Chapter 2 Quadratic Equations

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2.2 Solving Quadratic Equations by Factoring

2.3 Solving Quadratic Equations by Completing the Square

2.4 Solving Quadratic Equations by the Quadratic Formula

2.5 Roots of Quadratic Equations

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INTRODUCTION

The Hindus accepted the facts about negative and irrational numbers and recognized that a quadratic equation (having real answers) has two formal roots. They unified the algebraic solution of quadratic equations by the familiar method of completing the square. Today, this method is often referred to as the **Hindu Method.** Bhaskara gave the two remarkable identities

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Which are sometimes employed for finding the square root of a binomial surd.

In this, chapter we will learn how to solve quadratic equations using different methods in section 2.1 to 2.4: the square property, factoring, completing the square, and the quadratic formula. The roots of quadratic equations will be discussed in Section 2.5 and we will learn to solve problems using quadratic equations in Section 2.6

Definition

Any equation that can be written in the form

Where a, b and c are real numbers and a 0, is called a **quadratic equation.**

is called the *standard form* for quadratic equations. For a quadratic equation written in standard form, the first term is called the *quadratic*,

the second term *bx* is the *linear term*, and the last term *c* is called the *constant*

Each of the following is a quadratic equation:

Note:

The first equation is clearly a quadratic equation since it is in standard form. Quadratic equations are divided into two classes: complete quadratic equations (such as ) and incomplete quadratic equations (such as

We know that the number 0 is a special number. Here is another property of 0 that holds the key to solving quadratic equations. It is called the *zero factor property.*

**Zero Factor Property**

For all real numbers *a* and *b,*

*a • b* = 0 if and only if *a* = 0 or *b* = 0 (or both).

To prove that a = 0, we have

Statements Reasons

1. ab = 0; b0 1. Given
2. (ab) • = 0 2. Multiplication Property of Equality
3. (ab) • = 0 3. Any number multiplied by 0 is 0.
4. a • (b • ) = 0 4. Associative axiom for Multiplication
5. a • 1 = 0 5. Multiplicative Inverse
6. a = 0 6. Multiplicative Identity

To prove that b = 0, we have

Statements Reasons

1. ab = 0; a0 1. Given
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4. ( • a) • b = 0 4. Associative axiom for Multiplication
5. 1 • b = 0 5. Multiplicative Inverse
6. b = 0 6. Multiplicative Identity

**2.1 Solving incomplete quadratic equations**

**Objectives**

**A** solve incomplete quadratic equations of the form   
; and

**B** solve incomplete quadratic equations of form

An incomplete quadratic is of the form (where the constant term is zero) or of the form ich lacks the term containing the first power of the unknown). Here are some incomplete quadratic equations.

**A Solving Quadratic Equation of the form**

An incomplete quadratic equation with the constant term equal to zero can be represented by the general form . This type can always be solved by factoring. However, if we examine the equation we can learn some interesting facts about the roofs of such an equation.

1. If we factor the left member, we obtain *x* (*ax* + *b*) = 0 by the distributive law.
2. Since both factors have variable x, we have two linear equations: *x* = 0 and *ax* + *b* = 0.
3. If we solve the equation *x* = 0, the root is 0. If we solve the equation *ax* + *b* = 0, the root is -.

Form this, we established that one solution to an equation of the type is *x* = 0 and the other solution is *x* = -.

**Example 1**

Solve the following equations using either the factoring method or the method discussed above.



**Solution**

Although these can be solved by factoring we will use the fact that the solution to equation of the type are x =.

a = 1 and b = -3

1. By changing to general form, we have,

a = 3 and b = 4

x = 0, x =

**B Solving Quadratic Equations of the**

When an incomplete quadratic equation is of the type (where the coefficient, of the first degree term is zero), factoring may or may not work. Let us consider the case where factoring is not possible.

If  can be solved by factoring, we can use the extraction of roots method. The procedure is given below.

1. Solve for the square of the variable. This will give an equation of the form .
2. The root of will be the roots of the two equations = and (if is positive). Note that there will be no real number solution if is negative.
3. Check the results in the original equation. Therefore, the solution to must include both the principal square root of and the negative square root of .

Here are some facts about square roots that we need to know before we can solve quadratic equations:

1. Every positive number *b* has two square roots; a positive number *a* and a negative number –*a*, where and
2. The radical symbol is used to represent the positive square root:. A negative sign is included to represent the negative square root: :.

We can use the radical symbol to represent the square roots of every nonnegative number.

Example: the square roots of 15 are and because and .

1. The square root of a product is the product of the square roots. That is, = where *a*  and *b* are nonnegative numbers.

Example: .

1. The square root of a quotient of the square roots. That is, where *a*  and  *b*  are nonnegative real numbers and *b* 0.

Example:

Example 1

Solve the equation

**Solution**

Although we can solve this by factoring we will use the examination of roots method. Thus, we have,

Example 2

Solve the solution

**Solution**

Divide both sides by 2.

Extract the square roots of both sides.

Simplify (note: = = )

**Check:**

Example 3

Solve the equation

**Solution**

Transpose the known quantity to the right side.

Extract the square roots of both sides.

**Check:**

Example 4

Solve the equation

**Solution**

Transpose the 18 and

Combine similar terms.

Divide both sides by 2.

Extract the square roots of both sides.

**Check:**

Example 5

Solve the equation

**Solution**

**Check:**

Example 6

Solve the equation

**Solution**

Transpose to the right side.

Divide both sides by 4.

Extract the square roots of both sides.

**Check:**

0 0

Example 7

Find the value of x so that length of PQ is 39.

**P R Q**

**Solution**

Definition of betweeneess

Substitute the given values.

Combine similar terms.

Transpose 3 to the left.

Divide both sides by 4.

Extract the square roots of both sides.

*Note:* We take only the positive root since distance can never be negative.

**Check:**

Example 8

Solve

Solution

Using the extraction of roots method.

`

and

We should now write the radicals in their simplest form. To simplify we multiply the both numerator and denominator by in order to make the denominator a perfect square. (this process is called ***rationalization of the denominator.***)

Thus, becomes or.

Thus, the roots are x = and x =.

**Check:**

Exercise 2.1

A. Solve each and check the solution.

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

B. Solve each and check the solution Use the extraction of roots method.

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

C. Solve each and check the solution. Use the extraction of roots method.

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

D. Solve for x and check the solution. Use the extraction of roots method.

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

2.2 SOLVING QUADRATIC EQUATIONS BY FACTORING

**Objectives**

Solve quadratic equations by factoring.

We have learned how to solve linear equations like . We have also learned how to factor quadratic trinomials. We can combine these skills to solve some quadratic equations by factoring. However, not all quadratic equations can be solved by factoring.

**Solving Quadratic Equations by Factoring**

**Procedure**  *To solve quadratic equations by factoring.*

1. Rewrite the equation with 0 on the right-hand side.

2. Factor the left-hand side completely.

3. Use the zero-factor property to get simple linear equations.

4. Solve the linear equations.

5. Check the answers in the original equations.

Example 1

Solve the equation

**Solution**

Factor completely.

or Equate each factor to zero.

**Check:**

If x = 0, we have if x =, we have

4

The values of x are 0 or . These values of X are the solutions to the equation .

*Note:* The given quadratic equation is of the form . This is an example of an incomplete quadratic equation. It must be remembered that we cannot divide by the common variable given. This will be the effect when we do so.

Divide all terms by x.

Addition Property of Equality and

Multiplication Property of Equality

Take note that there is only one root. This must not happen if we are dealing with quadratic equations.

Example 2

Solve the equation.

**Solution**

1. Equate to zero.
2. Factor the quadratic trinomial.
3. Equate each factor to zero.
4. Solve for the roots.
5. Check each root in the given equation

Example 3

Solve the equation

**Solution**

1. Equate to zero.
2. Factor the difference of two squares.
3. Write each factor equal to zero.
4. Solve for the roots.
5. Check each root In the given equation. 9(- 9(

*Note:* the equation is of the form hence, it can be solved by using the first method.

Example 4

Solve the equation

**Solution**

1. Apply distributive property to remove

The parentheses.

1. Equate to zero.
2. Factor the left side.
3. Write each factor equal to zero.
4. Solve the roots.
5. Check each root in the given equation

Example 5

Solve .

**Solution**

1. Multiply the binomial on the left and

Subtract 14 from each side.

1. Factor completely
2. Apply zero-factor Property
3. Solve for the roots.
4. Check each root in the given equation.

Example 6

Find x if the perimeter of the rectangle is 180 centimeters.

**Solution**

*Note:* we only take since length must always be positive.

**Exercise 2.2**

Solve each by factoring and check the solution.



Solve each by factoring.

Solve each by factoring.

**2.2 SOLVING QUADRATIC EQUATION BY COMPLETING THE SQUARE**

**OBJECTIVES**

1. Complete the square of the expression ; and
2. Solve quadratic equations by completing the square.

Not all quadratic equations can be solved by factoring and using the square root property. What do we do when we have an equation that cannot be solved using either of these methods? We can use a method called ***completing the square***. In this method, we write the equation in form , then we use the square root property. Completing the square can be use solve any quadratic equation.

1. **Completing the Square**

To write an equation in form , we must write the left side of the equation as a perfect square. We can learn how to do this with the example . The use of algebra tiles is of great help in the understanding of mathematical concepts.

Here is a model of the left side,

Try to rearrange the tiles to form a square by moving half of the x-tiles:

We see that the square is not complete.

To complete the square, we need to add 16 unit-tiles

By adding 16 to both sides of the equation, we have formed a square.

The equation is now in the form .

**Procedure**

*Without using tiles, here is how we complete the square:*

1. Make sure that the coefficient of is 1.
2. Multiply the coefficient of by.
3. Square the result.
4. Add this number to both sides of the equation.

Example 1

Complete the square given .

**Solution**

1. Make sure that the coefficient of is 1, then

Multiply the coefficient of x by.

1. Square the result. (
2. Add this number to both sides.

The left side of the equation is a perfect square:

Example 2

**Solution**

1. Make sure that the coefficient of is 1, then

Multiply the coefficient of x by.

1. Square the result.
2. Add this number to both sides.

The left side of the equation is a perfect square:

Example 3

Complete the square given .

**Solution**

1. Make sure that the coefficient of is 1, then

Multiply the coefficient of x by.

1. Square the result.
2. Add this number to both sides of the equation.

The left side of the equation is a perfect square: ()

Example 4

Find the missing term to complete the square : .

**Solution**

1. The coefficient of x is 10.
2. The missing term is

Hence,.

**B. Solving Quadratic Equations by Completing the Square**

The following are the steps in solving quadratic equations by completing the squares:

1. Transpose all terms containing the unknown to the left side of the equation and the constant term to the right side if necessary.
2. Divide each term of the equation in the numerical coefficient of term if necessary. This will change the equation in the form .
3. Divide the coefficient of x by 2, square it, then add the answer to both sides of the equation.
4. Factor the left side of the equation. This is a perfect square trinomial. Simplify the right side.
5. Take the square root of both sides. Write a sign to square root of the right side.
6. Write the square root of the left side equal to the positive square root of the right side in step 5. Then find the second root of the equation.
7. Write the square root of the left side equal to the negative square root of the right side in step 5. Then find the second root of the equation.
8. Check each root by substituting it to the original equation.

Example 1

Solve by completing the square.

**Solution**

1. Transpose 2 to the other side.

2. Make the left side

a perfect square.

3. Factor the left side.

4. Extract the roots.

5. Solve for x.

**Check:**

Substitute the values of x in the given equation.

If x =

[(

If x =

Therefore, x = and x=are the solutions of the equation.

Example 2

Solve by completing the square.

**Solution**

1. Subtract 3.

2. Divide by 4 so that the coefficient of is 1.

3. Add to both sides.

4. Rewrite.

5. Use the square root property to solve the resulting equation.

**Check:**

Substitute the values of x in the given equation.

If

If

Thus, or = are the solutions of the given quadratic equation.

Example 3

Solve by completing the square.

**Solution**

1. Add 12 to both sides of the equation.

2. Divide by 2. =6

3. Add to each side.

4. Rewrite as a perfect square.

5. Use the square root property.

**EXERCISE 2.3**

A. Make each quadratic expression a perfect square trinomial to complete the square.

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

B. Solve each equation by completing the square.

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

**2.4 Solving Quadratic Equation by the Quadratic Formula**

A. Write a quadratic equation in the form and identify *a, b, and c*;

B. Solve a quadratic equation using the quadratic formula; and

C. Solve equations that are in quadratic form

There is a sure-fire method for solving any quadratic equation. This new method incorporates the completing the square method. We will use completing the square to solve the equation , and to derive the quadratic formula.

The solution is given below.

|  |  |
| --- | --- |
| 1. ; a, b, c are real numbers, a | 1. Given |
| 2. | 2. Divide each term in the equation by a, the numerical coefficient of |
| 3. | 3. Isolate all terms with x on the left side of the equation. |
| 4. | 4. Complete the square by adding or to both sides of the equation. |
| 5. | 5. Factor the left side of the equation and simplify the right side. |
| 6. | 6. Find the square root of both sides of the equation. |
| 7. | 7. Add to both sides of the equation. |
| 8. | 8. This is the **quadratic formula**. It can be used to solve any quadratic equation by substituting the numbers to the corresponding letters in the formula. |

*Note:* The roots of any quadratic equation of the form , where a, b, c are real numbers and *a*

Where

1. **Writing Quadratic Equations in Standard Form**

The formula

is so important that it should be memorized. Before using it, we must remember two things:

1. Write the given equation in standard form.

2. Determine the values of *a, b,* and *c* in the given equation.

Example 1

Write the equation in standard form and identify a, b, and c.

**Solution**

1. Write in standard form.

1. When the equation is in standard form, it is a = 1, b = -7, and c = 5

Easy to find the values of a, b, and c.

Example 2

Write the equation in standard form and identify *a, b,* and *c.*

**Solution**

1. Multiply each term by 10, the LCD.
2. Write the equation is standard form.

Hence, a = 2, b = -5, and c = 3.

1. **Solving Quadratic Equations Using the Quadratic Formula**

**Procedure**

*To solve a quadratic equation using the quadratic formula follow these steps:*

1. Isolate all terms on one side of the equation, if necessary. Then combine similar terms.
2. Clear the equations of all fractions. Then transpose and combine similar terms.
3. Remove parentheses. Then transpose and combine similar terms.
4. When the quadratic equation is in the form ,
5. Represent the numerical coefficient of as *a;*
6. Represent the numerical coefficient of *x* as *b;* and
7. Represent the constant as *c.*
8. Substitute the values of *a, b,* and *c*  in the formula
9. Get the two values of x.
10. Simplify the roots.
11. Check each root by substituting it in the original equation.

Example 1

Solve using the quadratic formula: .

**Solution**

1. The equation is written in standard form: .

It is clear that a = 1, b = 6, and c = 5.

1. Substitute the values of a, b, and c in the formula and solve.

Thus,

Or.

**Check:**

Substitute the values of x in the given equation.

If x = - 1,

If x = -5,

Therefore, the roots are – 1 and – 5.

*Note:*  we can also solve by factoring. is rewritten and we will get the same answer.

Example 2

Solve using the quadratic formula:

**Solution**

1. Write the equation in standard form.
2. Identify the values of a, b, and c. a = 2, b = 4, and c = 2
3. Substitute these values into the quadratic formula simplify.

Thus, the solution is -1.

The checking is left for you to do.

Note that if we value of is zero, we have only one root.

Example 3

Solve using the quadratic formula:

**Solution**

1. Write the equation in standard form.
2. Identify the values of a, b, and c.
3. Substitute these values into the quadratic formula.

Example 3

Find the Values of n that will make the roots of the equation 9x2 – nx + 25 =0 equal.

Solution

1. Determine the values of a, b and c

a = 9, b =-n and c =4

1. Substitute these in the discriminant and let it be equal to zero because the roots are equal.

b2 – 4ac = 0

(-n) 2 – 4(9) (25) = 0

3. Solve for n.

n2 = 900

4. Take the square root of both sides of the equation.

n = ± 30

Check:

Substitute ± 30 in place of n in the given equation

If n = 30, then

9x2 – nx + 25 = 0

9x2 – 30x + 25 = 0

(3x-5)(3x-5) = 0

3x-5=0 or 3x-5=0

x= or x=

For t <, we have t = -5, -6

Thus, if t = -5, we have

2x2 – 6x – t = 0

2x2- 6x – (-5) = 0

2x2 – 6x + 5 =0

Hence, a = 2, b= -6, and c =5

\substituting these in the expression b2 – 4ac, we have

B2- 4ac = (-6)2 – 4(2)(5)

= 36 – 40

= -4

Since 4 is less than zero, then the roots of the equation 2x2 – 6x +5 = 0 are nonreal.

**B. relation of roots and coefficients of a quadratic equation**

We can always verify the solution set of any equation is correct by substituting the roots in the originals equation. We have done this earlier part of this chapter. However, when we work with quadratic equations whose general form is ax2 +bx + c = 0, we can use an easier method. Let us first us examine the table below.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **Equation** | **a** | **b** | **c** | **Roots** | | **r1+r2** | **r1. r2** |
| 1. X2 + x-20=0 | 1 | 1 | -20 | 4 | -5 | -1 | -20 |
| 1. X2 +7x – 18 = 0 | 1 | 7 | -18 | 2 | -9 | -7 | -18 |
| 1. 2x2 – 13x +15 =0 | 1 | -13 | 15 | 5 |  |  |  |
| 1. 15x2 – 14x – 8 =0 | 15 | -14 | -8 |  |  |  |  |
| 1. 4x2 + 9x = 0 | 4 | 9 | 0 | 0 |  |  | 0 |
| 1. 5x2 – 10 =0 | 5 | 0 | -10 |  |  | 0 | -2 |