Tight Upper Bounds on the Error Probability of Spinal Codes over Fading Channels

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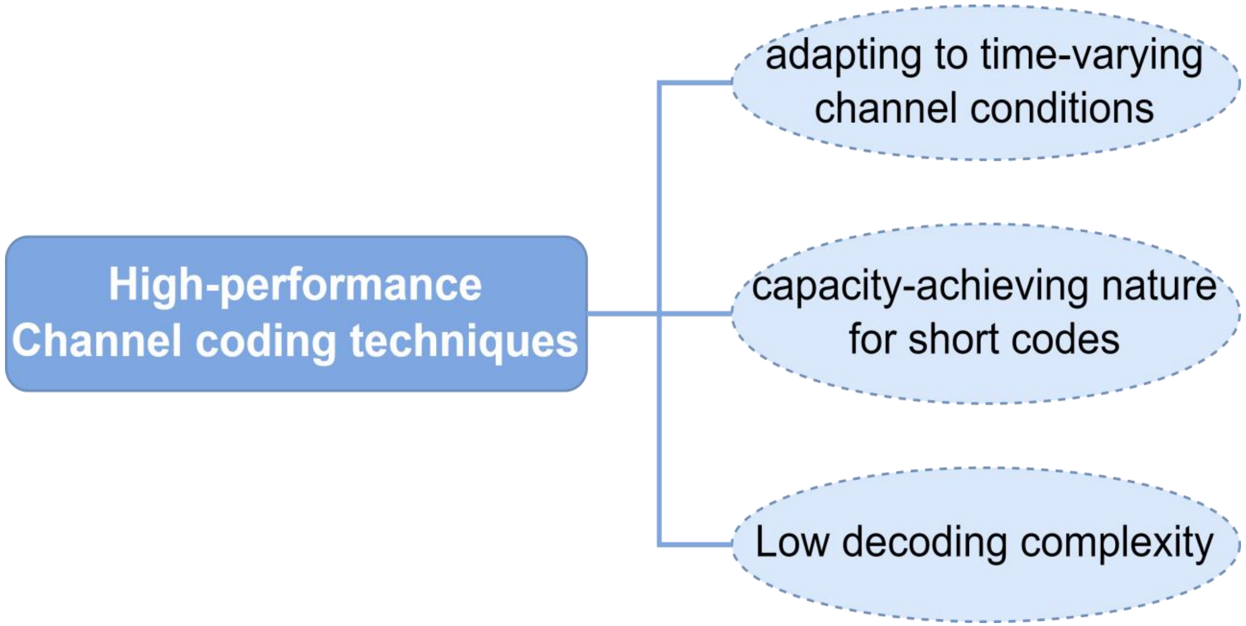
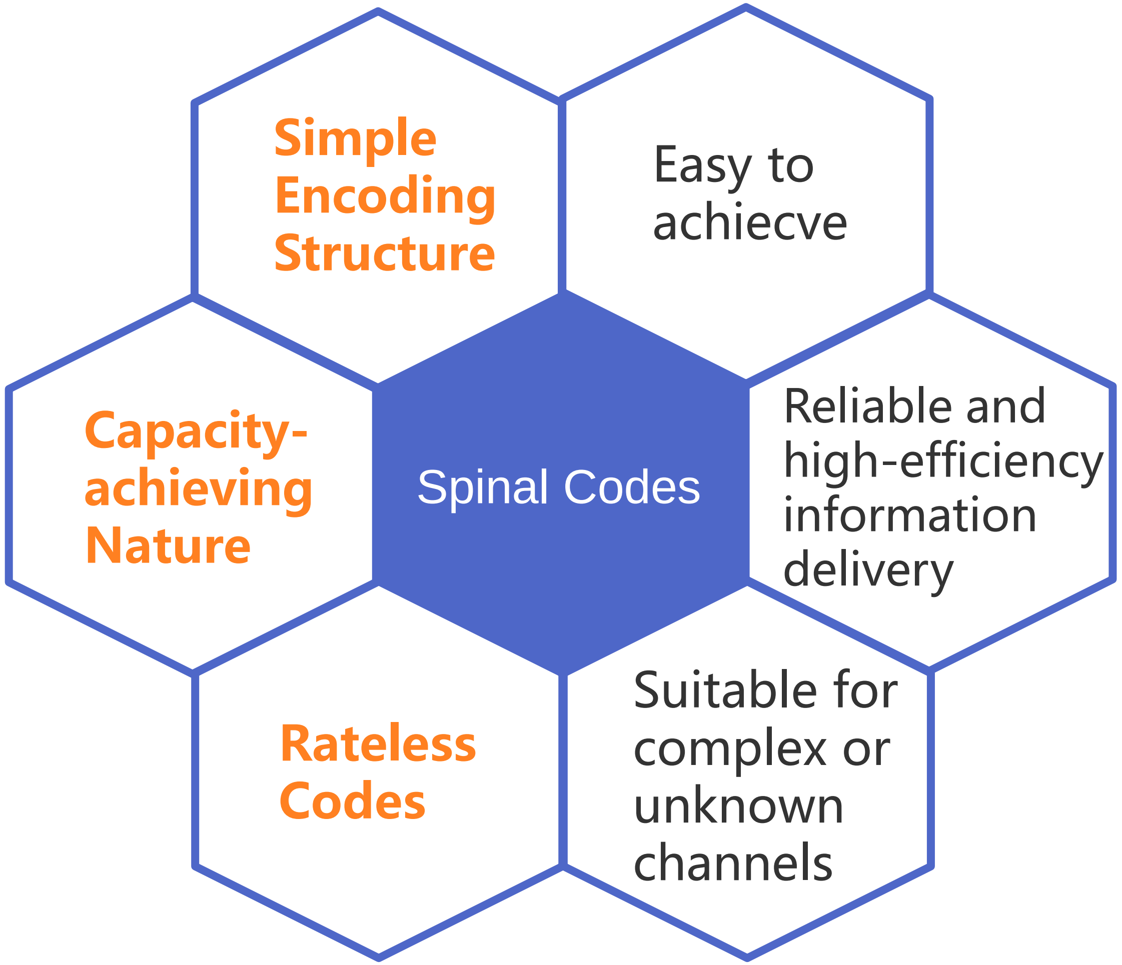
Spinal Codes

Spinal codes are a family of rateless codes that have been proved to achieve Shannon capacity over both the AWGN channel and the BSC.

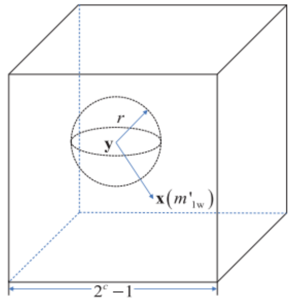




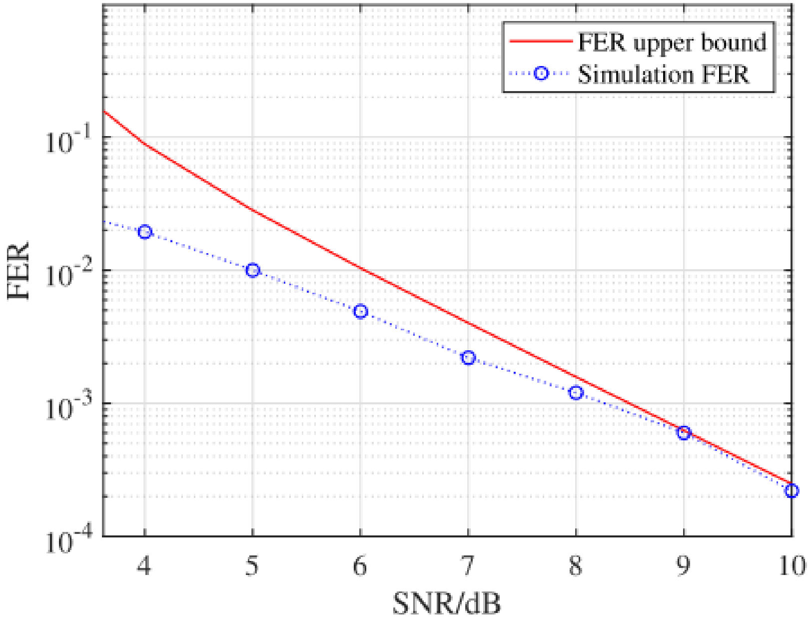
Motivation: Error probability analysis of Spinal codes

Current Analysis[1]

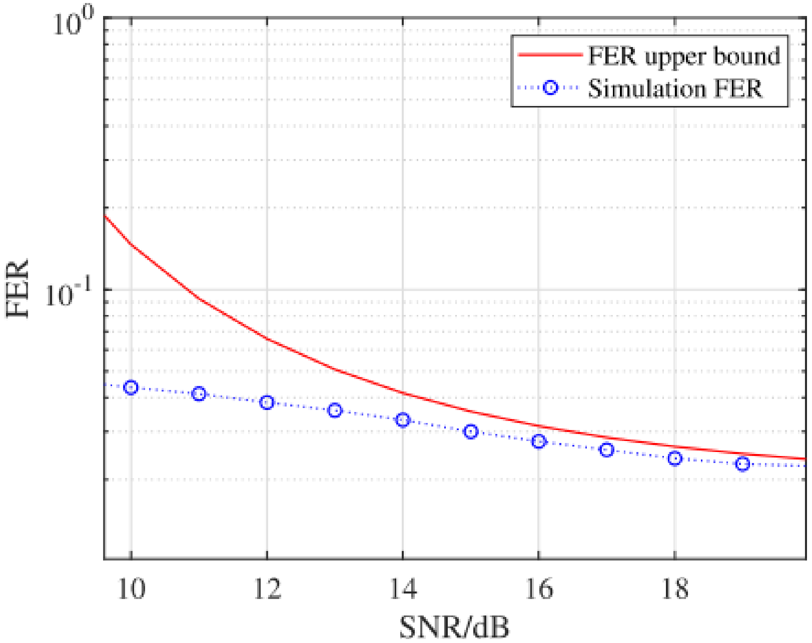
- ✓ Sequentially encoding under a tree structure for Spinal codes
- ✓ ML decoding rules and other decoding process
- ✓ Characterizing the error probability **as the volume of a hypersphere divided by the volume of a hypercube**
- ✓ Over **AWGN** channels and **Rayleigh** fading channels (no CSI)



- ➤ AWGN channels



- ➤ Rayleigh fading channels (no CSI)



At a low SNR, the approximation of the upper bound needs to be improved.

[1] LI A，WU S，JIAO J，et al. Spinal codes over fading channel: error probability analysis and encoding structure improvement［J］. IEEE Transactions on Wireless Communications，2021，20（12）：8288-8300.

Motivation

- **The derived bound over the Rayleigh fading channel is not strictly explicit**

  The convergence of the upper bound is probability-dependent

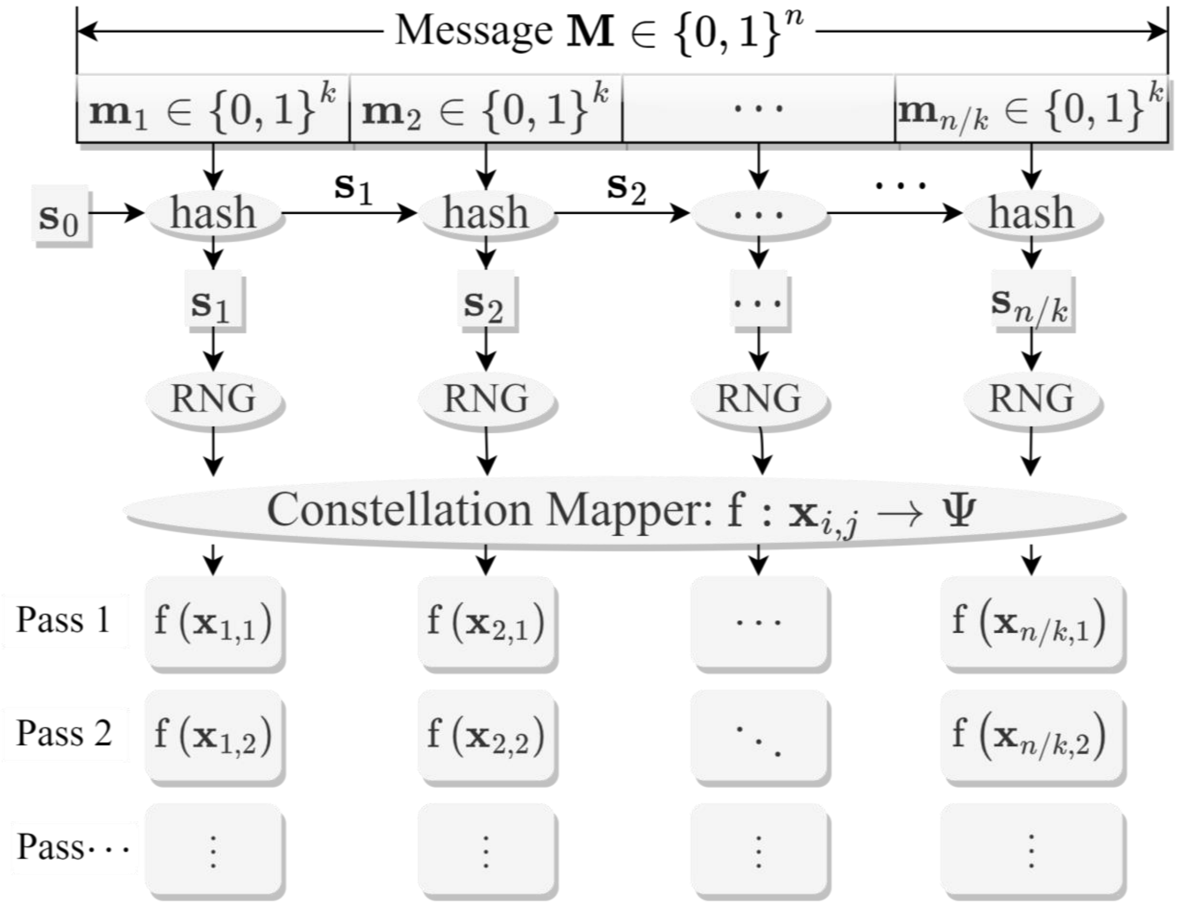
- **Nakagami-m and Rician channels have not been considered in the available error probability analyses**

  Both of them are common in practical wireless communication scenarios

- **There is not yet an upper bound that achieves uniform tight approximations over a wide range of signal-to-noise ratio (SNR)**

  Either over the fading channel or over the AWGN channel

Encoding process of Spinal codes



The encoding process of Spinal codes

- **Segmentation**: Divide an n-bit message $\mathbf{M}$ into k-bit segments $\mathbf{m}_i \in \{0,1\}^k$, where $i = 1, 2, \cdots, n/k$
- **Sequentially Hashing**: The hash function sequentially generates v-bit spine values

$$\mathbf{s}_i = h(\mathbf{s}_{i-1}, \mathbf{m}_i), i = 1, 2, \cdots, n/k, \mathbf{s}_0 = \mathbf{0}^v$$

- **RNG**: Each spine value s_i is used to seed an RNG to generate a binary pseudo-random uniform distributed sequence

$$\mathrm{RNG}: \mathbf{s}_i \to \{\mathbf{x}_{i,j}\}, j = 1, 2, 3, \cdots$$

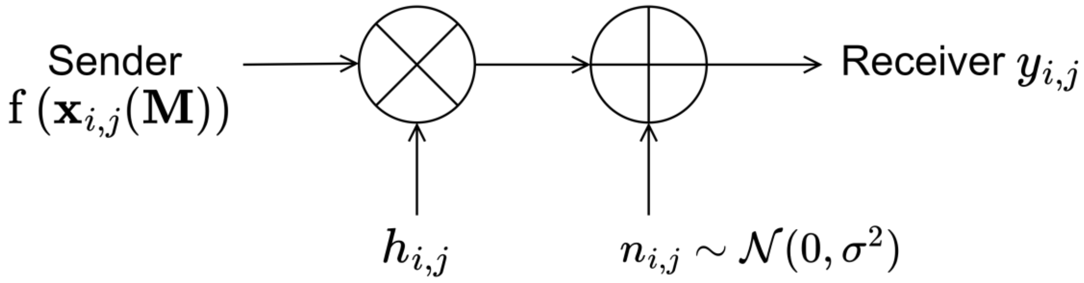
- **Constellation Mapping**: The constellation mapper maps each c-bit symbol $\mathbf{x}_{i,j}$ to a channel input set Ψ:

$$\mathrm{f}: \mathbf{x}_{i,j} \to \Psi$$

In this paper, f is a uniform constellation mapping function.

Transmission and decoding process of Spinal codes

Transmission over fading channels



Received signal $\mathrm{y}_{i,j} = h_{i,j}\,\mathrm{f}\left(\mathbf{x}_{i,j}(\mathbf{M})\right) + n_{i,j}$

ML decoding

Selecting the one with the lowest decoding cost from the candidate sequence space.

$$\hat{\mathbf{M}} \in \underset{\bar{\mathbf{M}} \in \{0,1\}^n}{\arg\min} \sum_{i=1}^{n/k} \sum_{j=1}^{L} \left(y_{i,j} - h_{i,j}\,\mathrm{f}(\mathbf{x}_{i,j}(\bar{\mathbf{M}}))\right)^2$$

$\hat{\mathrm{M}}$ is the decoding result, $\bar{\mathrm{M}}$ is the candidate sequence, L is the number of pass.

Cost of the correct decoding sequence

$$\mathscr{D}(\mathbf{M}) \triangleq \sum_{i=1}^{n/k} \sum_{j=1}^{L} \left(y_{i,j} - h_{i,j}\,\mathrm{f}(\mathbf{x}_{i,j}(\mathbf{M}))\right)^2 = \sum_{i=1}^{n/k} \sum_{j=1}^{L} n_{i,j}^2.$$

Cost of wrong decoding sequences

$$\mathscr{D}(\mathbf{M}') \triangleq \sum_{i=1}^{n/k} \sum_{j=1}^{L} \left(y_{i,j} - h_{i,j}\,\mathrm{f}(\mathbf{x}_{i,j}(\mathbf{M}'))\right)^2.$$

where $\mathbf{M}' = \left(\mathbf{m}_1', \mathbf{m}_2', \cdots, \mathbf{m}_{n/k}'\right) \in \mathcal{W}$

$$\mathcal{W} \triangleq \left\{\left(\mathbf{m}_1', \mathbf{m}_2', \cdots, \mathbf{m}_{n/k}'\right) : \exists 1 \le i \le n/k, \mathbf{m}_i' \neq \mathbf{m}_i\right\}$$

Analysis of Error Probabilities under finite code length conditions

$\mathcal{E}_a$ The event that there exists an error in the a^{th} segment

$\overline{\mathcal{E}}_a$ The complement of $\mathcal{E}_a$

The Error Probability of Spinal codes

$$P_e = \Pr\left(\bigcup_{a=1}^{n/k} \mathcal{E}_a\right) = 1 - \Pr\left(\bigcap_{a=1}^{n/k} \overline{\mathcal{E}}_a\right)$$
$$= 1 - \prod_{a=1}^{n/k}\left[1 - \underline{\Pr\left(\mathcal{E}_a \middle| \bigcap_{i=1}^{a-1} \overline{\mathcal{E}}_i\right)}\right].$$

The probability that the a^{th} segment is wrong while the previous (a − 1) segments are correct.

$$\Pr\left(\mathcal{E}_a \middle| \bigcap_{i=1}^{a-1} \overline{\mathcal{E}}_i\right) = \Pr\left(\exists \mathbf{M}' \in \mathcal{W}_a : \mathscr{D}(\mathbf{M}') \leq \mathscr{D}(\mathbf{M})\right)$$
$$\leq \sum_{\mathbf{M}' \in \mathcal{W}_a} \Pr\left(\mathscr{D}(\mathbf{M}') \leq \mathscr{D}(\mathbf{M})\right)$$

where

$$\mathcal{W}_a \triangleq \{(\mathbf{m}_1', \cdots, \mathbf{m}_a') : \mathbf{m}_1' = \mathbf{m}_1, \cdots, \mathbf{m}_{a-1}' = \mathbf{m}_{a-1}, \mathbf{m}_a' \neq \mathbf{m}_a\} \subseteq \mathcal{W}$$

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where $\mathcal{W}_a \triangleq \{(\mathbf{m}'_1, \cdots, \mathbf{m}'_a) : \mathbf{m}'_1 = \mathbf{m}_1, \cdots, \mathbf{m}'_{a-1} = \mathbf{m}_{a-1}, \mathbf{m}'_a \neq \mathbf{m}_a\} \subseteq \mathcal{W}$

DERIVATION

$$\Pr\left(\mathscr{D}(\mathbf{M}') \le \mathscr{D}(\mathbf{M})\right)$$

Applying the cost function

$$\Pr\left(\sum_{i=a}^{n/k}\sum_{j=1}^{L} V_{i,j}^2 + 2\sum_{i=a}^{n/k}\sum_{j=1}^{L} V_{i,j} n_{i,j} \le 0\right)$$

Simplifying to a vector form

$$\Pr\left(\mathbf{V}^{L_a}\left(\mathbf{V}^{L_a} + 2\mathbf{N}^{L_a}\right)^{\mathrm{T}} \le 0\right)$$

Introducing rotation matrices

$$\int_{R^{L_a}} Q\left(\frac{\left\|\mathbf{v}^{L_a}\right\|}{2\sigma}\right) \cdot \Pr\left(\mathbf{V}^{L_a} = \mathbf{v}^{L_a}\right) \mathrm{d}\mathbf{v}^{L_a}$$

Adopting a transformation of Q(·)

$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \exp\left(\frac{-x^2}{2\sin^2\theta}\right) \mathrm{d}\theta$$

$$P_e \le 1 - \prod_{a=1}^{n/k}\left(1 - \min\left(1, \frac{1}{\pi}\sum_{\mathbf{M}' \in \mathcal{W}_a}\int_0^{\frac{\pi}{2}} \mathcal{F}(\theta;\sigma,L_a)\mathrm{d}\theta\right)\right)$$

where $\mathcal{F}(\theta;\sigma,L_a) = \left(\sum_u p_U(u)\int_{\mathbb{R}} \exp\left(-\frac{h^2u^2}{8\sigma^2\sin^2\theta}\right)\underline{g(h)}\mathrm{d}h\right)^{L_a}$

The Average Error Probability on Fading Channels

Upper Bounds on Rayleigh Fading Channels

Rayleigh fading: $g_1(h) = \frac{2h}{\Omega}\exp\left(\frac{-h^2}{\Omega}\right)$

Ω is the mean square

$$\mathcal{F}(\theta;\sigma,L_a) = \left(\sum_u p_U(u)\int_{\mathbb{R}} \exp\left(-\frac{h^2u^2}{8\sigma^2\sin^2\theta}\right)\boxed{g(h)}\mathrm{d}h\right)^{L_a}$$

For Rayleigh fading channels: $P_e \le 1 - \prod_{a=1}^{n/k}(1-\epsilon_a)$

$$\epsilon_a = \min\left\{1, \left(2^k-1\right)2^{n-ak}\cdot\mathscr{F}_{\text{Rayleigh}}(L_a,\sigma)\right\}$$

$$\mathscr{F}_{\text{Rayleigh}}(L_a,\sigma) = \sum_{r=1}^{N} b_r \mathcal{F}_{\text{Rayleigh}}(\theta_r;\sigma,L_a)$$

$$\mathcal{F}_{\text{Rayleigh}}(\theta;\sigma,L_a) = \left(\sum_{i\in\Psi}\sum_{j\in\Psi}2^{-2c}\frac{8\sigma^2\sin^2\theta}{\Omega(i-j)^2+8\sigma^2\sin^2\theta}\right)^{L_a}$$

Upper Bounds on Nakagami-m Fading Channels

Nakagami-m fading: $g_2(h) = \frac{2m^m}{\Gamma(m)\Omega^m}h^{2m-1}\exp\left(\frac{-mh^2}{\Omega}\right)$

$$\mathcal{F}_{\text{Nakagami}}(\theta_r;\sigma,L_a) = \left(\sum_{i\in\Psi}\sum_{j\in\Psi}2^{-2c}\left(\frac{8m\sigma^2\sin^2\theta_r}{\Omega(i-j)^2+8m\sigma^2\sin^2\theta_r}\right)^m\right)^{L_a}$$

Upper Bounds on Rician Fading Channels

Rician fading: $g_3(h) = \frac{2(K+1)h}{\Omega\exp\left(K+\frac{(K+1)h^2}{\Omega}\right)}I_0\left(2\sqrt{\frac{K(K+1)}{\Omega}}h\right)$

$$\mathcal{F}_{\text{Rician}}(\theta_r;\sigma,L_a) = \left(\sum_{i\in\Psi}\sum_{j\in\Psi}2^{-2c}\frac{8(K+1)\sigma^2\sin^2\theta_r}{\Omega(i-j)^2+8(K+1)\sigma^2\sin^2\theta_r}\cdot\exp\left(\frac{8K(K+1)\sigma^2\sin^2\theta_r}{\Omega(i-j)^2+8(K+1)\sigma^2\sin^2\theta_r}-K\right)\right)^{L_a}$$

Analysis of Error Probabilities under finite code length conditions

Fading channels

Rayleigh fading

$$g_1(h) = \frac{2h}{\Omega}\exp\left(\frac{-h^2}{\Omega}\right)$$

$m = 1$

Nakagami-m fading

$$g_2(h) = \frac{2m^m}{\Gamma(m)\Omega^m}h^{2m-1}\exp\left(\frac{-mh^2}{\Omega}\right)$$

$K = 0$

Rician fading

$$g_3(h) = \frac{2(K+1)h}{\Omega\exp\left(K+\frac{(K+1)h^2}{\Omega}\right)}I_0\left(2\sqrt{\frac{K(K+1)}{\Omega}}h\right)$$

Verifying the correctness of the results in another way

Upper bounds

$$\mathcal{F}_{\text{Rayleigh}}(\theta_r;\sigma,L_a) = \left(\sum_{i\in\Psi}\sum_{j\in\Psi}2^{-2c}\frac{8\sigma^2\sin^2\theta_r}{\Omega(i-j)^2+8\sigma^2\sin^2\theta_r}\right)^{L_a}$$

$m = 1$

$$\mathcal{F}_{\text{Nakagami}}(\theta_r;\sigma,L_a) = \left(\sum_{i\in\Psi}\sum_{j\in\Psi}2^{-2c}\left(\frac{8m\sigma^2\sin^2\theta_r}{\Omega(i-j)^2+8m\sigma^2\sin^2\theta_r}\right)^m\right)^{L_a}$$

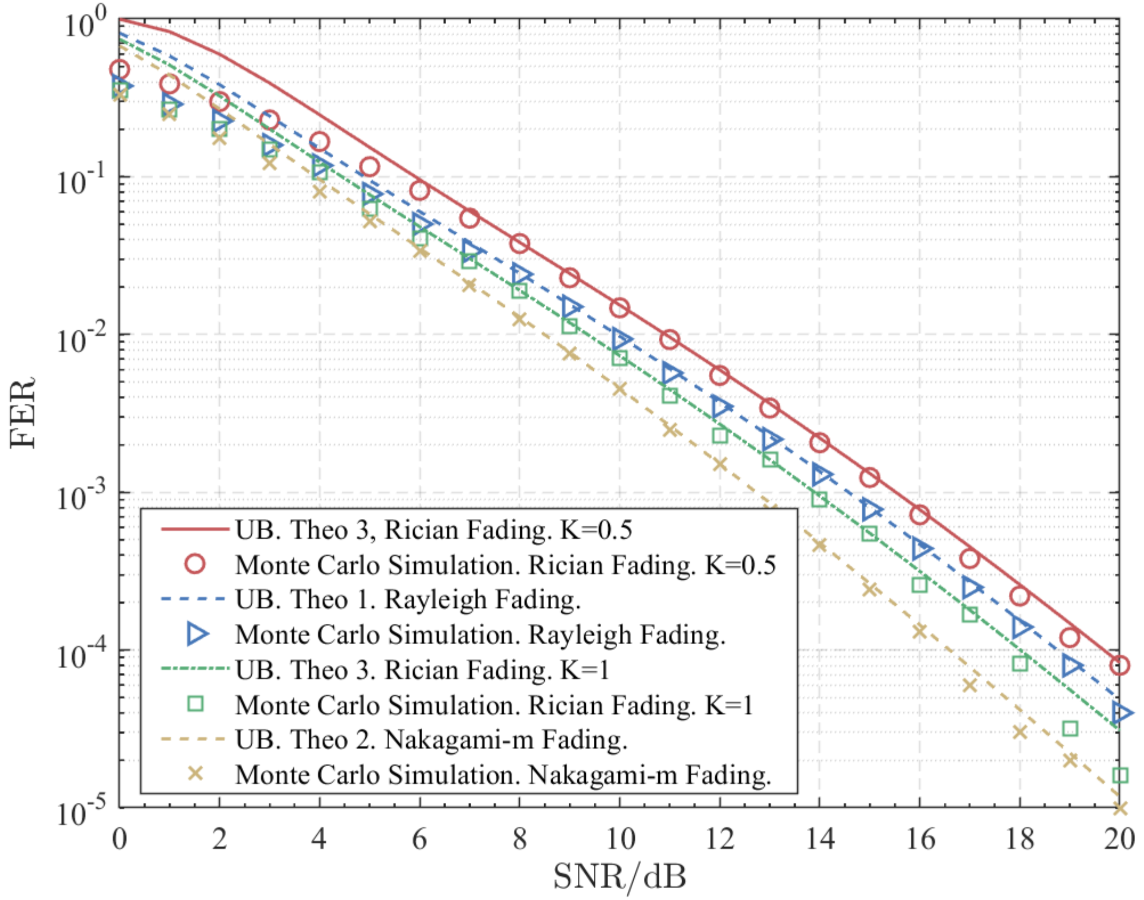
$K = 0$

$$\mathcal{F}_{\text{Rician}}(\theta_r;\sigma,L_a) = \left(\sum_{i\in\Psi}\sum_{j\in\Psi}2^{-2c}\frac{8(K+1)\sigma^2\sin^2\theta_r}{\Omega(i-j)^2+8(K+1)\sigma^2\sin^2\theta_r}\cdot\exp\left(\frac{8K(K+1)\sigma^2\sin^2\theta_r}{\Omega(i-j)^2+8(K+1)\sigma^2\sin^2\theta_r}-K\right)\right)^{L_a}$$

Simulation Result

New Analysis

- ✓ Sequentially encoding under a tree structure for Spinal codes
- ✓ ML decoding rules and other decoding process
- ✓ Introducing **Rotation Matrix** to simplify the derivation
- ✓ Considering **three typical fading channels**：**Rayleigh**、**Rician**、**Nakagami** (CSI)



➢ Conclusion

All approximations are close to simulated values.

1、The derived upper bounds is right.

2、Better estimation of the upper bound.

Achieving uniform tight approximations over a wide range of SNR

Thank you!
Q&A