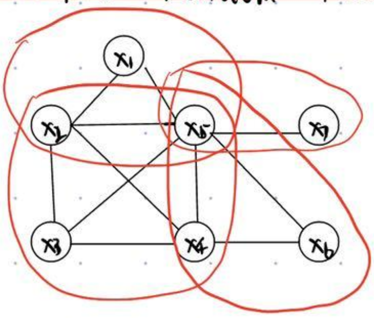
Ex 3

Pen and Paper

→ Markov Random Fields



a).

$$p(x_1, \cdots, x_7) = \frac{1}{Z} \cdot \phi(x_1, x_2, x_5) \cdot \phi(x_5, x_7) \cdot \phi(x_2, x_3, x_4, x_5) \cdot \phi(x_4, x_6)$$

b). ① $A = \{x_1, x_7\}$, $B = \{x_3, x_4\}$ $\qquad (A \cup B \cup C = V)$

② $A = \{x_3, x_4, x_7\}$, $B = \{x_1\}$

③ $A = \{x_1, x_3, x_4\}$, $B = \{x_7\}$

Markov Blanket of x_4: $\{x_2, x_3, x_5, x_6\}$

c). ① $p(x_4 | x_1, x_2, x_3, x_5, x_6, x_7) = p(x_4 | x_2, x_3, x_5, x_6)$:

✓, because of the local Markov property

② $x_1 \perp\!\!\!\perp x_3 | x_5$:

X, x_5 doesn't separate x_1 from x_3

③ $X_2 \perp\!\!\!\perp X_6 \mid \{X_4, X_5\}$:

✓, $\{X_4, X_5\}$ separates X_2 from X_6

④ Marginalize X_5 makes X_1 and X_7 dependent:

✓, by marginalizing over X_5, $p(X_1, X_7) \neq p(X_1) \cdot p(X_7)$

d).

$$p(a,b|c) = \frac{p(a,b,c)}{p(c)} = \frac{\phi_1(a,c)\cdot\phi_2(b,c)}{\sum_{a,b}\phi_1(a,c)\cdot\phi_2(b,c)}$$

$$p(a|c)\cdot p(b|c) = \frac{p(a,c)}{p(c)} \cdot \frac{p(b,c)}{p(c)}$$

$$= \frac{\sum_b \phi_1(a,c)\cdot\phi_2(b,c) \cdot \sum_a \phi_1(a,c)\cdot\phi_2(b,c)}{\left[\sum_{a,b}\phi_1(a,c)\cdot\phi_2(b,c)\right]^2}$$

$$= \frac{\phi_1(a,c)\cdot\sum_b \phi_2(b,c) \cdot \phi_2(b,c)\cdot\sum_a \phi_1(a,c)}{\left[\sum_{a,b}\phi_1(a,c)\cdot\phi_2(b,c)\right]^2}$$

$$= \frac{\phi_1(a,c)\cdot\phi_2(b,c)\cdot \sum_b\sum_a \phi_1(a,c)\cdot\phi_2(b,c)}{\left[\sum_{a,b}\phi_1(a,c)\cdot\phi_2(b,c)\right]^2}$$

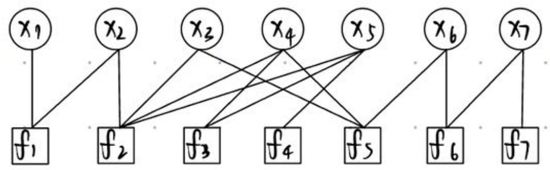
$$= \frac{\phi_1(a,c)\cdot\phi_2(b,c)}{\sum_{a,b}\phi_1(a,c)\cdot\phi_2(b,c)}$$

$\therefore p(a,b|c) = p(a|c)\cdot p(b|c) \quad \therefore a \perp\!\!\!\perp b \mid c$

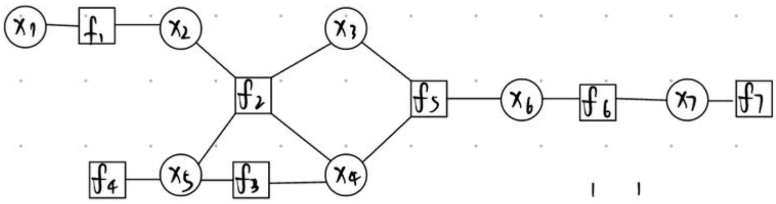
→ Factor Graph

a).

$$f(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = f_1(x_1,x_2)\cdot f_2(x_2,x_3,x_4,x_5)\cdot f_3(x_4,x_5)$$
$$\cdot f_4(x_5)\cdot f_5(x_3,x_4,x_6)\cdot f_6(x_6,x_7)\cdot f_7(x_7)$$



solution:



b).

$$Z = \sum_{x} \phi_1(x_1)\,\phi_2(x_1,x_2,x_3,x_4)\cdot\phi_3(x_3,x_5,x_6)$$
$$= \sum_{x/\{x_1\}} \phi_2(1,x_2,x_3,x_4)\cdot\phi_3(x_3,x_5,x_6)$$
$$= \sum_{x_5,x_6} \phi_3(0,x_5,x_6) + \sum_{x_5,x_6} \phi_3(1,x_5,x_6)$$
$$= 0+1+1+2+\ 2+3+3+4$$
$$= 16$$

c).

$$f(\text{state}) \propto \exp\left(-\frac{E}{kt}\right) \Rightarrow P(x_1,x_2,x_3,x_4,x_5,x_6) > 0$$

$\because$ $P(0,1,1,1,1,1) = 0$

$\therefore$ is not Gibbs Distribution

→ Belief Propagation

a). use dp to calculate the marginal distribution efficiently by storing the "message" ✗

solution: assumes a singly-connected graph

b).

$$\mu_{x\to f}(x) = \prod_{g\in\{ne(x)/f\}} \mu_{g\to x}(x)$$

$$\mu_{f\to x}(x) = \sum_{x_f/x}\left(f(x_f)\cdot \prod_{y\in\{ne(f)\backslash x\}} \mu_{y\to f}(y)\right)$$

$\mu_{a\to f_1}(a) = 1$

$\mu_{f_3\to c}(c) = f_3(c) = [c=1]$

$\mu_{d\to f_2}(d) = 1$

$\mu_{c\to f_2}(c) = \mu_{f_3\to c}(c) = f_3(c) = [c=1]$

$\mu_{f_2\to b}(b) = \sum_{c,d} f_2(b,c,d)\cdot \mu_{c\to f_2}(c)\cdot \mu_{d\to f_2}(d) = \sum_{c,d} f_2(b,c,d)\cdot f_3(c)$

$= \sum_{d}(0.5b+0.3+0.2d) = b+0.b+0.2 = b+0.8$

$\mu_{b\to f_1}(b) = \mu_{f_2\to b}(b) = \sum_{c,d}(0.5b+0.3c+0.2d)\cdot[c=1] = b+0.8$

$\mu_{f_1\to a}(a) = \sum_{b} f_1(a,b)\cdot \mu_{b\to f_1}(b) = \sum_{b}[a=b](b+0.8) = 0.8[a=0]$
$+1.8[a=1]$

$\mu_{f_1\to b}(b) = \sum_{a} f_1(a,b)\cdot \mu_{a\to f_1}(a) = [b=1]+[b=0] = 1$

$\mu_{f_2\to c}(c) = \sum_{b,d} f_2(b,c,d)\cdot \underbrace{\mu_{b\to f_2}(b)}_{1}\cdot \underbrace{\mu_{d\to f_2}(d)}_{1} = 1.2c+1.4$

$$\mu_{f_2 \to d}(d) = \sum_{b,c} f_2(b,c,d) \cdot \mu_{b \to f_2}(b) \cdot \mu_{c \to f_2}(c)$$

$$= \sum_{b} (0.5b + 0.3 + 0.2d) \cdot$$

$$= 0.6 + 0.5 + 0.4d = 0.4d + 1.1$$

$$\therefore p(a) = \begin{cases} 0.308, & a=0 \\ 0.692, & a=1 \end{cases}$$

$$p(b=0) = \frac{\mu_{f_1 \to b}(0) \cdot \mu_{f_2 \to b}(0)}{\mu_{f_1 \to b}(0) \cdot \mu_{f_2 \to b}(0) + \mu_{f_1 \to b}(1) \cdot \mu_{f_2 \to b}(1)}$$

$$= \frac{1 \times 0.8}{0.8 + 1.8}$$

$$\approx 0.308$$

$$p(b=1) = 1 - 0.308 = 0.692$$

$$p(c=0) = \frac{\mu_{f_2 \to c}(0) \cdot \mu_{f_3 \to c}(0)}{\mu_{f_2 \to c}(0) \cdot \mu_{f_3 \to c}(0) + \mu_{f_2 \to c}(1) \cdot \mu_{f_3 \to c}(1)}$$

$$= 0$$

$$p(c=1) = 1$$

$$p(d=0) = \frac{\mu_{f_2 \to d}(0)}{\mu_{f_2 \to d}(0) + \mu_{f_2 \to d}(1)} = \frac{1.1}{1.1 + 1.5}$$

$$\approx 0.423$$

$$p(d=1) = \frac{\mu_{f_2 \to d}(1)}{\mu_{f_2 \to d}(0) + \mu_{f_2 \to d}(1)} \approx 0.577$$

$\therefore\ p(a) = \begin{cases} 0.308, & a=0 \\ 0.692, & a=1 \end{cases}$ $\quad p(c) = \begin{cases} 0, & c=0 \\ 1, & c=1 \end{cases}$

$p(b) = \begin{cases} 0.308, & b=0 \\ 0.692, & b=1 \end{cases}$ $\quad p(d) = \begin{cases} 0.423, & d=0 \\ 0.577, & d=1 \end{cases}$

c). $$\max_{a,b,c}\left(p(a,b,c)\right) = \frac{1}{Z}\max_{a,b,c}\left(f_1(a,b)\cdot f_2(b,c)\cdot f_3(c)\right)$$

$$= \frac{1}{Z}\cdot\max_{a,b}\left(f_1(a,b)\cdot \max_{c} f_2(b,c)\cdot f_3(c)\right)$$

$$= \frac{1}{Z}\cdot\max_{a,b}\left(f_1(a,b)\cdot \mu_{c\to b}(b)\right)$$

$$= \frac{1}{Z}\cdot\max_{a}\left(\max_{b} f_1(a,b)\cdot \mu_{c\to b}(b)\right)$$

$$= \frac{1}{Z}\max_{a}\cdot \mu_{b\to a}(a)$$

$\mu_{a\to f_1}(a) = 1$

$\mu_{f_1\to b}(b) = \max_{a} f_1(a,b)\cdot \mu_{a\to f_1}(a) = \max_{a} f_1(a,b)$

$\mu_{b\to f_2}(b) = \max_{a}\left(\max_{a} f_1(a,b)\right) = \max_{a}\cdot f_1(a,b)$

$\mu_{f_2\to c}(c) = \max_{b}\left(f_2(b,c)\cdot \mu_{b\to f_2}(b)\right)$

$\mu_{c\to f_3}(c) = \mu_{f_2\to c}(c)$

$\mu_{f_3\to c}(c) = f_3(c)$

$\mu_{c\to f_2}(c) = f_3(c)$

$$\mu_{f_2 \to b}(b) = \max_c \cdot f_2(b,c) \cdot \mu_{c \to f_2}(c) = \max_c \left(f_2(b,c) \cdot f_3(c) \right)$$

$$\mu_{b \to f_1}(b) = \mu_{f_2 \to b}(b) = \begin{cases} 0.2, & b=0 \\ 0.15, & b=1 \end{cases}$$

$$\mu_{f_1 \to a}(a) = \max_b \left(f_1(a,b) \cdot \mu_{b \to f_1}(b) \right)$$

$$= \max_b \left(f_1(a,b) \cdot \max_c \left(f_2(b,c) \cdot f_3(c) \right) \right)$$

$$= \begin{cases} 0.1, & a=0 \\ 0.045, & a=1 \end{cases}$$

$\therefore$ $a^* = \operatorname{argmax} \mu_{f_1 \to a}(a) = 0$

$b^* = \operatorname{argmax} \left(\mu_{f_1 \to b}(b) \cdot \mu_{f_2 \to b}(b) \right)$

$= 0$

$c^* = \operatorname{argmax} \left(\mu_{f_3 \to c}(c) \cdot \mu_{f_2 \to c}(c) \right)$

$= 1$

d). ① Yes. if we condition on a node, the ring will be turned into a singly linked graph, so it is possible to compute the marginal inference effectively.

② Yes, for the similar reason as ①

solution: $p(X_2) = \sum_{X_1 \cdots X_{100}} p(X_1, \cdots, X_{100})$

$= \sum_{X_1 \cdots X_{100}} p(X_1) \cdot p(X_2 \cdots X_{100} | X_1)$

$= \sum_{X_1} p(X_1) \cdot \sum_{X_3 \cdots X_{100}} p(X_2 \cdots X_{100} | X_1)$

singly linked

$$p(x_L) = \max_{x_1, x_3 \cdots x_{100}} p(x_1, \cdots, x_{100})$$

$$= \max_{x_1, \cdots x_{100}} \left(p(x_1) \cdot \quad p(x_2, \cdots, x_{100} | x_1) \right)$$

$$= \max_{x_1} (p(x_1)) \cdot \max_{x_3, \cdots, x_{100}} \left(p(x_2 \cdots x_{100} | x_1) \right)$$

$$x_1^* = \arg\min_{x_1} p(x_1) \cdot \left(\underbrace{\max_{x_2 \cdots x_{100}} p(x_2 \cdots x_{100} | x_1)}_{\text{singly linked}} \right)$$

→ Graphical Models for Multi-view Reconstruction

a).

Advantages: (1). tractable without exponential complexity

(2). Non-local constraints via joint inference in 2D and 3D

(3). CAD priors can help disambiguate textureless regions

(4) Using octrees, reconstructions up to 1024^3 voxels are possible

Disadvantage:

(1) only approximate inference possible when loopy

(2) slow

(3) appearance term is very simplistic and not robust

(4). resolution is limited by discrete voxels

b). ~~①~~ the image of the pixel is determined by the intensity of the first voxel occupied along the ray.

~~②~~ violated when the surfaces are specular.

Solution:

① color constancy between pixels showing the same 3D points

② violated when:

1° non-Lambertion surfaces

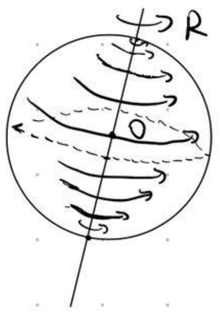
2° basically for all surfaces

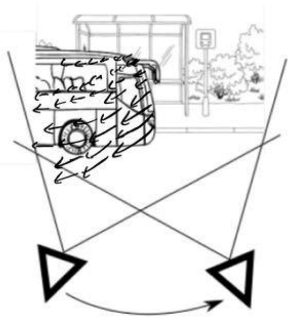
3° illumination changes

4° camera noise is assumed to be negligible

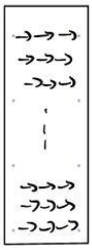
→ Graphic Models for Optical Flow

a). Flow fields





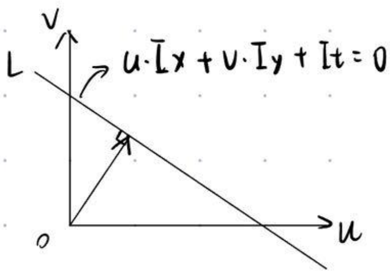
b). motion field



optical flow field:

c). Apenture Problem

for the u.v of one specific point, the constraint can be written as $I(x+u(x,y), y+v(x,y), t+1) = I(x,y,t)$ which can be transformed to 「$u \cdot I_x + v \cdot I_y + I_t = 0$」 by Taylor Series. Because x.y are scalar, I_x, I_y, I_t are all fixed so that the constraint can be seen as a function of u, v.



So, the points on line L are all possible pair of (u, v) that satisfy the constraint, which means that the optical flow can't be determined only with this constraint.

solution: we can only measure the normal velocity or normal flow

But getting the normal flow just needs to calculate the distance between O and L, which is $\frac{-I_t}{\sqrt{I_x^2+I_y^2}}$