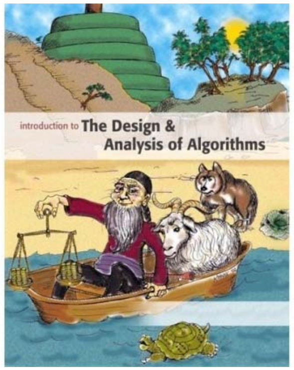
Introduction to

Algorithm Design and Analysis

[11] Graph Traversal



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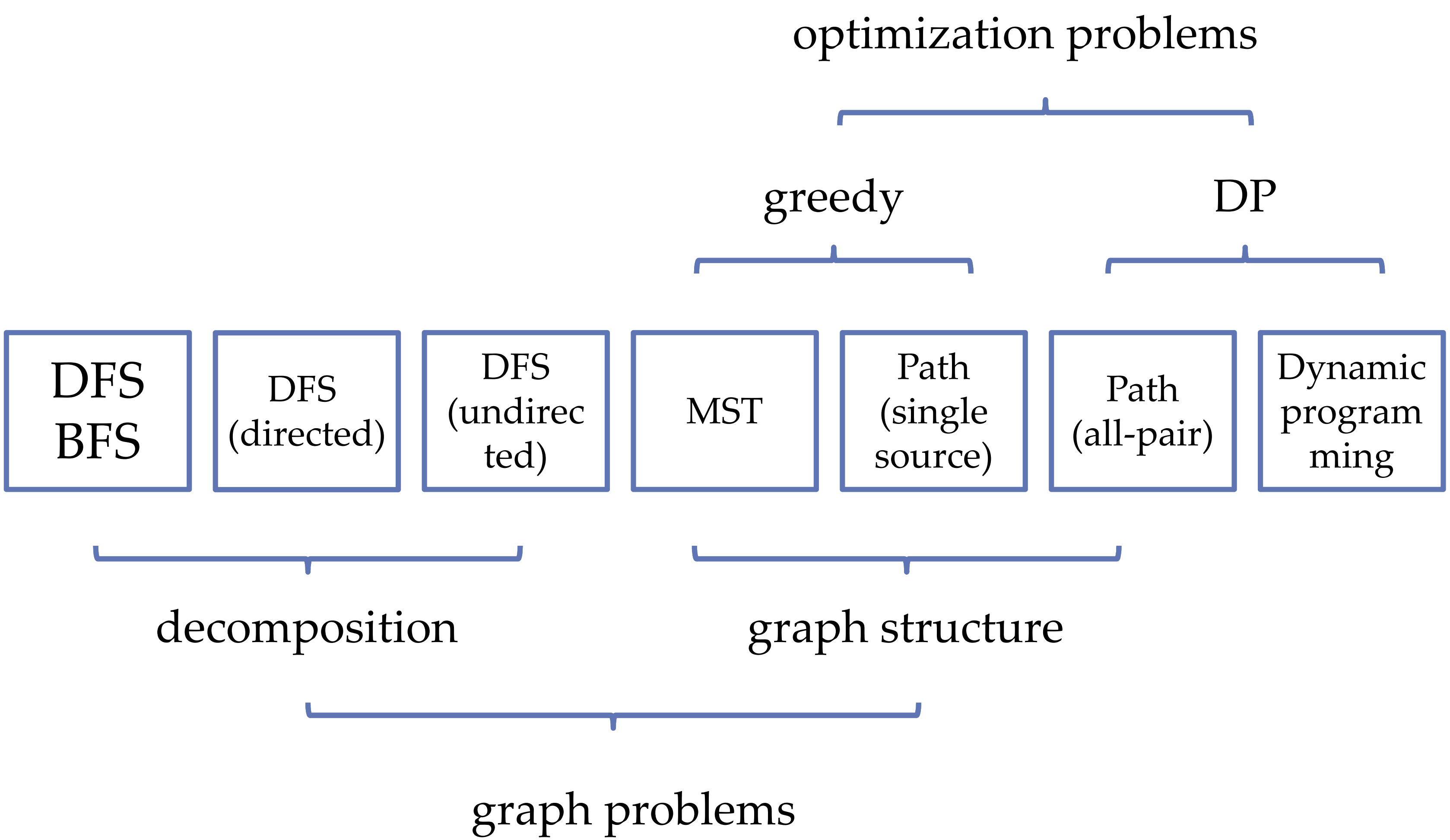
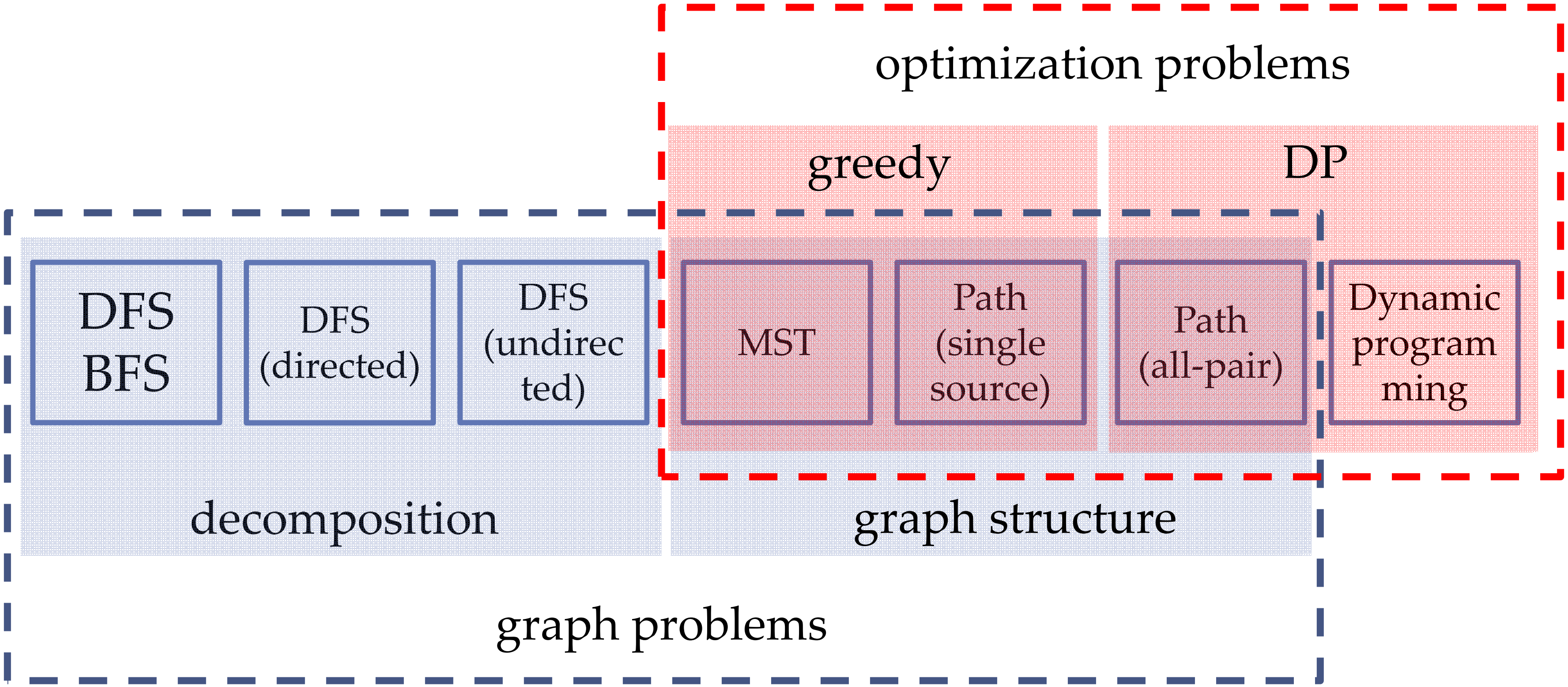
Course Contents



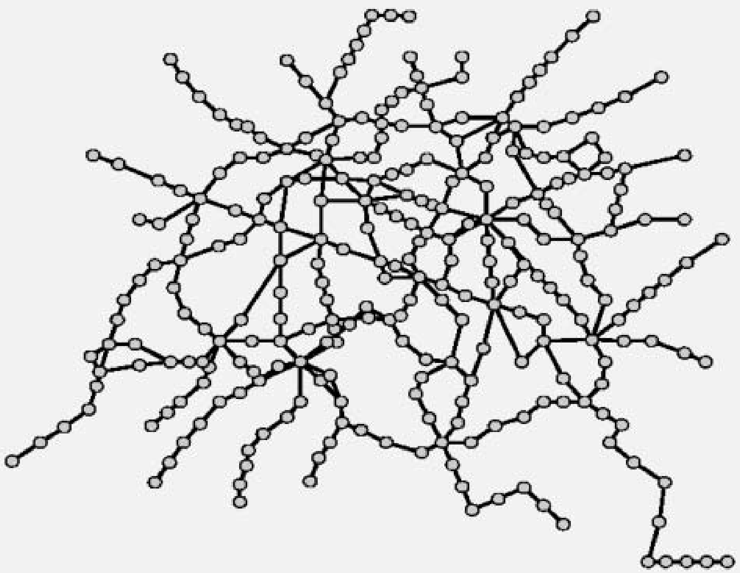
Course Contents



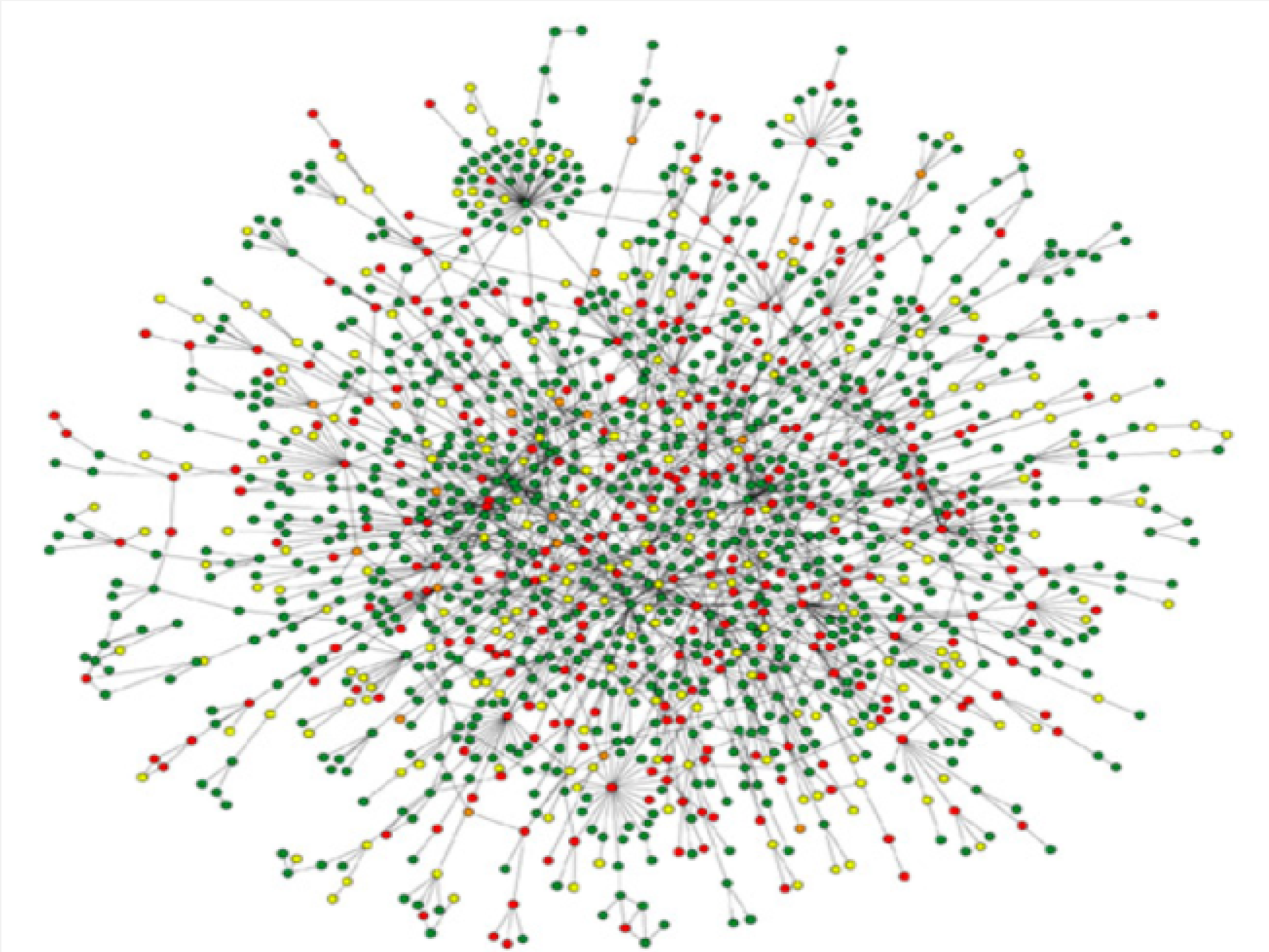
In the Last Class…

- **Dynamic Equivalence Relation**
- **Implementing *disjoint set* by Union-Find**
  - Straight Union-Find
  - Making Shorter Tree by Weighted Union
  - Compressing Path by Compressing Find
    - Amortized analysis of *wUnion-cFind*

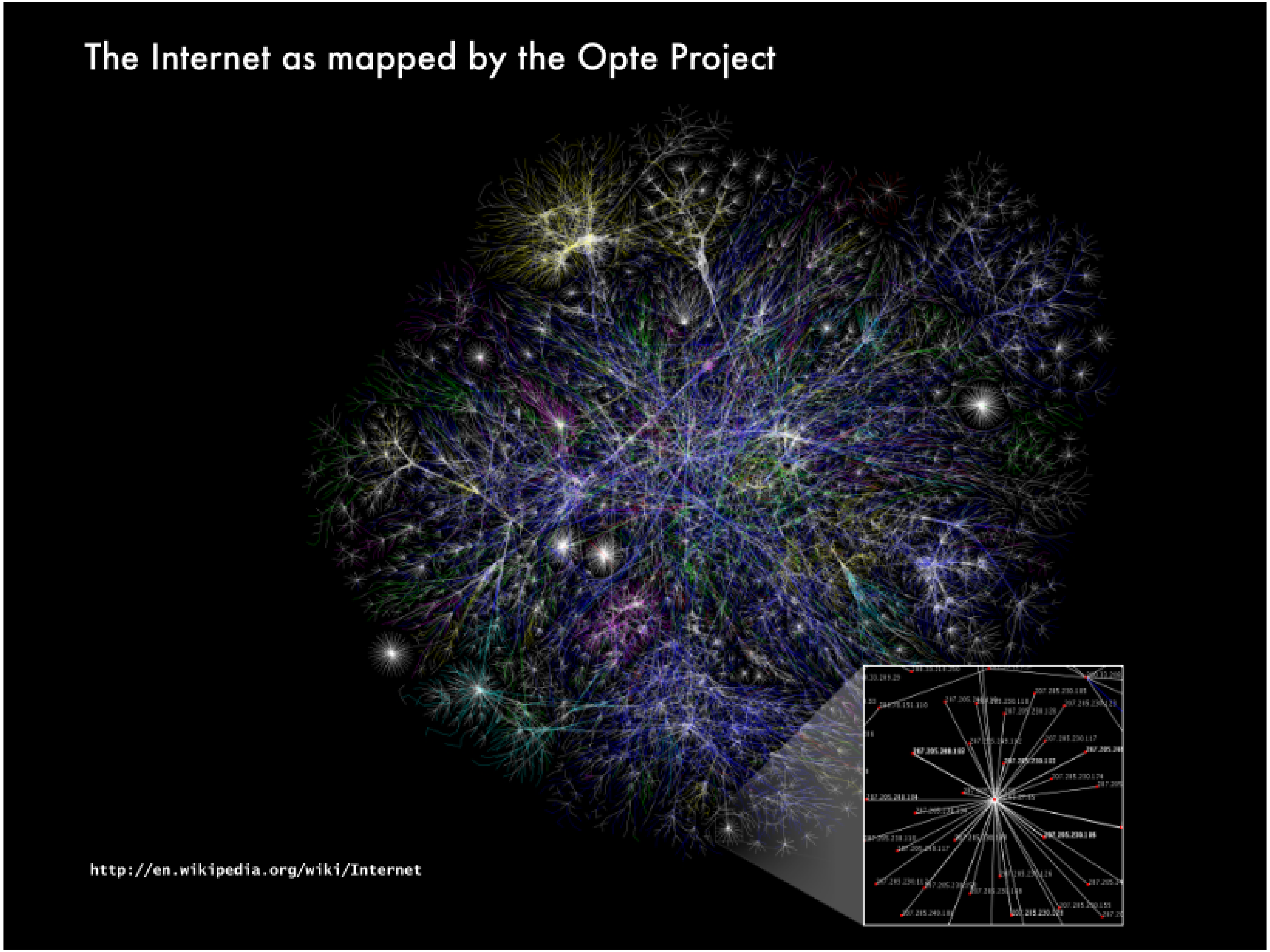
Graph Everywhere



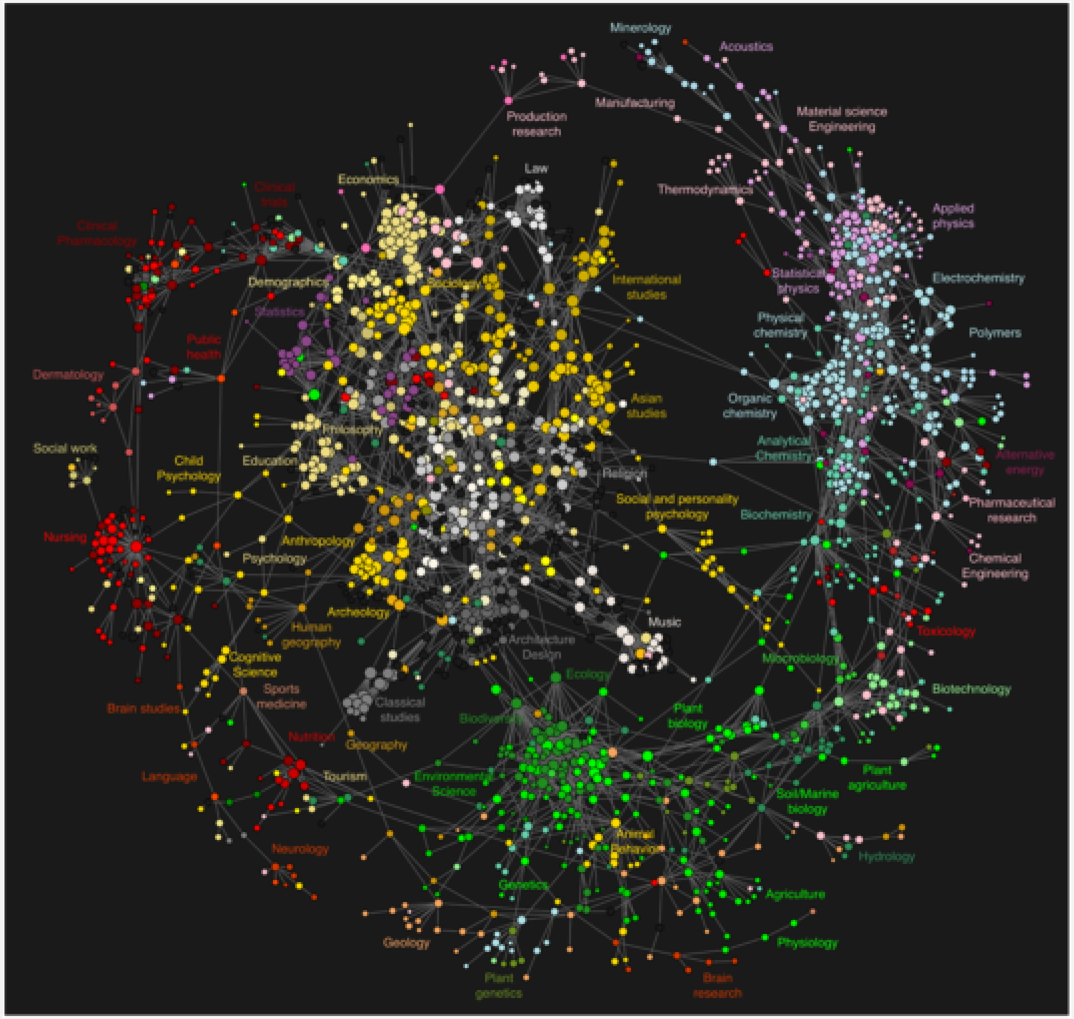
Protein-protein interaction network



Reference: Jeong et al, Nature Review | Genetics


The Internet as mapped by the Opte Project
http://en.wikipedia.org/wiki/Internet

Map of science clickstreams

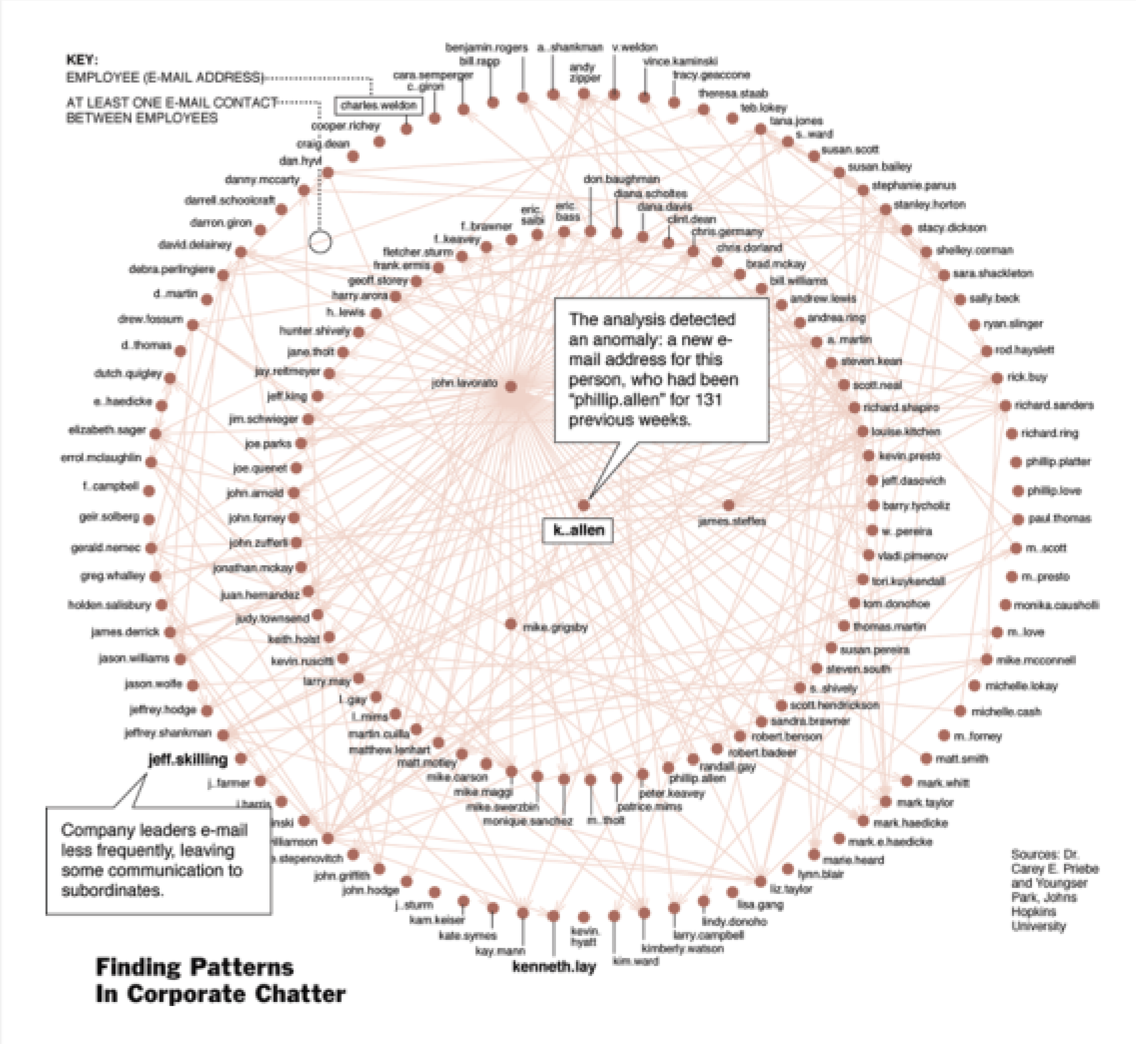


http://www.plosone.org/article/info:doi/10.1371/journal.pone.0004803

10 million Facebook friends

"Visualizing Friendships" by Paul Butler

One week of Enron emails



Framingham heart study

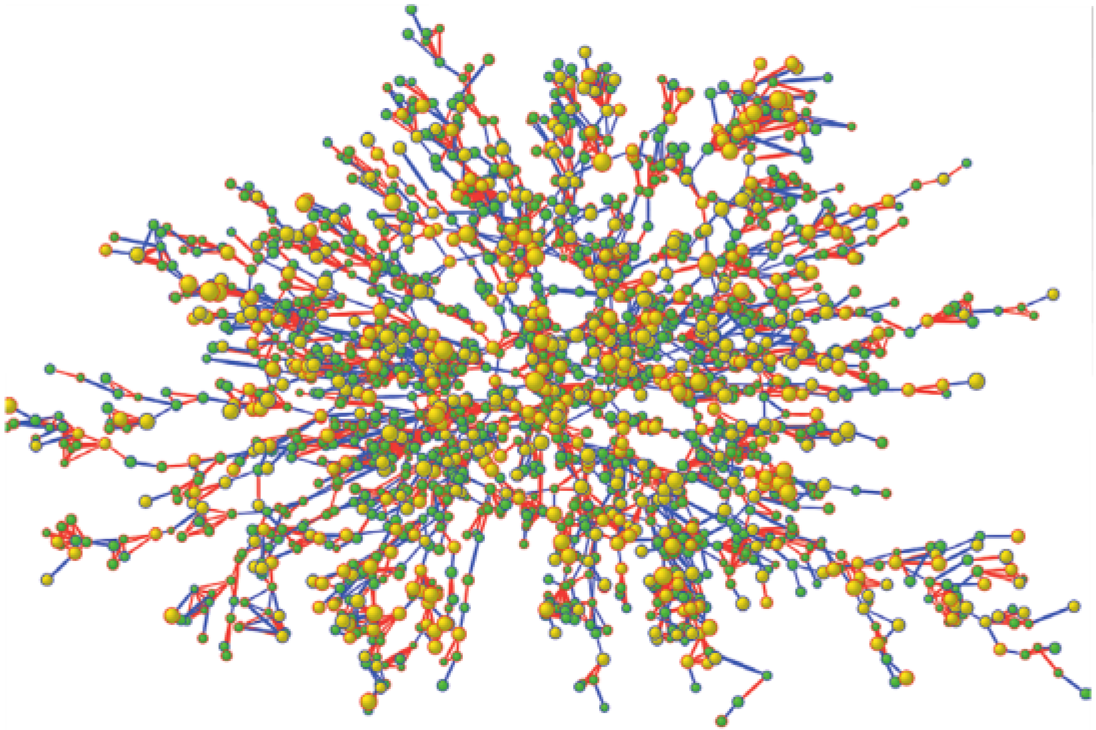


Figure 1. **Largest Connected Subcomponent of the Social Network in the Framingham Heart Study in the Year 2000.**
Each circle (node) represents one person in the data set. There are 2200 persons in this subcomponent of the social network. Circles with red borders denote women, and circles with blue borders denote men. The size of each circle is proportional to the person's body-mass index. The interior color of the circles indicates the person's obesity status: yellow denotes an obese person (body-mass index, ≥30) and green denotes a nonobese person. The colors of the ties between the nodes indicate the relationship between them: purple denotes a friendship or marital tie and orange denotes a familial tie.

"The Spread of Obesity in a Large Social Network over 32 Years" by Christakis and Fowler in New England Journal of Medicine, 2007

Graph Basics

- **Node**
  - Entities of interest
  - $V(G)$
- **Edge**
  - Relations of interest
  - $E(G) \subseteq V \times V$

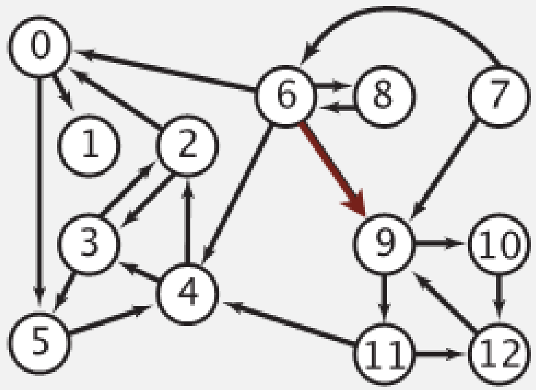
Graph Traversals

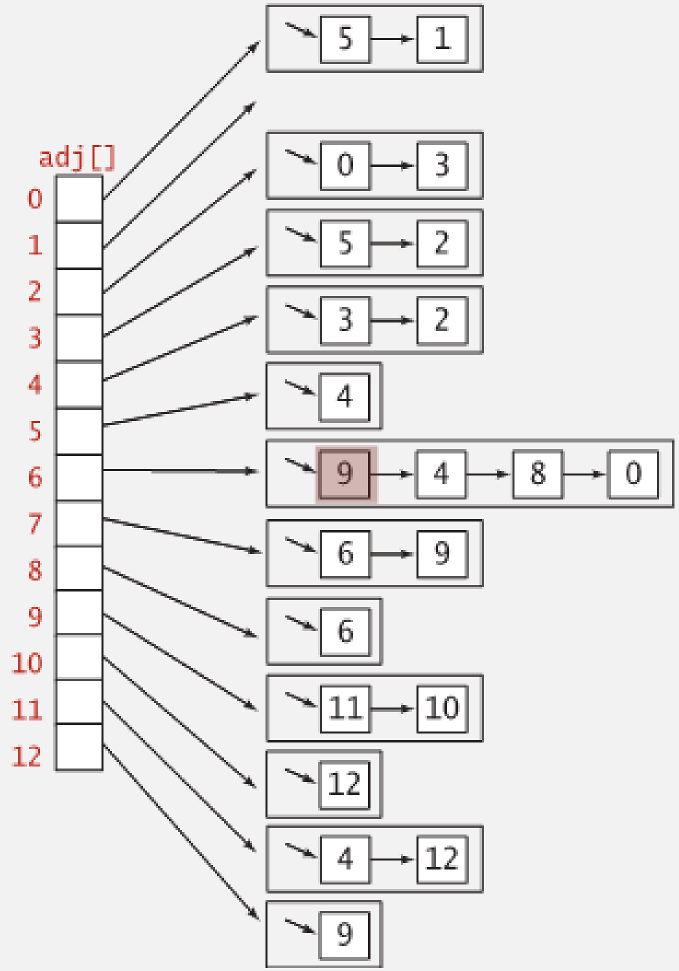
- Depth-First and Breadth-First Search
- Finding Connected Components

- General Depth-First Search Skeleton
- Depth-First Search Trace

Adjacency-lists digraph representation

Maintain vertex-indexed array of lists.



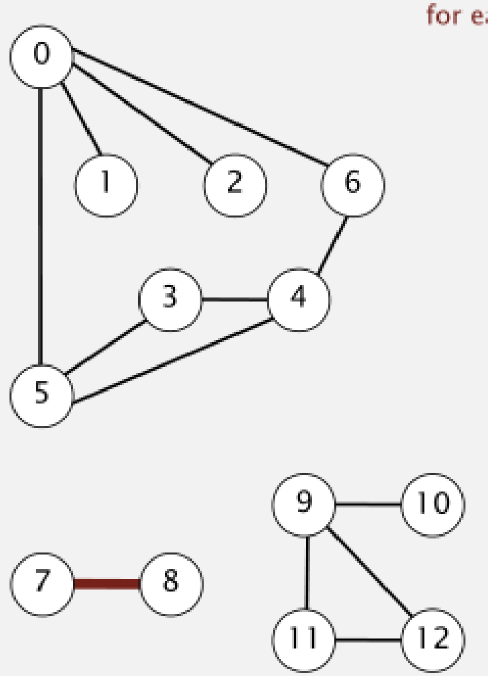


Directed vs. **Undirected** graphs

Adjacency-matrix graph representation

Maintain a two-dimensional V-by-V boolean array;

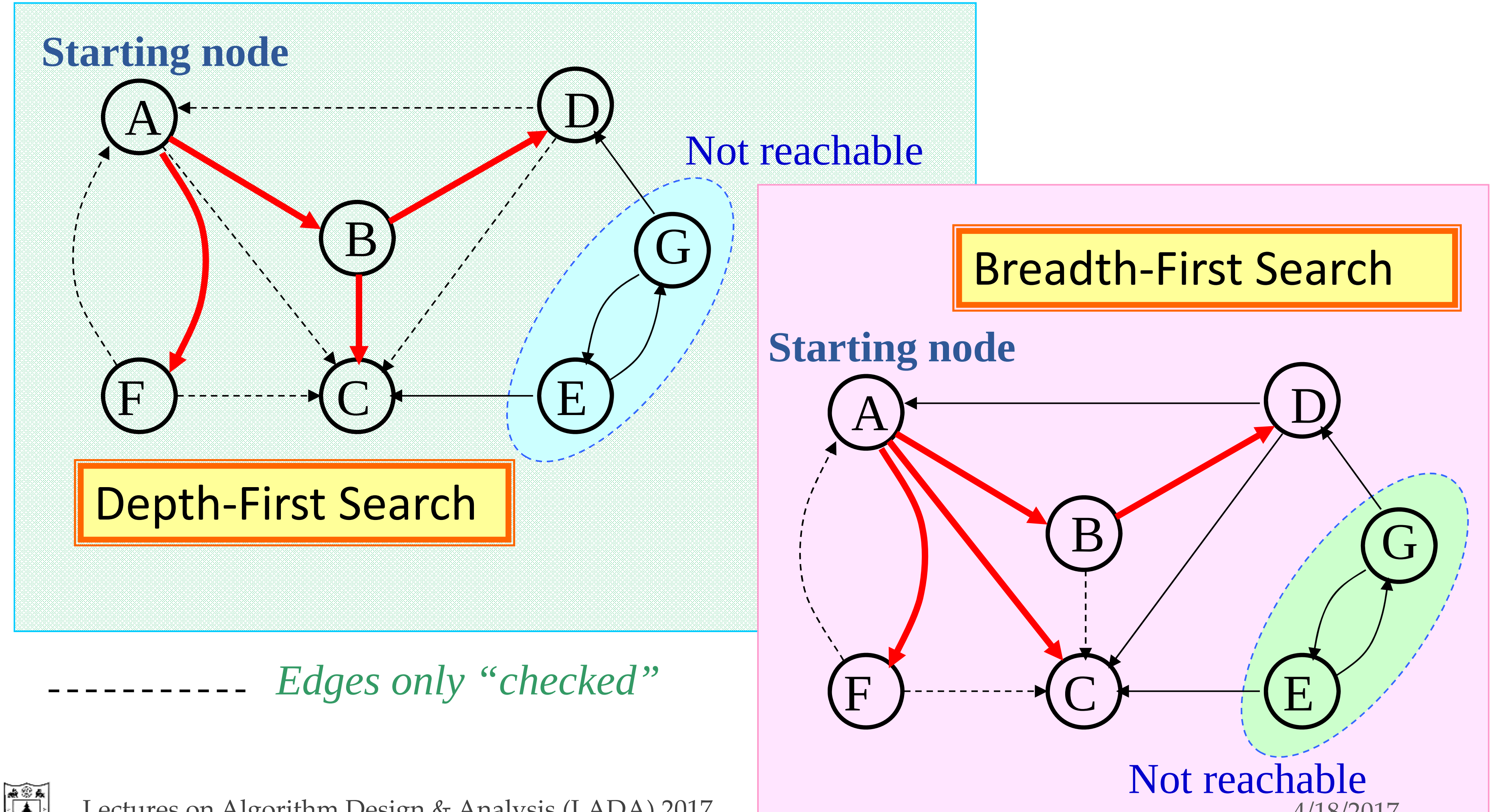
for each edge v–w in graph: `adj[v][w] = adj[w][v] = true.`



two entries for each edge

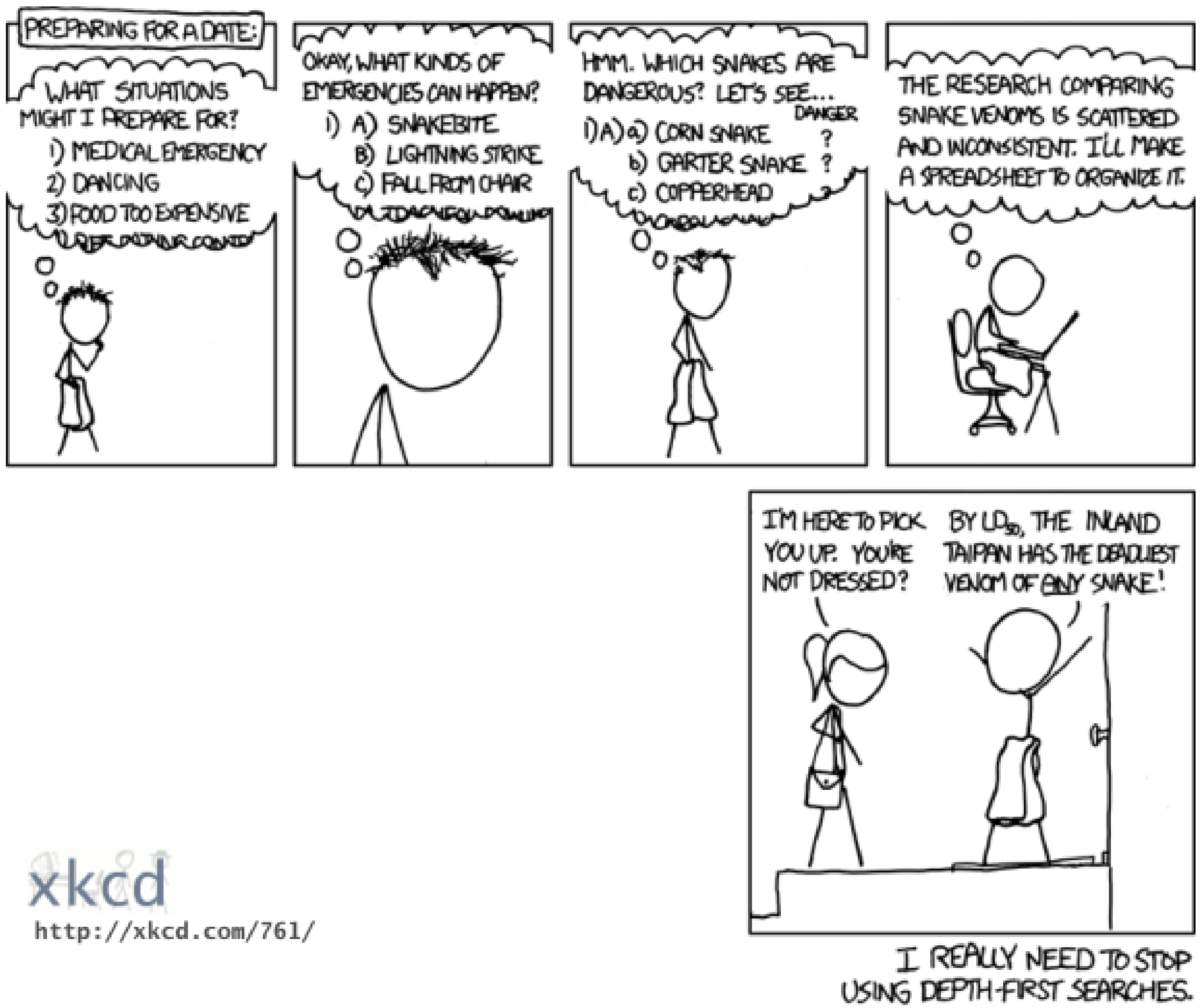
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1	0	0	1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0	0	0	0	0	0
4	0	0	0	1	0	1	1	0	0	0	0	0	0
5	1	0	0	1	1	0	0	0	0	0	0	0	0
6	1	0	0	0	1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	1	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	1	1	1
10	0	0	0	0	0	0	0	0	0	1	0	0	0
11	0	0	0	0	0	0	0	0	0	1	0	0	1
12	0	0	0	0	0	0	0	0	0	1	0	1	0

Graph Traversal



- - - - - - - - - - *Edges only "checked"*

Depth-first search application: preparing for a date



xkcd
http://xkcd.com/761/

Outline of Depth-First Search

- dfs(G,v)
-     Mark v as "discovered".
-     For each vertex w that edge vw is in G:
-         If w is undiscovered:
-             dfs(G,w)
-         Otherwise:
-             "Check" vw without visiting w.
-     Mark v as "finished".

A vertex must be exact one of three different status:
- undiscovered
- discovered but not finished
- finished

That is: exploring vw, visiting w, exploring from there as much as possible, and backtrack from w to v.

Outline of Breadth-First Search

- **Bfs(G,*s*)**
- **Mark *s* as "discovered";**
- **enqueue(pending,*s*);**

- **while (pending is nonempty)**
- **dequeue(pending, *v*);**
- **For each vertex *w* that edge *vw* is in G:**
- **If *w* is "undiscovered"**
- **Mark *w* as "discovered" and enqueue(pending, *w*)**
- **Mark *v* as "finished";**

Finding Connected Components

- **Input: a symmetric digraph G, with *n* nodes and 2*m* edges(interpreted as an undirected graph), implemented as a array *adjVertices*[1,…*n*] of adjacency lists.**
- **Output: an array *cc*[1..*n*] of component number for each node v_i**

```
• void connectedComponents(Intlist[ ] adjVertices, int n,
        int[ ] cc) // This is a wrapper procedure
•     int[ ] color=new int[n+1];
•     int v;
•     <Initialize color array to white for all vertices>
•     for (v=1; v≤n; v++)
•         if (color[v]==white)
•             ccDFS(adjVertices, color, v, v, cc);
•     return
```

Depth-first search

ccDFS: the procedure

- void ccDFS(IntList[ ] *adjVertices*, int[ ] *color*, int *v*, int *ccNum*, int [ ] *cc*)//*v* as the code of current connected component
- int *w*;
- IntList remAdj;

- color[v]=gray;
- cc[v]=ccNum;
- remAdj=adjVertices[v];
- while (remAdj≠nil)
- w=first(remAdj);
- if (color[w]==white)
- ccDFS(adjVertices, color, w, ccNum, cc);
- remAdj=rest(remAdj);
- color[v]=black;
- return

The elements of *remAdj* are neighbors of *v*

Processing the next neighbor, if existing, another depth-first search to be incurred

v finished

Analysis of CC Algorithm

- **connectedComponents, the wrapper**
  - Linear in n (color array initialization+for loop on *adjVertices*)
- **ccDFS, the depth-first searcher**
  - In one execution of ccDFS on v, the number of instructions(rest(*remAdj*)) executed is proportional to the size of *adjVertices*[v].
  - Note: Σ(size of *adjVertices*[v]) is $2m$, and the adjacency lists are traveresed **only once**.

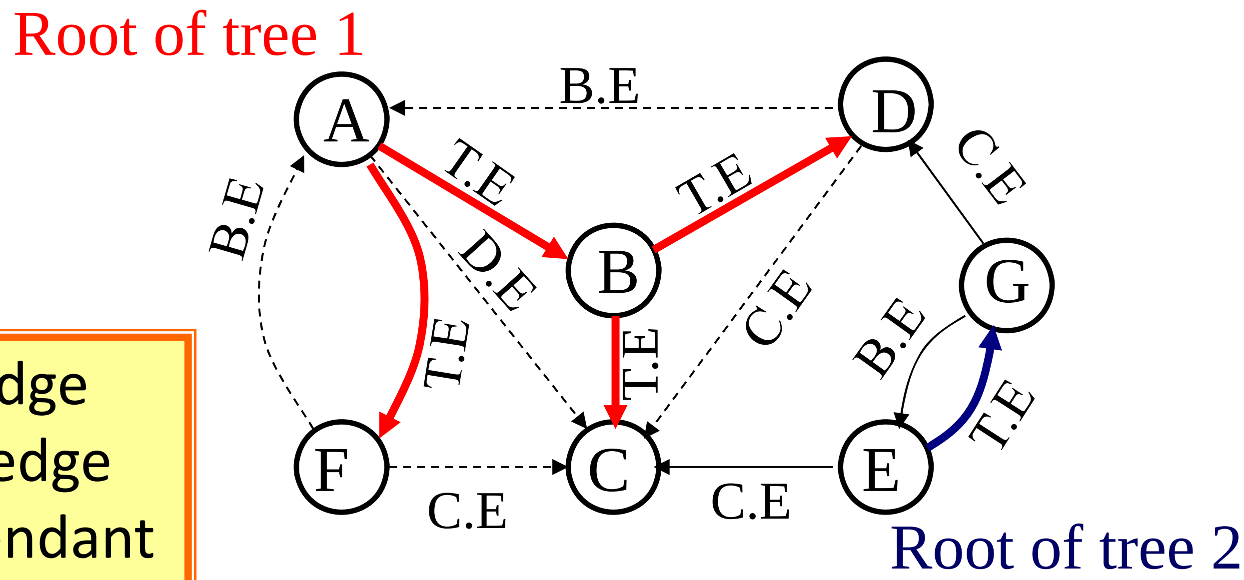
- **So, the time complexity is in Θ(m+n)**
  - Extra space requirements:
    - Color array
    - Activation frame stack for recursion

Visits On a Vertex

- **Classification for the visits on a vertex**
  - First visit(exploring): status: white→gray
  - (Possibly) multi-visits by backtracking to: status keeps gray
  - Last visit(no more branch-finished): status: gray→black
- **Different operations can be done, during the different visits on a specific vertex**
  - On the vertex
  - On (selected) incident edges

Depth-first Search Trees

DFS forest={(DFS tree1), (DFS tree2)}



T.E: tree edge
B.E: back edge
D.E: descendant edge
C.E: cross edge

A finished vertex is never revisited, such as C

Depth-First Search – Generalized Skeleton

- Input: Array *adjVertices* for graph G
- Output: Return value depends on application.
- int dfsSweep(IntList[] *adjVertices*,int n, …)
-     int ans;
-     <Allocate color array and initialize to white>
-     For each vertex *v* of G, in some order
-         if (color[v]==white)
-             int vAns=dfs(*adjVertices*, color, v, …);
-             <Process vAns>
-         // Continue loop
-     return ans;

Depth-First Search – Generalized Skeleton

```
int dfs(IntList[] adjVertices, int[] color, int v, …)
    int w;
    IntList remAdj;
    int ans;
    color[v]=gray;
    <Preorder processing of vertex v>
    remAdj=adjVertices[v];
    while (remAdj≠nil)
        w=first(remAdj);
        if (color[w]==white)
            <Exploratory processing for tree edge vw>
            int wAns=dfs(adjVertices, color, w, …);
            < Backtrack processing for tree edge vw , using wAns>
        else
            <Checking for nontree edge vw>
        remAdj=rest(remAdj);
    <Postorder processing of vertex v, including final computation of ans>
    color[v]=black;
    return ans;
```

If partial search is used for a application, tests for termination may be inserted here.

Specialized for connected components:
- parameter added
- preorder processing inserted – cc[v]=ccNum

Breadth-First Search - Skeleton

- **Input: Array *adjVertices* for graph G**
- **Output: Return value depends on application.**
- **void bfsSweep(IntList[] *adjVertices*,int n, …)**
- **int ans;**
- **<Allocate color array and initialize to white>**
- **For each vertex *v* of G, in some order**
- **if (color[v]==white)**
- **void bfs(*adjVertices*, color, v, …);**
- **// Continue loop**
- **return;**

Breadth-First Search - Skeleton

- void bfs(IntList[] *adjVertices*, int[] color, int v, …)
-     int w; IntList remAdj; Queue pending;
-     color[v]=gray;   enqueue(pending, *v*);
-     while (pending is nonempty)
-         *w*=dequeue(pending); remAdj=*adjVertices*[*w*];
-         while (remAdj≠nil)
-             *x*=first(remAdj);
-             if (color[*x*]==white)
-                 color[*x*]=gray; enqueue(pending, *x*);
-             remAdj=rest(remAdj);
-         <processing of vertex w>
-         color[*w*]=black;
-     return ;

can be further generalized

DFS vs. BFS Search

- **Processing opportunities for a node**
  - Depth-first: 2
    - At discovering
    - At finishing
  - Breadth-first: only 1, when de-queued
  - At the second processing opportunity for the DFS, the algorithm can make use of information about the descendants of the current node.

Time Relation on Changing Color

- **Keeping the order in which vertices are encountered for the first or last time**
  - A global interger time: 0 as the initial value, incremented with each color changing for *any* vertex, and the final value is $2n$
  - Array *discoverTime*: the i th element records the time vertex v_i turns into gray
  - Array *finishTime*: the i th element records the time vertex v_i turns into black
  - The active interval for vertex v, denoted as *active*(v), is the duration while v is gray, that is:

$$discoverTime[v], \ldots, finishTime[v]$$

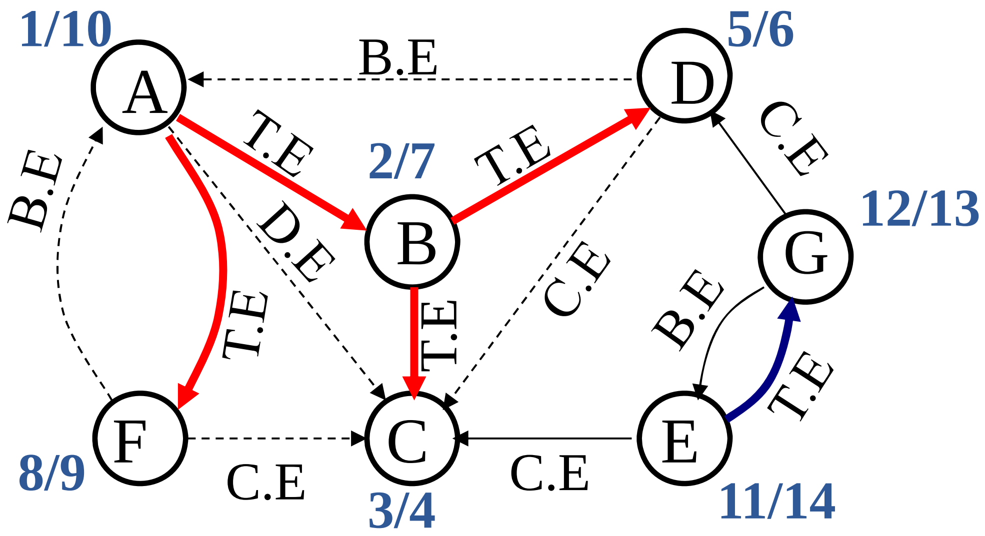
Depth-First Search Trace

- General DFS skeleton modified to compute discovery and finishing times and "construct" the depth-first search forest.
- int dfsTraceSweep(IntList[ ] *adjVertices*,int n, int[ ] *discoverTime, *int[ ] *finishTime,* int[ ] *parent*)
- int ans; int *time*=0
- <Allocate color array and initialize to white>
- For each vertex *v* of G, in some order
- if (color[v]==white)
- parent[v]=-1
- int vAns=dfsTrace(*adnVertices*, color, v, *discoverTime, finishTime, parent, time*);
- // Continue loop
- return ans;

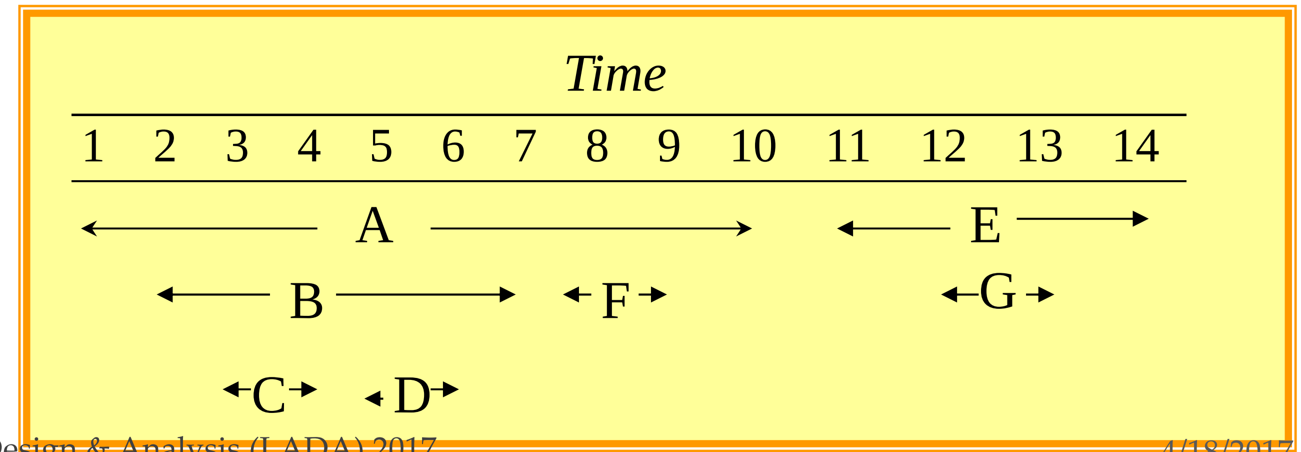
Depth-First Search Trace

- int dfsTrace(intList[ ] *adjVertices*, int[ ] color, int v, int[ ] *discoverTime,*
- int[ ] *finishTime*, int[ ] *parent* int *time*)
- int w; IntList remAdj; int ans;
- color[v]=gray; time++; *discoverTime*[v]=time;
- remAdj=*adjVertices*[v];
- while (remAdj≠nil)
- w=first(remAdj);
- if (color[w]==white)
- parent[w]=v;
- int wAns=dfsTrace(*adjVertices*, color, w, *discoverTime, finishTime,* *parent, time*);
- else <Checking for nontree edge vw>
- remAdj=rest(remAdj);
- time++; *finishTime*[v]=time; color[v]=black;
- return ans;

Active Interval



The relations are summarized in the next frame



Properties of Active Intervals(1)

- **If w is a descendant of v in the DFS forest, then $active(w) \subseteq active(v)$, and the inclusion is proper if w≠v.**
- **Proof:**
  - Define a partial order <: w<v iff. w is a proper descendants of v in its DFS tree. The proof is by induction on <.
  - If v is minimal. The only descendant of v is itself. Trivial.
  - Assume that for all x<v, if w is a descendant of x, then active(w)⊆active(x).
  - Let w be any proper descendant of v in the DFS tree, there must be some x such that vx is a tree edge on the tree path to w, so w is a descendant of x. According to dfsTrace, we have $active(x) \subset active(v)$, by inductive hypothesis, $active(w) \subset active(v)$.

Properties of Active Intervals(2)

- **If $active(w) \subseteq active(v)$, then w is a descendant of v. And if $active(w) \subset active(v)$, then w is a proper descendant of v.**

  That is: *w* is discovered while *v* is active.

- **Proof:**
  - If w is **not** a descendant of v, there are two cases:
    - v is a proper descendant of w, then $active(v) \subset active(w)$, so, it is impossible that $active(w) \subseteq active(v)$, contradiction.
    - There is no ancestor/descendant relationship between v and w, then $active(w)$ and $active(v)$ are disjoint, contradiction.

Properties of Active Intervals(3)

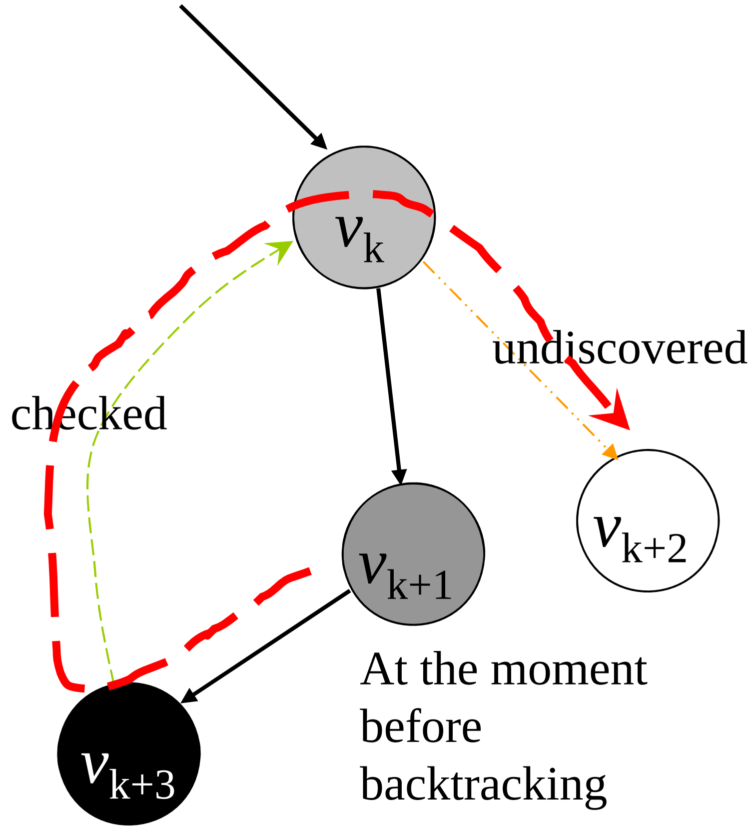
- **If v and w have no ancestor/descendant relationship in the DFS forest, then their active intervals are disjoint.**
- **Proof:**
  - o If v and w are in different DFS tree, it is trivially true, since the trees are processed one by one.
  - o Otherwise, there must be a vertex c, satisfying that there are tree paths c to v, and c to w, without edges in common. Let the leading edges of the two tree path are cy, cz, respectively. According to dfsTrace, $active(y)$ and $active(z)$ are disjoint.
  - o We have $active(v) \subseteq active(y)$, $active(w) \subseteq active(z)$. So, $active(v)$ and $active$(w) are disjoint.

Properties of Active Intervals(4)

- **If edge vw$\in E_G$, then**
  - vw is a **cross edge** iff. *active*(w) entirely precedes *active*(v).
  - vw is a **descendant edge** iff. there is some third vertex x, such that *active*(w)$\subset$*active*(x)$\subset$*active*(v),
  - vw is a **tree edge** iff. *active*(w)$\subset$*active*(v), and there is no third vertex x, such that *active*(w)$\subset$*active*(x) $\subset$*active*(v),
  - vw is a **back edge** iff. *active*(v)$\subset$*active*(w),

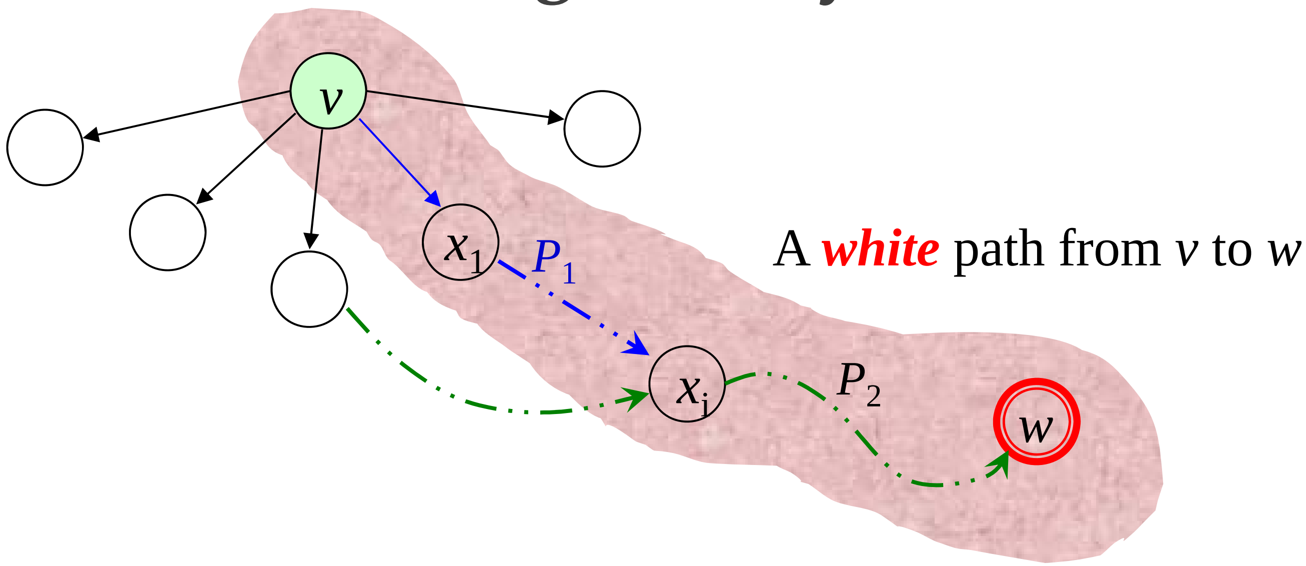
Ancestor and Descendant

- That w is a descendant of v in the DFS forest means that there is a direct path from v to w in some DFS tree.
- The path is also a path in G.
- **However, if there is a direct path from v to w in G, is w necessarily a descendant of v in *the* DFS forest?**



DFS Tree Path

- [**White Path Theorem**] w is a descendant of v in a DFS tree iff. at the time v is discovered(just to be changing color into gray), there is a path in G from v to w consisting entirely of white vertices.



Proof of White Path Theorem

- **Proof**
  - o $\Rightarrow$ All the vertices in the path are descendants of v.
  - o $\Leftarrow$ by induction on the length *k* of a white path from v to w.
    - When k=0, v=w.
    - For k>0, let P=(v, x_1,x_2,…x_k=w). There must be some vertex on P which is discovered during the active interval of v, e.g. x_1, Let x_i is earliest discovered among them. Divide P into P_1 from v to x_i, and P_2 from x_i to w. P_2 is a white path with length less than k, so, by inductive hypothesis, w is a descendant of x_i. Note: $active(x_i) \subseteq active(v)$, so x_i is a descendant of v. By transitivity, w is a descendant of v.

Thank you!

Q & A

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