Decompositions of Graphs

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Decompositions of Graphs

Turing Award

John Hopcroft

Robert Tarjan

"For fundamental achievements in the design and analysis of algorithms and data structures."

— *Turing Award, 1986*

Depth-first search

SIAM J. COMPUT.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

Abstract. The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an undirect graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants k_1, k_2, and k_3, where V is the number of vertices and E is the number of edges of the graph being examined.

Key words. Algorithm, backtracking, biconnectivity, connectivity, depth-first, graph, search, spanning tree, strong-connectivity.

"We have seen how the depth-first search method may be used in the construction of very efficient graph algorithms. ...
Depth-first search is a powerful technique with many applications."

Reference

- "Depth-First Search And Linear Graph Algorithms" by Robert Tarjan.

The POWER of DFS

Graph decomposition vs. Graph traversal

Structures!

1. states of vertices
2. types of edges
3. lifetime of vertices (DFS)
   - v : $\mathsf{d}[v], \mathsf{f}[v]$
   - $\mathsf{f}[v]$: DAG, SCC
   - $\mathsf{d}[v]$: biconnectivity

Types of edges

Definition (Classifying edges)

Given a DFS/BFS traversal $\Rightarrow$ DFS/BFS tree:

Tree edge: $\to$ child

Back edge: $\to$ ancestor

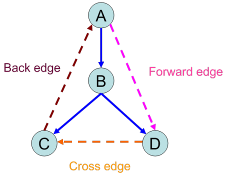
Forward edge: $\to$ *nonchild* descendant

Cross edge: $\to$ neither ancestor nor descendant

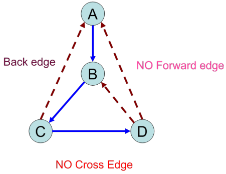
Remarks

- applicable to both DFS and BFS
- w.r.t. DFS/BFS trees

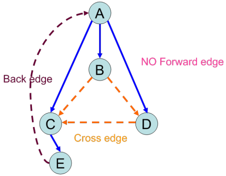
Types of edges (Problem 5.18)



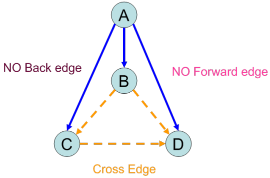
(a) DFS on directed graph.



(b) DFS on undirected graph.



(c) BFS on directed graph.



(d) BFS on undirected graph.

Types of edges

DFS tree and BFS tree coincide (Additional Problem)

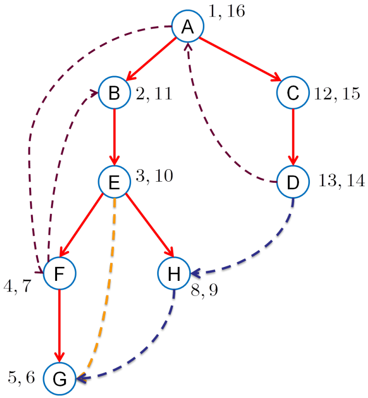
- undirected connected graph $G = (V, E), v \in V$
- DFS tree T from $v \equiv$ BFS tree T' from v
- prove: $G = T$

G_{DFS}: tree + back *vs.* G_{BFS}: tree + cross

Question

- DFS&BFS from different $v's$?
- What if G is a digraph?

Lifttime of vertices in DFS

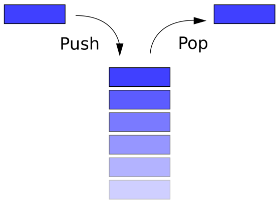


Lifttime of vertices in DFS

Theorem (Disjoint or contained)

$$\forall u, v :$$
$$[_u \;]_u \cap [_v \;]_v = \emptyset$$
$$\bigvee$$
$$([_u \;]_u \subsetneq [_v \;]_v \vee [_v \;]_v \subsetneq [_u \;]_u)$$

Proof.



Ancestor/descendant relation

Preprocessing for ancestor/descendant relation (Problem 5.23)

- binary tree $T = (V, E)$
- $r \in V$

$$v : \mathsf{d}[v], \mathsf{f}[v]$$

Question

$\forall v$: how many descendants?

Remark

General (rooted) tree?

Edge types and lifetime of vertices in DFS

Edge types and lifetime of vertices in DFS (Problem 5.2)

$\forall u \to v$:

- tree/forward edge: $[_u \; [_v \;]_v \;]_u$
- back edge: $[_v \; [_u \;]_u \;]_v$
- cross edge: $[_v \;]_v \; [_u \;]_u$

Remark

- $\mathsf{f}[v] < \mathsf{d}[u]$: cross edge
- $\mathsf{f}[u] < \mathsf{f}[v]$: back edge

$$u \to v \iff \mathsf{f}[v] < \mathsf{f}[u]$$

Height and diameter of tree

Height and diameter of tree (Problem 5.21)

Binary tree $T = (V, E)$ with $|V| = n$:

- height $(O(n))$
- diameter $(O(n))$

throught root or not?

Question

Diameter of a tree *without* a designated root?

Perfect subtree

Perfect subtree (Problem 5.22)

- binary tree $T = (V, E)$
- root $r \in V$
- goal: find all perfect subtrees

Counting shortest paths

Counting shortest paths (Problem 5.26)

Counting # of shortest paths in (un)directed graphs using BFS.

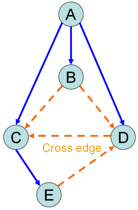
Maybe in the next class...

Decompositions of Graphs

Cycle detection

Cycle detection (Problem 5.24–1)

	Digraph	Undirected graph
DFS	back edge $\Longleftrightarrow$ cycle	back edge $\Longleftrightarrow$ cycle
BFS	back edge $\Longrightarrow$ cycle cycle $\not\Longrightarrow$ back edge	cross edge $\Longleftrightarrow$ cycle



Remark

How to identify back edges?

Evasiveness of acyclicity

Evasiveness of acyclicity (Problem 5.24–2)

$$\text{Evasiveness} \triangleq \text{check} \binom{n}{2} \text{edges (adjacency matrices)}$$

Is acyclicity evasive?

Hint: Kruskal

Evasiveness of connectivity

Evasiveness of connectivity (Additional Problem)

$$\text{Evasiveness} \triangleq \text{check} \binom{n}{2} \text{edges}$$

Is connectivity evasive?

Hint: Anti-Kruskal

Edge deletion

Edge deletion (Problem 5.20)

- connected, undirected graph G
- $\exists ? e \in E : G \setminus e$ is connected?
- $O(|V|)$

$$\exists \text{ cycle } \iff \exists \text{ such } e$$

$$O(m+n)$$

$$\text{tree: } |E| = |V| - 1 \implies \text{ check } |E| \geq |V|$$

Orientation of undirected graph

Orientation of undirected graph (Problem 5.9)

- undirected (connected) graph G
- edges oriented *s.t.*

$$\forall v, \text{in}[v] \geq 1$$

$$\text{orientation} \iff \exists \text{ cycle } C$$

$$\text{BFS/DFS from } v \in C$$

Shortest cycle of undirected graph

Shortest cycle of undirected graph (Problem 5.8)

Shortest cycle of G:

- DFS on G
- $\forall v$: level$[v]$
- back edge $u \to v$: level$[u]$ $-$ level$[v] + 1$

Question

What about digraphs?

Decompositions of Graphs

DAG

no back edge $\iff$ DAG $\iff$ $\exists$ topo. ordering

Toposort algorithm by Tarjan (probably), 1976

DFS on digraph, $u \to v$:

- ~~back edge:~~ $\mathsf{f}[u] < \mathsf{f}[v]$
- others: $\mathsf{f}[u] > \mathsf{f}[v]$

$$u \to v \implies \mathsf{f}[u] > \mathsf{f}[v]$$

Toposort: sort vertices in *decreasing* order of their *finish* times.

Kahn's toposort algorithm

Kahn's toposort algorithm (1962; Problem 5.11)

- queue for source vertices (in$[v] = 0$)
- repeat: dequeue v, delete it, output it

Lemma

Every DAG has at least one source (and at least one sink vertex).

Question

What if G is not a DAG?

Taking courses

Taking courses in few semesters (Problem 5.14)

- n courses
- $c_1 \to c_2$
- goal: taking courses in few semesters

critical path *OR* longest path

Remark

For general digraph, LONGEST-PATH is NP-hard.

Line up

Line up (Problem 5.16)

1. i hates j: $i \prec j$
2. i hates j: $\#i < \#j$

~~BFS~~

Hamiltonian path in DAG

Hamiltonian path in DAG (Problem 5.10)

- DAG G
- HP: path visiting each vertex once

Remark

For general (di)graph, HP is NP-hard.

$$\text{DAG: } \exists \text{ HP} \iff \exists! \text{ topo. ordering}$$

Proof.

$\Longleftarrow$: By construction. $\square$

Algorithms:

1. toposort, check edges
2. the Kahn toposort algorithm

Digraph as DAG

Digraph as DAG (Problem 5.3)

Every digraph is a dag of its SCCs.

Remark

Two tiered structure of digraphs:

- digraph $\equiv$ a dag of SCCs
- SCC: equivalence class over reachability

SCC

Kosaraju SCC algorithm, 1978

"SCCs can be topo-sorted in decreasing order of their highest finish time."

The vertice with the highest finish time is in a source SCC.

Remark

- DFS on G; DFS/BFS on G^T
- DFS on G^T; DFS/BFS on G

SCC

Kosaraju SCC algorithm, 1978 (Problem 5.4)

- 1st DFS $\overset{?}{\Longrightarrow}$ BFS
- 2nd DFS $\overset{?}{\Longrightarrow}$ BFS

One-to-all reachability

One-to-all reachability (Problem 5.12)

Digraph $G = (V, E)$:

- given $v : v \leadsto^{?} \forall u$
- $\exists? \; v : v \leadsto \forall u$

$$\text{SCC; } \exists!\text{source vertex } v \iff v \leadsto \forall u$$

Proof.

- $\Longleftarrow$: (1) source (2) $\exists!$
- $\Longrightarrow$: By contradiction.

□

Impacts of vertices

Impacts of vertices (Problem 5.13)

Digraph G:

$$\text{impact}(v) = |\{w : v \rightsquigarrow w\}|$$

- $\arg\min_v \text{impact}(v)$
- $\arg\max_v \text{impact}(v)$

$$\arg\min_v \text{impact}(v) \in \text{SCC of smallest cardinality}$$

Question

$\forall v$: computing $\text{impact}(v)$.

One-way streets

One-way streets (Problem 5.15)

Digraph G for city:

1. $\forall u, v : u \leftrightsquigarrow v$
2. $s : s \rightsquigarrow v \rightsquigarrow s$

$$(2)\ \{v \mid s \rightsquigarrow v\} \text{ is an SCC}$$

Connectivity

Connectivity (Problem 5.7)

Prove: connected undirected graph G:

$$\exists v : G \setminus v \text{ is still connected}$$

Example: strongly connected digraph G:

$$\exists v : G \setminus v \text{ is not strongly connected}$$

Example: digraph G with 2 SCCs:

$$(G + e) \text{ is not strongly connected}$$

2SAT

2SAT (Problem 5.17)

$$I : (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (\overline{x_3} \vee x_4) \wedge (\overline{x_1} \vee x_4)$$

$$\alpha \vee \beta \equiv \overline{\alpha} \to \beta \equiv \overline{\beta} \to \alpha$$

Implication graph G_I.

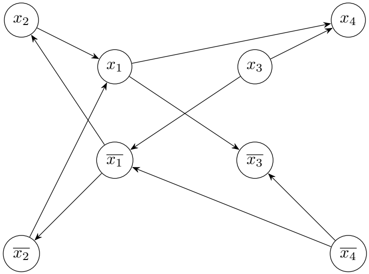
Theorem

$\exists$ SCC $\exists x : v_x \in$ SCC $\wedge\, v_{\overline{x}} \in$ SCC $\iff$ I is not satisfiable.

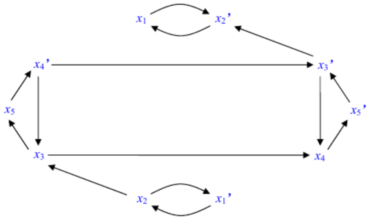
Reference

- "A Linear-time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas" by Bengt Aspvall, Michael Plass, and Robert Tarjan, 1979.

2SAT



x_2
x_4
x_1
x_3
$\overline{x_1}$
$\overline{x_3}$
$\overline{x_2}$
$\overline{x_4}$



x_1
x_2'
x_4'
x_3'
x_5
x_5'
x_3
x_4
x_2
x_1'

Odd cycle in digraph

Odd cycle in digraph (Additional Problem)

Find an odd cycle in a digraph G.

Lemma

A digraph G has an odd directed cycle $\iff \exists scc : scc$ is non-bipartite (when treated undirected).

Question

To prove the lemma and design an algorithm.

Decompositions of Graphs

Biconnectivity algorithm in one word

Back!

Biconnectivity algorithm in two questions

(1) When and how to update back$[v]$?

(2) When and how to identify a bicomponent?

Biconnectivity algorithm

Initialization of back$[v]$ (Problem 5.6)

$$\text{back}[v] = d[v] \textit{ vs. } \text{back}[v] = \infty, 2(n+1)$$

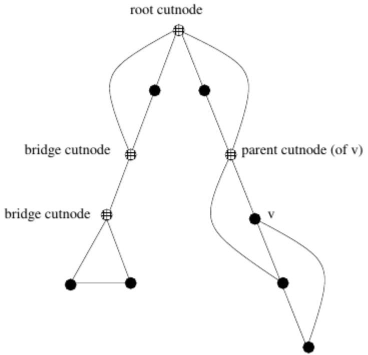
- if updated
- if never updated:

$$\text{wBack} = \infty > d[v] \textit{ vs. } \text{wBack} = d[w] > d[v]$$

Root cutnode

Root cutnode (Problem 5.5)

$$v \text{ is a cutnode} \iff \text{OutDegree}[v] \geq 2$$



K-core of a graph

Planning a party (Problem 5.27)

- undirected graph G
- subgraph $G' = (V', E')$:

$$\forall v' \in V : K(v') \geq 5 \wedge D(v') \geq 5$$

K-core of a graph:

$$\forall v' \in V' : \text{deg}[v'] \geq k$$

Iteratively delete nodes v of $K(v) < 5 \vee D(v) < 5$


THANK YOU