Decision Trees, Adversary Argument, and Amortized Analysis

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Algorithm analysis

- Given a problem P
- Design algorithms $A, A', \ldots$
- Input space $\mathcal{X}_n$: inputs of size n

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Algorithm analysis

Selection (median):

Sorting:

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$n!$

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$$n!$$

$$\Longrightarrow n^2$$

Algorithm analysis

Selection (median):

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$$
\begin{aligned}
& n! \\
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\end{aligned}
$$

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n \log n \Longleftarrow & \\
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Selection (median):

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$$\Longrightarrow n \log n$$

$$\Longrightarrow 16n$$

Algorithm analysis

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Selection (median):

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Algorithm analysis

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Algorithm analysis

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$$\frac{3n}{2} - \frac{3}{2} \log n \Longleftarrow$$

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Algorithm analysis

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$$2n \Longleftarrow$$
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K-sorted (Problem 2.9)

$$1,\ 2,\ 4,\ 3; \quad 7,\ 6,\ 8,\ 5; \quad 10,\ 11,\ 9,\ 12; \quad 15,\ 13,\ 16,\ 14$$

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1-sorted $\to$ 2-sorted $\to$ 4-sorted $\to \cdots \to$ n-sorted

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Quicksort (with median) stops after the $\log k$ recursions.

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$$L = \binom{n}{n/k, \ldots, n/k} = \frac{n!}{\left(\left(\frac{n}{k}\right)!\right)^k}$$

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$$\Omega(n \log k)$$

$$L = \binom{n}{n/k, \ldots, n/k} = \frac{n!}{\left((\frac{n}{k})!\right)^k}$$

$$H \geq \log\left(\frac{n!}{\left((\frac{n}{k})!\right)^k}\right)$$

K-sorted (Problem 2.9)

Sorting the k-sorted array.

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$$H \geq \log((\frac{n}{k})!)^k = \Omega(n \log \frac{n}{k})$$

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Horse Racing

Example (Horse Racing)

- 25 horses
- Round: ≤ 5 horses race
- Goal: Find $\#1, \#2, \#3$ fastest.

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Finding patterns in bit strings

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n is odd: checking $A[2, 4, \ldots, n-1]$

n is even: adversary argument

Decision Trees, Adversary Argument, and Amortized Analysis

Amortized analysis

Amortized analysis is
an algorithm analysis technique for
analyzing a sequence of operations
irrespective of the input to show that
the average cost per operation is small, even though
a single operation within the sequence might be expensive.

Methods for amortized analysis: the summation method

$$o_1, o_2, \ldots, o_n$$

$$c_1, c_2, \ldots, c_n$$

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$$(\sum_{i=1}^{n} c_i)/n$$

Summation method: array doubling revisited

On any sequence of n INSERT ops on an initially empty array.

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$$
\begin{array}{cccccccccccc}
o_i: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
c_i: & 1 & 2 & 3 & 1 & 5 & 1 & 1 & 1 & 8 & 1
\end{array}
$$

Summation method: array doubling revisited

On any sequence of n $\textsc{Insert}$ ops on an initially empty array.

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$$c_i = \begin{cases} (i-1)+1 = i & \text{if } i-1 \text{ is an exact power of } 2 \\ 1 & \text{o.w.} \end{cases}$$

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$$\sum_{i=1}^{n} c_i \le n + \sum_{j=0}^{\lceil \lg n \rceil - 1} 2^j = n + (2^{\lceil \lg n \rceil} - 1) \le n + 2n = 3n$$

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$$\forall i, \hat{c}_i = 3$$

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Key way of thinking:

Put the accounting cost on specific objects.

Accounting method: array doubling revisited

$$\hat{c}_i = 3 \textit{ vs. } \hat{c}_i = 2$$

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	$\hat{c}_i$	c_i (actual cost)	a_i (accounting cost)
INSERT (normal)	3	1	2
INSERT (expansion)	3	$1 + t$	$-t + 2$

Array merging (Problem 4.13): the summation method

$\textsc{Create}\ (1);\ \textsc{Merge}\ (2m)$

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$$\text{Create}\ (1);\ \text{Merge}\ (2m)$$

i	c_i
1	1
2	$1+2$
3	1
4	$1+2+4$
5	1
6	$1+2$
7	1
8	$1+2+4$
$\vdots$	$\ldots$

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$\vdots$	$\dots$

$$\sum_{i=1}^{n} c_i = \sum_{j=0}^{\lfloor \log n \rfloor} \lfloor \frac{n}{2^j} \rfloor 2^j \le n(\lfloor \log n \rfloor + 1)$$

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$$\forall i, \hat{c}_i = 1 + \lfloor \log n \rfloor$$

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What does it mean?

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$$\hat{c}_i = 1 + \lfloor \log n \rfloor$$

What does it mean?

$$\forall n, \sum_{i=1}^{n} a_i \geq 0$$

Two stacks, one queue (Problem 4.14)

Algorithm 1 Simulating a queue using two stacks S_1, S_2.

```
procedure Enq(x)
    Push(S_1, x)

procedure Deq()
    if S_2 = ∅ then
        while S_1 ≠ ∅ do
            Push(S_2, Pop(S_1))
    Pop(S_2)
```

Two stacks, one queue: the summation method

$$(\sum_{i=1}^{n} c_i)/n$$

Two stacks, one queue: the summation method

$$(\sum_{i=1}^{n} c_i)/n$$

The operation sequence is NOT known.

Two stacks, one queue: the accounting method

item:	Push into S_1	Pop from S_1	Push into S_2	Pop from S_2
	1	1	1	1

Two stacks, one queue: the accounting method

item:	Push into S_1	Pop from S_1	Push into S_2	Pop from S_2
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$$\hat{c}_{\text{ENQ}} = 3$$
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$$\hat{c}_{\text{ENQ}} = 3$$
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$$\sum_{i=1}^{n} a_i \geq 0$$

Two stacks, one queue: the accounting method

item:	Push into S_1	Pop from S_1	Push into S_2	Pop from S_2
	1	1	1	1

$$\hat{c}_{\text{ENQ}} = 3$$
$$\hat{c}_{\text{DEQ}} = 1$$

$$\sum_{i=1}^{n} a_i \geq 0 \Longleftarrow \sum_{i=1}^{n} a_i = \#S_1 \times 2$$


THANK YOU

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