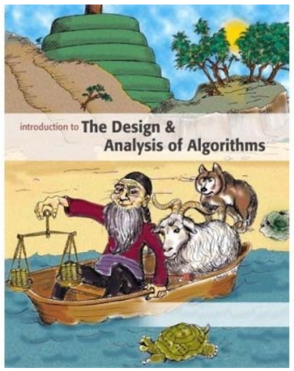
Introduction to

Algorithm Design and Analysis

[6] MergeSort



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In the Last Class…

- **Heap**
  - o Partial order property
    - FixHeap
    - ConstructHeap
  - o Heap structure
    - Array-based implementation
- **HeapSort**
  - o Complexity
  - o Accelerated HeapSort

MergeSort

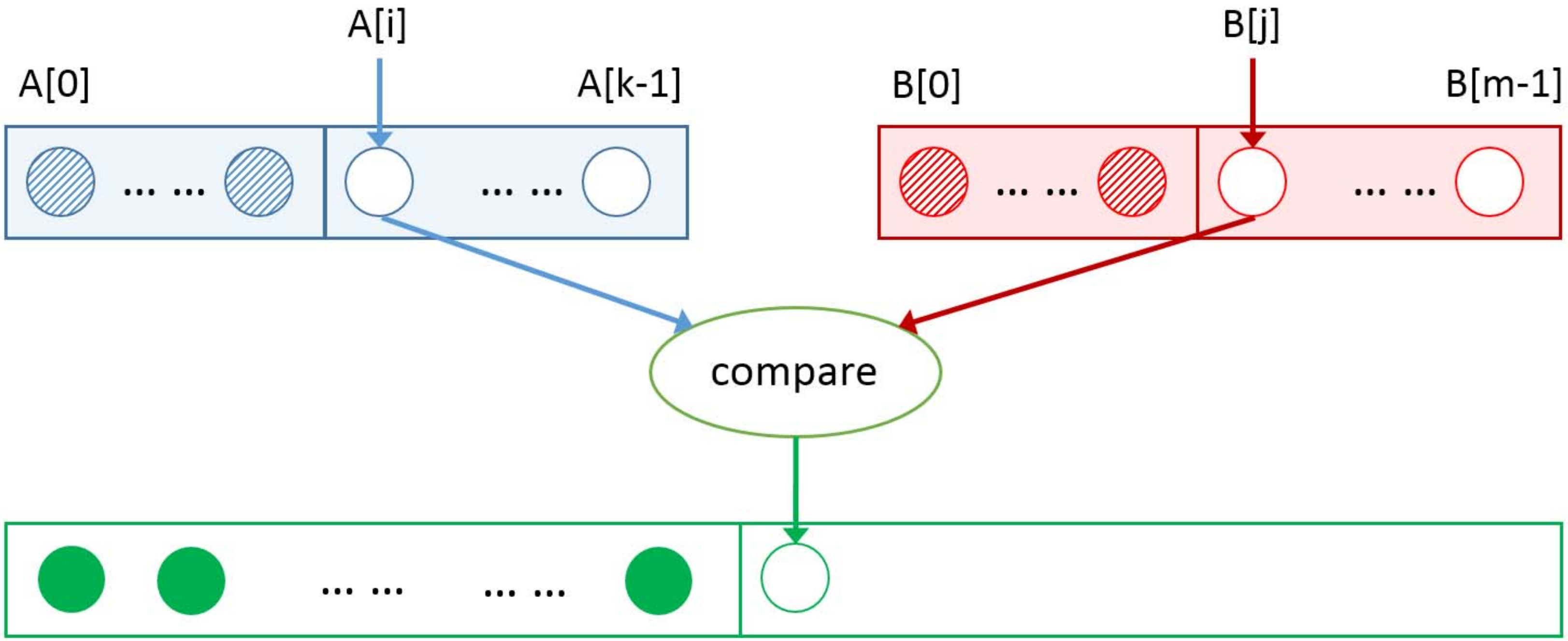
- **MergeSort**
  - Worst-case analysis of MergeSort

- **Lower Bounds for *comparison-based sorting***
  - Worst-case
  - Average-case

MergeSort: the Strategy

- **Easy division**
  - No comparison is conducted during the division
  - Minimizing the size difference between the divided subproblems
- **Merging two sorted subranges**
  - Using *Merge*

Merging Sorted Arrays



Merge: the Specification

- **Input**
  - Array A with k elements and B with m elements, whose keys are in non-decreasing order

- **Output**
  - Array C containing $n = k + m$ elements from A and B in non-decreasing order
  - C is passed in and the algorithm fills it

Merge: Recursive Version

merge(A,B,C)
  if (A is empty)
    rest of C = rest of B
  else if (B is empty)
    rest of C = rest of A
  else
    if (first of $A \leq$ first of B)
      first of C =first of A
      merge(rest of A, B, rest of C)
    else
      first of C =first of B
      merge(A, rest of B, rest of C)
  return

Base cases

Worst Case Complexity of Merge

- **Observations**
  - o Worst case is that the last comparison is conducted between A[k-1] and B[m-1]
    - After each comparison, one element is inserted into Array C, ***at least***.
    - After entering Array C, an element will never be compared again
    - After the last comparison, at least two elements (the two just compared) have not yet been moved to Array C. So at most n-1 comparisons are done.
- **In worst case, *n-1* comparisons are done, where n=k+m**

Optimality of Merge

- **Any algorithm to merge two sorted arrays, each containing k=m=n/2 entries, by comparison of keys, does at least n-1 comparisons in the worst case.**
  - Choose keys so that:

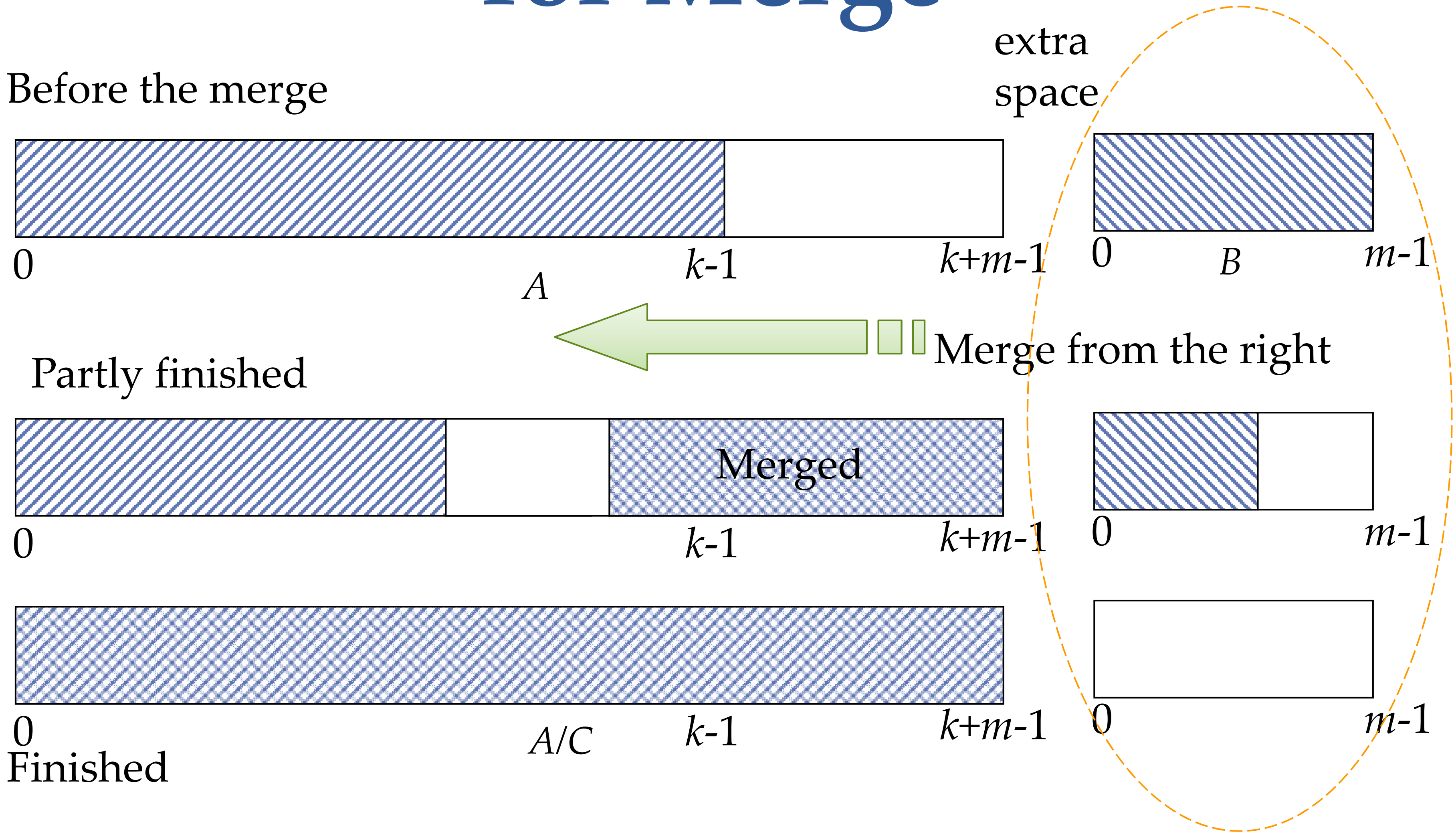
    $$b_0<a_0<b_1< a_1<\ldots<b_i<a_i<b_{i+1},\ldots,<b_{m-1}<a_{k-1}$$
  - Then the algorithm must compare a_i with b_i for every i in [0,m-1], and must compare a_i with b_{i+1} for every i in [0, m-2], so, there are n-1 comparisons.

Valid for $|k\text{-}m|\leq 1$, as well.

Space Complexity of Merge

- A algorithm is “in space”, if the extra space it has to use is in $\Theta(1)$
- Merge *is not* a algorithm “in space”, since it need enough extra space to store the merged sequence during the merging process.

Overlapping Arrays for Merge



MergeSort

- **Input: Array *E* and indexes *first*, and *last*, such that the elements of *E*[*i*] are defined for *first*≤*i*≤*last*.**
- **Output: E[*first*],…,E[*last*] is a sorted rearrangement of the same elements.**
- **Procedure**

```
void mergeSort(Element[] E, int first, int last)
    if (first<last)
        int mid=(first+last)/2;
        mergeSort(E, first, mid);
        mergeSort(E, mid+1, last);
        merge(E, first, mid, last)
    return
```

Analysis of MergeSort

- **The recurrence equation for Mergesort**
  - o $W(n)=W(\lfloor n/2 \rfloor)+W(\lceil n/2 \rceil)+n-1$
  - o $W(1)=0$

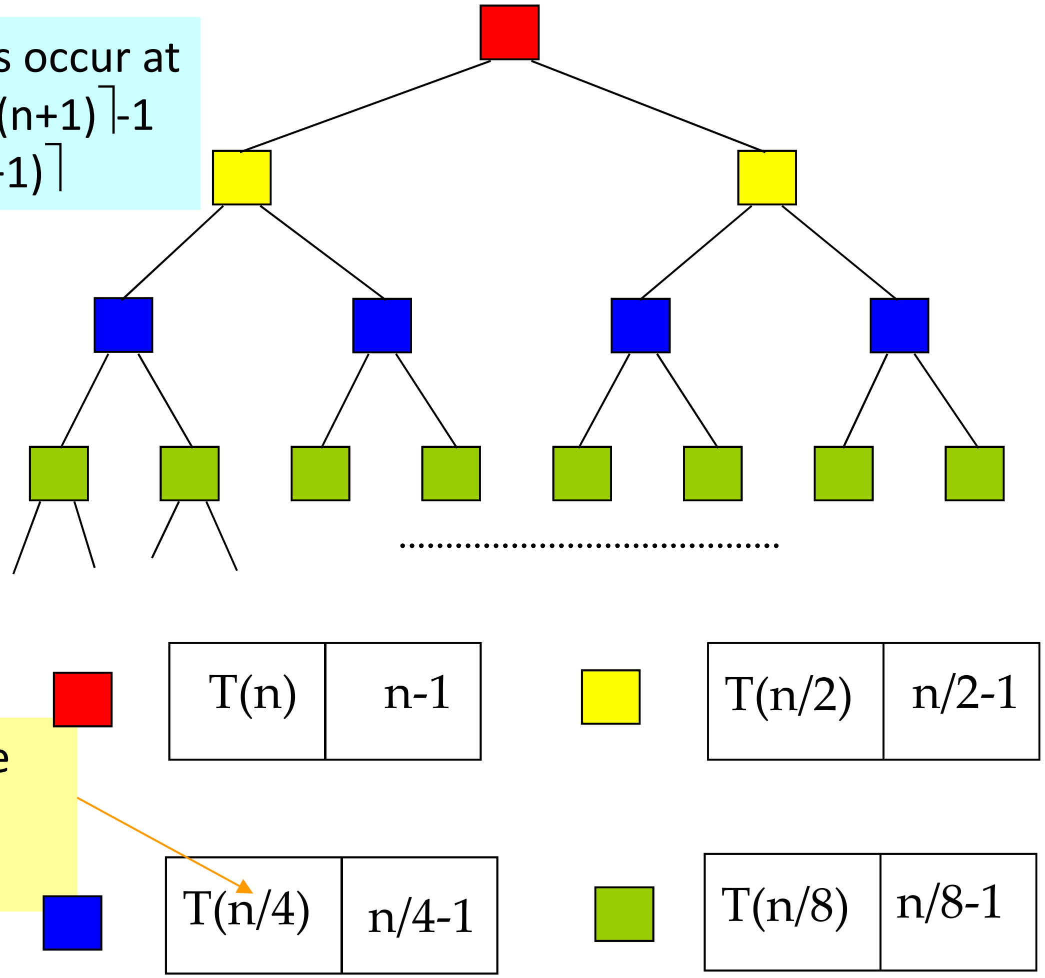
Where n=last-first+1, the size of range to be sorted

- **The *Master Theorem* applies for the equation, so:**

$$W(n) \in \Theta(n \log n)$$

Recursion Tree for Mergesort

Base cases occur at depth $\lceil \lg(n+1) \rceil$-1 and $\lceil \lg(n+1) \rceil$



n-1 Level 0

n-2 Level 1

n-4 Level 2

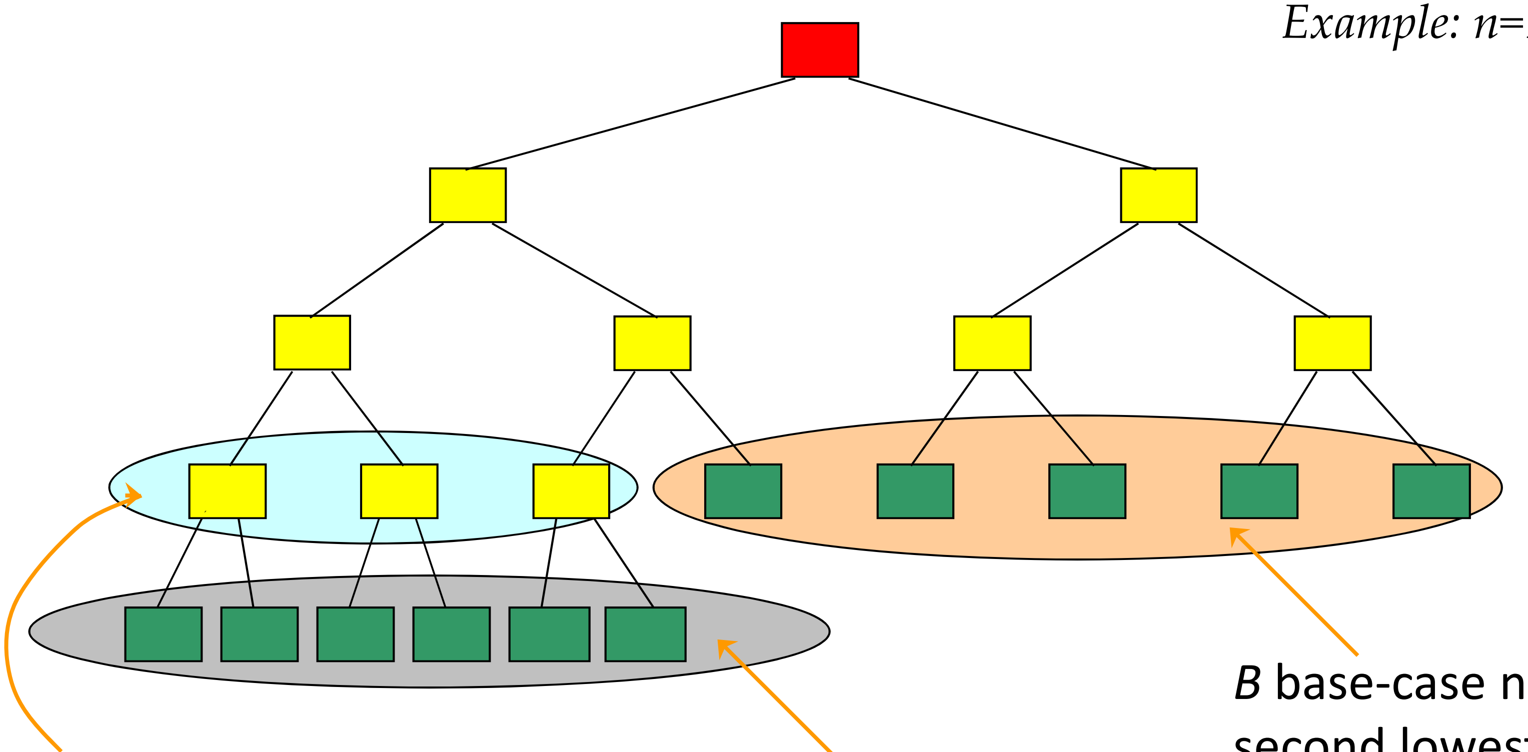
n-8 Level 3

Note:
nonrecursive costs on level k is n-2^k for all level without basecase node

k/2 may be $\lceil k/2 \rceil$ or $\lfloor k/2 \rfloor$

Non-complete Recursion Tree

Example: n=11



$2^{D\text{-}1}$ nodes

B base-case nodes on the second lowest level

Since each nonbase-case node has 2 children, there are (*n*-*B*)/2 nonbase-case nodes at depth *D*-1

n-*B* base-case nodes

No nonbase-case nodes

at this depth

Number of Comparison of MergeSort

- **The maximum depth *D* of the recursive tree is $\lceil \log(n+1) \rceil$.**
- **Let *B* base case nodes on depth *D*-1, and *n-B* on depth *D*, (Note: base case node has nonrecursive cost 0).**
- **(*n-B*)/2 nonbase case nodes at depth *D*-1, each has nonrecursive cost 1.**
- **So:**

$$W(n) = \sum_{d=0}^{D-2}(n - 2^d) + \frac{n-B}{2} = n(D-1) \quad (2^{D-1} - 1) + \frac{n-B}{2}$$

$$Since \ \ (2^D - 2B) + B = n, \ that \ \ is \ \ B = 2^D - n$$

$$So, W(n) = nD - 2^D + 1$$

$$Let \ \ \frac{2^D}{n} = 1 + \frac{B}{n} = \alpha, \ then \ \ 1 \leqslant \alpha < 2, \quad D = \log n + \log \alpha$$

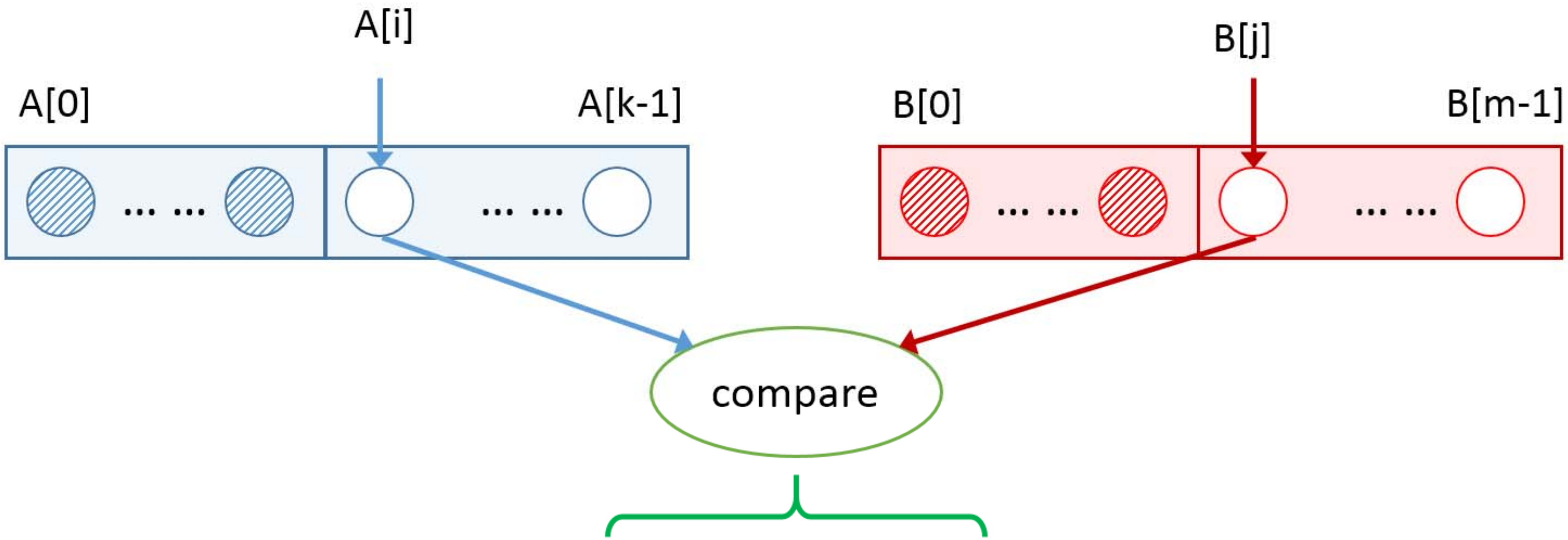
$$So, W(n) = n \log n - (\alpha - \log \alpha) n + 1$$

- **$\lceil n\log(n)-n+1 \rceil \leq$ number of comparison $\leq \lceil n\log(n)-0.914n \rceil$**

The MergeSort D&C

- **Counting the number of inversions**
  - o Brute force: O(n^2)
  - o Can we use divide & conquer
    - In O(*nlogn*) => combination in O(n)
- **MergeSort as the carrier**
  - o Sorted subarrays
    - A[0..k-1] and B[0..m-1]
  - o Compare the *left* and the *right* elements
    - A[i] vs. B[j]

The MergeSort D&C

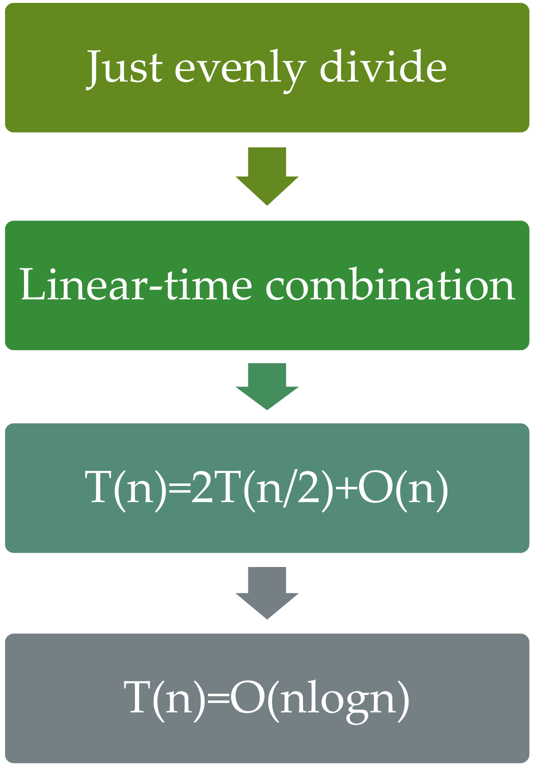


- If A[i]>A[j]
  - (i,j) is an inversion
  - All (i’, j) are inversions (i’>i)
  - A[j] is selected

- If A[i]<A[j]
  - No inversion found
  - A[i] is selected

The MergeSort D&C

- **Max-sum subsequence**
- **Counting inversions**
- **Finding the *frequent* element**
- **Find the nearest two points on the plane**
- **…**

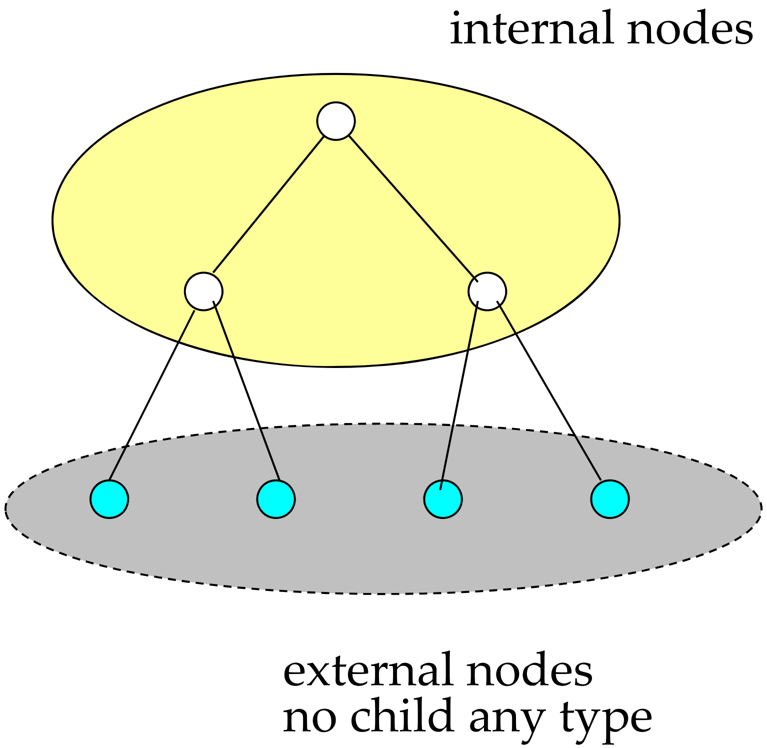


Lower Bounds for Comparison-based Sorting

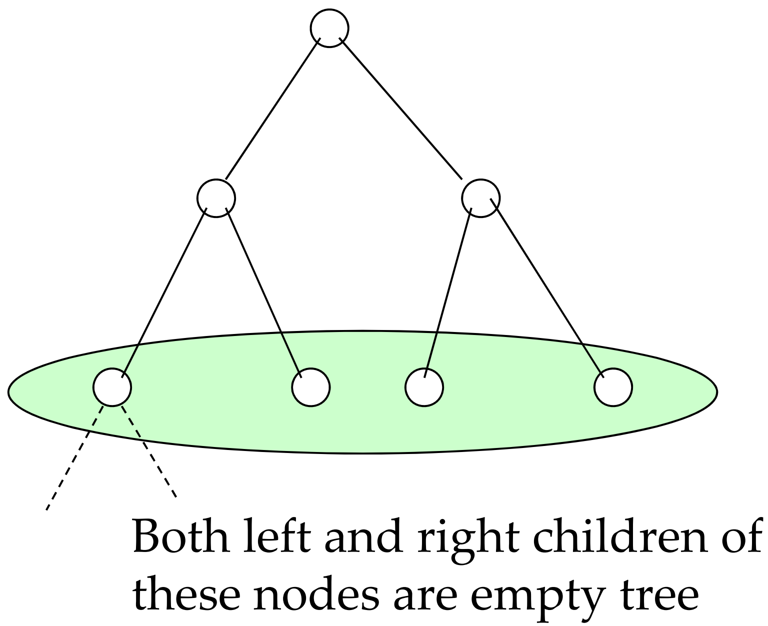
- **Upper bound, e.g., worst-case cost**
  - For any possible input, the cost of the specific algorithm A is no more than the *upper bound*
    - $Max\{Cost(i) \mid i \text{ is an input}\}$
- **Lower bound, e.g., comparison-based sorting**
  - For any possible (comparison-based) sorting algorithm A, the worst-case cost is no less than the *lower bound*
    - Min{*Worst-case*(a) | a is an algorithm}

2-Tree

- **2-Tree**

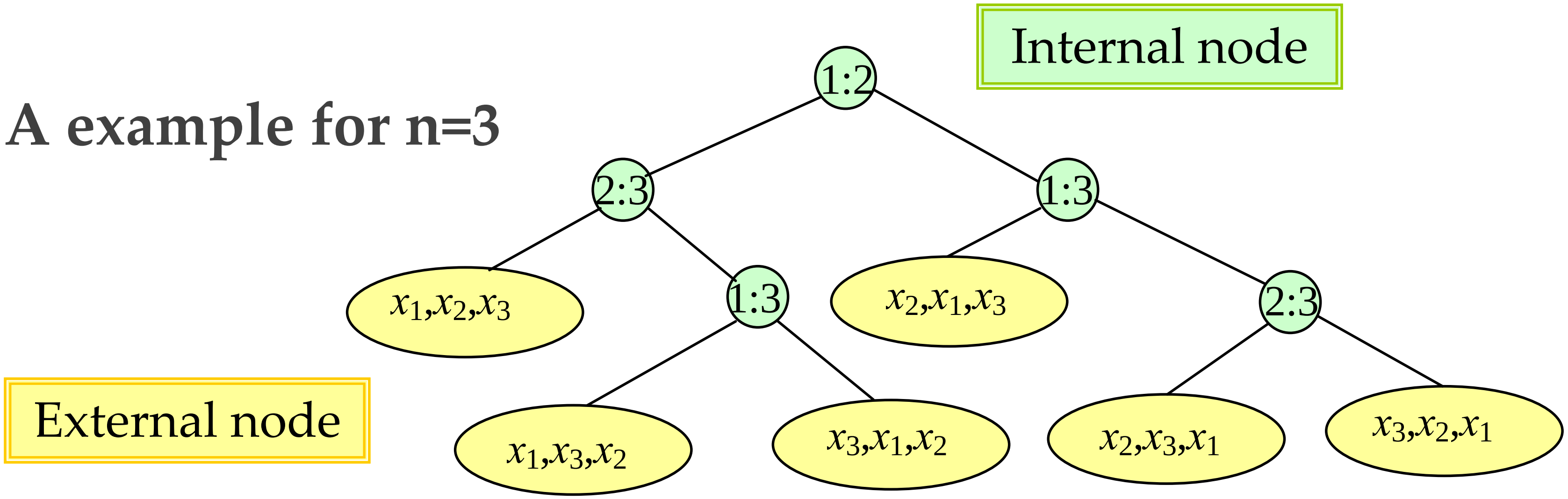


- **Common Binary Tree**



Decision Tree for Sorting

A example for n=3



- Decision tree is a 2-tree (assuming no same keys)
- The action of Sort on a particular input corresponds to following on path in its decision tree from the root to a leaf associated to the specific output

Characterizing the Decision Tree

- **For a sequence of n distinct elements, there are n! different permutation**
  - So, the decision tree has at least n! leaves, and exactly n! leaves can be reached from the root.
  - So, for the purpose of lower bounds evaluation, we use trees with exactly n! leaves.
- **The number of comparison done in the *worst case* is the height of the tree.**
- **The *average* number of comparison done is the average of the lengths of all paths from the root to a leaf.**

Lower Bound for Worst Case

- ***Theorem***: **Any algorithm to sort n items by comparisons of keys must do at least $\lceil \log n! \rceil$, or approximately $\lceil n\log n - 1.443n \rceil$, key comparisons in the worst case.**
  - Note: Let L=n!, which is the number of leaves, then L≤2^h, where h is the height of the tree, that is h≥$\lceil \log L \rceil = \lceil \log n! \rceil$
    - Lemma: let L be the number of leaves in a binary tree and h be its height. Then L≤2^h
  - For the asymptotic behavior:

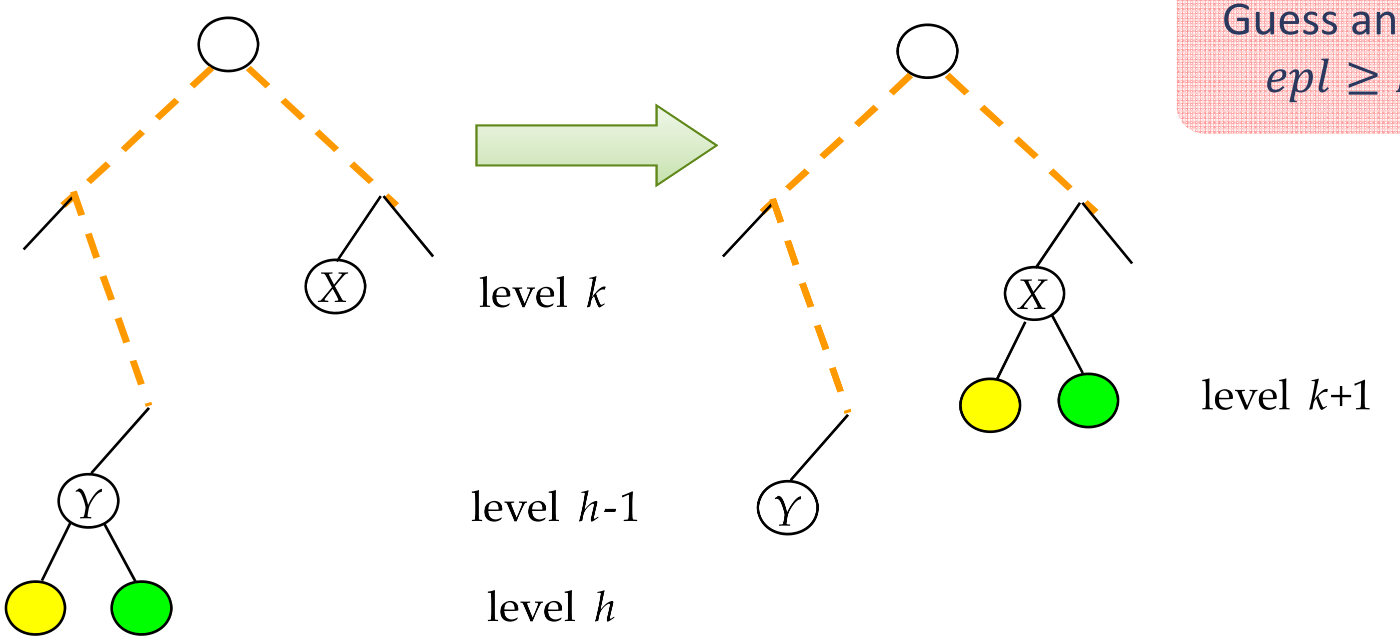
$$\log(n!) \geq \log(n(n-1)\cdots(\lceil \frac{n}{2} \rceil)) \geq \log(\frac{n}{2})^{\frac{n}{2}} = \frac{n}{2}\log(\frac{n}{2}) \in \Theta(n \log n)$$

External Path Length(EPL)

- **EPL – sum of path length to every leaf**
  - The EPL t is recursively defined as follows:
  - [Base case] 0 for a single external node
  - [Recursion] *t* is non-leaf with sub-trees *L* and *R*, then the sum of:
    - the external path length of *L*;
    - the number of external node of *L*;
    - the external path length of *R*;
    - the number of external node of *R*;

More Balanced 2-tree, Less EPL

Guess and prove:
$epl \geq L \log L$



Assuming that h-k>1, when calculating *epl*, h+h+k is replaced by (h-1)+2(k+1). The net change in *epl* is k-h+1<0, that is, the *epl* decreases.

Properties of EPL

- **Let t be a 2-tree, then the epl of t is the sum of the paths from the root to each external node.**
- **$epl \geq m\log(m)$, where m is the number of external nodes in t**
  - o $epl = epl_L + epl_R + m \geq m_L\log(m_L) + m_R\log(m_R) + m$,
    - note $f(x)+f(y) \geq 2f((x+y)/2)$ for $f(x)=x\log x$
  - o so,

    $epl \geq 2((m_L+m_R)/2)\log((m_L+m_R)/2)+m$

    $= m(\log(m)-1)+m = m\log m$.

p. 116, Lemma 3.7 of [Baase01]

Lower Bound for Average Behavior

- **Since a decision tree with L leaves is a 2-tree, the average path length from the root to a leaf is $\frac{epl}{L}$.**
  - Recall that $epl \geq L\log(L)$.

- **Theorem: The average number of comparison done by an algorithm to sort n items by comparison of keys is at least $\log(n!)$, which is about $n\log n$-1.443n.**

MergeSort Has Optimal Average Performance

- The **average** number of comparisons done by an algorithm to sort n items by comparison of keys is at least about $n\log n$-1.443n
- The **worst** complexity of MergeSort is in $\Theta(n\log n)$
- So, MergeSort is optimal as for its average performance

Thank you!

Q & A

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