Abstract

A novel recurrence formula for moments with respect to Müntz-Legendre polynomials is proposed and applied to construct a numerical method for solving generalized Gauss quadratures with power function weight for Müntz systems. These quadrature rules exhibit several properties similar to the classical Gaussian quadratures for polynomial systems, including positive weights, rapid convergence, and others. They are applicable to a wide range of functions, including smooth functions and functions with endpoint singularities, commonly found in integral equations with singular kernels, complex analysis, potential theory, and other areas.

1 Introduction

Let $\omega(x)$ be a continuous function that is positive almost everywhere in the interval $[a, b]$. Let $\varphi(x)$ be an integrable function on $[a, b]$. If $\varphi(x)\omega(x)$ is integrable, we refer to its integral as the integral of $\varphi(x)$ with respect to the weight function $\omega(x)$, denoted by

$$I[\varphi] = \int_a^b \varphi(x)\omega(x)\mathrm{d}x. \tag{1}$$

According to the definition of integration, we can select certain nodes x_k on the interval $[a, b]$, and then approximate the value of $I[\varphi]$ through a weighted average of $\varphi(x_k)$. This leads to a numerical quadrature with the following form:

$$Q_N[\varphi] = \sum_{k=0}^{N} \varphi(x_k)w_k, \tag{2}$$

where $x_k \in [a, b]$ and $w_k \in \mathbb{R}$ for $k = 0, 1, \cdots, N$. x_k are called the nodes of the quadrature eq. (2), and ω_k are the corresponding weights. The selection of $\{\omega_k\}_{k=0}^N$ depends only on $\{x_k\}_{k=0}^N$ and not on the specific form of the integrand $\varphi(x)$.

We use $Q_N[\varphi]$ as an approximation of $I[\varphi]$. When the nodes x_k are chosen as the zeros of orthogonal polynomials of degree $N + 1$ with respect to $\omega(x)$, and the weights ω_k are determined by the corresponding interpolation polynomials of the nodes, the resulting numerical quadrature $Q_N[\cdot]$, known as Gauss quadrature, achieves the highest algebraic accuracy of $2N + 1$.

Gauss quadrature offers several advantages. Firstly, all quadrature nodes x_k are located in the interior of $[a, b]$, and the weights ω_k are all positive, which ensures the stability of integration computations [17]. Secondly, the Gaussian rule with $N + 1$ points is exact for polynomials of degree up to $2N+1$, resulting in a rapid convergence for integrands that can be well approximated by polynomials. Moreover, Gaussian quadrature can be computed efficiently due to its connection with orthogonal polynomials, with computational costs scaling as $O(N^2)$ for the classic Golub-Welsch algorithm [10] or $O(N)$ for more specialized methods [9].

Although Gaussian rule can converge for some singular integrands, such as the log function or Müntz polynomials [3], which are commonly encountered in the boundary element method and the finite element method for solving partial differential equations, the convergence rate is typically low and using large nodes in quadrature is not recommended. While monomial transformation [18], graded meshes [26], or adaptive methods [7] can be used to address these issues, these methods often lack the stability, rapid convergence, and elegance of Gaussian rule.

Gaussian rule can be extended in a natural way to general functions. Let

$$\{\varphi_0, \varphi_1, \cdots, \varphi_{2N+1}\} \tag{3}$$

be a system of linearly independent functions that are usually chosen to be complete in some suitable space of functions [15, 19]. The generalized Gauss quadrature aims to select x_k and ω_k, $k = 0, 1, \cdots, N$, such that

$$Q_N[\varphi_j] = I[\varphi_j], \quad j = 0, 1, \cdots, 2N+1. \tag{4}$$

The work of Rokhlin and others [19, 30, 6, 4] has explored the use of generalized Gauss quadrature in numerical algorithms. The method proposed in [19] constructs a mapping from the Hermite system and identifies the zeros of this mapping as the Gaussian nodes. This approach involves a variant of Newton's algorithm coupled with a continuation scheme. Subsequent modifications have been proposed in [30, 6, 4] by incorporating preprocessing steps to enhance the robustness of the Gauss quadrature procedures. While these approaches can be used for almost any basis set, they may not be efficient when eq. (3) is restricted to a specific system, such as the Müntz system, partly due to the difficulty in estimating the value of this mapping.

Specifically, when constraining the functions in eq. (3) to trigonometric functions [23] or spline functions [5], the corresponding Gauss quadratures have been considered. For the Müntz system, a straightforward approach for computing Gauss quadrature was proposed in [22]. The proposed method utilizes a continuation technique and Newton's method to solve a set of $2N + 2$ nonlinear equations eq. (4) for the $2N + 2$ unknowns x_k and ω_k. However, the proposed method has some limitations. For instance, when considering a specific Müntz sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{2N+1}\}$ and choosing φ_j as the corresponding Müntz-Legendre polynomials:

1. The accuracy of the estimation of $\varphi_j(x)$ may be compromised when $\varphi_j(x)$ is singular at 0 and x approaches 0.

2. The method relies on calculating $I[\varphi_j]$ via orthogonality, which can only be done by choosing $\omega(x) = x^{\lambda_0}$.

3. The Jacobian in the Newton equation may not be accurately computed when $\varphi_j(x)$ becomes singular at 0 and x is close to 0.

In this paper, we propose a strategy for determining stable parameters for computing Müntz-Legendre polynomials, and optimize this process to minimize computational cost using dynamic programming. Subsequently, a recurrence formula (theorem 3.2) for the moments of Müntz-Legendre polynomials is presented, which enables efficient computation of these moments for arbitrary power weight functions. Finally, we introduce a modification to the original Newton equation by incorporating a damping factor for the step size, with the goal of improving stability and efficiency in the computation process.

This paper is organized as follows. In section 2, we provide a restatement of some preliminaries. In section 3, we introduce the concept of orthogonal Müntz polynomials and derive some of their recurrence formulae. The detailed computation of Müntz-Legendre polynomials will be presented in section 4. The numerical construction of the Gauss quadrature rule will be discussed in section 5. Finally, we will address the error estimation of the Gauss quadrature for a specific class of Müntz sequences in section 6, and present numerical examples in section 7.

2 Preliminaries

In this section, we summarize several classical results and numerical tools. They can be found, for example, in [15, 19, 1].

2.1 Existence and uniqueness of quadrature rule

Definition 2.1 ([15]). *A finite set of functions $\varphi_0, \varphi_1, \cdots, \varphi_n$ defined on the interval $[a, b]$ is called a Chebyshev system if and only if $\varphi_j \in C[a, b]$, $j = 0, 1, \cdots, n$, and the determinant of matrix $\mathbf{\Phi}$ is non-zero, where*

$$\mathbf{\Phi} = \begin{bmatrix} \varphi_0(x_0) & \varphi_0(x_1) & \cdots & \varphi_0(x_n) \\ \varphi_1(x_0) & \varphi_1(x_1) & \cdots & \varphi_1(x_n) \\ \vdots & \vdots & & \vdots \\ \varphi_n(x_0) & \varphi_n(x_1) & \cdots & \varphi_n(x_n) \end{bmatrix}, \tag{5}$$

and $x_0, x_1, \cdots, x_n$ are any distinct points in the interval $[a, b]$.

Theorem 2.1 ([15]). *Let $\varphi_0, \varphi_1, \cdots, \varphi_{2N+1}$ be a Chebyshev system of functions defined on the interval $[a, b]$. Then, there exist unique $N + 1$ Gaussian nodes $x_k \in (a, b)$ and weights $w_k > 0$, $k = 0, 1, \cdots, N$, such that*

$$I[\varphi_j] = Q_N[\varphi_j], \quad j = 0, 1, \cdots, 2N + 1.$$

The result of theorem 2.1 is a corollary of a more general geometric property of moment spaces, as derived from the Chebyshev system [15]. Theorem 2.1 can be extended to a class of Gauss quadratures involving functions with endpoint singularities.

Theorem 2.2 ([19]). *Let $\varphi_j : (a,b) \to \mathbb{R}$ be continuous and integrable on (a,b) for $j = 0, 1, \cdots, 2N+1$, and $r(x) > 0$ be continuous and integrable on (a,b). Define $\psi_j(x) = \varphi_j(x)/r(x)$. If*

$$\lim_{x \to a} \psi_j(x) < \infty, \tag{6}$$

and $\psi_0, \psi_1, \cdots, \psi_{2N+1}$ form a Chebyshev system on the interval $[a,b]$, and $\omega(x)r(x)$ is integrable, then there exists a unique Gaussian rule with $N+1$ nodes that is exact for $\varphi_0, \varphi_1, \cdots, \varphi_{2N+1}$, and all the Gaussian weights $w_0, w_1, \cdots, w_N$ are positive.

2.2 Continuation method

The continuation method [1], also known as the homotopy continuation method, is a numerical technique used to find roots of nonlinear equations. It is a path-following algorithm that traces a solution curve in parameter space from a known solution to the desired one.

Stated briefly, suppose that we are trying to solve a system of m nonlinear equations with m unknows,

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}, \tag{7}$$

where $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable. Certainly, if a good approximation $\mathbf{x}^{(0)}$ of the zero point $\mathbf{x}^*$ of $\mathbf{F}$ is available, it is advisable to calculate $\mathbf{x}^*$ by a Newton's method defined by an iteration formula as

$$\mathbf{x}^{(k+1)} \leftarrow \mathbf{x}^{(k)} - \mathbf{J}_k^{-1}\mathbf{F}(\mathbf{x}^{(k)}), \tag{8}$$

where $\mathbf{J}_k$ is the Jacobian of $\mathbf{F}$ at $\mathbf{x}^{(k)}$. In many cases, the iteration eq. (8) fails if a starting point is not available directly. As a possible remedy, we define a homotopy mapping $\mathbf{H} : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ such that

$$\mathbf{H}(\mathbf{x}, 0) = \mathbf{G}(\mathbf{x}), \quad \mathbf{H}(\mathbf{x}, 1) = \mathbf{F}(\mathbf{x}),$$

where $\mathbf{G} : \mathbb{R}^m \to \mathbb{R}^m$ is a trivial mapping, so that the solution $\mathbf{x}_0$ of $\mathbf{G}(\mathbf{x}) = \mathbf{0}$ is unique and known. We attempt to trace an implicitly defined curve $\mathcal{C} \in \mathbf{H}^{-1}(\mathbf{0})$ from a starting point $(\mathbf{x}_0, 0)$ to the solution point $(\mathbf{x}^*, 1)$.

Suppose that for any $\alpha \in [0,1]$, the system of equations $\mathbf{H}(\mathbf{x}, \alpha) = \mathbf{0}$ has a unique solution denoted as $\mathbf{x}(\alpha)$. Furthermore, assume that $\mathbf{x}(\alpha)$ is continuously differentiable with respect to α, and that the Jacobian matrix $\partial_{\mathbf{x}}\mathbf{H}$ evaluated at $\mathbf{x}(\alpha)$ is nonsingular.

Then, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p = 1$ such that the maximum interval length $\max_{1 \le i \le p} |\alpha_i - \alpha_{i-1}|$ is sufficiently small. We can obtain $\mathbf{x}(\alpha_i)$ by solving $\mathbf{H}(\mathbf{x}, \alpha_i) = \mathbf{0}$ with initial guess $\mathbf{x}(\alpha_{i-1})$. This process is repeated starting from $\mathbf{x}(\alpha_0)$, and incrementing i from 1 to p. The solution $\mathbf{x}^*$ is obtained when α_p is reached.

A detailed discussion of the continuation method can be found in [1], where the convergence of the continuation method under more general conditions is considered.

3 Orthogonal Müntz polynomials

Let a complex Müntz sequence $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \cdots\}$, and its first $N+1$ elements be denoted as $\Lambda_N = \{\lambda_0, \lambda_1, \ldots, \lambda_N\}$. The set of functions generated by the Müntz sequence, $\{x^{\lambda_0}, x^{\lambda_1}, \cdots, x^{\lambda_N}\}$, is called the Müntz system. The linear space over the field of real numbers generated by the Müntz sequence, denoted by $M(\Lambda_N) = \operatorname{span}\left\{x^{\lambda_0}, x^{\lambda_1}, \cdots, x^{\lambda_N}\right\}$, is called the Müntz space. That is, the Müntz space is the collection of all Müntz polynomials

$$p(x) = \sum_{i=0}^{N} a_i x^{\lambda_i}, \quad a_i \in \mathbb{R}.$$

For the L^2 theory of Müntz system [2], we consider Λ_N satisfying

$$\operatorname{Re}(\lambda_n) > -\frac{1}{2}, \quad n = 0, 1, 2 \cdots, N, \tag{9}$$

where $\operatorname{Re}(\lambda_n)$ is the real part of λ_n. eq. (9) ensures that every Müntz polynomial in $M(\Lambda_N)$ is in $L^2(0,1)$. The corresponding Müntz-Legendre polynomials [28], orthogonal with respect to Lebesgue measure, are defined as

$$L_n(x;\Lambda_n) = \frac{1}{2\pi i}\int_{\Gamma_n} W_n(t)x^t \mathrm{d}t, \quad W_n(t) = \prod_{k=0}^{n-1}\frac{t+\bar{\lambda}_k+1}{t-\lambda_k}\frac{1}{t-\lambda_n}, \tag{10}$$

where $n = 0, 1, 2, \cdots, N$, Γ_n is a simple contour on the complex plane that encircles $\lambda_0, \cdots, \lambda_n$, and $\bar{\lambda}_k$ denotes the conjugate of λ_k. It is obvious that $L_0(x;\Lambda_0) = x^{\lambda_0}$. Henceforth, without causing confusion, we will denote $L_n(x;\Lambda_n)$ as $L_n(x)$ or simply L_n. The definition of *eq.* (10) does not require the distinctness of the exponents λ_k. In fact, repeated λ_k results in $\log x$ coming into the picture.

Lemma 3.1. *Let $\Lambda_n = \{\lambda_0, \cdots, \lambda_0, \cdots, \lambda_s, \cdots, \lambda_s\}$ consist of r_k copies of λ_k, $r_k \geq 1$, $k = 0, 1, \cdots, s$, $s \leq n$, and $\sum_{k=0}^{s} r_k = n+1$. Then*

$$\operatorname{span}\{L_0, \cdots, L_n\} = \operatorname{span}\left\{x^{\lambda_0}, \cdots, x^{\lambda_0}\log^{r_0-1}x, \cdots, x^{\lambda_s}, \cdots, x^{\lambda_s}\log^{r_s-1}x\right\}.$$

The proof of lemma 3.1 requires lemma 3.2.

Lemma 3.2 ([14])**.** *Let $P(x)$ and $Q(x)$ be coprime polynomials with complex coefficients, where the degree of $P(x)$ is lower than the degree of $Q(x)$. Since irreducible polynomials over the field of complex numbers are linear, we suppose the irreducible factorization of $Q(x)$ as follows*

$$Q(x) = \prod_{\nu=0}^{s}(x-\lambda_\nu)^{r_\nu},$$

where $\lambda_\nu \in \mathbb{C}$ and $r_\nu \in \mathbb{Z}^+$. Then, $P(x)/Q(x)$ can be represented as the sum of simple fractions in the following form

$$\frac{P(x)}{Q(x)} = \sum_{\nu=0}^{s} \sum_{k=1}^{r_\nu} \frac{A_{\nu,k}}{(x-\lambda_\nu)^k},$$

where $A_{\nu,k}$ are complex constants for $0 \le \nu \le s$ and $1 \le k \le r_\nu$.

Proof of lemma 3.1. By lemma 3.2, $W_n(t)$ can be factored into

$$W_n(t) = \frac{1}{t+\bar{\lambda}_s+1} \prod_{\nu=0}^{s} \frac{(t+\bar{\lambda}_v+1)^{r_v}}{(t-\lambda_v)^{r_v}} = \sum_{v=0}^{s} \sum_{k=1}^{r_v} \frac{A_{v,k}}{(t-\lambda_v)^k}. \tag{11}$$

By multiplying $(t-\lambda_v)^{r_v}$ on both sides of eq. (11) and taking the (r_v-k)-th derivative, we can compute $A_{v,k}$ as follows

$$A_{v,k} = \frac{1}{(r_v-k)!} \lim_{t\to\lambda_v} \partial_t^{r_v-k} \left(\frac{\prod_{j=0}^{s}(t+\bar{\lambda}_j+1)^{r_j}}{(t+\bar{\lambda}_s+1)\prod_{j=0,j\neq v}^{s}(t-\lambda_j)^{r_j}} \right).$$

Then

$$L_n(x) = \frac{1}{2\pi i} \int_{\Gamma_n} \sum_{v=0}^{s} \sum_{k=1}^{r_v} \frac{A_{v,k}}{(t-\lambda_v)^k} x^t \mathrm{d}t = \sum_{v=0}^{s} \sum_{k=1}^{r_v} A_{v,k} \frac{1}{2\pi i} \int_{\Gamma_n} \frac{x^t}{(t-\lambda_v)^k} \mathrm{d}t \tag{12}$$

By the residue theorem [27], eq. (12) could be represented as

$$L_n(x) = \sum_{v=0}^{s} \sum_{k=1}^{r_v} \frac{A_{v,k}}{\Gamma(k)} x^{\lambda_v} \log^{k-1} x.$$

Therefore, it follows that

$$\mathrm{span}\{L_0, \cdots, L_n\} \subseteq \mathrm{span}\left\{ x^{\lambda_0}, \cdots, x^{\lambda_0}\log^{r_0-1} x, \cdots, x^{\lambda_s}, \cdots, x^{\lambda_s}\log^{r_s-1} x \right\}.$$

By a dimension argument, the conclusion is proved. □

To describe the case of repeated elements in the Müntz sequence Λ_N, we define

$$\hat{M}(\Lambda_n) = \mathrm{span}\{L_0, L_1, \cdots, L_n\}, \quad n = 0, 1, \cdots, N. \tag{13}$$

Thus, for any Müntz sequence Λ_N that satisfies (9), we always have

$$M(\Lambda_n) \subseteq \hat{M}(\Lambda_n), \quad n = 0, 1, \cdots, N. \tag{14}$$

The name Müntz-Legendre polynomial is justified by the following theorem, where the orthogonality of L_n allows repeated indices.

Theorem 3.1 ([3]). *Let Müntz sequence Λ_N satisfy eq.* (9) *and L_n be defined by eq.* (10). *Then*

$$\int_0^1 L_n(x)\overline{L_m(x)}\mathrm{d}x = \frac{\delta_{n,m}}{1+\lambda_n+\bar{\lambda}_n}, \quad n,m = 0,1,\cdots,N, \tag{15}$$

where $\delta_{n,m}$ is the Kronecker-Delta symbol.

Furthermore, we have

$$L_n(1) = 1, \quad L_n'(1) = \lambda_n + \sum_{k=0}^{n-1}(\lambda_k + \bar{\lambda}_k + 1), \quad n = 0,1,\cdots,N, \tag{16}$$

and the recurrence formula with derivatives for L_n,

$$xL_n'(x) - xL_{n-1}'(x) = \lambda_n L_n(x) + \left(1+\bar{\lambda}_{n-1}\right)L_{n-1}(x), \quad n = 1,2,\cdots,N. \tag{17}$$

Moreover, the moments with respect to L_n in the interval $[0,1]$ have the recurrence formula.

Theorem 3.2. *Let Müntz sequence Λ_N and $\lambda \in \mathbb{C}$ satisfying*

$$\mathrm{Re}(\lambda+\lambda_n) > -1, \quad n = 0,1,\cdots,N.$$

Müntz-Legendre polynomial L_n is defined by eq. (10). *Then the moments satisfy*

$$\int_0^1 L_n(x)x^\lambda \mathrm{d}x = \frac{\lambda - \bar{\lambda}_{n-1}}{1+\lambda+\lambda_n}\int_0^1 L_{n-1}(x)x^\lambda \mathrm{d}x, \quad n \geq 1, \tag{18}$$

and $\int_0^1 L_0(x)x^\lambda \mathrm{d}x = 1/(1+\lambda+\lambda_0)$.

Proof. The case of $n = 0$ is trivial and we subsequently consider $n \geq 1$. We assume $\lambda \notin \Lambda_N$, otherwise there is a limit argument since

$$\int_0^1 L_n(x)x^\lambda \mathrm{d}x, \quad n = 0,1,\cdots,N,$$

are continuous with respect to λ. Let Γ_n encircle $\lambda_0, \lambda_1, \cdots, \lambda_n$ and satisfy

$$\mathrm{Re}(t+\lambda) > -1, \quad \forall t \in \Gamma_n.$$

By eq. (10),

$$\int_0^1 L_n(x)x^\lambda \mathrm{d}x = \frac{1}{2\pi i}\int_0^1 \int_{\Gamma_n} W_n(t)x^{t+\lambda}\mathrm{d}t\mathrm{d}x.$$

For any $\varepsilon > 0$, there exists $0 < \delta < 1$, such that

$$\left|\int_0^\delta x^{t+\lambda}\mathrm{d}x\right| < \varepsilon, \quad \forall t \in \Gamma_n.$$

Since $W_n(t)$ is continuous on the closed curve Γ_n, $W_n(t)$ is bounded and achieves its upper and lower bounds on Γ_n. Therefore, the integral $\int_0^1 W_n(t)x^{t+\lambda}\mathrm{d}x$ converges uniformly for $t \in \Gamma_n$. Thus, the order of integration can be exchanged, i.e.,

$$\int_0^1 \int_{\Gamma_n} W_n(t)x^{t+\lambda}\mathrm{d}t\mathrm{d}x = \int_{\Gamma_n} \int_0^1 W_n(t)x^{t+\lambda}\mathrm{d}x\mathrm{d}t.$$

Thus,

$$\int_0^1 L_n(x)x^{\lambda}\mathrm{d}x = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_n} \frac{W_n(t)}{t+\lambda+1}\mathrm{d}t.$$

Note that Γ_n does not encircle the singularity $-\lambda-1$. We change the integration path to $|t| = R$, where $R > \max\{|\lambda_0|+1, |\lambda_1|+1, \cdots, |\lambda_n|+1, |\lambda|+1\}$. Then, using the Cauchy integral formula [27], we have

$$\int_0^1 L_n(x)x^{\lambda}\mathrm{d}x = \frac{1}{2\pi i} \int_{|t|=R} \frac{W_n(t)}{t+\lambda+1}\mathrm{d}t - W_n(-\lambda-1). \tag{19}$$

Since

$$\left| \int_{|t|=R} \frac{W_n(t)}{t+\lambda+1}\mathrm{d}t \right| = \left| \int_0^{2\pi} \frac{W_n(Re^{i\theta})}{Re^{i\theta}+\lambda+1} Rie^{i\theta}\mathrm{d}\theta \right| = O\left(\frac{1}{R}\right),$$

we take the limit $R \to \infty$ in eq. (19), then the integral term over $|t| = R$ vanishes, i.e.,

$$\int_0^1 L_n(x)x^{\lambda}\mathrm{d}x = -W_n(-\lambda-1). \tag{20}$$

Similarly, we have

$$\int_0^1 L_{n-1}(x)x^{\lambda}\mathrm{d}x = -W_{n-1}(-\lambda-1). \tag{21}$$

By the definition of $W_n(t)$, it follows that

$$W_n(t) = W_{n-1}(t)\frac{t+\bar{\lambda}_{n-1}+1}{t-\lambda_n}. \tag{22}$$

By combining eq. (20), eq. (21), and eq. (22), we obtain eq. (18). □

Let $\beta \in \mathbb{R}$ and Λ_N satisfy

$$\mathrm{Re}(\lambda_n) + \beta/2 > -1/2, \quad n = 0, 1, \cdots, N. \tag{23}$$

Putting $\lambda_k + \beta/2$ instead of λ_k, $k = 0, 1, \cdots, N$, in the Müntz sequence Λ_N, we can define a kind of Müntz-Jacobi polynomials $L_n^{\beta}(x;\Lambda_n) = x^{-\beta/2}L_n(x;\Lambda_n+\beta/2)$. Then

$$L_n^{\beta}(x;\Lambda_n) = \frac{x^{-\beta/2}}{2\pi i} \int_{\Gamma} \prod_{k=0}^{n-1} \frac{t+\bar{\lambda}_k+\beta/2+1}{t-\lambda_k-\beta/2} \frac{x^t}{t-\lambda_n-\beta/2}\mathrm{d}t. \tag{24}$$

Owing to the properties of Müntz-Legendre polynomials, the following results hold.

Theorem 3.3. *Let $\beta \in \mathbb{R}$, and Λ_N satisfy eq. (23). Then*

$$\int_0^1 L_n^\beta(x;\Lambda_n) L_m^\beta(x;\Lambda_m) x^\beta \mathrm{d}x = \frac{\delta_{n,m}}{\lambda_n + \bar{\lambda}_n + \beta + 1}, \quad n,m = 0,1,\cdots,N. \tag{25}$$

It is evident that $\hat{M}(\Lambda_{2N+1}) = \mathrm{span}\{L_0^\beta, \cdots, L_{2N+1}^\beta\}$. Let $\partial_x L_n^\beta$ denote the derivative of L_n^β. It follows from eq. (16) that

$$L_n^\beta(1) = 1, \quad \partial_x L_n^\beta(1) = \lambda_n + \sum_{k=0}^{n-1} (\lambda_k + \bar{\lambda}_k + \beta + 1), \quad n = 0,1,\cdots,N. \tag{26}$$

And the recurrence formula of L_n^β

$$x\partial_x L_n^\beta - x\partial_x L_{n-1}^\beta = \lambda_n L_n^\beta + (1 + \bar{\lambda}_{n-1} + \beta) L_{n-1}^\beta. \tag{27}$$

Moreover, the moments recurrence read

$$\int_0^1 L_n^\beta x^\beta \mathrm{d}x = \frac{-\lambda_{n-1}}{1 + \lambda_n + \beta} \int_0^1 L_{n-1}^\beta x^\beta \mathrm{d}x. \tag{28}$$

4 Computation of Müntz-Legendre polynomials

Let Λ_N be a Müntz sequence and the corresponding Müntz-Legendre polynomial be defined as in eq. (10). In contrast to the computation of algebraic polynomials, which can be efficiently achieved using a three-term recurrence formula, Müntz-Legendre polynomials lack a recurrence formula that enables high-efficiency computation. Therefore, we resort to the complex integration method [20, 21], which aims to directly compute the original definition given in eq. (10).

Theorem 4.1 ([20])**.** *For all $x \in (0,1)$, $L_n(x)$ can be written as*

$$L_n(x) = -\frac{x^\sigma}{2\pi} \left(\int_0^{+\infty} W_n(\sigma - it) e^{i\omega t} \mathrm{d}t + \int_0^{+\infty} W_n(\sigma + it) e^{-i\omega t} \mathrm{d}t \right), \tag{29}$$

where $\sigma < \min\{\mathrm{Re}(\lambda_0), \mathrm{Re}(\lambda_1), \cdots, \mathrm{Re}(\lambda_N)\}$, $\omega = -\log x$.

We only consider the real Müntz sequence. It is obvious that $\overline{W_n(t)} = W_n(\bar{t})$ and

$$L_n(x) = -\frac{x^\sigma}{\pi} \,\mathrm{Re}\left\{ \int_0^{+\infty} W_n(\sigma - it) e^{i\omega t} \mathrm{d}t \right\}. \tag{30}$$

Without loss of generality, we apply a scaling transformation to the integral in equation (30). Thus

$$\int_0^{+\infty} W_n(\sigma - it) e^{i\omega t} \mathrm{d}t = \frac{1}{\omega} \int_0^{+\infty} W_n\left(\sigma - i\frac{t}{\omega}\right) e^{it} \mathrm{d}t.$$

Let $f_n(t,\omega) = -\frac{i}{\omega} W_n(\sigma - \frac{it}{\omega})$, then

$$f_n(t,\omega) = \prod_{v=0}^{n-1} \frac{t + i\omega(\sigma + \bar{\lambda}_v + 1)}{t + i\omega(\sigma - \lambda_v)} \frac{1}{t + i\omega(\sigma - \lambda_n)}. \tag{31}$$

Hence

$$L_n(x) = \frac{x^\sigma}{\pi} \operatorname{Im}\left\{ \int_0^{+\infty} f_n(t,\omega) e^{it} \mathrm{d}t \right\}, \tag{32}$$

where $\operatorname{Im}\{u\}$ denotes the image part of u.

Theorem 4.2 ([20]). *For any $a > 0$, the integral of $f_n(t,\omega)$can be transformed into*

$$\int_0^{+\infty} f_n(t,\omega) e^{it} \mathrm{d}t = \int_0^a f_n(t,\omega) e^{it} \mathrm{d}t + i e^{ia} \int_0^{+\infty} f_n(a + iy, \omega) e^{-y} \mathrm{d}y. \tag{33}$$

Thus, we have

$$L_n(x) = \frac{x^\sigma}{\pi} \operatorname{Im}\left\{ I_1^n + I_2^n \right\}, \tag{34}$$

where

$$I_1^n = \int_0^a f_n(t,\omega) e^{it} \mathrm{d}t, \quad I_2^n = i e^{ia} \int_0^{+\infty} f_n(a + iy, \omega) e^{-y} \mathrm{d}y.$$

Therefore, it is need to accurately compute two parts: x^σ and $I_1^n + I_2^n$. By a simple scaling argument, it follows that

$$|f_n(t,\omega)| \le \prod_{v=0}^{n-1} \frac{|\sigma + \bar{\lambda}_v + 1|}{|\sigma - \lambda_v|} \frac{1}{\omega|\sigma - \lambda_n|}. \tag{35}$$

When σ approaches λ_v for some v or ω approaches 0, $f_n(t,\omega)$ exhibits singularity at 0. We aim to compute L_n with the objective of ensuring that:

1. $f_n(t,w)$ does not exhibit singularity at 0, so that $I_1^n + I_2^n$ can be accurately calculated numerically.
2. x^σ does not amplify the error of $I_1^n + I_2^n$.

Let $\sigma = \lambda_{min} - \theta/\omega$, where $\theta > 0$ is a parameter to be determined, and $\lambda_{min} = \min_v \lambda_v$. Then we have

$$\prod_{v=0}^{n-1} \frac{|\sigma + \bar{\lambda}_v + 1|}{|\sigma - \lambda_v|} \frac{1}{\omega|\sigma - \lambda_n|} = \prod_{v=0}^{n-1} \frac{|\theta - \omega(\lambda_{min} + \bar{\lambda}_v + 1)|}{|\theta + \omega(\lambda_{min} + \lambda_v)|} \frac{1}{|\theta + \omega(\lambda_{min} + \lambda_n)|},$$

and

$$x^\sigma = x^{\lambda_{min}} e^\theta.$$

The choice of σ effectively avoids the singularity of $f_n(t,\omega)$ as ω approaches 0. However, when $\lambda_{min}+\lambda_v$ approaches 0 for some v, we need to carefully choose the value of θ. On the one hand, we cannot choose too large θ as it will result in large x^σ and may cause numerical instability. On the other hand, we cannot choose a too small θ as it will make $f_n(t,\omega)$ singular near 0, which also leads to numerical issues. Hence, it is crucial to strike a balance between these two considerations. Let

$$\mathcal{R}_n(\theta)=e^{\sqrt{\omega}}\prod_{v=0}^{n-1}\frac{|\theta-\omega(\lambda_{min}+\bar{\lambda}_v+1)|}{|\theta+\omega(\lambda_{min}+\lambda_v)|}\frac{1}{|\theta+\omega(\lambda_{min}+\lambda_n)|}+\frac{e^{\theta-\lambda_{min}\omega}}{\sqrt{\theta}}.$$

We use the Nelder-Mead simplex algorithm [16] to calculate $\theta=\arg\min_{\theta>0}\mathcal{R}_n(\theta)$.

For the calculation of I_2^n, we choose an appropriate $a>0$ and use Gauss-Laguerre quadrature to approximate it.

For the calculation of I_1^n, note that $f_n(t,\omega)e^{it}$ oscillates in $(0,a)$. When $x\in[\epsilon,1]$ for some small $\epsilon>0$, the oscillation factor ω falls within the range of $[0,-\log\epsilon]$. Consequently, in double precision calculations, its value remains relatively small. Therefore, we use a piecewise approximation to calculate I_1^n. Let $a=mh$, where h is a small constant and $m\in\mathbb{N}^+$ is a positive integer. Then

$$I_1^n=h\int_0^1\sum_{k=1}^{m}f_n(h(y+k-1),\omega)e^{ih(y+k-1)}\mathrm{d}y \tag{36}$$

can be approximated by using Gauss-Legendre quadrature. We denote Q_1^n and Q_2^n as the numerical approximations of I_1^n and I_2^n, respectively.

In terms of implementation, the recurrence relation

$$f_{n+1}(t,\omega)=f_n(t,\omega)\frac{t+i\omega(\sigma+\bar{\lambda}_n+1)}{t+i\omega(\sigma-\lambda_{n+1})}$$

enables us to optimize the computation of all Müntz-Legendre polynomials by dynamic programming. As a result, the computational cost of $L_0(x),L_1(x),\cdots,L_N(x)$ is comparable to that of $L_N(x)$. This process is summarized in the following algorithm 1.

5 Numerical construction of quadrature rule

In this section, we present a method for the numerical construction of the generalized Gauss quadrature that is weighted by power function $\omega(x)=x^\beta$ for Müntz polynomials. Assume that the Müntz sequence Λ_{2N+1} satisfies (23) and consists solely of real indices. Our goal is to determine Gaussian nodes x_k and weights ω_k, $k=0,1,\cdots,N$, such that

$$\sum_{k=0}^{N}L_n^\beta(x_k;\Lambda_n)\omega_k=m_n,\quad n=0,1,\cdots,2N+1, \tag{37}$$

Algorithm 1 Compute all Müntz-Legendre polynomials for real Müntz sequence

Require: Λ_N and x.

Choose m, h in eq. (36) and $M \in \mathbb{N}^+$.

Compute $\theta = \arg\min_{\theta>0} \mathcal{R}_N(\theta)$, $\omega = -\log x$ and $\sigma = \lambda_{min} - \theta/\omega$.

Compute Gauss-Legendre rule $\{\xi_j, \psi_j\}_{j=0}^{M}$ and Gauss-Laguerre rule $\{\tau_j, \eta_j\}_{j=0}^{M}$.

Compute $H_{j,k}^0 = f_0(h(\xi_j + k - 1), \omega)e^{ih(\xi_j+k-1)}$ and $f_0(mh + i\tau_j, \omega)$, $j = 0, 1, \cdots, M$.

Let $L_0(x) \leftarrow x^{\lambda_0}$.

for $n = 1, 2, \cdots, N$ **do**

for $j = 1, 2, \cdots, M$ **do**

$$H_{j,k}^n = \frac{h(\xi_j + k - 1) + i\omega(\sigma + \lambda_{n-1} + 1)}{h(\xi_j + k - 1) + i\omega(\sigma - \lambda_n)} H_{j,k}^{n-1}, \quad k = 0, 1, \cdots, m.$$

$$f_n(mh + i\tau_j, \omega) = \frac{mh + i\tau_j + i\omega(\sigma + \lambda_{n-1} + 1)}{mh + i\tau_j + i\omega(\sigma - \lambda_n)} f_{n-1}(mh + i\tau_j, \omega).$$

end for

Compute $Q_1^n \leftarrow h \sum_{j=1}^{M} \sum_{k=0}^{m} H_{j,k}^n \psi_j$ and $Q_2^n \leftarrow ie^{imh} \sum_{j=1}^{M} f_n(mh + i\tau_j, \omega)\eta_j$.

Update $L_n(x) \leftarrow \frac{x^\sigma}{\pi} \mathrm{Im}\{Q_1^n + Q_2^n\}$.

end for

return $L_0(x), L_1(x), \cdots, L_N(x)$.

where the moments m_n are defined as

$$m_n = \int_0^1 L_n^\beta(x;\Lambda_n)\omega(x)\mathrm{d}x.$$

The existence and uniqueness of the Gauss quadrature eq. (37) are guaranteed by theorem 2.1 and theorem 2.2, as the set $\{L_0^\beta, \cdots, L_{2N+1}^\beta\}$ constitutes a Chebyshev system.

We define $\mathbf{c} = [m_0, \cdots, m_N]^\mathrm{T}$, $\mathbf{d} = [m_{N+1}, \cdots, m_{2N+1}]^\mathrm{T}$, $\mathbf{m} = [\mathbf{c}^\mathrm{T}, \mathbf{d}^\mathrm{T}]^\mathrm{T}$, $\boldsymbol{\omega} = [\omega_0, \cdots, \omega_N]^\mathrm{T}$, $\mathbf{x}^{-\frac{\beta}{2}} = [x_0^{-\frac{\beta}{2}}, \cdots, x_N^{-\frac{\beta}{2}}]^\mathrm{T}$, and the matrices $\mathbf{U}$ and $\mathbf{V}$

$$\mathbf{U} = \begin{bmatrix} L_0(x_0;\Lambda_0+\beta/2), & \cdots, & L_0(x_N;\Lambda_0+\beta/2) \\ L_1(x_0;\Lambda_1+\beta/2), & \cdots, & L_1(x_N;\Lambda_1+\beta/2) \\ \vdots & & \vdots \\ L_N(x_0;\Lambda_N+\beta/2), & \cdots, & L_N(x_N;\Lambda_N+\beta/2) \end{bmatrix}, \tag{38}$$

$$\mathbf{V} = \begin{bmatrix} L_{N+1}(x_0;\Lambda_{N+1}+\beta/2), & \cdots, & L_{N+1}(x_N;\Lambda_{N+1}+\beta/2) \\ L_{N+2}(x_0;\Lambda_{N+2}+\beta/2), & \cdots, & L_{N+2}(x_N;\Lambda_{N+2}+\beta/2) \\ \vdots & & \vdots \\ L_{2N+1}(x_0;\Lambda_{2N+1}+\beta/2), & \cdots, & L_{2N+1}(x_N;\Lambda_{2N+1}+\beta/2) \end{bmatrix}. \tag{39}$$

Then, the equation eq. (37) can be written in matrix form as

$$\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} \mathrm{diag}(\mathbf{x}^{-\frac{\beta}{2}})\boldsymbol{\omega} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}, \tag{40}$$

where $\mathrm{diag}(\mathbf{x})$ is a diagonal matrix with $\mathbf{x}$ as its entries. Let $\boldsymbol{\Psi} = [\mathbf{U}^\mathrm{T}, \mathbf{V}^\mathrm{T}]^\mathrm{T}$ and $\mathbf{z} = [\mathbf{x}^\mathrm{T}, \boldsymbol{\omega}^\mathrm{T}]^\mathrm{T}$. We construct a mapping $\mathbf{F} : \mathbb{R}^{2N+2} \to \mathbb{R}^{2N+2}$ such that

$$\mathbf{F}(\mathbf{z}) = \boldsymbol{\Psi}\mathrm{diag}(\mathbf{x}^{-\frac{\beta}{2}})\boldsymbol{\omega} - \mathbf{m}. \tag{41}$$

Then, there exists a unique solution $\mathbf{z}^*$ of $\mathbf{F}(\mathbf{z}) = 0$, which corresponds to the Gaussian nodes $\mathbf{x}^*$ and weights $\boldsymbol{\omega}^*$. For a given $\mathbf{z}$, computing $\mathbf{F}(\mathbf{z})$ requires both $\boldsymbol{\Psi}$ and $\mathbf{m}$. As described in section 4, $\boldsymbol{\Psi}$ can be computed by algorithm 1, while $\mathbf{m}$ can be recursively computed using eq. (28).

Consider the Jacobian of $\mathbf{F}$

$$\mathbf{J}(\mathbf{z}) = \partial_\mathbf{z}\mathbf{F} = \tilde{\mathbf{J}}(\mathbf{z}) \begin{bmatrix} \mathrm{diag}(\mathbf{x}^{-\frac{\beta}{2}-1}) & \\ & \mathrm{diag}(\mathbf{x}^{-\frac{\beta}{2}}) \end{bmatrix} \begin{bmatrix} \mathrm{diag}(\boldsymbol{\omega}) & \\ & \mathrm{diag}(\mathbf{1}) \end{bmatrix}, \tag{42}$$

where

$$\tilde{\mathbf{J}}(\mathbf{z}) = \begin{bmatrix} -\frac{\beta}{2}\mathbf{U} + \mathbf{U}'\mathrm{diag}(\mathbf{x}) & \mathbf{U} \\ -\frac{\beta}{2}\mathbf{V} + \mathbf{V}'\mathrm{diag}(\mathbf{x}) & \mathbf{V} \end{bmatrix}, \tag{43}$$

$$\mathbf{U}' = \begin{bmatrix} L_0'(x_0;\Lambda_0+\beta/2), & \cdots, & L_0'(x_N;\Lambda_0+\beta/2) \\ L_1'(x_0;\Lambda_1+\beta/2), & \cdots, & L_1'(x_N;\Lambda_1+\beta/2) \\ \vdots & & \vdots \\ L_N'(x_0;\Lambda_N+\beta/2), & \cdots, & L_N'(x_N;\Lambda_N+\beta/2) \end{bmatrix}, \tag{44}$$

and

$$\mathbf{V}' = \begin{bmatrix} L_{N+1}'(x_0;\Lambda_{N+1}+\beta/2), & \cdots, & L_{N+1}'(x_N;\Lambda_{N+1}+\beta/2) \\ L_{N+2}'(x_0;\Lambda_{N+2}+\beta/2), & \cdots, & L_{N+2}'(x_N;\Lambda_{N+2}+\beta/2) \\ \vdots & & \vdots \\ L_{2N+1}'(x_0;\Lambda_{2N+1}+\beta/2), & \cdots, & L_{2N+1}'(x_N;\Lambda_{2N+1}+\beta/2) \end{bmatrix}. \tag{45}$$

Clearly, the Jacobian matrix $\mathbf{J}(\mathbf{z})$ is a continuous mapping, and it is also nonsingular, as demonstrated in theorem 5.1. A zero of the function $\mathbf{F}$ where the Jacobian is nonsingular is referred to as a non-degenerate zero.

Theorem 5.1. *Let Müntz sequence Λ_{2N+1} consist solely of real indices and $\beta \in \mathbb{R}$ satisfy* (23)*. For any distinct $x_k \in (0,1)$ and $\omega_k > 0$, $k = 0, 1, \cdots, N$, $\mathbf{J}(\mathbf{z})$ is nonsingular.*

The proof of theorem 5.1 requires lemma 5.1.

Lemma 5.1. *For any distinct real numbers $z_0, z_1, \cdots, z_n$, we define*

$$w_n(y) = \sum_{j=0}^{n} \left(h_j e^{z_j y} + l_j y e^{z_j (y-1)} \right), \quad y \in (-\infty, +\infty).$$

Thus, $w_n(y)$ contains at most $2n+1$ zeros if $\sum_{j=0}^{n}(h_j^2 + l_j^2) > 0$.

Proof. By induction, when $n = 0$, it is obvious that $h_0 e^{z_0 y} + l_0 y e^{z_0(y-1)}$ contains at most 1 zero if h_0 and l_0 are not all equal to zero. Assume that $w_{n-1}(y)$ contains at most $2n-1$ zeros if $\sum_{j=0}^{n-1}(h_j^2 + l_j^2) > 0$.

We claim that $w_n(y)$ contains at most $2n$ zeros if $l_n = 0$ and $\sum_{j=0}^{n}(h_j^2 + l_j^2) > 0$. Let $F(y) = w_n(y)\exp(-z_n y)$, then

$$F'(y) = \sum_{j=0}^{n-1} \left[\left(l_j e^{-z_j} + (z_j - z_n) h_j \right) e^{(z_j - z_n)y} + (z_j - z_n) l_j e^{-z_n} y e^{(z_j - z_n)(y-1)} \right].$$

By the induction hypothesis, $F'(y)$ contains at most $2n-1$ zeros. Applying the inverse proposition of Rolle's theorem [25], both $F(y)$ and $w_n(y)$ contain at most $2n$ zeros.

When $l_n \neq 0$ and $\sum_{j=0}^{n}(h_j^2 + l_j^2) > 0$,

$$F'(y) = \sum_{j=0}^{n} \left[(l_j e^{-z_n} + h_j(z_j - z_n)) e^{(z_j - z_n)y} + l_j (z_j - z_n) e^{-z_n} y e^{(z_j - z_n)(y-1)} \right].$$

Let $\tilde{h}_j = (l_j e^{-z_n} + h_j(z_j - z_n))$ and $\tilde{l}_j = l_j(z_j - z_n)e^{-z_n}$. It can be observed that $\tilde{l}_n = 0$ and $\sum_{j=0}^{n}(\tilde{h}_j^2 + \tilde{l}_j^2) > 0$. By the previously established claim, $F'(y)$ contains at most $2n$ zeros, which implies that both $F(y)$ and $w_n(y)$ contain at most $2n+1$ zeros. □

Proof of theorem 5.1. Without loss of generality, it suffices to consider the case when $\beta = 0$. We begin our proof by assuming that all the elements in Λ_{2N+1} are distinct. Let $\varphi_j(x) = x^{\lambda_j}$ for $j = 0, 1, \cdots, 2N+1$. It suffices to prove that the determinant $\det(\mathbf{Q})$ is non-zero, where

$$\mathbf{Q} = \begin{bmatrix} \varphi_0(x_0) & \varphi_0'(x_0) & \cdots & \varphi_0(x_{n-1}) & \varphi_0'(x_{n-1}) \\ \varphi_1(x_0) & \varphi_1'(x_0) & \cdots & \varphi_1(x_{n-1}) & \varphi_1'(x_{n-1}) \\ \vdots & \vdots & & \vdots & \vdots \\ \varphi_{2n-1}(x_0) & \varphi_{2n-1}'(x_0) & \cdots & \varphi_{2n-1}(x_{n-1}) & \varphi_{2n-1}'(x_{n-1}) \end{bmatrix}. \tag{46}$$

By lemma 5.1, for any distinct real numbers $z_0, z_1, \cdots, z_N$ and $y_0, y_1, \cdots, y_N$, the determinant of the matrix

$$\begin{bmatrix} e^{z_0 y_0} & y_0 e^{z_0(y_0-1)} & \cdots & e^{z_N y_0} & y_0 e^{z_N(y_0-1)} \\ e^{z_0 y_1} & y_1 e^{z_0(y_1-1)} & \cdots & e^{z_N y_1} & y_1 e^{z_N(y_1-1)} \\ \vdots & \vdots & & \vdots & \vdots \\ e^{z_0 y_{2N+1}} & y_{2N+1} e^{z_0(y_{2N+1}-1)} & \cdots & e^{z_N y_{2N+1}} & y_{2N+1} e^{z_N(y_{2N+1}-1)} \end{bmatrix}$$

does not vanish. By taking $z_j = \log x_j$ and $y_j = \lambda_j$, we conclude that $\det(\mathbf{Q}) \neq 0$.

When there are repeated indices in Λ_{2N+1}, the conclusion still holds due to the properties of determinant [29] and lemma 3.1. □

To compute the matrix $\tilde{\mathbf{J}}$, it is necessary to first calculate the matrix $\mathbf{U}$ and $\mathbf{V}$ using algorithm 1. These matrices are then combined with the recurrence relation

$$\begin{aligned} xL_n'(x;\Lambda_n + \beta/2) = xL_{n-1}'(x;\Lambda_n + \beta/2) + \left(\lambda_n + \frac{\beta}{2}\right) L_n(x;\Lambda + \beta/2) \\ + \left(1 + \lambda_{n-1} + \frac{\beta}{2}\right) L_{n-1}(x;\Lambda_{n-1} + \beta/2), \end{aligned} \tag{47}$$

which can be directly obtained from eq. (17), to obtain both $\mathbf{U}'\mathrm{diag}(\mathbf{x})$ and $\mathbf{V}'\mathrm{diag}(\mathbf{x})$ in a stable and accurate manner. This process enables an efficient and reliable numerical computaition of $\tilde{\mathbf{J}}$.

We employ Newton's method to determine the zero of $\mathbf{F}$. Let $\mathbf{z}^{(0)}$ denote the initial guess, and $\mathbf{z}^{(k)}$ represent the kth iteration point. At the kth iteration, it is essential to solve the Newton equation

$$\mathbf{J}(\mathbf{z}^{(k)})\mathbf{p}^{(k)} = -\mathbf{F}(\mathbf{z}^{(k)}). \tag{48}$$

We denote $\tilde{\mathbf{p}}^{(k)}$ as

$$\tilde{\mathbf{p}}^{(k)} = \begin{bmatrix} \mathrm{diag}(\mathbf{x}^{-\frac{\beta}{2}-1}) & \\ & \mathrm{diag}(\mathbf{x}^{-\frac{\beta}{2}}) \end{bmatrix} \begin{bmatrix} \mathrm{diag}(\boldsymbol{\omega}) & \\ & \mathrm{diag}(\mathbf{1}) \end{bmatrix} \mathbf{p}^{(k)}. \tag{49}$$

Then $\tilde{\mathbf{J}}(\mathbf{z}^{(k)})\tilde{\mathbf{p}}^{(k)} = \mathbf{J}(\mathbf{z}^{(k)})\mathbf{p}^{(k)}$. The Newton equation is transformed into

$$\tilde{\mathbf{J}}(\mathbf{z}^{(k)})\tilde{\mathbf{p}}^{(k)} = -\mathbf{F}(\mathbf{z}^{(k)}). \tag{50}$$

In numerical computations, we solve eq. (50) to obtain $\tilde{\mathbf{p}}^{(k)}$ instead of $\mathbf{p}^{(k)}$ as in eq. (48). The reason for this modification is that when x is close to 0 — commonly encountered when $\hat{M}(\Lambda_{2N+1})$ consists of endpoint singular functions — the computation of $\mathbf{J}$ can result in a loss of significant digits, which can cause the Newton iteration to converge slowly or even diverge.

Let $\tilde{\mathbf{p}}_1^{(k)}$ and $\mathbf{p}_1^{(k)}$ denote the first $N+1$ elements of $\tilde{\mathbf{p}}^{(k)}$ and $\mathbf{p}^{(k)}$ respectively. Similarly, let $\tilde{\mathbf{p}}_2^{(k)}$ and $\mathbf{p}_2^{(k)}$ denote their last $N+1$ elements, respectively. Thus, we have

$$\mathbf{p}_1^{(k)} = \mathrm{diag}(\mathbf{x}^{(k)})^{\frac{\beta}{2}+1}\mathrm{diag}(\boldsymbol{\omega}^{(k)})^{-1}\tilde{\mathbf{p}}_1^{(k)}, \quad \mathbf{p}_2^{(k)} = \mathrm{diag}(\mathbf{x}^{(k)})^{\frac{\beta}{2}}\tilde{\mathbf{p}}_2^{(k)}. \tag{51}$$

From the definition of $\mathbf{z}$, one can see that $\mathbf{p}_1^{(k)}$ and $\mathbf{p}_2^{(k)}$ correspond to the Newton descent directions of $\mathbf{x}^{(k)}$ and $\boldsymbol{\omega}^{(k)}$, respectively. Update the Gaussian nodes and weights at kth iteration by

$$\boldsymbol{\omega}^{(k+1)} \leftarrow \boldsymbol{\omega}^{(k)} + \mathbf{p}_2^{(k)}, \quad \mathbf{x}^{(k+1)} \leftarrow \mathbf{x}^{(k)} + \mathbf{p}_1^{(k)}. \tag{52}$$

We can prove that our method for this modification is convergent locally. The classic theorem on the Newton's method can be summarized as follows.

Theorem 5.2 ([13]). *Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$ be a differentiable mapping with a non-degenerate zero $\mathbf{z}^*$ such that $\mathbf{F}(\mathbf{z}^*) = \mathbf{0}$ and $\det(\mathbf{J}(\mathbf{z}^*)) \neq 0$, where $\mathbf{J}(\mathbf{z})$ denotes the Jacobian matrix of $\mathbf{F}$ evaluated at $\mathbf{z}$. Then, there exists $\varepsilon > 0$ such that for any initial guess $\mathbf{z}^{(0)}$ satisfying $\|\mathbf{z}^{(0)} - \mathbf{z}^*\| < \varepsilon$, the iterates generated by the update rule*

$$\mathbf{z}^{(k+1)} = \mathbf{z}^{(k)} - \mathbf{J}(\mathbf{z}^{(k)})^{-1}\mathbf{F}(\mathbf{z}^{(k)}), \quad k = 0, 1, 2, \cdots.$$

satisfy $\mathbf{z}^{(k+1)} - \mathbf{z}^* = o(\|\mathbf{z}^{(k)} - \mathbf{z}^*\|)$. Moreover, if $\mathbf{J}(\mathbf{z})$ is Lipschitz continuous in a neighborhood of $\mathbf{z}^*$ with radius ε, then $\mathbf{z}^{(k+1)} - \mathbf{z}^* = O(\|\mathbf{z}^{(k)} - \mathbf{z}^*\|^2)$.

As all the Gaussian nodes lie in the interior of the integration interval and all the Gaussian weights are positive, it follows that there exists some $\varepsilon > 0$ such that the Jacobian $\mathbf{J}(\mathbf{z})$ is Lipschitz continuous in a neighborhood of $\mathbf{z}^*$ with radius ε. Thus the following result holds.

Lemma 5.2. *Provided that the starting values are sufficiently good, the presented form of the Newton's method eq.* (52) *is convergent quadratically.*

In practical implementation, Newton's method usually converges after only a few iterations. However, in cases where slow convergence occurs, we use a damped Newton's method to dampen the step size as $s^{(k)} = \gamma^{\max(0,k-k_0)}$, where $0 < \gamma < 1$ and $k_0 \in \mathbb{N}$.

Additionally, in terms of the local convergence property of the Newton's method, it is imperative to select an optimal initial guess. However, obtaining a suitable initial point can be a challenging endeavor in various scenarios. As a result, we resort to the Continuation method as a means of obtaining the initial guess.

Let $\alpha \in [0,1]$. Set

$$\Lambda_{2N+1}^{(\alpha)} = \left\{\lambda_0^{(\alpha)}, \lambda_1^{(\alpha)}, \cdots, \lambda_{2N+1}^{(\alpha)}\right\},$$

where

$$\lambda_n^{(\alpha)} = \alpha\lambda_n + (1-\alpha)n, \quad n = 0, 1, \cdots, 2N+1. \tag{53}$$

Then $L_n^{\beta}(x; \Lambda_n^{(\alpha)}) \in \hat{M}\left(\Lambda_{2N+1}^{(\alpha)}\right)$ are the orthogonal Müntz polynomials with respect to weight function $\omega(x)$. Again, by applying the theorem 2.1 and theorem 2.2, there exists the unique Gaussian nodes $x_k(\alpha) \in (0,1)$ and weights $\omega_k(\alpha) > 0$, $k = 0, 1, \cdots, N$, such that

$$\sum_{k=0}^{N} L_n^{\beta}(x_k(\alpha); \Lambda_n^{(\alpha)})\omega_k(\alpha) = m_n(\alpha), \quad n = 0, 1, \cdots, 2N+1, \tag{54}$$

where $m_n(\alpha) = \int_0^1 L_n^{\beta}(x; \Lambda_n^{(\alpha)})\omega(x)\mathrm{d}x$. Using the Newton's method outlined above with an appropriate initial guess, we are able to compute Gaussian nodes $x_k(\alpha)$ and weights $\omega_k(\alpha)$, moreover, both of which are continuous with respect to α.

Theorem 5.3. *The Müntz sequence Λ_{2N+1} and $\beta \in \mathbb{R}$ are defined in* (23)*. For any $\alpha \in [0,1]$, we define $\Lambda_{2N+1}^{(\alpha)}$ according to* (53)*. The Gaussian nodes $x_k(\alpha)$ and weights $\omega_k(\alpha)$ satisfy* (54) *and exhibit continuity with respect to α.*

Proof. Let $\mathbf{z}(\alpha) = [x_0(\alpha), \cdots, x_N(\alpha), \omega_0(\alpha), \cdots, \omega_N(\alpha)]^{\mathrm{T}}$. Our goal is to show that $\mathbf{z}(\alpha)$, or its every component, is continuous. Based on (41), we can similarly establish a mapping $\mathbf{F}(\mathbf{z}, \alpha)$ associated with the Müntz sequence $\Lambda_{2N+1}^{(\alpha)}$ such that

$$\mathbf{F}(\mathbf{z}(\alpha), \alpha) = \mathbf{0}. \tag{55}$$

As $L_n^{\beta}(x; \Lambda_n^{(\alpha)})$ is differentiable with respect to α, it follows that $\mathbf{F}(\mathbf{z}, \alpha)$ is differentiable with respect to α, i.e., $\partial_\alpha \mathbf{F}$ exists. Taking the derivative of (55) with respect to α, we obtain

$$\frac{\partial \mathbf{F}}{\partial \alpha} + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}\frac{\partial \mathbf{z}}{\partial \alpha} = \mathbf{0},$$

where $\partial_\alpha \mathbf{F}$ and $\partial_\alpha \mathbf{z}$ are $2N+2$ dimensional column vectors. $\partial_{\mathbf{z}} \mathbf{F}$ is a $2N+2$ order matrix, which is the Jacobian matrix in (42). Therefore, $\partial_{\mathbf{z}} \mathbf{F}$ is nonsingular, implying that $\partial_\alpha \mathbf{z}$ exists and hence $\mathbf{z}(\alpha)$ is continuous. □

Remark 5.1. *If $\alpha = 0$, Gaussian nodes $\mathbf{x}(\alpha)$ and weights $\boldsymbol{\omega}(\alpha)$ reduce to the classic Gauss-Jacobi nodes and weights, which can be obtained by various well-known methods in [10, 9]. When α equals* 1*, $\mathbf{x}(\alpha)$ and $\boldsymbol{\omega}(\alpha)$ are the desired ones.*

To obtain the Gaussian nodes and weights corresponding to a specific Müntz sequence, we first compute the initial iterate $\mathbf{z}(0)$. We then incrementally increase α by a small step size $\Delta\alpha$ and use Newton's method with the previous iterate $\mathbf{z}(\alpha-\Delta\alpha)$ as the initial guess to compute $\mathbf{z}(\alpha)$. By repeating this process with sufficiently small $\Delta\alpha$, the continuity of $\mathbf{z}(\alpha)$ ensures that the desired Gaussian nodes and weights can be obtained when $\alpha=1$. This process is summarized in the following algorithm 2.

Algorithm 2 Compute Gauss quadrature

Require: Λ_{2N+1} and $\beta\in\mathbb{R}$ satisfy (23).
  Compute $\mathbf{z}(0)$ and set $\alpha\leftarrow\Delta\alpha$.
  while $\alpha\leq 1$ **do**
    Compute moments $\mathbf{m}(\alpha)$ by eq. (28).
    Determine the initial guess in Newton's method $\mathbf{z}^{(0)}\leftarrow\mathbf{z}(\alpha-\Delta\alpha)$.
    for $k=1,2,\cdots$ **do**
      Compute $\tilde{\mathbf{J}}(\mathbf{z}^{(k)})$ and $\mathbf{F}(\mathbf{z}^{(k)})$ by algorithm 1 and *eq.* (47).
      Compute the descent directions $\tilde{\mathbf{p}}_1^{(k)}$ and $\tilde{\mathbf{p}}_1^{(k)}$ by (50).
      Update weights: $\boldsymbol{\omega}^{(k+1)}\leftarrow\boldsymbol{\omega}^{(k)}+s^{(k)}\mathrm{diag}(\mathbf{x}^{(k)})^{\frac{\beta}{2}}\tilde{\mathbf{p}}_2^{(k)}$.
      Update nodes: $\mathbf{x}^{(k+1)}\leftarrow\mathbf{x}^{(k)}+s^{(k)}\mathrm{diag}(\mathbf{x}^{(k)})^{\frac{\beta}{2}+1}\mathrm{diag}(\boldsymbol{\omega}^{(k)})^{-1}\tilde{\mathbf{p}}_1^{(k)}$.
      $\mathbf{z}^*\leftarrow(\mathbf{x}^{(k+1)},\boldsymbol{\omega}^{(k+1)})$.
    end for
    Obtain $\mathbf{z}(\alpha)\leftarrow\mathbf{z}^*$.
    Update $\alpha\leftarrow\alpha+\Delta\alpha$.
  end while
  return $\mathbf{z}(1)$.

Remark 5.2. *In algorithm 2, an unordered Müntz sequence is allowed. Moreover, the continuation step size $\Delta\alpha$ is not necessarily the same at each iteration.*

Specifically, Gauss quadrature with $\omega(x)=1$ can be obtained via a transformation of that with $\omega(x)=x^\beta$.

Theorem 5.4. *Let $\{x_j,\omega_j\}_{j=0}^N$ be the Gaussian nodes and weights with weight function $\omega(x)=x^\beta$ that is exact for functions in $\hat{M}(\Lambda_{2N+1})$. Let $\kappa=1/(\beta+1)$,*

$$\tau_j=x_j^{\frac{1}{\kappa}},\quad \chi_j=\frac{\omega_j}{\kappa},\quad j=0,1,\cdots,N.$$

Then $\{\tau_j,\chi_j\}_{j=0}^N$ are the Gaussian nodes and weights with weight function $\omega(x)=1$ over $\hat{M}\left(\kappa\Lambda_{2N+1}\right)$, i.e.,

$$\int_0^1 p(x)\mathrm{d}x=\sum_{j=0}^N p(\tau_j)\chi_j,\quad \forall p(x)\in\hat{M}\left(\kappa\Lambda_{2N+1}\right). \tag{56}$$

Proof. For any $p(x) \in \hat{M}(\kappa\Lambda_{2N+1})$, consider the variable transformation $x = t^{\frac{1}{\kappa}}$. Then, $p(t^{\frac{1}{\kappa}}) \in \hat{M}(\Lambda_{2N+1})$, and as a result,

$$\int_0^1 p(x)\mathrm{d}x = \frac{1}{\kappa}\int_0^1 p(t^{\frac{1}{\kappa}})t^{\frac{1}{\kappa}-1}\mathrm{d}t = \frac{1}{\kappa}\int_0^1 p(t^{\frac{1}{\kappa}})t^{\beta}\mathrm{d}t = \frac{1}{\kappa}\sum_{j=0}^{N} p(x_j^{\frac{1}{\kappa}})\omega_j = \sum_{j=0}^{N} p(\tau_j)\chi_j.$$

□

6 Error estimation

This section discusses the error estimation for a specific type of Müntz Gauss quadrature and shows that its convergence rate is independent of the integrand's singularity.

For interpolation type numerical quadrature, error estimation is usually given in terms of the derivative of the integrand, and the order of the derivative depends on the number of quadrature nodes. Such estimates can be obtained from polynomial interpolation error estimates or the Peano kernel theorem. Estimating the interpolation error for general functions that form a Chebyshev system is difficult. For special Müntz polynomials, however, we can use the Peano kernel theorem to obtain an error estimate for the quadrature.

Let $\mathcal{V}[a,b]$ be the collection of all bounded variation real-valued functions on the interval $[a,b]$. Suppose that $\mathscr{F} : \mathcal{V}[a,b] \to \mathbb{R}$ is a bounded linear functional , and $K(\theta)$ is the Peano kernel [24, 12] with respect to $\mathscr{F}$ and $n \in \mathbb{N}$:

$$K(\theta) = \frac{1}{n!}\mathscr{F}_x[(x-\theta)_+^n], \quad \theta \in [a,b], \tag{57}$$

where $\mathscr{F}_x[\cdot]$ denotes the action of the functional $\mathscr{F}$ on a function with respect to x, and

$$(x-\theta)_+^n = \begin{cases} (x-\theta)^n, & x \geq \theta, \\ 0, & x < \theta. \end{cases}$$

Based on Taylor's theorem and the expression for the remainder in Taylor's series, the Peano kernel theorem can be proved [24]. The Peano kernel theorem provides a useful expression: the action of a linear functional on a function can be represented as the integral of the function's derivative and the Peano kernel.

Theorem 6.1 ([24, 12, 8])**.** *Let $n \in \mathbb{N}$ and $\mathscr{F} : \mathcal{V}[a,b] \to \mathbb{R}$ be a bounded linear functional satisfying $\mathscr{F}[p(x)] = 0$ for all $p(x) \in \mathbb{P}_n$, where $\mathbb{P}_n$ represents the collection of all polynomials with degree less than or equal to n. $K(\theta)$ is defined as in* (57) *and $K(\theta) \in \mathcal{V}[a,b]$. Then, for any $f(x) \in C^{n+1}[a,b]$, we have*

$$\mathscr{F}[f] = \int_0^1 K(\theta)f^{(n+1)}(\theta)\mathrm{d}\theta. \tag{58}$$

From eq. (58), it is evident that the following estimate holds:

$$|\mathscr{F}[f]| \leq \|K\|_1 \|f^{(n+1)}\|_\infty. \tag{59}$$

Therefore, if we choose a specific $\mathscr{F}$ that satisfies the conditions of theorem 6.1, we can obtain an estimate for $|\mathscr{F}[f]|$ using eq. (59). In particular, we utilize it to estimate the error of a certain class of Müntz Gauss quadratures.

Lemma 6.1 ([12]). *Let $\omega(x)$ be a weight function. The functional*

$$\mathscr{F}[u] = I[u] - Q_N[u] \tag{60}$$

defines the error in the numerical approximation of the exact integral $I[u]$ as defined in eq. (1), using Gauss quadrature $Q_N[u]$ as defined in eq. (2). If $\mathscr{F}[p(x)] = 0$, $\forall p(x) \in \mathbb{P}_N$, then for any $u \in C^{N+1}[a,b]$, it follows that

$$|I[u] - Q_N[u]| \leq \frac{(b-a)^{N+1}}{N!} \int_a^b \omega(x)\mathrm{d}x \|u^{(N+1)}\|_\infty. \tag{61}$$

We consider the interval $[0,1]$ and Müntz sequence

$$\Lambda_{2N+1} = \left\{ \lambda_v = \left\lfloor \frac{v}{2} \right\rfloor : v = 0, 1, \cdots, 2N+1 \right\}, \tag{62}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Let $\omega(x) = x^\beta$ satisfy eq. (23) and $Q_N[\cdot]$ be the associated Gaussian rule. We have the following estimation.

Theorem 6.2. *Let Λ_{2N+1} be defined as in (62). For any $u, v \in C^{N+1}[0,1]$, we set $f(x) = u(x) + v(x)\log x$, then*

$$|I[f] - Q_N[f]| \leq \frac{1}{N!} \left(\frac{1}{1+\beta} \|u^{(N+1)}\|_\infty + \frac{1}{(1+\beta)^2} \|v^{(N+1)}\|_\infty \right). \tag{63}$$

Proof. We firstly take $\mathscr{F}[u] = I[u] - Q_N[u]$ and $\omega(x) = x^\beta$. Then we have $\mathscr{F}[p] = 0$, $\forall p \in \mathbb{P}_N$. By lemma 6.1, we have

$$|I[u] - Q_N[u]| \leq \frac{1}{N!} \frac{1}{1+\beta} \|u^{(N+1)}\|_\infty. \tag{64}$$

Secondly, we denote $\tilde{I}[v]$ and $\tilde{Q}_N[v]$ by

$$\tilde{I}[v] = I[v(-\log x)], \quad \tilde{Q}_N[v] = Q_N[v(-\log x)].$$

Let $\mathscr{F}[v] = \tilde{I}[v] - \tilde{Q}_N[v]$. Then we have $\mathscr{F}[p] = 0$, $\forall p \in \mathbb{P}_N$. Taking replace $\omega(x)$ with $\omega(x)(-\log x)$ in lemma 6.1, we have

$$|I[v(-\log x)] - Q_N[v(-\log x)]| \leq \frac{1}{N!} \frac{1}{(1+\beta)^2} \|v^{(N+1)}\|_\infty. \tag{65}$$

Hence eq. (63) follows from combining

$$|I[v(-\log x)] - Q_N[v(-\log x)]| = |I[v\log x] - Q_N[v\log x]|$$

with

$$|I[f] - Q_N[f]| \le |I[u] - Q_N[u]| + |I[v\log x] - Q_N[v\log x]|\,.$$

□

According to theorem 6.2, the error in Gauss quadrature for singular functions at endpoints, of the form $u(x) + v(x)\log x$, is not influenced by $\log x$, but rather determined by the smoothness of $u(x)$ and $v(x)$.

Generally, let $T \in \mathbb{N}^+$, we consider Müntz sequence as

$$\Lambda_{T(N+1)} = \left\{\lambda_v = \left\lfloor \frac{v}{T} \right\rfloor : v = 0, 1, \cdots, T(N+1)\right\}. \tag{66}$$

Let $\omega(x) = x^\beta$ and $S = \left\lceil \frac{T(N+1)}{2} \right\rceil$. $\lceil\cdot\rceil$ denotes the ceil function. $Q_S[\cdot]$ are the associated Gauss quadrature. Similar to the result stated in theorem 6.2, the following estimation holds.

Theorem 6.3. *Let $\Lambda_{T(N+1)}$ be defined as in eq.* (66) *and $\beta \in \mathbb{R}$ satisfy eq.* (23)*. For any $u_j \in C^{N+1}[0,1]$, we set $f(x) = \sum_{j=0}^{T-1} u_j(x)\log^j x$, then*

$$|I[f] - Q_S[f]| \le \frac{1}{N!}\sum_{j=0}^{T-1} \frac{\Gamma(j+1)}{(1+\beta)^{j+1}} \|u_j^{(N+1)}\|_\infty, \tag{67}$$

where Γ is the Gamma function and $S = \left\lceil \frac{T(N+1)}{2} \right\rceil$.

7 Numerical examples

In this section we present some numerical examples.

Example 7.1. *Suppose that $\beta = -1/4$ and Müntz sequence Λ_{2N+1} satisfies*

$$\lambda_{2k} = k + \frac{2}{3}, \quad \lambda_{2k+1} = k - \frac{2}{3}, \quad k = 0, 1, \cdots, N.$$

Gaussian nodes x_k and weights ω_k, $k = 0, 1, \cdots, N$, as shown in table 1, such that

$$\int_0^1 \varphi(x)\omega(x)\mathrm{d}x = \sum_{k=0}^{N} \varphi(x_k)\omega_k, \quad \forall\varphi(x) \in \hat{M}(\Lambda_{2N+1}).$$

Note that Λ_{2N+1} is not monotonous and its minimum index is $-2/3$. Therefore, some functions in $\hat{M}(\Lambda_{2N+1})$ may not even be in $L^2[0,1]$. From table 1, it can be observed that the distribution of Gaussian nodes is more concentrated around 0, which is suitable for handling cases where the integrand grows or decreases rapidly near the left endpoint of the interval. In table 2, we present a list of relative errors, denoted as

$$R[\varphi] = \left| \frac{I[\varphi] - Q_N[\varphi]}{I[\varphi]} \right|, \tag{68}$$

where $I[\varphi]$ represents the exact integral as defined in eq. (1), and $Q_N[\varphi]$ denotes the quadrature as given in eq. (2) with $N+1$ nodes and weights listed in table 1. The results in table 2 demonstrate that the $N+1$ quadrature rule $Q_N[\cdot]$ achieves exactness, within machine accuracy, for any $\varphi \in \hat{M}(\Lambda_{2N+1})$.

Table 1: Gauss quadrature of example 7.1.

$N+1$	Nodes x_k	Weights ω_k
20	2.3157766972828912(-6)	9.4222433583541251(-4)
	2.7233174824378183(-4)	5.2428252534242620(-3)
	1.8028430213927918(-3)	1.2949690421473462(-2)
	6.2586172324495502(-3)	2.3479387650276507(-2)
	1.5778280059386831(-2)	3.6207147638052821(-2)
	3.2714449729529464(-2)	5.0373872558283253(-2)
	5.9338006505081552(-2)	6.5125430600342371(-2)
	9.7535823413401598(-2)	7.9558300013059469(-2)
	1.4853234267777532(-1)	9.2767975279634124(-2)
	2.1266338866366832(-1)	1.0389748261138633(-1)
	2.8922591041325307(-1)	1.1218341981654856(-1)
	3.7642038226828578(-1)	1.1699708689403455(-1)
	4.7139401605805348(-1)	1.1787854613502737(-1)
	5.7038355874684765(-1)	1.1456183436550083(-1)
	6.6894714365725705(-1)	1.0699002517468645(-1)
	7.6226630027383013(-1)	9.5319381997936783(-2)
	8.4549259057549253(-1)	7.9912437335814088(-2)
	9.1410908658850543(-1)	6.1320533998553618(-2)
	9.6427580053292161(-1)	4.0258021409587424(-2)
	9.9313650659281172(-1)	1.7593770011811099(-2)
40	1.5187265199530925(-7)	1.2213355322124739(-4)
	1.7992560515276967(-5)	6.8605007097713726(-4)
	1.2069214706402429(-4)	1.7228923928849940(-3)
	4.2706310022074624(-4)	3.2002598939306478(-3)
	1.1039676484141875(-3)	5.0956596491180742(-3)
	2.3612508697757515(-3)	7.3804450403362623(-3)
	4.4453368850422828(-3)	1.0020032335832917(-2)
	7.6316415093306460(-3)	1.2974358916941377(-2)
	1.2215994413850436(-2)	1.6198429091116944(-2)
	1.8505288925579538(-2)	1.9642930054004918(-2)

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	2.6807597817191973(-2)	2.3254907906113849(-2)
	3.7422007865292671(-2)	2.6978494103994850(-2)
	5.0628434205556849(-2)	3.0755672338654855(-2)
	6.6677677381083530(-2)	3.4527075372775690(-2)
	8.5781981413664360(-2)	3.8232800993517967(-2)
	1.0810634033123627(-1)	4.1813235992838205(-2)
	1.3376078361624985(-1)	4.5209876987019809(-2)
	1.6279384840367642(-1)	4.8366136936514245(-2)
	1.9518741849553753(-1)	5.1228126426811106(-2)
	2.3085307803670563(-1)	5.3745399118535722(-2)
	2.6963009178752406(-1)	5.5871651265625516(-2)
	3.1128508519477094(-1)	5.7565365827494788(-2)
	3.5551345683402841(-1)	5.8790392455852776(-2)
	4.0194251424969035(-1)	5.9516455508932177(-2)
	4.5013628275188522(-1)	5.9719583223261812(-2)
	4.9960189633891738(-1)	5.9382452242441777(-2)
	5.4979744157007604(-1)	5.8494642849110580(-2)
	6.0014108983850478(-1)	5.7052801454937990(-2)
	6.5002132193804196(-1)	5.5060708157983945(-2)
	6.9880802184137081(-1)	5.2529248460949217(-2)
	7.4586419486102140(-1)	4.9476289542301162(-2)
	7.9055804937950391(-1)	4.5926462772571036(-2)
	8.3227517151117658(-1)	4.1910855467128326(-2)
	8.7043051867701615(-1)	3.7466616191765861(-2)
	9.0447996135064512(-1)	3.2636479428631744(-2)
	9.3393111257234906(-1)	2.7468217697410070(-2)
	9.5835320406140556(-1)	2.2014035553052494(-2)
	9.7738580706529599(-1)	1.6329952018268864(-2)
	9.9074634991357979(-1)	1.0475501181546670(-2)
	9.9823822268456675(-1)	4.5199615467215945(-3)

Example 7.2. *Suppose that* $\beta = -1/3$ *and Müntz sequence* Λ_{2N+1} *satisfies*

$$\lambda_{2k} = \lambda_{2k+1} = k - \frac{1}{2}, \quad k = 0, 1, \cdots, N.$$

Gaussian nodes x_k *and weights* ω_k, $k = 0, 1, \cdots, N$, *as shown in table 3, are exact for any function in* $\hat{M}(\Lambda_{2N+1})$.

Due to repeated indices, $\hat{M}(\Lambda_{2N+1})$ *contains logarithmic functions. From table 3, it is evident that the Gaussian nodes are densely clustered around 0, making them well-suited for accurately handling integrands that exhibit rapid growth or decline near the left endpoint of the interval. Similar to example 7.1, in table 4, we provide a list of relative errors* $R[\varphi]$ *as defined in eq.* (68)*, from which we observe that* $N + 1$ *quadrature rule* $Q_N[\cdot]$ *achieve exactness, within machine accuracy, for a collection of basis functions belonging to* $\hat{M}(\Lambda_{2N+1})$.

Table 2: Relative error of Gauss quadrature in example 7.1.

$N+1$	$\varphi(x)$	Relative error $R[\varphi]$	$\varphi(x)$	Relative error $R[\varphi]$
20	$x^{2/3}$	1.1102230246251565(-15)	$x^{32/3}$	1.8873791418627661(-15)
	$x^{-2/3}$	7.7715611723760958(-16)	$x^{28/3}$	1.7763568394002505(-15)
	$x^{5/3}$	1.5543122344752192(-15)	$x^{35/3}$	1.8873791418627661(-15)
	$x^{1/3}$	2.2204460492503131(-16)	$x^{31/3}$	1.8873791418627661(-15)
	$x^{8/3}$	1.6653345369377348(-15)	$x^{38/3}$	1.8873791418627661(-15)
	$x^{4/3}$	1.2212453270876722(-15)	$x^{34/3}$	1.6653345369377348(-15)
	$x^{11/3}$	1.5543122344752192(-15)	$x^{41/3}$	1.9984014443252818(-15)
	$x^{7/3}$	1.5543122344752192(-15)	$x^{37/3}$	1.9984014443252818(-15)
	$x^{14/3}$	1.6653345369377348(-15)	$x^{44/3}$	1.8873791418627661(-15)
	$x^{10/3}$	1.3322676295501878(-15)	$x^{40/3}$	1.9984014443252818(-15)
	$x^{17/3}$	1.7763568394002505(-15)	$x^{47/3}$	1.9984014443252818(-15)
	$x^{13/3}$	1.8873791418627661(-15)	$x^{43/3}$	1.8873791418627661(-15)
	$x^{20/3}$	1.6653345369377348(-15)	$x^{50/3}$	1.8873791418627661(-15)
	$x^{16/3}$	1.8873791418627661(-15)	$x^{46/3}$	1.8873791418627661(-15)
	$x^{23/3}$	1.6653345369377348(-15)	$x^{53/3}$	2.2204460492503131(-15)
	$x^{19/3}$	1.5543122344752192(-15)	$x^{49/3}$	1.9984014443252818(-15)
	$x^{26/3}$	1.9984014443252818(-15)	$x^{56/3}$	2.1094237467877974(-15)
	$x^{22/3}$	1.7763568394002505(-15)	$x^{52/3}$	1.9984014443252818(-15)
	$x^{29/3}$	1.7763568394002505(-15)	$x^{59/3}$	2.1094237467877974(-15)
	$x^{25/3}$	1.6653345369377348(-15)	$x^{55/3}$	1.9984014443252818(-15)

Table 3: Gauss quadrature of example 7.2.

$N+1$	Nodes x_k	Weights ω_k
20	1.7885486758102558(-8)	1.1523469504263825(-4)
	7.1551616529140532(-5)	5.8503872337450366(-3)
	8.3561615801303415(-4)	1.6215461369396420(-2)
	3.7289725070694703(-3)	2.9742159038797088(-2)
	1.0841226296519608(-2)	4.5322220327996356(-2)
	2.4650360296325360(-2)	6.1907685033178526(-2)
	4.7699277205001729(-2)	7.8483315017420313(-2)
	8.2247917905893128(-2)	9.4076870863819106(-2)
	1.2993765604008317(-1)	1.0778485377987003(-1)
	1.9150387312943998(-1)	1.1880283936542314(-1)
	2.6656819611889676(-1)	1.2645525913197742(-1)
	3.5353427020110151(-1)	1.3022161026435305(-1)
	4.4960088886379523(-1)	1.2975708380221534(-1)
	5.5089479647303985(-1)	1.2490623592648446(-1)
	6.5271364954315925(-1)	1.1570882831668874(-1)
	7.4985864107441091(-1)	1.0239741728426736(-1)
	8.3702725462981842(-1)	8.5386716624398701(-2)
	9.0923046967895849(-1)	6.5255288538808617(-2)
	9.6219650408072144(-1)	4.2721655477618713(-2)
	9.9273163921294560(-1)	1.8641580804642555(-2)
40	1.1093514362142486(-9)	1.8057444657681708(-5)
	4.4555423015369943(-6)	9.2111663330328875(-4)
	5.2578791974959081(-5)	2.5842500435975871(-3)
	2.3857885803177525(-4)	4.8339802247692066(-3)
	7.0975130683830704(-4)	7.5702889670536764(-3)
	1.6619379108385249(-3)	1.0712736841684651(-2)
	3.3333984874852280(-3)	1.4189619366383679(-2)
	5.9972993504626114(-3)	1.7933242068639596(-2)
	9.9530127416244830(-3)	2.1877894697564508(-2)
	1.5516454181706534(-2)	2.5959008876182795(-2)
	2.3009710582646729(-2)	3.0112906557680737(-2)
	3.2750231378357329(-2)	3.4276871917926208(-2)
	4.5039867503105624(-2)	3.8389410738086611(-2)
	6.0154048467500588(-2)	4.2390620986094932(-2)
	7.8331385908669379(-2)	4.6222627958035944(-2)
	9.9763982860287634(-2)	4.9830053211846963(-2)
	1.2458871181630048(-1)	5.3160495584589983(-2)
	1.5287970184084276(-1)	5.6165008075362670(-2)
	1.8464224606269530(-1)	5.8798557911169114(-2)
	2.1980830659385084(-1)	6.1020459538421661(-2)
	2.5823375507081436(-1)	6.2794772061574217(-2)
	2.9969744458992525(-1)	6.4090654045891315(-2)
	3.4390216383989641(-1)	6.4882669767053980(-2)
	3.9047747783555770(-1)	6.5151042017447869(-2)
	4.3898441297273816(-1)	6.4881847522119687(-2)
	4.8892189830990607(-1)	6.4067151908927977(-2)

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	5.3973483117197907(-1)	6.2705082035888510(-2)
	5.9082359445178412(-1)	6.0799834313980346(-2)
	6.4155481636620071(-1)	5.8361618480053577(-2)
	6.9127313181855365(-1)	5.5406537072951248(-2)
	7.3931367871632625(-1)	5.1956401646260371(-2)
	7.8501504324459204(-1)	4.8038487514455615(-2)
	8.2773235570215697(-1)	4.3685229585270061(-2)
	8.6685023342696965(-1)	3.8933862619944019(-2)
	9.0179526983967162(-1)	3.3826010239928678(-2)
	9.3204777915725145(-1)	2.8407228822337316(-2)
	9.5715252676017615(-1)	2.2726518444260720(-2)
	9.7672821701962775(-1)	1.6835845666401086(-2)
	9.9047567078037091(-1)	1.0790014028473644(-2)
	9.9818651979791806(-1)	4.6532712355365508(-3)

Table 4: Relative error of Gauss quadrature in example 7.2.

$N+1$	$\varphi(x)$	Relative error $R[\varphi]$	$\varphi(x)$	Relative error $R[\varphi]$
20	$x^{-1/2}$	6.6613381477509392(-16)	$x^{-1/2}\log x$	1.3322676295501878(-15)
	$x^{1/2}$	1.6653345369377348(-15)	$x^{1/2}\log x$	5.5511151231257827(-15)
	$x^{3/2}$	4.4408920985006262(-16)	$x^{3/2}\log x$	8.8817841970012523(-16)
	$x^{5/2}$	5.5511151231257827(-16)	$x^{5/2}\log x$	6.6613381477509392(-16)
	$x^{7/2}$	5.5511151231257827(-16)	$x^{7/2}\log x$	9.9920072216264089(-16)
	$x^{9/2}$	3.3306690738754696(-16)	$x^{9/2}\log x$	8.8817841970012523(-16)
	$x^{11/2}$	2.2204460492503131(-16)	$x^{11/2}\log x$	1.3322676295501878(-15)
	$x^{13/2}$	2.2204460492503131(-16)	$x^{13/2}\log x$	1.5543122344752192(-15)
	$x^{15/2}$	6.6613381477509392(-16)	$x^{15/2}\log x$	2.1094237467877974(-15)
	$x^{17/2}$	6.6613381477509392(-16)	$x^{17/2}\log x$	2.6645352591003757(-15)
	$x^{19/2}$	8.8817841970012523(-16)	$x^{19/2}\log x$	2.8865798640254070(-15)
	$x^{21/2}$	1.3322676295501878(-15)	$x^{21/2}\log x$	3.4416913763379853(-15)
	$x^{23/2}$	1.7763568394002505(-15)	$x^{23/2}\log x$	3.7747582837255322(-15)
	$x^{25/2}$	2.4424906541753444(-15)	$x^{25/2}\log x$	3.8857805861880479(-15)
	$x^{27/2}$	2.6645352591003757(-15)	$x^{27/2}\log x$	4.2188474935755949(-15)
	$x^{29/2}$	3.3306690738754696(-15)	$x^{29/2}\log x$	4.4408920985006262(-15)
	$x^{31/2}$	3.7747582837255322(-15)	$x^{31/2}\log x$	4.7739590058881731(-15)
	$x^{33/2}$	4.4408920985006262(-15)	$x^{33/2}\log x$	4.5519144009631418(-15)
	$x^{35/2}$	5.1070259132757201(-15)	$x^{35/2}\log x$	4.5519144009631418(-15)
	$x^{37/2}$	5.5511151231257827(-15)	$x^{37/2}\log x$	4.7739590058881731(-15)

Example 7.3. *In this example, we present an application of Gauss quadrature. Consider the integrand*

$$\psi(x) = \sin(4\pi x) + \frac{\log x(1-x)}{1+x}.$$

Thus its exact integration value [11] is

$$\int_0^1 \psi(x)\mathrm{d}x = 1 - \frac{\pi^2}{6}. \tag{69}$$

Moreover, the integral of the Bessel function $J_0(x)$ is given by

$$\int_0^1 J_0(x)(1+\log x)\mathrm{d}x = -0.053108037589511873046848618697817 2\cdots. \tag{70}$$

For obtaining exactly the integrals in numerical, we choose $\beta = 0$ and Müntz sequences Λ_{2N+1} that are categorized into three cases for a comparison.

- *Case 1. $\lambda_k = k$ for $k = 0, 1, \cdots, 2N+1$.*
- *Case 2. $\lambda_k = \lfloor k/2 \rfloor$ for $k = 0, 1, \cdots, 2N+1$.*
- *Case 3. $\lambda_k = \lfloor k/3 \rfloor$ for $k = 0, 1, \cdots, 2N+1$.*

By lemma 3.1, it is evident that in case 1, the set $\hat{M}(\Lambda_{2N+1})$ exclusively consists of algebraic polynomials, that in case 2, it includes algebraic polynomials as well as algebraic polynomials multiplied by a logarithmic function, and that in case 3, the set comprises algebraic polynomials, algebraic polynomials multiplied by a logarithmic function, and algebraic polynomials multiplied by a squared logarithmic function. It is important to note that neither $\psi(x)$ nor $J_0(x)(1+\log x)$ belongs to $\hat{M}(\Lambda_{2N+1})$ in any of the three cases considered.

By employing algorithm 2, we can calculate the Gaussian nodes and weights required for the three quadrature rules $Q_N[\cdot]$ to approximate the integrals in eq. (69) and eq. (70). It is noteworthy that in case 1, the quadrature rule is equivalent to the classical Gauss-Legendre quadrature. The result of the approximations is depicted in fig. 1, where the term Error represents the absolute error, and N denotes the number of Gaussian nodes utilized.

In fig. 1, It can be observed that despite the singularity of both integrands $\psi(x)$ and $J_0(x)(1+\log x)$ at 0, their integrals can be well approximated by the quadrature rules of case 2 and case 3, which is consistent with the result obtained in theorem 6.3. However, classic Gauss-Legendre quadrature fails to provide accurate approximations.

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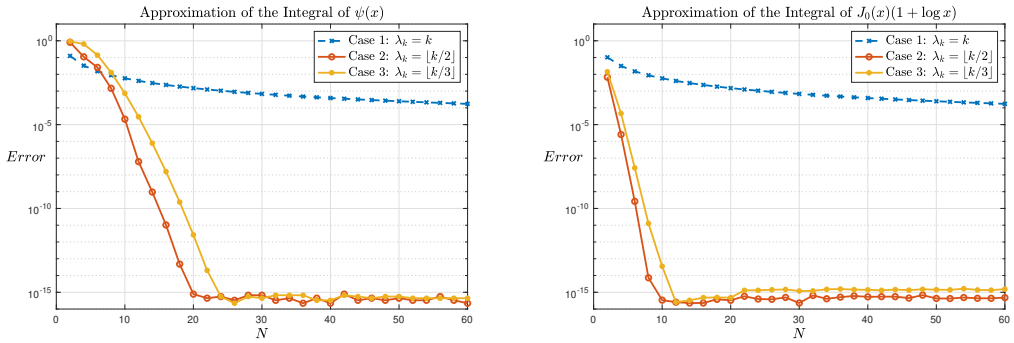
Figure 1: A comparison of different Gauss quadrature rules.



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