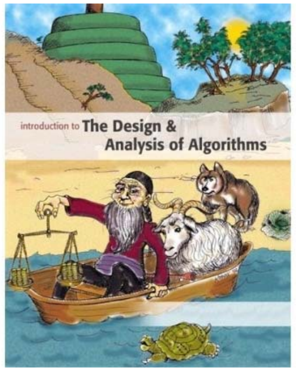
Introduction to

Algorithm Design and Analysis

[17] Dynamic Programming 2



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In the Last Class…

- **Basic idea of DP**
- **Least cost matrix multiplication**
  - BF1, BF2
  - A DP solution
- **Weighted binary search tree**
  - The same DP solution

DP - II

- All-pair shortest path
- Edit distance
- Highway restaurants
- Separating sequence of words
- Changing coins

- Elements of DP

All-pair Shortest Path

- **“BF 2”**
  - Path length k
    - k in [1, n]
- **Floyd algorithm**
  - Index range k
    - k in [1, n]

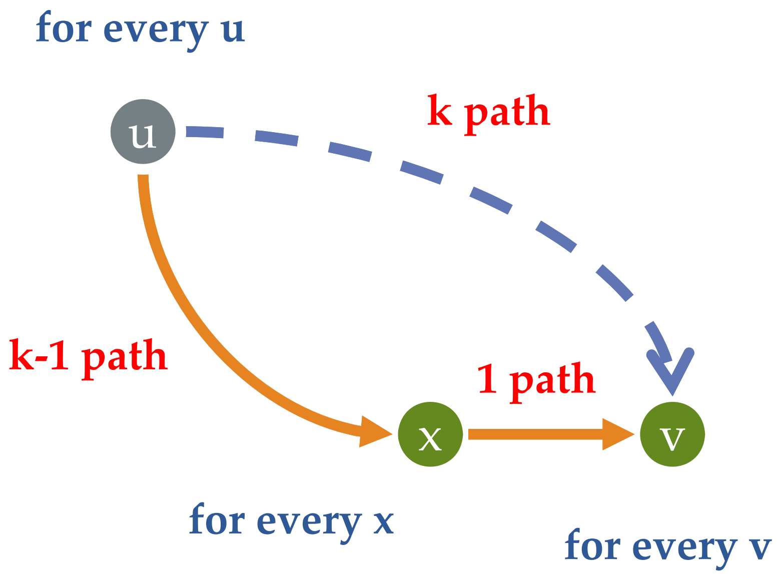
BF2

Length of the shortest of at most k edges

$$dist(u,v,k) = \begin{cases} 0 & \text{if } u = v \\ \infty & \text{if } k = 0 \text{ and } u \neq v \\ \min\limits_{x} \left(dist(u,x,k-1) + w(x \rightarrow v) \right) & \text{otherwise} \end{cases}$$

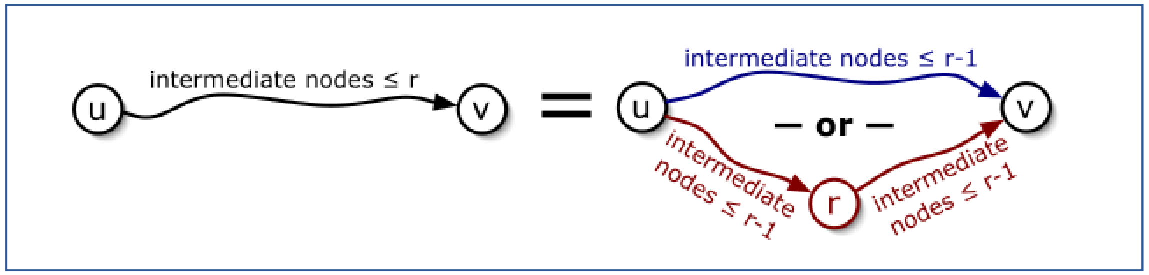
```
APSP(V, E, w):
  for all vertices $u$
      for all vertices $v$
          if $u = v$
              $dist[u, v, 0] \leftarrow 0$
          else
              $dist[u, v, 0] \leftarrow \infty$
  for $k \leftarrow 1$ to $V - 1$
      for all vertices $u$
          for all vertices $v$
              $dist[u, v, k] \leftarrow \infty$
              for all vertices $x$
                  if $dist[u, v, k] > dist[u, x, k-1] + w(x \rightarrow v)$
                      $dist[u, v, k] \leftarrow dist[u, x, k-1] + w(x \rightarrow v)$
```

$O(n^4)$



Floyd Algorithm

- **Basic idea**

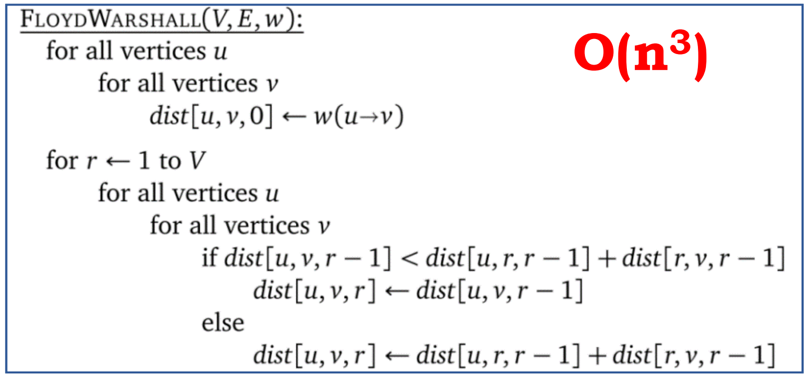


- **Smart recursion**

$$dist(u,v,r) = \begin{cases} w(u \rightarrow v) & \text{if } r = 0 \\ \min\left\{dist(u,v,r-1),\ dist(u,r,r-1) + dist(r,v,r-1)\right\} & \text{otherwise} \end{cases}$$

Floyd Algorithm

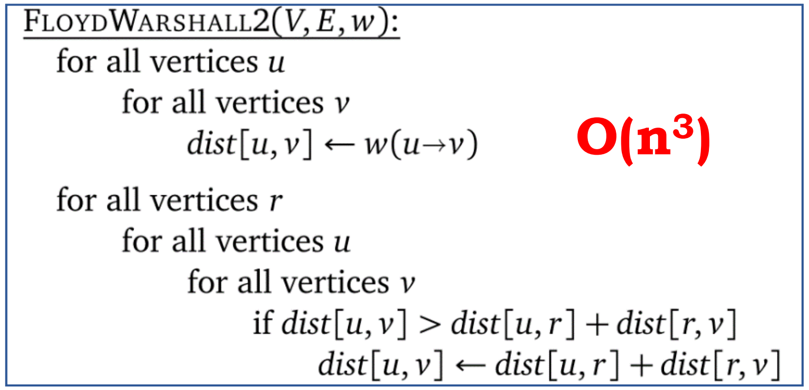
- **Basic DP (3-dimensional)**



```
FLOYDWARSHALL($V, E, w$):
    for all vertices $u$
        for all vertices $v$
            $dist[u, v, 0] \leftarrow w(u \to v)$
    for $r \leftarrow 1$ to $V$
        for all vertices $u$
            for all vertices $v$
                if $dist[u, v, r-1] < dist[u, r, r-1] + dist[r, v, r-1]$
                    $dist[u, v, r] \leftarrow dist[u, v, r-1]$
                else
                    $dist[u, v, r] \leftarrow dist[u, r, r-1] + dist[r, v, r-1]$
```

$O(n^3)$

- **Improved DP (2-dimensional)**



```
FLOYDWARSHALL2($V, E, w$):
    for all vertices $u$
        for all vertices $v$
            $dist[u, v] \leftarrow w(u \to v)$
    for all vertices $r$
        for all vertices $u$
            for all vertices $v$
                if $dist[u, v] > dist[u, r] + dist[r, v]$
                    $dist[u, v] \leftarrow dist[u, r] + dist[r, v]$
```

$O(n^3)$

Edit Distance

- **You can edit a word by**
  - Insert, Delete, Replace
- **Edit distance**
  - Minimum number of edit operations
- **Problem**
  - Given two strings, compute the edit distance

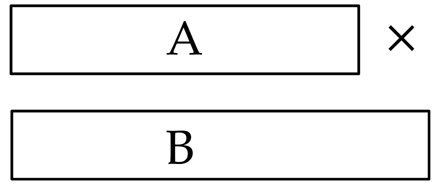
The edit distance is 4

	F	O	O		D
	M	O	N	E	Y
4 op:	**R**		**R**	**I**	**R**
3 op:	not possible				

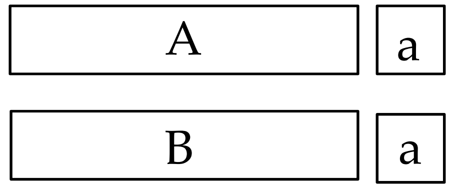
"BF" Recursion

- **Case 1**
  - 1.1 Insert
  - 1.2: dual of case 1.1
- **Case 2**
  - 2.1 a=a
  - 2.2 a≠b

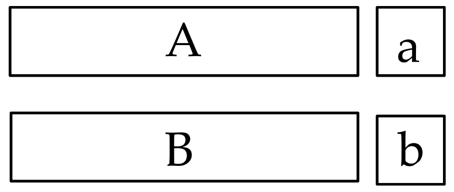
Case 1.1



Case 2.1



Case 2.2



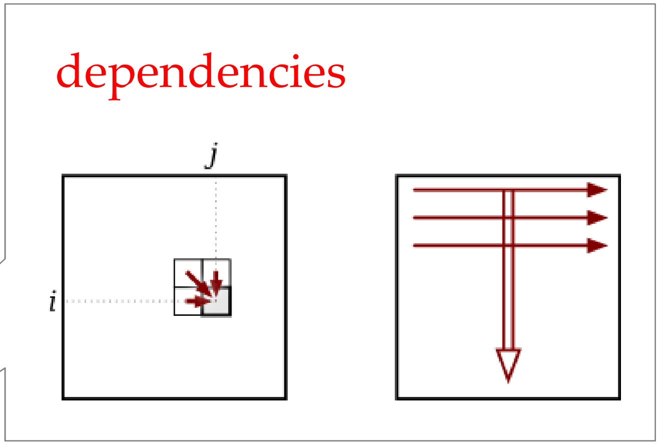
“BF” Recursion

- **EditDis(i,j)**
  - Base case:
    - If i=0, EditDis(i,j)=j
    - If j=0, EditDis(i,j)=i
  - Recursion:

$$EditDis(A[1..m], B[1..n]) = min \begin{cases} EditDis(A[1..m-1], B[1..n]) + 1 \\ EditDis(A[1..m], B[1..n-1]) + 1 \\ EditDis(A[1..m-1], B[1..n-1]) + I\{A[m] \neq B[n]\} \end{cases}$$

Smart Programming

- **DP dict**
  - EditDis[1..m, 1..n]
- **DP algorithm**



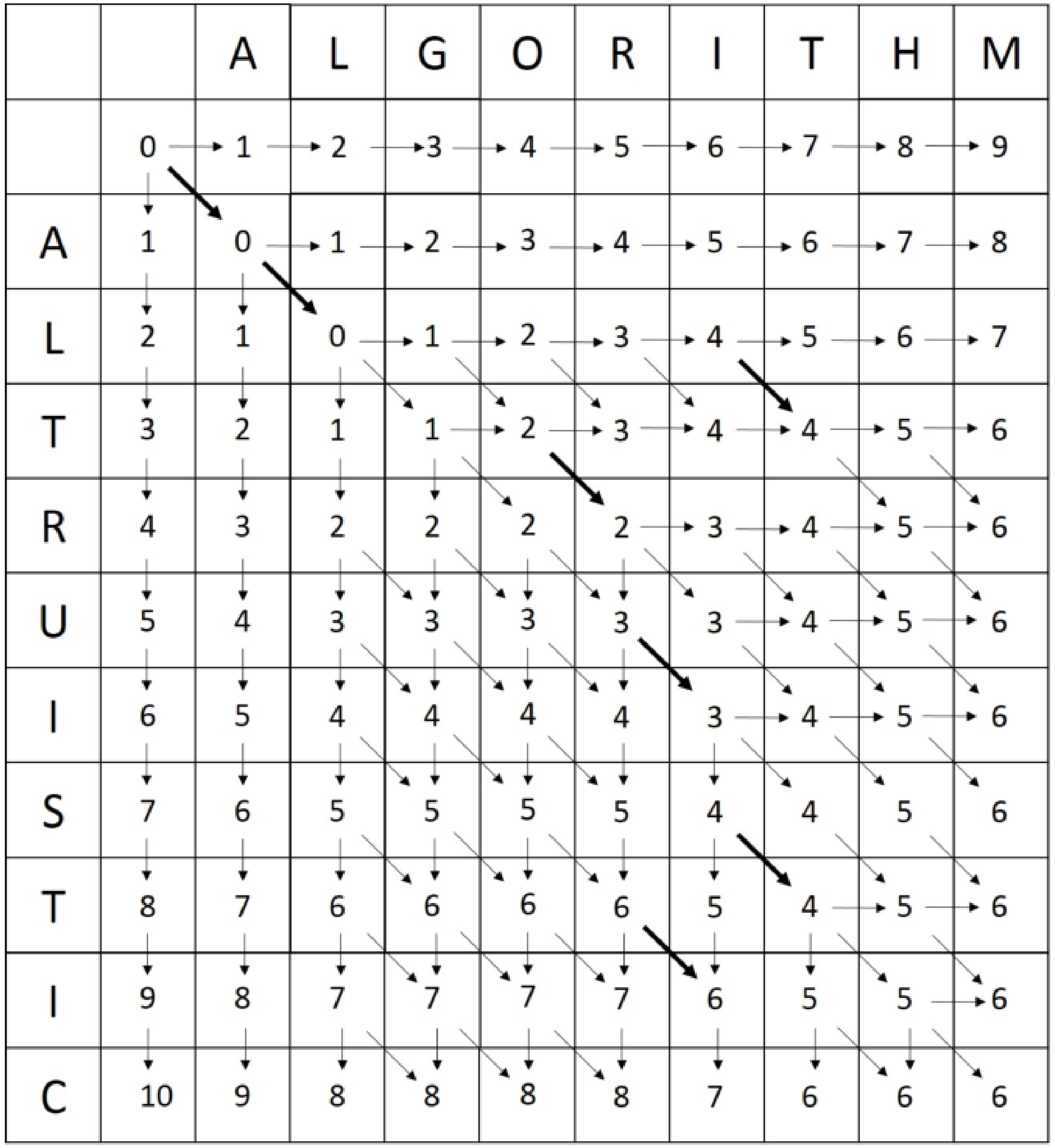
```
EDITDISTANCE($A[1..m], B[1..n]$):
  for $j \leftarrow 1$ to $n$
      $Edit[0, j] \leftarrow j$
  for $i \leftarrow 1$ to $m$
      $Edit[i, 0] \leftarrow i$
      for $j \leftarrow 1$ to $n$
          if $A[i] = B[j]$
              $Edit[i, j] \leftarrow \min\{Edit[i-1, j] + 1,\ Edit[i, j-1] + 1,\ Edit[i-1, j-1]\}$
          else
              $Edit[i, j] \leftarrow \min\{Edit[i-1, j] + 1,\ Edit[i, j-1] + 1,\ Edit[i-1, j-1] + 1\}$
  return $Edit[m, n]$
```

Example

algorithm

vs.

altruistic



		A	L	G	O	R	I	T	H	M
	0	1	2	3	4	5	6	7	8	9
A	1	0	1	2	3	4	5	6	7	8
L	2	1	0	1	2	3	4	5	6	7
T	3	2	1	1	2	3	4	4	5	6
R	4	3	2	2	2	2	3	4	5	6
U	5	4	3	3	3	3	3	4	5	6
I	6	5	4	4	4	4	3	4	5	6
S	7	6	5	5	5	5	4	4	5	6
T	8	7	6	6	6	6	5	4	5	6
I	9	8	7	7	7	7	6	5	5	6
C	10	9	8	8	8	8	7	6	6	6

DP in One Dimension

- **Highway restaurants**
  - n possible locations on a straight line
    - $m_1, m_2, m_3, \ldots, m_n$
  - At most one restaurant at one location
    - Expected profit for location i is p_i
  - Any two restaurants should be at least k miles apart
- **How to arrange the restaurants**
  - To obtain the maximum expected profit

Highway Restaurants

- **The recursion**
  - P(j): the max profit achievable using only first j locations
    - P(0)=0
  - prev[j]: largest index before j and k miles away

$$P(j) = \max(p_j + P(\texttt{prev}[j]),\ P(j-1))$$

Highway Restaurants

- **One dimension DP algorithm**
  - Fill in P[0], P[1], …, P[n]

```
(First compute the prev[$\cdot$] array)
$i = 0$
for $j = 1$ to $n$:
    while $m_{i+1} \leq m_j - k$:
        $i = i + 1$
    prev[$j$] $= i$
(Now the dynamic programming begins)
$P[0] = 0$
for $j = 1$ to $n$:
    $P[j] = \max(p_j + P[$prev$[j]], P[j-1])$
return $P[n]$
```

Words into Lines

- **Words into lines**
  - Word-length $w_1, w_2, \ldots, w_n$ and line-width: W
- **Basic constraint**
  - If $w_i, w_{i+1}, \ldots, w_j$ are in one line, then $w_i + w_{i+1} + \ldots + w_j \leq W$
- **Penalty for one line: some function of *X*. *X* is:**
  - 0 for the last line in a paragraph, and
  - $W - (w_i + w_{i+1} + \ldots + w_j)$ for other lines
- **The problem**
  - How to make the penalty of the paragraph, which is the sum of the penalties of individual lines, minimized

Greedy Solution

i	word	w
1	Those	6
2	who	4
3	cannot	7
4	remember	9
5	the	4
6	past	5
7	are	4
8	condemned	10
9	to	3
10	repeat	7
11	it.	4

W is 17, and penalty is X^3

Solution by greedy strategy

words	(1,2,3)	(4,5)	(6,7)	(8,9)	(10,11)
X	0	4	8	4	0
penalty	0	64	512	64	0

Total penalty is **640**

An improved solution

words	(1,2)	(3,4)	(5,6,7)	(8,9)	(10,11)
X	7	1	4	4	0
penalty	343	1	64	64	0

Total penalty is **472**

Problem Decomposition

- **Representation of subproblem: a pair of indexes (*i*,*j*), breaking words *i* through *j* into lines with minimum penalty.**
- **Two kinds of subproblem**
  - o (k, n): the penalty of the last line is 0
  - o all other subproblems
- **For some *k*, the combination of the optimal solution for (1,*k*) and (*k*+1,*n*) gives a optimal solution for (1,*n*).**
- **Subproblem graph**
  - o About n^2 vertices
  - o Each vertex (i,j) has a edge to about $j - i$ other vertices, so, the number of edges is in $\Theta(n^3)$

Simpler Identification of Subproblems

- **If a subproblem concludes the paragraph, then (*k*,*n*) can be simplified as (*k*)**
  - About *k* subproblems
- **Can we eliminate the use of (*i*,*j*) with *j*<*n*?**
  - Put the first *k* words in the first line(with the basic constraint satisfied), the subproblem to be solved is (*k*+1,*n*)
  - Optimizing the solution over all *k*'s. (*k* is at most *W*/2)

Breaking into Lines

```
lineBreak($w,W,i,n,L$)
    if ($w_i + w_{i+1} + \dots + w_n \le W$)
        <Put all words on line $L$, set penalty to 0>
    else
        for (k=1; $w_i + \dots + w_{i+k-1} \le W$; k++)
            $X = W - (w_i + \dots + w_{i+k-1})$;
            kPenalty=lineCost($X$)+lineBreak($w,W, i+k, n, L+1$)
            <Set penalty always to the minimum k Penalty>
            <Updating $k_{min}$, which records the $k$ that produced
                                the minimum penalty>
            <Put words $i$ through $i+k_{min}-1$ on line $L$>
    return penalty
```

In *DP* version this is replaced by "**Recursion or Retrieve**"

In DP version, "**Storing**" inserted

Converted to DP

- **Subproblem Dictionary: P[1], P[2], …, P[n]**
- **lineBreakDP**

```
for(i=n; i>=1;i--){
    if (all words in one line){
        P[i]=0;
        continue;
    } // fi
    while (sum of word length <= W){
        // assume we have k words put in the current line
        calculate the penalty $Cost_{cur}$ of putting k words in this line;
        minCost=min{minCost, $Cost_{cur}$+P[i+k]} if necessary;
    } // while
    P[i]=minCost;
} // for
```

Analysis of lineBreak()

- **Each subproblem is identified by only one integer *k*, for (*k*,*n*)**
  - Number of vertex in the subproblem graph: at most *n*
  - So, in DP version, the recursion is executed at most *n* times.
- **So, the running time is in** $\Theta(Wn)$
  - The loop is executed at most *W*/2 times.
  - In fact, *W*, the line width, is usually a constant. So, $\boldsymbol{\Theta(n)}$.
  - The extra space for the dictionary is in $\boldsymbol{\Theta(n)}$.

Making Change: Revisited

- **How to pay a given amount of money?**
  - Using the smallest possible number of coins
  - With certain systems of coinage

- **We have known that the greedy strategy fails sometimes**

Subproblems

- **Assumptions**
  - Given n different denotations
  - A coin of denomination i has d_i units
  - The amount to be paid: N.
- **Subproblem [*i*,*j*]**
  - The minimum number of coins required to pay an amount of j units, using only coins of denominations 1 to i.
- **The problem**
  - Figure out subproblem [n, N] (as $c[n,N]$)

Dependency of Subproblems

- $c[i,0]$ is 0 for all i
- When we are to pay an amount j using coins of denominations 1 to i, we have two choices:
  - No coins of denomination i is used: $c[i\text{-}1, j]$
  - One coins of denomination i is used: $1+c[i, j\text{-}d_{\mathrm{i}}]$
- So, $c[i,j] = \min\,(c[i\text{-}1, j], 1+c[i, j\text{-}d_{\mathrm{i}}])$

Data Structure

Define a array coin[1..n, 0..N] for all $c[i, j]$

an example

$$\begin{array}{c|ccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline d_1=1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ d_2=4 & 0 & 1 & 2 & 3 & 1 & 2 & 3 & 4 & 2 \\ d_3=6 & 0 & 1 & 2 & 3 & 1 & 2 & 1 & 2 & 2 \end{array}$$

direction of computation

The Procedure

```
int coinChange(int $N$, int $n$, int[] coin)
    int denomination[]=[$d_1,d_2,\ldots,d_n$];
    for ($i$=1; $i \le n$; $i$++)
        coin[$i$,0]=0;
    for ($i$=1; $i \le n$; $i$++)
        for ($j$=1; $i \le N$; $j$++)
            if ($i$= =1 && $j$<denomination[$i$])  coin[$i,j$]=$+\infty$ ;
            else if ($i$= =1) coin[$i,j$]=1+coin[1, $j$-denomination[1]];
            else if ($j$<denomination[$i$]) coin[$i,j$]=cost[$i$-1, $j$];
            else coin[$i,j$]=min(coin[$i$-1, $j$], 1+coin[$i$, $j$-denomination[$i$];
    return coin[$n,N$];
```

in $\Theta(nN)$,

n is usually a constant

Other DP Problems

- **Text string problems**
  - Longest common subsequence, …
  - Variations of standard text string problems
- **One dimensional problems**
  - Arrangements along the road, …
- **Graph problems**
  - Shortest paths over DAGs, …
- **Hard problems**
  - Knapsack problems and variations, …

Principle of Optimality

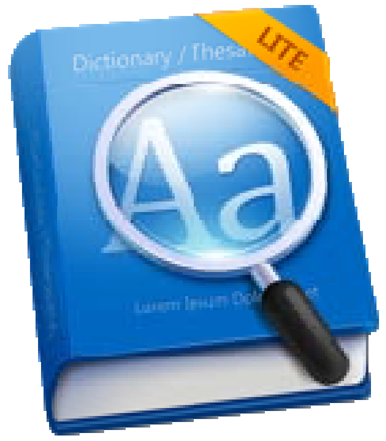
- **Given an optimal sequence of decisions, each *subsequence* must be *optimal by itself*.**
  - Positive example: shortest path
  - Counterexample: [illegible] e) path
- **DP relies on the principle of optimality**
  - The optimal solution to any nontrivial instance of a problem is a combination of optimal solutions to some of its sub-instances.
  - It is often not obvious which sub-instances are relevant to the instance under consideration.

Optimal Substructure

Elements of Dynamic Programming

- **Symptoms of DP**
  - Overlapping subproblems
  - Optimal substructure
- **How to use DP**
  - **"Brute force"** recursion
    - Overlapping subproblems
  - **Smart** programming
    - Topological ordering of subproblems

DP Dictionary



Thank you!

Q & A

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