1.16}$$

equations (1.7) with one of (1.11) and one of (1.15) result in a $2(N+1)$-by-$2(N+1)$ system of linear equations, which corresponds to a banded diagonal sparse matrix with at most 6 non-zero elements in each row. The rest coefficients $\boldsymbol{c}_3^{(j)}$, $\boldsymbol{c}_4^{(j)}$ and $\boldsymbol{c}_5^{(j)}$ can be reconstruct using equation (1.6).

It is convenient to re-parametrize each local spline with a new variable $t \in [0, 1]$,

$$\boldsymbol{s}^{(j)}(t) = \boldsymbol{x}_j + \sum_{k=1}^{5} \boldsymbol{c}_k^{(j)} t^k, \quad \text{for} \quad j = 0, \ldots, N-1 \tag{1.17}$$

by redefining $\boldsymbol{c}_k^{(j)} \rightarrow \boldsymbol{c}_k^{(j)} h_j^k$. We can construct Lagrange interpolations along the arc-length of the global spline by the local arc-length L_j and its local fraction ξ,

$$L_j = \int_0^1 \dot{\boldsymbol{s}}^{(j)}\, \mathrm{d}t', \quad \xi(t) = \frac{1}{L_j} \int_0^t \dot{\boldsymbol{s}}^{(j)}\, \mathrm{d}t', \tag{1.18}$$

Convention for normal vector $\boldsymbol{n}$,

$$\boldsymbol{n} = \frac{(-\dot{z}, \dot{r})}{\|\dot{\boldsymbol{s}}\|} \tag{1.19}$$

2. Numerical integration and approximation

Standard m-point Gauss-Legendre quadrature,

$$\int_0^1 f(t)\,\mathrm{d}t \approx \sum_{k=1}^{m} w_k f(x_k), \quad \int_a^b f(t)\,\mathrm{d}t \Rightarrow \begin{array}{l} x_k \to a + (b-a)x_k \\ w_k \to (b-a)w_k \end{array} \tag{2.1}$$

Logarithmic-weighted Gauss quadrature,

$$\int_0^1 f(t)\ln t\,\mathrm{d}t \approx -\sum_{k=1}^{m} w_k^{\log} f(x_k), \quad \int_0^1 f(t)\ln(1-t)\,\mathrm{d}t \approx -\sum_{k=1}^{m} w_k^{\log} f(1-x_k) \tag{2.2}$$

When logarithmic singularity $\tau \in (0,1)$,

$$\begin{aligned}
\int_0^1 f(t)\ln|t-\tau|\,\mathrm{d}t &= \int_0^\tau f(t)\ln(\tau-t)\,\mathrm{d}t + \int_\tau^1 f(t)\ln(t-\tau)\,\mathrm{d}t \\
&= \tau \int_0^1 f(\tau s)\ln(\tau - \tau s)\,\mathrm{d}s + (1-\tau)\int_0^1 f(\tau + (1-\tau)s)\ln((1-\tau)s)\,\mathrm{d}s \\
&= \tau \int_0^1 f(\tau s)\ln(1-s)\,\mathrm{d}s + (1-\tau)\int_0^1 f(\tau + (1-\tau)s)\ln s\,\mathrm{d}s \\
&\quad + \tau\ln\tau \int_0^1 f(\tau s)\,\mathrm{d}s + (1-\tau)\ln(1-\tau)\int_0^1 f(\tau + (1-\tau)s)\,\mathrm{d}s
\end{aligned}$$

Logarithmic-weighted Gauss quadrature for singularity $\tau \in (0,1)$,

$$\int_0^1 f(t)\ln|t-\tau|\,\mathrm{d}t \approx \left\{ \begin{array}{r} \tau\ln\tau \displaystyle\sum_{k=1}^{m} w_k f\left[x_k\right] \\ +(1-\tau)\ln(1-\tau)\displaystyle\sum_{k=1}^{m} w_k f\left[\tau + (1-\tau)x_k\right] \\ -\tau \displaystyle\sum_{k=1}^{m} w_k^{\log} f\left[\tau(1-x_k^{\log})\right] \\ -(1-\tau)\displaystyle\sum_{k=1}^{m} w_k^{\log} f\left[\tau + (1-\tau)x_k^{\log}\right] \end{array} \right. \tag{2.3}$$

Elliptic integral of the first and second kind,

$$K(m) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}, \quad E(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\theta}\,\mathrm{d}\theta \tag{2.4}$$

Elliptic integral of the first and second kind,

$$K(m) \approx P_K(m) - \ln(1-m)Q_K(m), \quad E(m) \approx P_E(m) - \ln(1-m)Q_E(m) \tag{2.5}$$

where P_K, P_E, Q_K, Q_E are tenth order polynomials. Associate Legendre polynomial of 1/2 order can be conveniently expressed as,

$$P_{1/2}(x) = \frac{2}{\pi}\left\{2E\left(\frac{1-x}{2}\right) - K\left(\frac{1-x}{2}\right)\right\} \tag{2.6}$$

3. Axisymmetric boundary integral

Axisymmetric Green's function $G(\boldsymbol{\eta}';\boldsymbol{\eta})$,

$$G(\boldsymbol{\eta}';\boldsymbol{\eta}) = \frac{1}{\pi\sqrt{a+b}}K(m) \tag{3.1}$$

$$\frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}} = \frac{1}{\pi r\sqrt{a+b}}\left\{\left[\frac{n_r}{2} + \frac{\boldsymbol{n}\cdot(\boldsymbol{x}'-\boldsymbol{x})}{\|\boldsymbol{x}'-\boldsymbol{x}\|^2}r\right]E(m) - \frac{n_r}{2}K(m)\right\} \tag{3.2}$$

Auxiliary variables,

$$a = r'^2 + r^2 + (z'-z)^2, \quad b = 2r'r, \quad m = \frac{2b}{a+b} \tag{3.3}$$

Note $\eta \to \eta'$ implies $m \to 1$. Axisymmetric boundary integral equation,

$$\frac{\beta(\boldsymbol{\eta}')}{2\pi}\phi(\boldsymbol{\eta}') = \int_\gamma \left\{G(\boldsymbol{\eta}';\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{n}}(\boldsymbol{\eta}) - \phi(\boldsymbol{\eta})\frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial\boldsymbol{n}}(\boldsymbol{\eta})\right\} r\,\mathrm{d}\gamma(\boldsymbol{\eta}) \tag{3.4}$$

Lagrange interpolation along arc-length,

$$\phi(\boldsymbol{\eta}) \approx \sum_e\sum_k p_{e_k}\mathcal{N}_k^{(e)}(\xi), \quad \frac{\partial\phi}{\partial\boldsymbol{n}}(\boldsymbol{\eta}) \approx \sum_e\sum_k q_{e_k}\mathcal{N}_k^{(e)}(\xi) \tag{3.5}$$

where $\mathcal{N}_k$ is the Lagrange basis of the local fractional arc-length variable ξ,

$$\mathcal{N}_0 = (1-\xi)(1-2\xi), \quad \mathcal{N}_1 = 4\xi(1-\xi), \quad \mathcal{N}_2 = \xi(2\xi-1) \tag{3.6}$$

3.1. *Single-layer integral*

Single-layer integral reduces to a summation of local integrals over local arc γ_e of element e,

$$\begin{aligned}\int_\gamma G(\boldsymbol{\eta}';\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{n}}(\boldsymbol{\eta})r\,\mathrm{d}\gamma(\boldsymbol{\eta}) &\approx \sum_e\sum_k q_{e_k}\int_{\gamma_e} G(\boldsymbol{\eta}';\boldsymbol{\eta})\mathcal{N}_k^{(e)}(\xi)r\,\mathrm{d}\gamma(\boldsymbol{\eta})\\ &= \sum_e\sum_k q_{e_k}\int_0^1 \frac{rJ}{\pi\sqrt{a+b}}\mathcal{N}_k^{(e)}(\xi)K(m)\,\mathrm{d}t\end{aligned} \tag{3.7}$$

When singularity $\boldsymbol{\eta}' \in \gamma_e$ is located at $t = \tau \in [0,1]$,

$$\int_0^1 \frac{rJ}{\pi\sqrt{a+b}}\mathcal{N}_k^{(e)}K(m)\,\mathrm{d}t = \int_0^1 \frac{rJ}{\pi\sqrt{a+b}}\mathcal{N}_k^{(e)}\left[P_K - Q_K\ln\frac{1-m}{(t-\tau)^2} - 2Q_K\ln|t-\tau|\right]\mathrm{d}t$$

Introduce helper functions,

$$f_K^{\text{single}}(t) = \frac{rJ}{\pi\sqrt{a+b}} \tag{3.8}$$

$$R_{K/E}(m,\tau) = P_{K/E}(m) - Q_{K/E}(m)\ln\frac{1-m}{(t-\tau)^2} \tag{3.9}$$

If we define integrand $I^{(o)}$ for source location $\boldsymbol{\eta}'$ outside of γ_e and $I^{(i)}$ for $\boldsymbol{\eta}'$ interior to γ_e,

$$I^{(o)}(t) = f_K^{\text{single}}(t)K(m)\mathcal{N}_k^{(e)}(\xi) \tag{3.10}$$

$$I^{(i)}(t,\tau) = f_K^{\text{single}}(t)R_K(m,\tau)\mathcal{N}_k^{(e)}(\xi) \tag{3.11}$$

$$I^{(i)}_{\log}(t) = -2f_K^{\text{single}}(t)Q_K(m)\mathcal{N}_k^{(e)}(\xi) \tag{3.12}$$

then the local single-layer integral can be compactly represented with a regular part and a logarithmic-singular one,

$$\boldsymbol{\eta}' \notin \gamma_e: \text{ compute} \int_0^1 I^{(o)}(t)\,\mathrm{d}t \tag{3.13}$$

$$\boldsymbol{\eta}' \text{ at } t=0: \text{ compute} \int_0^1 I^{(i)}(t,0)\,\mathrm{d}t + \int_0^1 I^{(i)}_{\log}(t)\ln t\,\mathrm{d}t \tag{3.14}$$

$$\boldsymbol{\eta}' \text{ at } t=\tau: \text{ compute} \int_0^1 I^{(i)}(t,\tau)\,\mathrm{d}t + \int_0^1 I^{(i)}_{\log}(t)\ln|\tau-t|\,\mathrm{d}t \tag{3.15}$$

$$\boldsymbol{\eta}' \text{ at } t=1: \text{ compute} \int_0^1 I^{(i)}(t,1)\,\mathrm{d}t + \int_0^1 I^{(i)}_{\log}(t)\ln(1-t)\,\mathrm{d}t \tag{3.16}$$

3.2. *Double-layer integral*

Similar to singular-layer integral, double-layer integral reduces to a summation of local integrals over local arc γ_e of element e,

$$\begin{aligned}
&\int_\gamma \frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}}\phi(\boldsymbol{\eta}) r\,\mathrm{d}\gamma(\boldsymbol{\eta}) \approx \sum_e\sum_k p_{e_k}\int_{\gamma_e}\frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial \boldsymbol{n}}\mathcal{N}_k^{(e)}(\xi) r\,\mathrm{d}\gamma(\boldsymbol{\eta})\\
&= \sum_e\sum_k p_{e_k}\int_0^1 \mathcal{N}_k^{(e)}(\xi)\frac{J}{\pi\sqrt{a+b}}\left\{\left[\frac{n_r}{2}+\frac{\boldsymbol{n}\cdot(\boldsymbol{x}'-\boldsymbol{x})}{\|\boldsymbol{x}'-\boldsymbol{x}\|^2}r\right]E(m)-\frac{n_r}{2}K(m)\right\}\mathrm{d}t\\
&= \sum_e\sum_k p_{e_k}\int_0^1 \frac{\mathcal{N}_k^{(e)}(\xi)}{\pi\sqrt{a+b}}\left\{\left[\frac{\dot r(z'-z)-\dot z(r'-r)}{(r'-r)^2+(z'-z)^2}r-\frac{\dot z}{2}\right]E(m)+\frac{\dot z}{2}K(m)\right\}\mathrm{d}t
\end{aligned} \tag{3.17}$$

Technically we don't need to treat integrand involving elliptic integral of the second kind $E(m)$. However, convergence rate of standard Gauss-Legendre quadrature depends magnitude of derivatives.

$$f_E^{\text{double}} = \frac{1}{\pi\sqrt{a+b}}\left[\frac{\dot r(z'-z)-\dot z(r'-r)}{a-b}r-\frac{\dot z}{2}\right], \quad f_K^{\text{double}} = \frac{1}{\pi\sqrt{a+b}}\frac{\dot z}{2} \tag{3.18}$$

If we define integrand $I^{(o)}$ for source location $\boldsymbol{\eta}'$ outside of γ_e and $I^{(i)}$ for $\boldsymbol{\eta}'$ interior to γ_e,

$$I^{(o)}(t) = \left[f_E^{\text{double}}E(m)+f_K^{\text{double}}K(m)\right]\mathcal{N}_k^{(e)}(\xi) \tag{3.19}$$

$$I^{(i)}(t,\tau) = \left[f_E^{\text{double}}R_E(m,\tau)+f_K^{\text{double}}R_K(m,\tau)\right]\mathcal{N}_k^{(e)}(\xi) \tag{3.20}$$

$$I^{(i)}_{\log}(t) = -2f_E^{\text{double}}Q_E(m)-2f_K^{\text{double}}Q_K(m) \tag{3.21}$$

When $\boldsymbol{\eta}'$ is located the symmetry axis where $r'=0$, we can simplify the elliptic integrals,

$$\int_\gamma G(\boldsymbol{\eta}';\boldsymbol{\eta})\frac{\partial\phi}{\partial\boldsymbol{n}}(\boldsymbol{\eta})r\,\mathrm{d}\gamma(\boldsymbol{\eta}) \approx \sum_e\sum_k q_{e_k}\int_0^1 \mathcal{N}_k^{(e)}(\xi)\frac{J}{2\sqrt{1+[(z-z')/r]^2}}\,\mathrm{d}t \tag{3.22}$$

$$\int_\gamma \frac{\partial G(\boldsymbol{\eta}';\boldsymbol{\eta})}{\partial\boldsymbol{n}}\phi(\boldsymbol{\eta})r\,\mathrm{d}\gamma(\boldsymbol{\eta}) \approx \sum_e\sum_k p_{e_k}\int_0^1 \mathcal{N}_k^{(e)}(\xi)\frac{\dot z/r+\dot r(z-z')/r^2}{2\{1+[(z-z')/r]^2\}^{3/2}}\,\mathrm{d}t \tag{3.23}$$

The above integrands have well-definite limit as $\boldsymbol{\eta}\to\boldsymbol{\eta}'$ as long as $\ddot z/\dot r$ at $r=0$ is finite.

Element order	o	
Number of elements	n	
Number of nodes	$n \times o + 1$	
Nodal index of source point	i	
Elemental index of receiving point	j	
Local node index	k	$0 \leqslant k \leqslant o$
Indices of elements $\ni$ i-th node if $i \bmod o = 0$	$\lfloor i/o \rfloor - 1,\ \lfloor i/o \rfloor$	check $0 \leqslant \mathrm{id} \leqslant n-1$
Indices of elements $\ni$ i-th node if $i \bmod o \neq 0$	$\lfloor i/o \rfloor - 1$	check $0 \leqslant \mathrm{id} \leqslant n-1$
Nodal indices of j-th element	$j \times o + k$	$0 \leqslant k \leqslant o$

TABLE 1. Indicesfffffffk

3.3. *Assembly*

REFERENCES

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