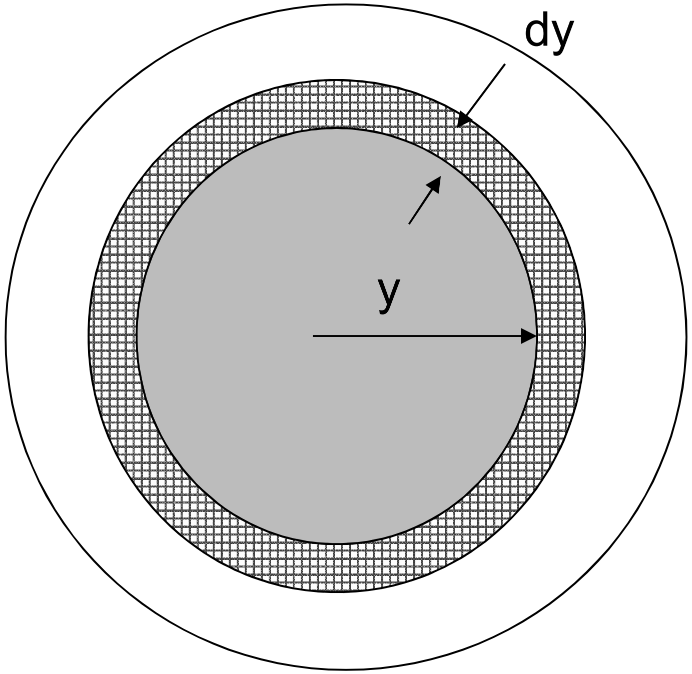
Blade Element Momentum Theory

- Blade Element Theory has a number of assumptions.
- The biggest (and worst) assumption is that the inflow is uniform.
- In reality, the inflow is non-uniform.
- It may be shown that uniform inflow yields the lowest induced power consumption.

Model



- Annulus
  - At a distance *y*
  - Width *dy*:
  - area $dA=2\pi y dy$
  - Mass flow rate

$$d\dot{m} = \rho dA\left(V_C + v_i\right) =$$
$$= 2\pi y \rho\left(V_C + v_i\right)dy$$

Thrust and Power

- The thrust for the annulus is :

$$dT = d\dot{m}2v_i = 4\pi\rho\left(V_C + v_i\right)v_i y dy$$

- Or in the non-dimensional form:

$$dC_T = \frac{dT}{\rho\left(\pi R^2\right)\left(\Omega R\right)^2} = 4\frac{V_C + v_i}{\Omega R}\left(\frac{v_i}{\Omega R}\right)\left(\frac{y}{R}\right)d\left(\frac{y}{R}\right) =$$

$$dC_T = 4\lambda\lambda_i r dr = 4\lambda\left(\lambda - \lambda_C\right)r dr$$

- The induced power coefficient is:

$$dC_{P_i} = \lambda dC_T = 4\lambda^2\lambda_i r dr = 4\lambda^2\left(\lambda - \lambda_C\right)r dr$$

Thrust and Power

- For the hover case $\lambda_C=0$.
  - The thrust coefficient is then:

$$dC_T = 4\lambda^2 r dr \Rightarrow C_T = \int_0^1 dC_T = \int_0^1 4\lambda^2 r dr$$

  - The power coefficient is:

$$dC_{P_i} = 4\lambda^3 r dr \Rightarrow C_{P_i} = \int_0^1 dC_{P_i} = \int_0^1 4\lambda^3 r dr$$

  - Valid for any form of λ

Thrust Coefficient

- Let's assume that $\lambda(r) = \lambda_{tip} r^n$ for $n \geq 0$.

$$C_T = \int_0^1 4\lambda^2 r dr = 4\lambda_{tip}^2 \int_0^1 r^{2n+1} dr = \frac{4\lambda_{tip}^2}{2n+2}$$

- We can then relate λ_{tip} with C_T

$$\lambda_{tip} = \sqrt{n+1}\sqrt{\frac{C_T}{2}}$$

Power Coefficient

- Calculating the power coefficient

$$C_{P_i} = \int_0^1 4\lambda^3 r dr = 4\lambda_{tip}^3 \int_0^1 r^{3n+1} dr \quad = \frac{4\lambda_{tip}^3}{3n+2}$$

- Substituting in the λ_{tip} equation :

$$C_{P_i} = \frac{2(n+1)^{3/2}}{(3n+2)} \frac{C_T^{3/2}}{\sqrt{2}}$$

- Since we know that

$$C_{P_i} = \kappa \frac{C_T^{3/2}}{\sqrt{2}} \quad \Rightarrow \kappa = \frac{2(n+1)^{3/2}}{(3n+2)}$$

Induced Inflow

- In the previous equation when $n=0$ then $\kappa=1$, uniform inflow which has the lowest induced power.
- For $n>0$ then $\kappa>1$. With increasing n the inflow is more heavily biased towards the blade tip.

- Therefore what we want to achieved is, through the blade properties (pitch angle and chord), the first situation

Blade Element Momentum Theory

- From BET we have found

$$dC_T = \tfrac{1}{2}\sigma C_l r^2 dr = \frac{\sigma C_{l_\alpha}}{2}\left(\theta r^2 - \lambda r\right)dr$$

- Then we have

$$dC_T = \frac{\sigma C_{l_\alpha}}{2}\left(\theta r^2 - \lambda r\right)dr = 4\lambda\left(\lambda - \lambda_C\right)r\,dr$$

- Which can be expressed:

$$\frac{\sigma C_{l_\alpha}}{8}\theta r - \frac{\sigma C_{l_\alpha}}{8}\lambda = \lambda^2 - \lambda_C \lambda$$

Blade Element Momentum Theory

- Or in another way:

$$\lambda^2 + \left(\frac{\sigma C_{l_\alpha}}{8} - \lambda_C \right)\lambda - \frac{\sigma C_{l_\alpha}}{8}\theta r = 0$$

- This quadratic equation has the solution:

$$\lambda(r, \lambda_C) = \sqrt{\left(\frac{\sigma C_{l_\alpha}}{16} - \frac{\lambda_C}{2} \right)^2 + \frac{\sigma C_{l_\alpha}}{8}\theta r} - \left(\frac{\sigma C_{l_\alpha}}{16} - \frac{\lambda_C}{2} \right)$$

Ideal Twist

- In hover ($\lambda_C=0$) it simplifies to

$$\lambda(r)=\frac{\sigma C_{l_\alpha}}{16}\left[\sqrt{1+\frac{32}{\sigma C_{l_\alpha}}\theta r}-1\right]$$

- Since the objective is uniform inflow then we need $\theta r = const. = \theta_{tip}$. The ideal twist is given by:

$$\theta(r)=\frac{\theta_{tip}}{r}$$

Thrust coefficient in Ideal Twist

- We can then calculate the thrust coefficient with this ideal twist $\theta r = \theta_{tip}$

$$C_T = \frac{\sigma C_{l_\alpha}}{2} \int_0^1 \left(\theta_{tip} - \lambda\right) r dr = \frac{\sigma C_{l_\alpha}}{2} \left(\frac{\theta_{tip}}{2} - \frac{\lambda}{2}\right)$$

- Recalling that

$$\lambda = const. = r\phi = \phi_{tip}$$

$$C_T = \frac{\sigma C_{l_\alpha}}{4} \left(\theta_{tip} - \phi_{tip}\right) = \frac{\sigma C_{l_\alpha}}{4} \alpha_{tip}$$

Blade Pitch angle

- In hover

$$\lambda = cont. = \frac{\sigma C_{l_\alpha}}{16}\left[\sqrt{1+\frac{32}{\sigma C_{l_\alpha}}\theta_{tip}}-1\right] = \sqrt{\frac{C_T}{2}}$$

- We can then obtain θ_{tip}

$$\theta_{tip} = \frac{4C_T}{\sigma C_{l_\alpha}} + \sqrt{\frac{C_T}{2}} = \frac{4C_T}{\sigma C_{l_\alpha}} + \lambda$$

Final notes

- Recall that: $dC_T = \frac{\sigma C_{l_\alpha}}{2}\left(\theta_{tip} - \lambda\right) r dr$

- Therefore the thrust varies linearly with r. We have therefore a linear (triangular) variation of lift over the blade.

$$dC_T = \frac{\sigma C_{l_\alpha}}{2}\left(\alpha_{tip}\right) r dr = \frac{\sigma C_{l_{tip}}}{2} r dr \Rightarrow$$

$$\Rightarrow C_T = \frac{\sigma C_{l_{tip}}}{4}$$

Ideal Rotor vs. Optimum Rotor

- Ideal rotor has a non-linear twist: $\theta = \theta_{tip}/r$
- This rotor will, according to the BEM theory, have a uniform inflow, and the <u>lowest induced power</u> possible.
- The rotor blade will have very high local pitch angles θ near the root, which may cause the rotor to stall.
- Ideally Twisted rotor is also hard to manufacture.
- For these reasons, helicopter designers strive for optimum rotors that minimize <u>total power</u>, and maximize figure of merit.
- This is done by a combination of twist, and taper, and the use of low drag airfoil sections.

Optimum Hovering Rotor

- Minimum P_i requires λ=*const*. (uniform inflow)
- Minimum P_0 requires $\alpha = \alpha(min\ C_d/C_l) = \alpha_1$
- Then for minimum induce power $\theta = \theta_{tip}/r$ and each blade element must operate at α_1

$$dC_T = \frac{\sigma C_{l_\alpha}}{2}\left(\frac{\theta_{tip}}{r} - \frac{\lambda}{r}\right) r^2 dr = \frac{\sigma C_{l_\alpha}}{2}\alpha_1 r^2 dr$$

- With (BEMT) $dC_T=4\lambda^2 r dr$ then: $\lambda = \sqrt{\frac{\sigma r C_{l_\alpha}\alpha_1}{8}}$

Optimum Hovering Rotor

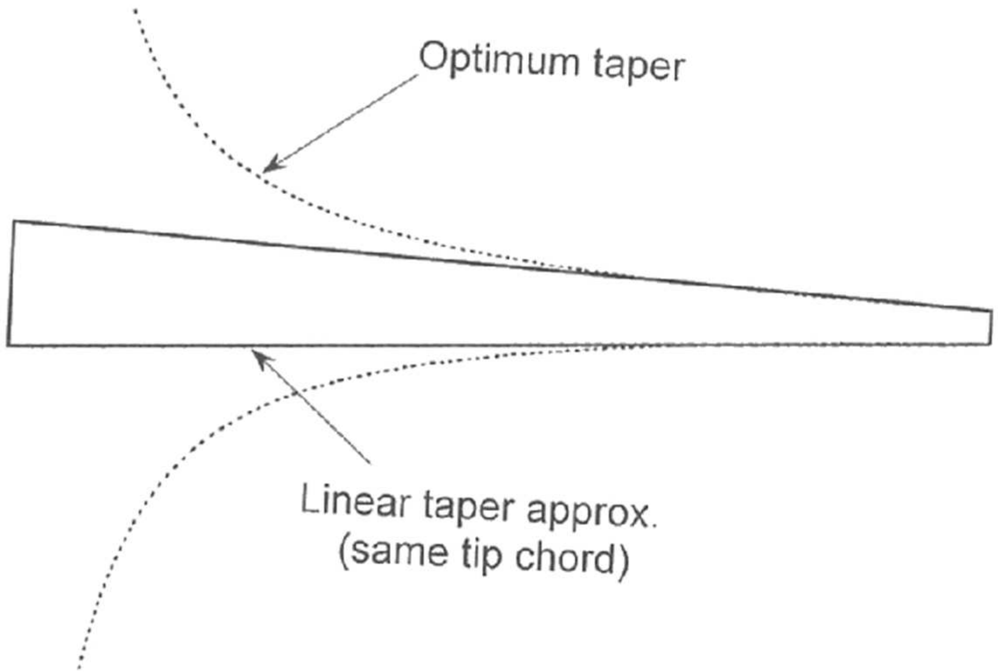
- We have seen that the minimum induced power requires a uniform inflow. Therefore the previous equation is constant over the disk.
- Let's assume that α_1 is the same for all airfoils along the blade and is independent of *Re* and M
- From the equation since α_1 =const and we now that λ=*const* then σr must be constant too.

$$\sigma r = const = \left(\frac{N_b}{\pi R}\right) c r$$

Optimum Hovering Rotor

- The previous situation is achieved when

$$c(r)=\frac{c_{tip}}{r} \quad \text{or} \quad \sigma(r)=\frac{\sigma_{tip}}{r}$$



Optimum Hovering Rotor

- Let's now study the blade twist for uniform inflow and constant AOA:

$$\alpha = \left(\theta - \frac{\lambda}{r} \right) = \alpha_1 = const$$

- Expressing now in terms of the blade pitch

$$\theta(r) = \alpha_1 + \frac{\lambda}{r} = \alpha_1 + \sqrt{\frac{\sigma_{tip} C_{l_\alpha} \alpha_1}{8}} \frac{1}{r}$$

Optimum Hovering Rotor

- The total thrust can be calculated

$$C_T = \tfrac{1}{2} C_{l_\alpha} \int_0^1 \sigma \alpha_1 r^2 dr = \left(\frac{\sigma_{tip} C_{l_\alpha}}{4} \right) \alpha_1$$

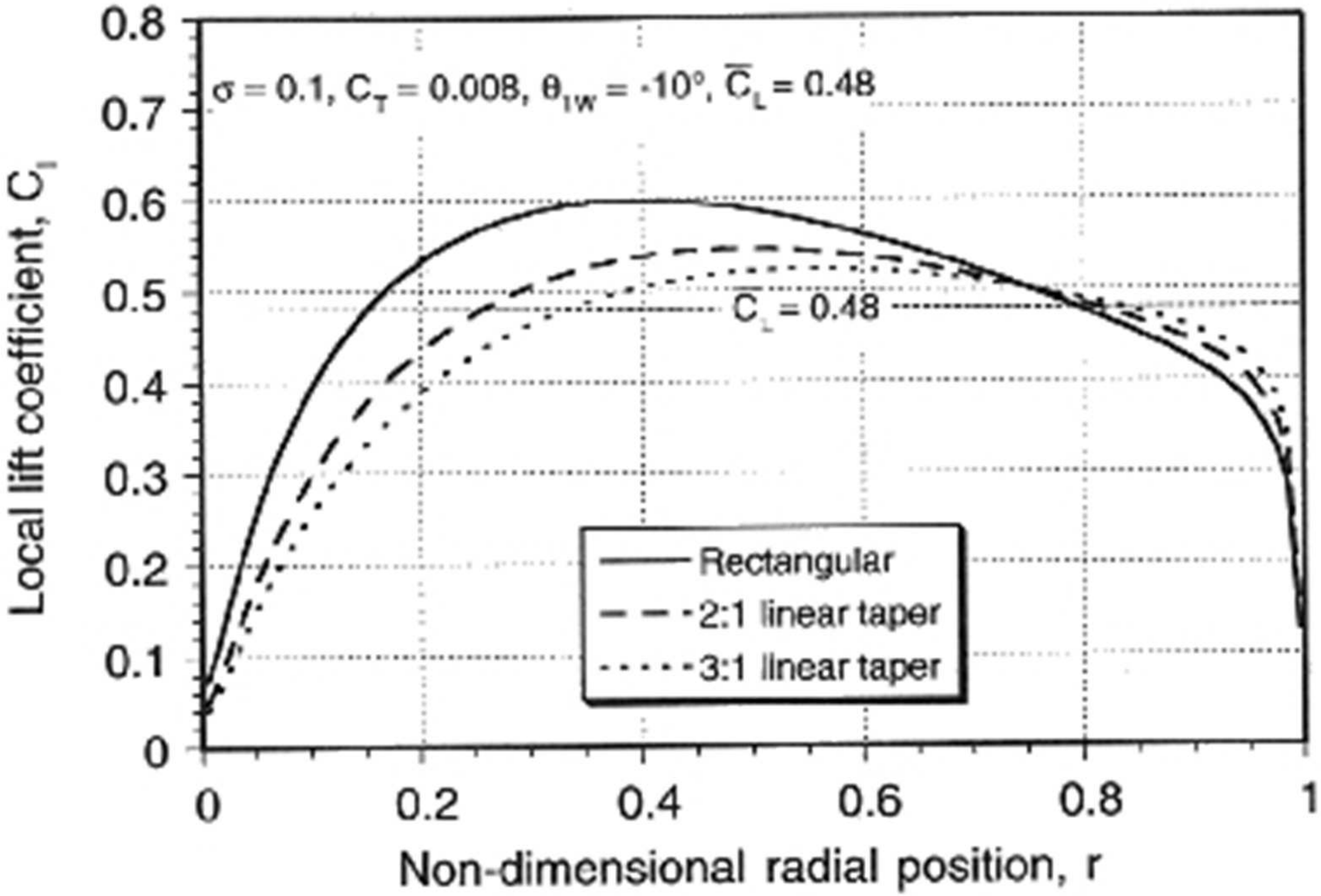
- The local solidity of the optimum rotor can be written:

$$\sigma(r) = \left(\frac{4 C_T}{C_{l_\alpha} \alpha_1} \right) \frac{1}{r}$$

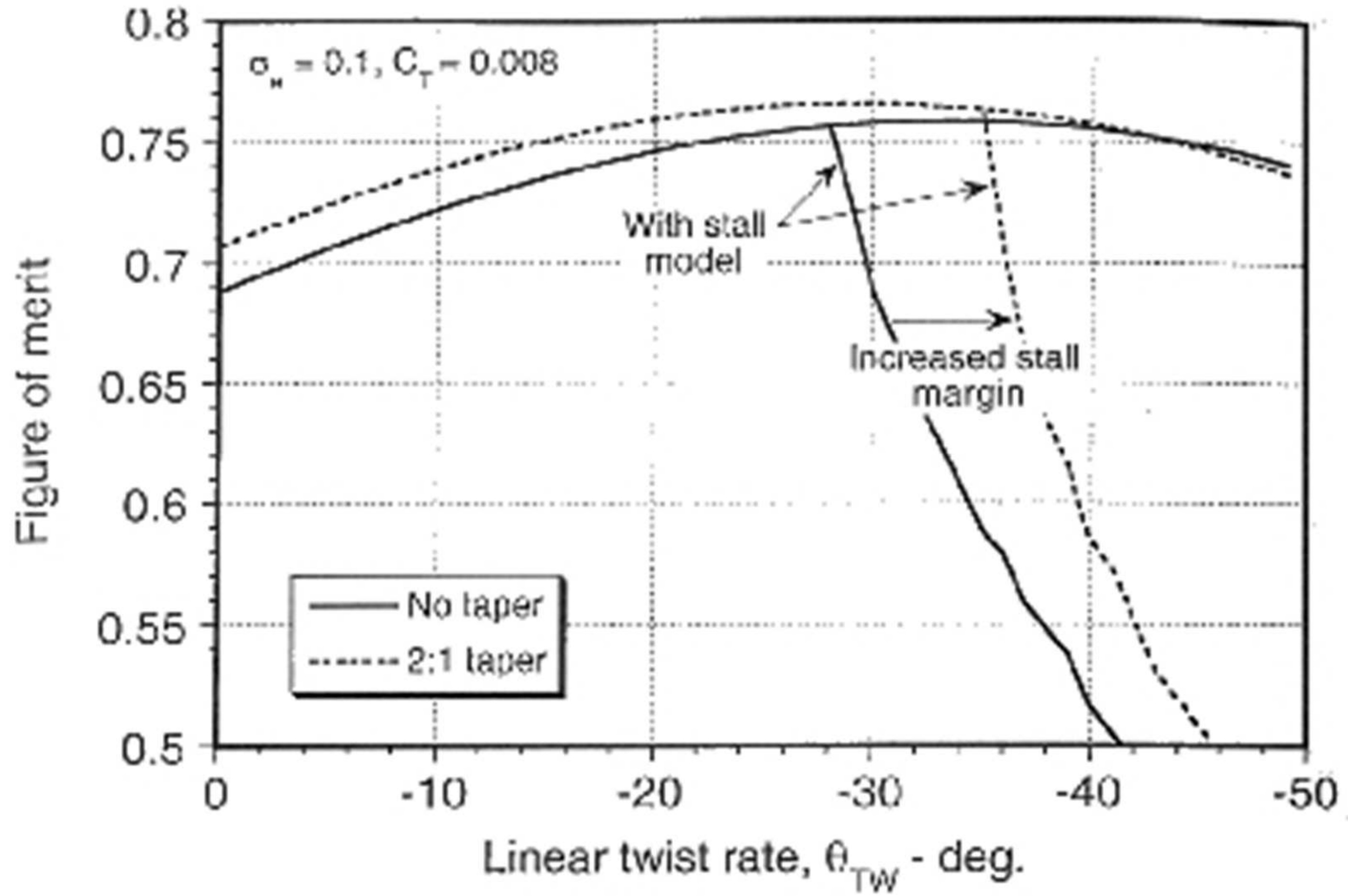
- And the blade twist

$$\theta(r) = \alpha_1 + \frac{\lambda}{r} = \alpha_1 + \sqrt{\frac{C_T}{2}} \left(\frac{1}{r} \right)$$

Optimum Hovering Rotor



Optimum Hovering Rotor



Power estimates

- In a real rotor the non uniformity of λ means that we must calculate numerically:

$$C_{P_i} = \int_{r=0}^{r=1} \lambda dC_t$$

- We also saw that we could then calculate the induce power factor κ :

$$\kappa = \frac{C_{P_i}}{C_T^{3/2} / \sqrt{2}}$$

Power estimates

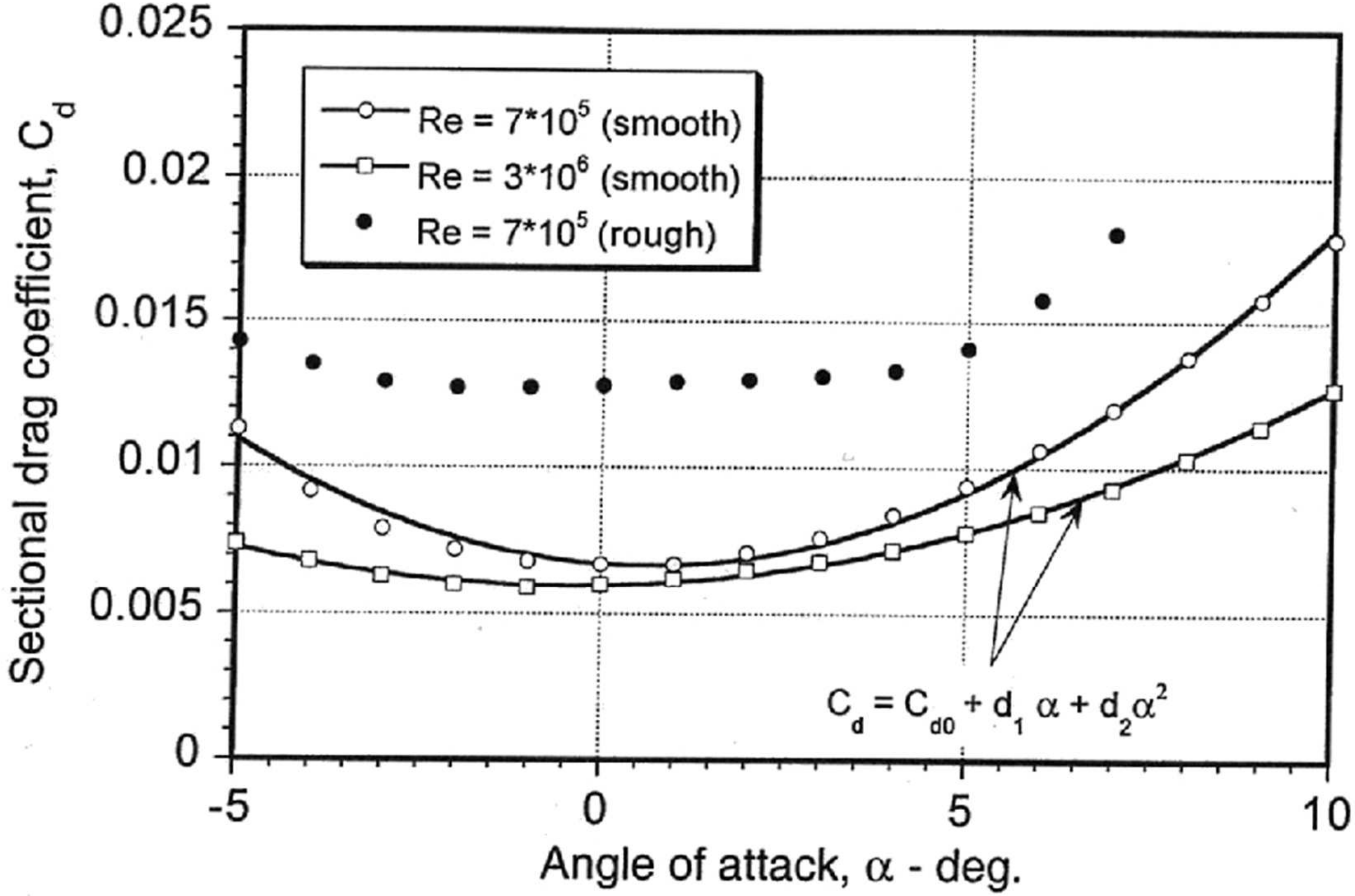
- Let's now calculate the rotor profile power. Remember that (BET):

$$C_{P_0} = \int_0^1 \tfrac{1}{2}\sigma C_d r^3 dr \;\; = \tfrac{1}{2}\sigma \int_0^1 C_d r^3 dr$$

- We can use for C_d the expression:

$$C_d = C_{d_0} + d_1\alpha + d_2\alpha^2$$

Power estimates



Power estimates

- The profile power is then:

$$C_{P_0} = \tfrac{1}{2}\sigma\int_0^1 \left(C_{d_0} + d_1\alpha + d_2\alpha^2\right) r^3 dr$$

$$= \tfrac{1}{2}\sigma\int_0^1 \left(C_{d_0} + d_1(\theta-\phi) + d_2(\theta-\phi)^2\right) r^3 dr$$

$$= \tfrac{1}{2}\sigma\int_0^1 \left(C_{d_0} + d_1\left(\theta-\frac{\lambda}{r}\right) + d_2\left(\theta-\frac{\lambda}{r}\right)^2\right) r^3 dr$$

Power estimates

- For a ideally twisted blade λ=*const* and $\theta r= \theta_{tip}$:

$$C_{P_0} = \frac{\sigma C_{d_0}}{8} + \frac{\sigma d_1}{6}\left(\theta_{tip} - \lambda\right) + \frac{\sigma d_2}{4}\left(\theta_{tip} - \lambda\right)^2$$

- Since $\left(\theta_{tip} - \lambda\right) = \dfrac{4C_T}{\sigma C_{l_\alpha}}$

$$C_{P_0} = \frac{\sigma C_{d_0}}{8} + \frac{2d_1}{3C_{l_\alpha}} C_T + \frac{4d_2}{\sigma C_{l_\alpha}^2} C_T^2$$

Power estimates

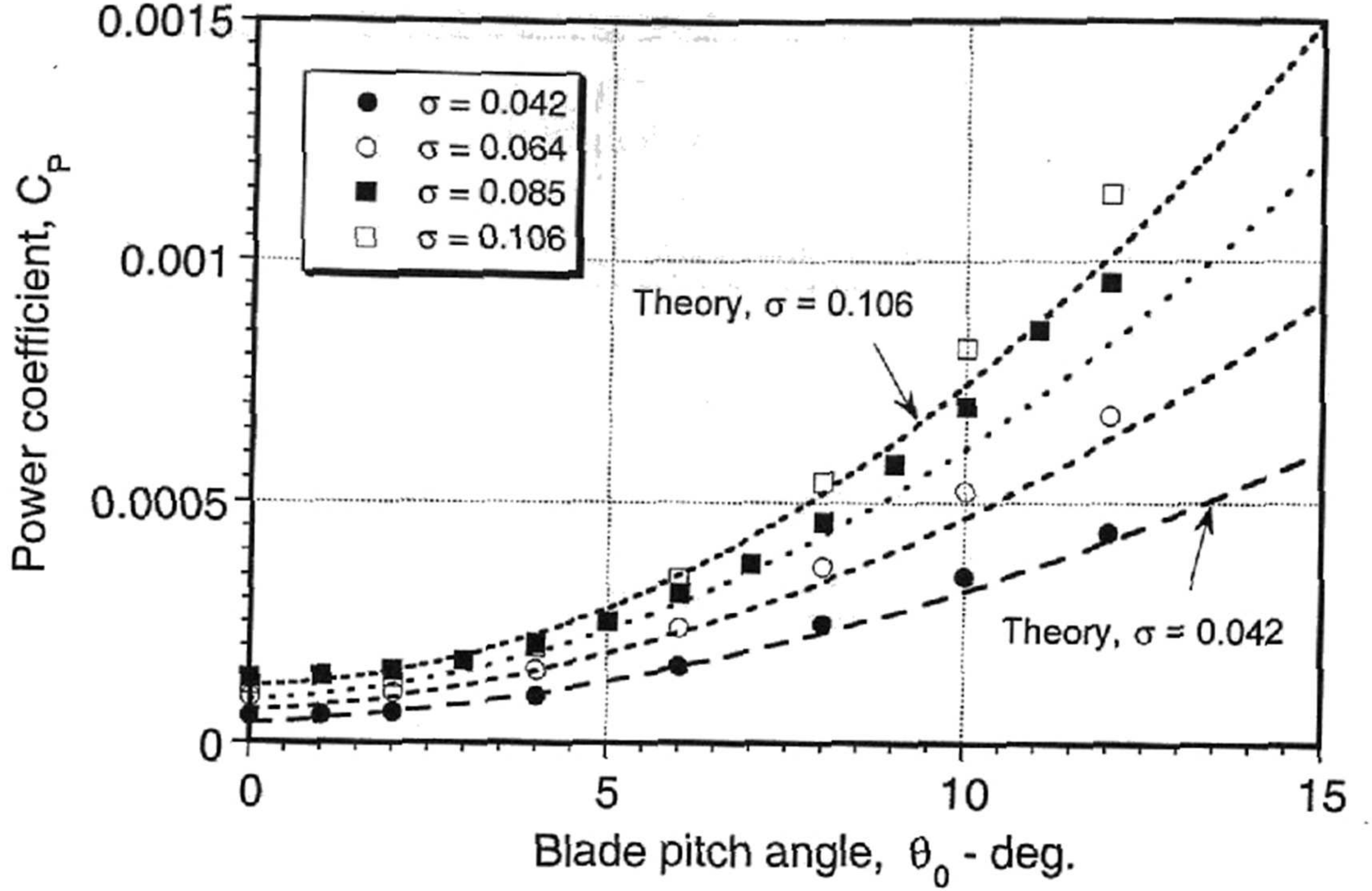


Figure of Merit

- Since we now have the expressions for all power coefficients we can calculate the rotor FM:

$$FM = \frac{C_{P_{ideal}}}{C_{P_i} + C_{P_0}} = \frac{C_T^{3/2} \big/ \sqrt{2}}{\kappa C_T^{3/2} \big/ \sqrt{2} + C_{P_0}}$$

$$= \frac{C_T^{3/2} \big/ \sqrt{2}}{\kappa C_T^{3/2} \big/ \sqrt{2} + \left[\frac{\sigma C_{d_0}}{8} + \frac{2d_1}{3C_{l_\alpha}} C_T + \frac{4d_2}{\sigma C_{l_\alpha}^2} C_T^2 \right]}$$

Prandtl's Tip-loss Function

- The idea of tip losses was already been introduced with the factor B
- Prandtl provided a solution to the problem of loss of lift near the blade tip.
- In this solution the curved helical vortex sheets of the rotor wake were replaced by a series of 2D sheets, meaning that the blade tip radius of curvature is large

Prandtl's Tip-loss Function

- Prandtl's final result can be expressed in terms of an induced velocity correction factor *F*:

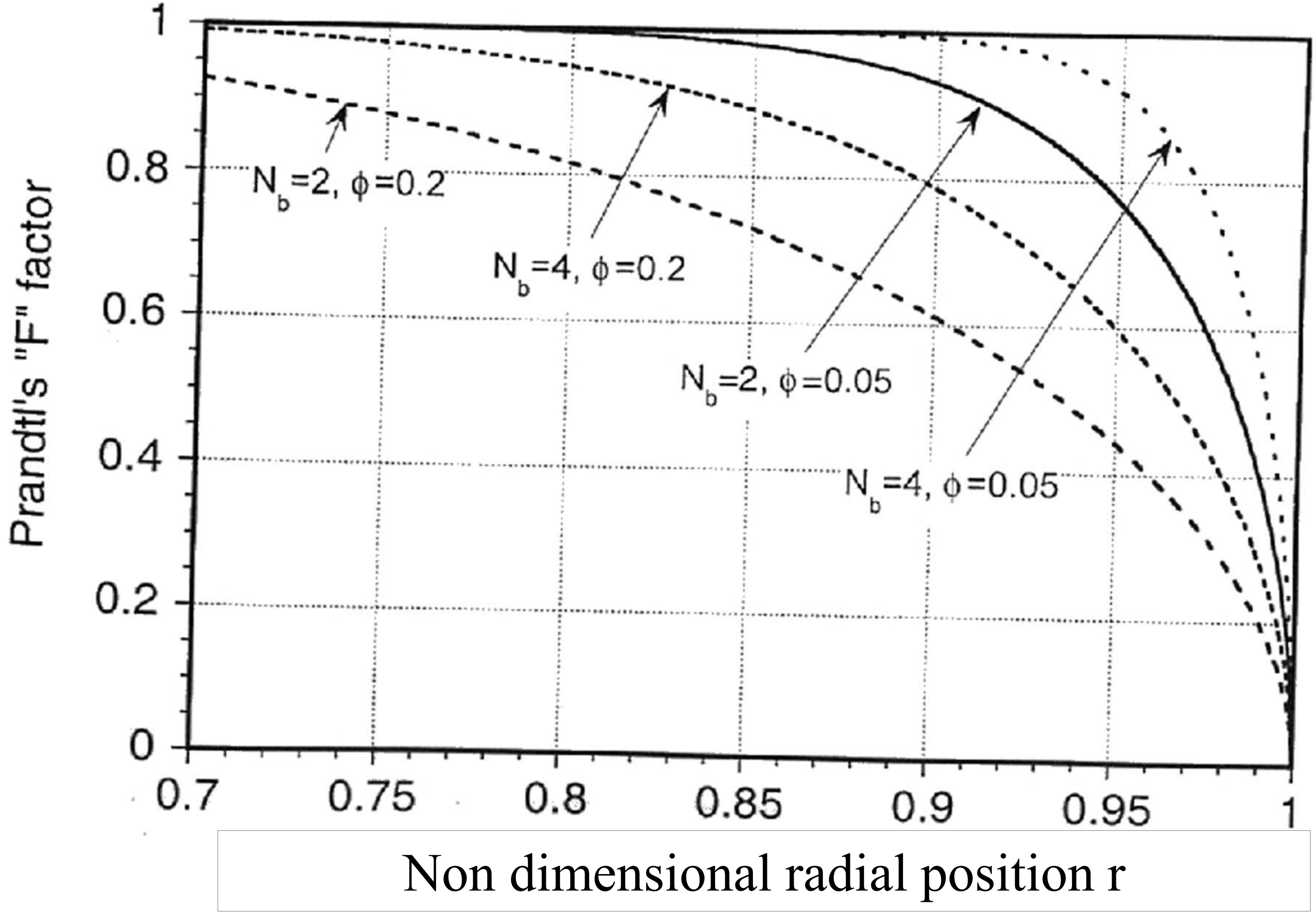
$$F = \frac{2}{\pi}\cos^{-1} e^{-f}$$

- Where *f*:

$$f = \frac{N_b}{2}\left(\frac{1-r}{r\phi}\right)$$

- Remember that $\phi = \dfrac{\lambda(r)}{r}$

Prandtl's Tip-loss Function



Prandtl's Tip-loss Function

- Note that when $N_b \rightarrow \infty$ (actuator disk) then $F \rightarrow 1$
- The function can be incorporated in the BEMT:

$$dC_T = 4F\lambda^2 r dr$$

- Since from BET

$$dC_T = \frac{\sigma C_{l_\alpha}}{2}\left(\theta r^2 - \lambda r\right)dr$$

- We can write

$$\frac{\sigma C_{l_\alpha}}{2}\left(\theta r^2 - \lambda r\right)dr = 4F\lambda^2 r dr$$

Prandtl's Tip-loss Function

$$\lambda^2 + \left(\frac{\sigma C_{l_\alpha}}{8F}\right)\lambda - \frac{\sigma C_{l_\alpha}}{8F}\theta r = 0$$

- With the solution

$$\lambda(r) = \frac{\sigma C_{l\alpha}}{16F}\left[\sqrt{1 + \frac{32F}{\sigma C_{l_\alpha}}\theta r} - 1\right]$$

- Since F is a function of λ the solution must be found numerically

Compressibility Corrections

- So far we have considered that all aerodynamic characteristics independent of Mach number.

- To introduce a correction to take into account the influence of M, let's used Glauert's rule:

$$C_{l_\alpha}(M) = \frac{C_{l_\alpha}\Big|_{M=0.1}}{\sqrt{1-M^2}}$$

Compressibility Corrections

- The local blade M is: $$M(y) = \frac{U_T}{a} = \frac{\Omega y}{a}$$

- Which gives a lift-curve-slope correction of:

$$\frac{1}{\sqrt{1-M^2}} = \frac{1}{\sqrt{1-\left(\Omega/a\right)^2 y^2}} = \frac{1}{\sqrt{1-M_{tip}^2 r^2}}$$

- We had obtained:

$$dC_T = \tfrac{1}{2}\sigma C_l r^2 dr = \tfrac{1}{2}\sigma \frac{\left.C_{l_\alpha}\right|_{M=0.1}}{\sqrt{1-M_{tip}^2 r^2}} \left(\theta - \frac{\lambda}{r}\right) r^2 dr$$

Compressibility Corrections

- Assuming the ideal twist (uniform inflow):

$$dC_T = \tfrac{1}{2}\sigma C_{l_\alpha}\Big|_{M=0.1}\left(\theta_{tip}-\lambda\right)\frac{1}{\sqrt{1-M_{tip}^2 r^2}}\,rdr$$

- Calculating the total thrust coefficient:

$$C_T = \tfrac{1}{2}\sigma C_{l_\alpha}\Big|_{M=0.1}\left(\theta_{tip}-\lambda\right)\int_0^1 \frac{r}{\sqrt{1-M_{tip}^2 r^2}}\,dr$$

$$= \tfrac{1}{2}\sigma K C_{l_\alpha}\Big|_{M=0.1}\left(\theta_{tip}-\lambda\right)$$

Compressibility Corrections

- In the previous expression:

$$K = \frac{2}{1+\sqrt{1-M_{tip}^2}}$$

- If $M \to 0$ then $K \to 1$ and we obtain the incompressible result

Weighted Solidity

- We have seen that for a optimum rotor c varies with r.
- In these cases where c varies along the blade span the rotor solidity is different from the local blade solidity:

$$\sigma_{rotor} = \frac{\text{Blade area}}{\text{Rotor area}} = \int_0^1 \sigma(r)dr$$

- The objective of weighted solidity is to help compare performance of different rotors with different blade planforms.

Trust Weighted Solidity

- The thrust coefficient is:

$$C_T = \tfrac{1}{2}\int_0^1 \sigma r^2 C_l dr = \tfrac{1}{2}\sigma_e \int_0^1 r^2 C_l dr$$

- Assuming a constant C_l:

$$\int_0^1 \sigma r^2 dr = \sigma_e \int_0^1 r^2 dr = \sigma_e / 3$$

$$\sigma_e = 3\int_0^1 \sigma(r) r^2 dr$$

- With the equivalent chord $c_e = \dfrac{3\pi R}{N_b}\int_0^1 \sigma r^2 dr$

Power Weighted Solidity

- Extending the study to the power coefficient

$$C_P = C_Q = \int_{r=0}^{r=1} \lambda dC_T + \tfrac{1}{2}\int_0^1 \sigma(r) r^3 C_d dr$$

$$= \int_{r=0}^{r=1} \lambda dC_T + \tfrac{1}{2}\sigma_e \int_0^1 r^3 C_d dr$$

- Therefore

$$\sigma_e = 4\int_0^1 \sigma(r) r^3 dr$$