RESTRUCTURING LATTICE THEORY: AN APPROACH BASED ON HIERARCHIES OF CONCEPTS

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ABSTRACT

Lattice theory today reflects the general status of current mathematics: there is a rich production of theoretical concepts, results, and developments, many of which are reached by elaborate mental gymnastics; on the other hand, the connections of the theory to its surroundings are getting weaker and weaker, with the result that the theory and even many of its parts become more isolated. Restructuring lattice theory is an attempt to reinvigorate connections with our general culture by interpreting the theory as concretely as possible, and in this way to promote better communication between lattice theorists and potential users of lattice theory.

The approach reported here goes back to the origin of the lattice concept in nineteenth-century attempts to formalize logic, where a fundamental step was the reduction of a concept to its "extent". We propose to make the reduction less abstract by retaining in some measure the "intent" of a concept. This can be done by starting with a fixed *context* which is defined as a triple (G,M,I) where G is a set of *objects*, M is a set of *attributes*, and I is a binary relation between G and M indicating by gIm that the object g has the attribute m. There is a natural Galois connection between G and M defined by $A' = \{m \in M \mid gIm \text{ for all } g \in A\}$ for $A \subseteq G$ and $B' = \{g \in G \mid gIm \text{ for all } m \in B\}$ for $B \subseteq M$. Now, a *concept* of the context (G,M,I) is introduced as a pair (A,B) with $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$, where A is called the *extent* and B the *intent* of the concept (A,B). The *hierarchy of concepts* given by the relation subconcept-superconcept is captured by the definition $(A_1,B_1) \leq (A_2,B_2) \Leftrightarrow A_1 \subseteq A_2 (\Leftrightarrow B_1 \supseteq B_2)$ for concepts (A_1,B_1) and (A_2,B_2) of (G,M,I). Let $L(G,M,I)$ be the

I. Rival (ed.), Ordered Sets, 445–470.

set of all concepts of (G,M,I). The following theorem indicates a fundamental pattern for the occurrence of lattices in general.

THEOREM: *Let (G,M,I) be a context. Then $(\mathfrak{L}(G,M,I),\leq)$ is a complete lattice* (called the *concept lattice* of (G,M,I)) *in which infima and suprema can be described as follows:*

$$\bigwedge_{i\in J} (A_i,B_i) = \left(\bigcap_{i\in J} A_i, \left(\bigcap_{i\in J} A_i\right)'\right),$$

$$\bigvee_{i\in J} (A_i,B_i) = \left(\left(\bigcap_{i\in J} B_i\right)', \bigcap_{i\in J} B_i\right).$$

Conversely, if L is a complete lattice then $L \cong (\mathfrak{L}(G,M,I),\leq)$ if and only if there are mappings $\gamma : G \to L$ and $\mu : M \to L$ such that γG is supremum-dense in L, μM is infimum-dense in L, and gIm is equivalent to $\gamma g \leq \mu m$ for all $g \in G$ and $m \in M$; in particular, $L \cong (\mathfrak{L}(L,L,\leq),\leq)$.

Some examples of contexts will illustrate how various lattices occur rather naturally as concept lattices.

(i) $(S,S,\neq)$ where S is a set.
(ii) $(\mathbb{N},\mathbb{N},1)$ where $\mathbb{N}$ is the set of all natural numbers.
(iii) $(V,V^*,\perp)$ where V is a finite-dimensional vector space.
(iv) $(\mathcal{V},\mathrm{Eq}(\mathcal{V}),\models)$ where $\mathcal{V}$ is a variety of algebras.
(v) $(G\times G,\mathbb{R}^G,\sim)$ where G is a set of objects, $\mathbb{R}^G$ is the set of all real-valued functions on G, and $(g_1,g_2) \sim \alpha$ iff $\alpha g_1 = \alpha g_2$.

Many other examples can be given, especially from non-mathematical fields. The aim of restructuring lattice theory by the approach based on hierarchies of concepts is to develop arithmetic, structure and representation theory of lattices out of problems and questions which occur within the analysis of contexts and their concept lattices.

1. RESTRUCTURING LATTICE THEORY

Lattice theory today reflects the general status of current mathematics: there is a rich production of theoretical concepts, results, and developments, many of which are reached by elaborate mental gymnastics; on the other hand, the connections of the theory to its surroundings are getting weaker: the result is that the theory and even many of its parts become more isolated. This is not only a problem of lattice theory or of mathematics. Many sciences suffer from this effect of specialization. Scientists and philosophers are, of course, aware of the danger of this growing isolation. In [17], H. von Hentig extensively discusses the status of the humanities and sciences today. One consequence is his charge to "restructure" theoretical developments in order to integrate and rationalize origins, connections, interpretations, and applications. In particular, abstract developments should be brought back to the commonplace in perception, thinking, and action. Thus, *restructuring lattice theory* is understood as an attempt to unfold lattice-theoretical concepts, results, and methods in a continuous relationship with their surroundings (cf. Wille [40], [41]). One basic aim is to promote better communication between lattice theorists and potential users of lattice theory.

The important current monographs on lattice theory are by G. Birkhoff [3], P. Crawley and R.P. Dilworth [8], and G. Grätzer [15]. They are primarily concerned with "systematically developing a body of results at the heart of the subject" [8, p. V]. In an approach to restructure lattice theory we are first confronted with the question: *Why develop lattice theory*? G. Birkhoff in [3] refers to an earlier article "What can lattices do for you?" [4], where he attempts to rationalize the statement: "lattice theory has helped to simplify, unify and generalize mathematics, and it has suggested many interesting problems". In his first survey paper on "lattices and their applications" [2], G. Birkhoff set up a more general viewpoint for lattice theory: "lattice theory provides a proper vocabulary for discussing order, and especially systems which are in any sense hierarchies". Notwithstanding this broad point of view, it was the abstract level of accepting lattice theory as a theory of "substructures" in mathematics which brought the breakthrough in the 1930's and established lattice theory as a mathematical discipline (cf. Mehrtens [24]).

The approach to lattice theory outlined in this paper is based on an attempt to reinvigorate the general view of order. For this purpose we go back to the origin of the lattice concept in nineteenth-century attempts to formalize logic, where the study of hierarchies of concepts played a central rôle (cf. Schröder [35]). Traditional philosophy considers a *concept* to be determined by its "extent" [extension] and its "intent" [intension, comprehension]: the *extent* consists of all objects (or entities) belonging to the concept while the *intent* is the multitude of all attributes (or properties) valid for all those objects (cf. Wagner [36]). The *hierarchy* of concepts is given by the relation of "subconcept" to "superconcept", i.e. the extent of the subconcept is contained in the extent of the superconcept while, reciprocally, the intent of the superconcept is contained in the intent of the subconcept. Paradigmatic is the example: "human being" is a subconcept of the superconcept "being". Listing all objects and attributes belonging to a given concept is almost never possible. For practical reasons then, there is the proposal to limit a "context" by fixing the set of objects and the set of attributes (cf. Deutsches Institut für Normung [11], [12]). In set-theoretical language, this gives rise to lattices whose elements correspond to the concepts of the context and whose order comes from the hierarchy of concepts. The approach reported here takes these "concept lattices" as the basis and discusses how parts of arithmetic, structure and representation theory of lattices may be developed out of problems and questions which occur within the analysis of contexts and their hierarchies of concepts.

2. CONCEPT LATTICES

We start defining a *context* as a triple (G,M,I) where G and M are sets, and I is a binary relation between G and M; the elements of G and M are called *objects* [Gegenstände] and *attributes* [Merkmale], respectively. If gIm for $g \in G$ and $m \in M$ we say: the object g *has* the attribute m. For $A \subseteq G$ and $B \subseteq M$ we define

$$A' := \{m \in M \mid gIm \text{ for all } g \in A\},$$
$$B' := \{g \in G \mid gIm \text{ for all } m \in B\}.$$

The mappings given by $A \mapsto A'$ and $B \mapsto B'$ are said to form a *Galois connection* between the power sets of G and M, i.e. they fulfill the following basic properties (cf. Birkhoff [3; pp. 122-125]).

PROPOSITION. *For a context* (G,M,I):

(1) $A_1 \subseteq A_2$ *implies* $A_1' \supseteq A_2'$ *for* $A_1, A_2 \subseteq G$,

(1') $B_1 \subseteq B_2$ *implies* $B_1' \supseteq B_2'$ *for* $B_1, B_2 \subseteq M$,

(2) $A \subseteq A''$ *and* $A' = A'''$ *for* $A \subseteq G$,

(2') $B \subseteq B''$ *and* $B' = B'''$ *for* $B \subseteq M$.

Now, a *concept* [Begriff] of the context (G,M,I) may be defined as a pair (A,B) where $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$; A and B are called the *extent* and the *intent* of the concept (A,B), respectively. The hierarchy of concepts is captured by the definition

$$(A_1,B_1) \leq (A_2,B_2) \;:\Leftrightarrow\; A_1 \subseteq A_2 \quad (\Leftrightarrow B_1 \supseteq B_2)$$

for concepts (A_1,B_1) and (A_2,B_2) of (G,M,I); (A_1,B_1) is called the *subconcept* of (A_2,B_2), and (A_2,B_2) is called the *superconcept* of (A_1,B_1). To formulate the basic theorem (cf. Banaschewski [1], Schmidt [34]) which indicates a fundamental pattern for the occurrence of lattices in general, we need some further definitions. Let $\mathfrak{B}(G,M,I)$ be the set of all concepts of the context (G,M,I) and let $\underline{\mathfrak{B}}(G,M,I) := (\mathfrak{B}(G,M,I),\leq)$. A subset D of a complete lattice L is called *infimum-dense* (*supremum-dense*) if $L = \{\bigwedge X \mid X \subseteq D\}$ $(L = \{\bigvee X \mid X \subseteq D\})$.

THEOREM. *Let (G,M,I) be a context. Then $\underline{\mathfrak{B}}(G,M,I)$ is a complete lattice, called the* concept lattice *[Begriffsverband] of (G,M,I) in which infima and suprema can be described as follows:*

$$\bigwedge_{j \in J} (A_j,B_j) = \Big(\bigcap_{j \in J} A_j, \big(\bigcap_{j \in J} A_j\big)'\Big),$$

$$\bigvee_{j \in J} (A_j,B_j) = \Big(\big(\bigcap_{j \in J} B_j\big)', \bigcap_{j \in J} B_j\Big).$$

Conversely, if L is a complete lattice then $L \cong \underline{\mathfrak{B}}(G,M,I)$ if and only if there are mappings $\gamma : G \to L$ and $\mu : M \to L$ such that γG is supremum-dense in L, μM is infimum-dense in L, and gIm is equivalent to $\gamma g \leq \mu m$ for all $g \in G$ and $m \in M$; in particular, $L \cong \underline{\mathfrak{B}}(L,L,\leq)$.

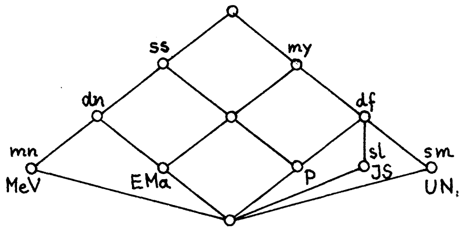
PROOF. Obviously, the relation $\leq$ is an order on $\mathfrak{B}(G,M,I)$. Let $(A_j,B_j) \in \mathfrak{B}(G,M,I)$ for $j \in J$. By the preceding proposition, $\bigcap_{j \in J} A_j \subseteq (\bigcap_{j \in J} A_j)''$ and $(\bigcap_{j \in J} A_j)'' \subseteq A_k'' = A_k$ for each $k \in J$ wherefore $(\bigcap_{j \in J} A_j)'' = \bigcap_{j \in J} A_j$ and hence $(\bigcap_{j \in J} A_j, (\bigcap_{j \in J} A_j)') \in \mathfrak{B}(G,M,I)$. If $(A,B) \in \mathfrak{B}(G,M,I)$ such that $(A,B) \leq (A_j,B_j)$ for all $j \in J$, i.e. $A \subseteq A_j$ for all $j \in J$, then $A \subseteq \bigcap_{j \in J} A_j$ and hence $(A,B) \leq (\bigcap_{j \in J} A_j, (\bigcap_{j \in J} A_j)')$. Thus, $(\bigcap_{j \in J} A_j, (\bigcap_{j \in J} A_j)')$ is the infimum of the (A_j,B_j) $(j \in J)$. The proof for the supremum goes analogously. Now, let φ be an isomorphism from $\underline{\mathfrak{B}}(G,M,I)$ onto a complete lattice L. We define $\gamma g := \varphi(\{g\}'',\{g\}')$ for $g \in G$ and $\mu m := \varphi(\{m\}',\{m\}'')$ for $m \in M$. Since $A = \bigcup_{g \in A} \{g\}''$ and $B = \bigcup_{m \in B} \{m\}''$ for every $(A,B) \in \mathfrak{B}(G,M,I)$, γG is supremum-dense in L and μM is infimum-dense in L, respectively. Furthermore, $gIm \Leftrightarrow \{g\}'' \subseteq \{m\}' \Leftrightarrow \gamma g \leq \mu m$. Conversely, let L be a complete lattice and let $\gamma : G \to L$ and $\mu : M \to L$ arbitrary mappings having the desired properties. We define a mapping $\varphi: \mathfrak{B}(G,M,I) \to L$ by $\varphi(A,B) := \bigvee\gamma A$ for all $(A,B) \in \mathfrak{B}(G,M,I)$. Obviously, φ is order-preserving. For $x \in L$, let $\psi x := (\{g \in G \mid \gamma g \leq x\}, \{m \in M \mid x \leq \mu m\})$. Since γG is supremum-dense in L and μM is infimum-dense in L, $\bigvee\gamma\{g \in G \mid \gamma g \leq x\} = x = \bigwedge\mu\{m \in M \mid x \leq \mu m\}$. Therefore, because of $gIm \Leftrightarrow \gamma g \leq \mu m$, ψx is a concept of (G,M,I) and $\varphi\psi x = x$. Clearly, $(A,B) \leq \psi\varphi(A,B)$. As $\gamma g \leq \bigvee\gamma A \leq \mu m$ always implies gIm, we even have $(A,B) = \psi\varphi(A,B)$. Furthermore, ψ is order-preserving. Now, we can summarize: φ is an isomorphism from $\mathfrak{B}(G,M,I)$ onto L and $\varphi^{-1} = \psi$. If we choose $G := L$, $M := L$, $I := \leq$, γ and μ as the identity on L, we get $L \cong \mathfrak{B}(L,L,\leq)$.

For a finite lattice L with $J(L)$ as its set of $\vee$-irreducible elements and $M(L)$ as its set of $\wedge$-irreducible elements, the theorem yields $L \cong \mathfrak{B}(J(L),M(L),\leq)$, and the context $(J(L),M(L),\leq)$ is smallest with the property that its concept lattice is isomorphic to L. In general, the proposition and the theorem makes apparent a dual situation which often gets a concrete meaning by interchanging objects and attributes. Formally, we consider (M,G,I^{-1}) as the *dual context* of the context (G,M,I) where the mapping given by $(A,B) \mapsto (B,A)$ is an antiisomorphism from $\mathfrak{B}(G,M,I)$ onto $\mathfrak{B}(M,G,I^{-1})$.

3. EXAMPLES

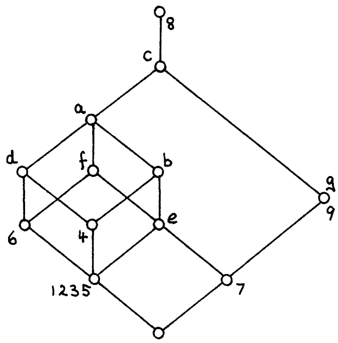
Some examples may illuminate how contexts give rise to concept lattices. The first example, a context of our planets, is described by the following table (cf. [25]) in which crosses indicate the relation between the objects and the attributes; the concept lattice is represented by its Hasse diagram where the labels indicate the mappings γ and μ of the preceding theorem.

	size			distance from sun		moon	
	small	medium	large	near	far	yes	no
Mercury	×			×			×
Venus	×			×			×
Earth	×			×		×	
Mars	×			×		×	
Jupiter			×		×	×	
Saturn			×		×	×	
Uranus		×			×	×	
Neptune		×			×	×	
Pluto	×				×	×	

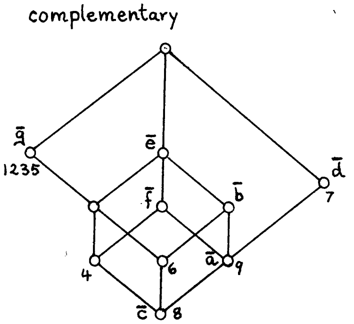


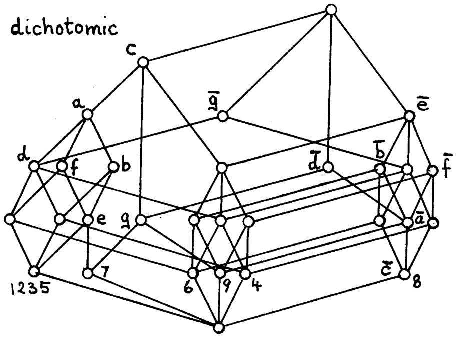
The second example is taken from the German national division of health and welfare (see Podlech [27]). The context indicates which personal information is needed by law for the different tasks of a local medical subdivision; tasks (objects) and information (attributes) are the following: 1. confirmation of requests for preventive care, 2. evaluations of regulations of insurance benefits, 3. confirmation of incapacity to work to insure success of treatment, 4. confirmation of correct diagnosis of incapacity to work, 5. work preliminary to rehabilitation, 6. verification of sickness, 7. advice to clients, 8. advice on general preventives, 9. carrying on the statistics of the medical service, a. name and address of the client ..., b. career history ..., c. kind of membership ..., d. name of responsible agency ..., e. family medical history, f. vocational education ..., g. number of certificates.

	a	b	c	d	e	f	g
1	×	×	×	×	×	×	
2	×	×	×	×	×	×	
3	×	×	×	×	×	×	
4	×	×	×	×			
5	×	×	×	×	×	×	
6	×		×	×		×	
7	×	×	×		×	×	×
8							
9			×				×



The concept lattice of a task-information-context (G,M,I) gives a natural hierarchical classification of the tasks and makes apparent the dependencies between information. But for certain purposes the concept lattice of the *complementary context* $(G,\bar{M},\bar{I})$ with $\bar{M} := \{\bar{m} | m \in M\}$ and $g\bar{I}\bar{m} :\Leftrightarrow \neg(gIm)$ is more suitable, for instance, if one wants to assign the tasks to different parties such that the access to personal information is mostly limited. Both lattices are combined in the concept lattice of the *dichotomic context* $(G, M\cup\overline{M}, I\cup\overline{I})$, and it might be interesting to analyse how this lattice is dependent on the two smaller ones (cf. proposition in Section 4). Here we only present the Hasse diagrams with respect to the context above.





Experience has shown that concept lattices of "empirical" contexts which occur in rich multitude in many fields usually do not belong to the mostly studied classes of lattices as those of distributive, modular, semimodular, orthomodular etc. lattices. The situation becomes different if we consider contexts in mathematics (cf. Birkhoff [3]). For a set S, the concept lattice of $(S,S,\neq)$ is isomorphic to the Boolean lattice of all subsets of S since $(A,S\setminus A)$ is a concept of $(S,S,\neq)$ for every $A \subseteq S$. If $|$ is the divisibility relation on the set $\mathbb{N}$ of all natural numbers, $\underline{\mathfrak{L}}(\mathbb{N},\mathbb{N},|)$ is distributive. Let V be a finite-dimensional vector space, let V^* be its dual space, and let $a\perp\varphi :\Leftrightarrow \varphi a = 0$ for $a \in V$ and $\varphi \in V^*$. Then $\underline{\mathfrak{L}}(V,V^*,\perp)$ is isomorphic to the complemented modular lattice of all linear subspaces of V. If H is a Hilbert space and $\perp$ its orthogonality relation, then $\underline{\mathfrak{L}}(H,H,\perp)$ is isomorphic to the orthomodular lattice of all closed linear subspaces of H. For a variety $\mathcal{V}$ of algebras and the set $Eq(\mathcal{V})$ of all equations valid in each algebra of $\mathcal{V}$, the concept lattice of $(\mathcal{V}, Eq(\mathcal{V}), \models)$ is isomorphic to the coalgebraic lattice of all subvarieties of $\mathcal{V}$. Many further examples could be given. These examples may suggest an extensive study of special classes of lattices. Here we want to continue with the general approach. We only mention one further class of examples which will be discussed more in Section 8: If $(P,\leq)$ is a (partially) ordered set then $\underline{\mathfrak{L}}(P,P,\leq)$ is up to isomorphism the Dedekind-MacNeille completion of $(P,\leq)$.

4. THE DETERMINATION PROBLEM

The concept lattice can be understood as a basic answer to two fundamental questions concerning a context, namely the question of an appropriate classification of the objects and the question about the dependencies between the attributes. Hence an important problem is: *How can one determine the concept lattice of a given context* (G,M,I)? We get the most simple answer by forming (X'',X') for all $X \subseteq G$ (or (Y',Y'') for all $Y \subseteq M$). Of course, this method is far too costly and should only be used to determine single concepts. More successful, in particular if no computer is involved, is the approach using the following formulas for $(A,B) \in \mathfrak{B}(G,M,I)$:

$$A = \bigcap_{m\in B} \{m\}' \quad , \qquad\qquad B = \bigcap_{g\in A} \{g\}' \quad .$$

For instance, in the context of our planets one first determines the extents $\{ss\}'$, $\{sm\}'$, $\{sl\}'$, $\{dn\}'$, $\{df\}'$, $\{my\}'$, and $\{mn\}'$; then one gets all the other extents as intersections from those. From the extents it is easy to obtain the intents. For a larger context one may use both formulas simultaneously.

For the use of computers, it would be desirable to have "good" algorithms determining the concept lattice of a concept. Of course, designing those algorithms one has to consider how the concept lattice is handled afterwards. For instance, it might be presented by its Hasse diagram (see Section 4), it might be used to establish a measurement concerning the objects (see Section 8), or it might be a tool for a description of the dependencies between the attributes. In particular, one may elaborate the classification of objects further in the line of cluster analysis (for a recent survey on cluster analysis see Bock [6]): For many contexts one would like to have suitable partitions of the set of objects whose blocks are extents, i.e. they can be described by attributes; first experiences with randomly chosen contexts have shown that there are usually few such partitions if the cardinalities of their blocks are not becoming too small.

In certain situations, the determination problem can be attacked by a structural method which idea is to construct the concept lattice of a context by the concept lattices of some suitable subcontexts. We recall that the order R of the *direct product* of the two ordered sets (P,R_1) and (P,R_2) is defined by $(x_1,x_2)R(y_1,y_2) :\Leftrightarrow x_1R_1y_1$ and $x_2R_2y_2$. The *horizontal sum* of two bounded ordered sets (P_1,R_1) and (P_2,R_2) is obtained from their *cardinal sum* $(P_1\dot{\cup}P_2, R_1\dot{\cup}R_2)$ by identifying the smallest elements and the greatest elements of both ordered sets, respectively. The *vertical sum* of (P_1,R_1) and (P_2,R_2) is obtained from their *ordinal sum* $(P_1\dot{\cup}P_2, R_1\dot{\cup}R_2\dot{\cup}P_1\times P_2)$ by identifying the greatest element of

(P_1,R_1) with the smallest element of (P_2,R_2).

THEOREM. *Let* (G_1,M_1,I_1) *and* (G_2,M_2,I_2) *be contexts with* $G_1\cap G_2 = \emptyset$, $M_1\cap M_2 = \emptyset$, $G_i' = \emptyset$ *and* $M_i' = \emptyset$ *for* $i = 1,2$; *let* $L_i := \underline{\mathfrak{B}}(G_i,M_i,I_i)$ *for* $i = 1,2$. *Then*

(1) $\underline{\mathfrak{B}}(G_1\dot\cup G_2, M_1\dot\cup M_2, I_1\dot\cup I_2)$ *is isomorphic to the horizontal sum of* L_1 *and* L_2,

(2) $\underline{\mathfrak{B}}(G_1\dot\cup G_2, M_1\dot\cup M_2, I_1\dot\cup I_2\dot\cup G_1\times M_2)$ *is isomorphic to the vertical sum of* L_1 *and* L_2,

(3) $\underline{\mathfrak{B}}(G_1\dot\cup G_2, M_1\dot\cup M_2, I_1\dot\cup I_2\dot\cup G_1\times M_2\dot\cup G_2\times M_1)$ *is isomorphic to the direct product of* L_1 *and* L_2.

PROOF. (1) Obviously, (A_i,B_i) with $\emptyset \neq A_i \subseteq G_i$ and $\emptyset \neq B_i \subseteq M_i$ is a concept of $(G_1\dot\cup G_2, M_1\dot\cup M_2, I_1\dot\cup I_2)$ if and only if it is a concept of (G_i,M_i,I_i) $(i = 1,2)$. If $A \subseteq G_1\cup G_2$ and $A\cap G_i \neq \emptyset$ for $i = 1,2$ then $A' = \emptyset$ and hence $A'' = G_1\cup G_2$; furthermore $\emptyset'' = \emptyset$. This proves (1).
(2) $(G_1\dot\cup G_2, M_1\dot\cup M_2, I_1\dot\cup I_2\dot\cup G_1\times M_2)$ has exactly the concepts $(\emptyset, G_1\cup G_2)$, $(A_1, B_1\cup M_2)$ with $A_1 \neq \emptyset$ and $(A_1,B_1) \in \underline{\mathfrak{B}}(G_1,M_1,I_1)$, and $(A_2\cup G_1, B_2)$ with $A_2 \neq \emptyset$ and $(A_2,B_2) \in \underline{\mathfrak{B}}(G_2,M_2,I_2)$.
(3) $(G_1\dot\cup G_2, M_1\dot\cup M_2, I_1\dot\cup I_2\dot\cup G_1\times M_2\dot\cup G_2\times M_1)$ has exactly the concepts $(A_1\cup A_2, B_1\cup B_2)$ with $(A_1,B_1) \in \underline{\mathfrak{B}}(G_1,M_1,I_1)$ and $(A_2,B_2) \in \underline{\mathfrak{B}}(G_2,M_2,I_2)$.

In cases where the preceding theorem cannot be applied, one may still reduce a given context to smaller contexts to apply the following proposition which can be easily verified using (3).

PROPOSITION. *Let* (G,M,I) *be a context, let* $\{G_j \mid j \in J\}$ *be a partition of* G, *and let* $\{M_k \mid k \in K\}$ *be a partition of* M. *Then*

(4) *a* $\wedge$*-embedding of* $\underline{\mathfrak{B}}(G,M,I)$ *into the direct product of the* $\underline{\mathfrak{B}}(G_j,M,I\cap G_j\times M)$ $(j \in J)$ *is given by* $(A,B) \mapsto (A\cap G_j, (A\cap G_j)')_{j\in J}$,

(4') *a* $\vee$*-embedding of* $\underline{\mathfrak{B}}(G,M,I)$ *into the direct product of the* $\underline{\mathfrak{B}}(G,M_k,I\cap G\times M_k)$ $(k \in K)$ *is given by* $(A,B) \mapsto ((B\cap M_k)', B\cap M_k)_{k\in K}$.

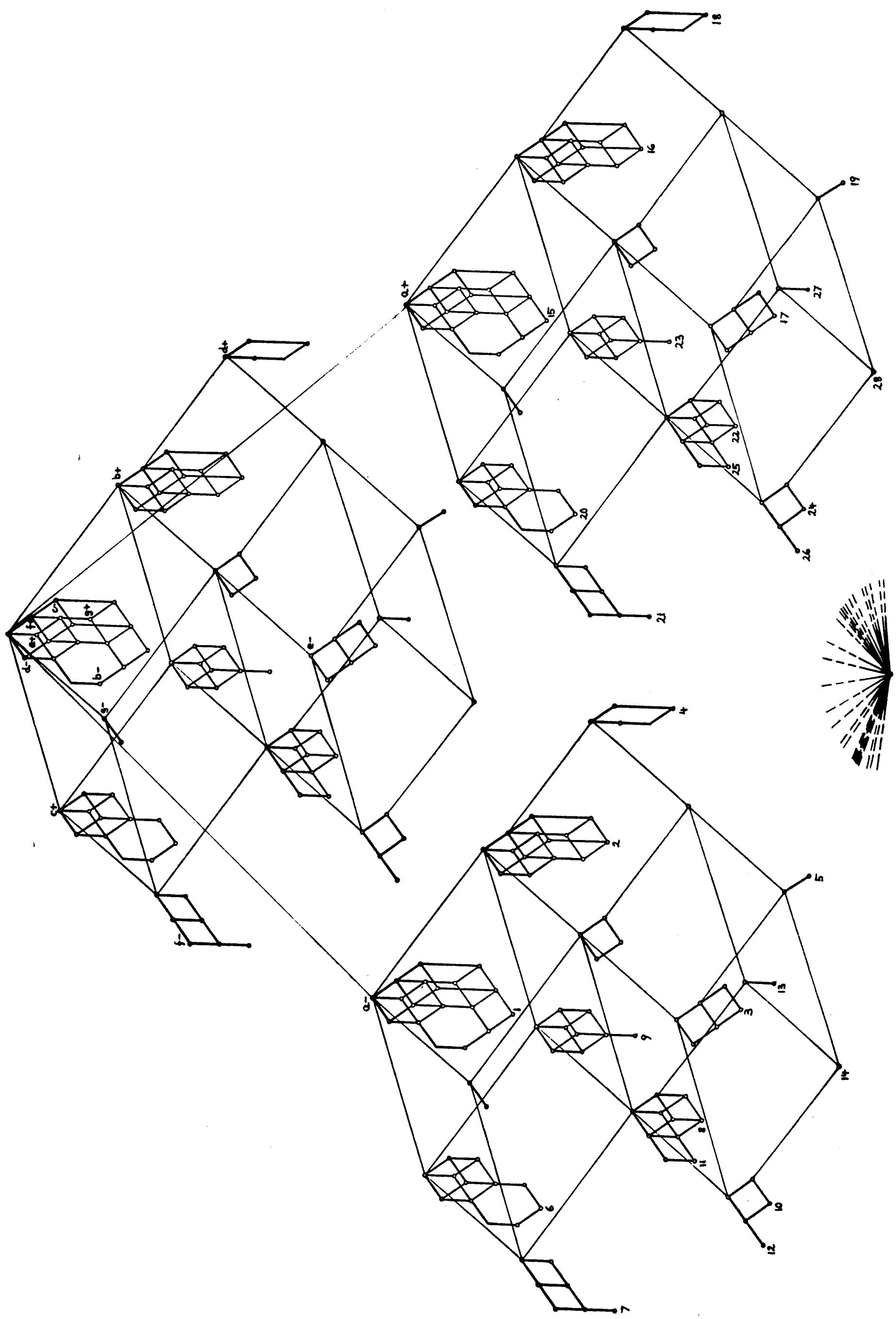
One hint is to apply (4) and (4') several times to reach subcontexts whose concept lattices can be easily determined; then in the direct product of these lattices one may find the images of the concepts of the original context under the order embedding given by (4) and (4'). The concept lattice of a context (G,M,I) is a chain if and only if G and M can be linearly ordered such that I becomes an order filter in $G\times M$. Thus, in the light of the described procedure, it might be a challenging problem to find for a given context (G,M,I) the smallest number of applications of (4) and (4') which yields an order embedding of $\underline{\mathfrak{B}}(G,M,I)$ into a direct product of (two-element) chains.

5. THE DESCRIPTION PROBLEM

Connected with the description problem is the basic problem: *How can one describe the concept lattice of a given context* (G,M,I)? "Good" solutions of this problem are helpful for further analysis and communication, especially with non-mathematicians. The most communicative description is given by Hasse diagrams. If we label in a Hasse diagram of $\underline{\mathfrak{B}}(G,M,I)$ the circle representing $(\{g\}'',\{g\}')$ by a name for g ($g \in G$) and the circle representing $(\{m\}',\{m\}'')$ by a name for m ($m \in M$), as in the examples in Section 3, the diagram retains the full information about (G,M,I) because gIm is equivalent to $(\{g\}'',\{g\}') \leq (\{m\}',\{m\}'')$ by the theorem in Section 2. Although Hasse diagrams are frequently used, it is surprising that there is no "theory of readable Hasse diagrams". Drawing a Hasse diagram is still more an art than a mechanical skill that even a computer can work out. Because of the absence of a theory, we shall only discuss the development of a Hasse diagram by an example.

As an example, we consider the context of human Rhesus blood types by Wiener and Wechsler (see Diem, Lentner [13]). Its objects are the 28 Rhesus phenotypes which have been established up to 1956, and its attributes are the following seven test serums: a. Anti-Rh_0, b. Anti-rh', c. Anti-rh'', d. Anti-rh'^{W}, e. Anti-hr', f. Anti-hr'', g. Anti-hr. The symbol + in the table below indicates when a blood type reacts positively with a test serum; no reaction is indicated by the symbol −. We analyse the context as a dichotomic context (cf. Section 3), i.e. we split each attribute x into the two attributes x+ and x−. The concept lattice of the dichotomic context is represented on the next page by a (partial) Hasse diagram in which several line segments are left out to allow easier reading. Besides the circle representing the least element, the diagram consists of three congruent components. If one moves the component above to one of the components below parallel along the line segment joining the top circles, then the traces of the circles will just add the missing line segments between the two components. Analogously, one has to move the sub-components in each component parallel along the joining line segments; but the trace of a circle is a missing line segment only if the circle reaches another. To get more familiar with the diagram, one may do the exercise to find

	a	b	c	d	e	f	g
1	−	−	−	−	+	+	+
2	−	+	−	−	+	+	+
3	−	+	−	−	−	+	−
4	−	+	−	+	+	+	+
5	−	+	−	+	−	+	−
6	−	−	+	−	+	+	+
7	−	−	+	−	+	−	−
8	−	+	+	−	+	+	−
9	−	+	+	−	+	+	+
10	−	+	+	−	−	+	−
11	−	+	+	−	+	−	−
12	−	+	+	−	−	−	−
13	−	+	+	+	+	+	−
14	−	+	+	+	−	+	−
15	+	−	−	−	+	+	+
16	+	+	−	−	+	+	+
17	+	+	−	−	−	+	−
18	+	+	−	+	+	+	+
19	+	+	−	+	−	+	−
20	+	−	+	−	+	+	+
21	+	−	+	−	+	−	−
22	+	+	+	−	+	+	−
23	+	+	+	−	+	+	+
24	+	+	+	−	−	+	−
25	+	+	+	−	+	−	−
26	+	+	+	−	−	−	−
27	+	+	+	+	+	+	−
28	+	+	+	+	−	+	−


d+
b+
a+
c+
f-
a-
d-
e+
f+
c-
g+
b-
g-
e-
1
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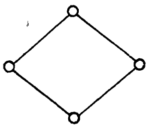
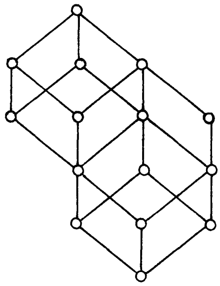
out which attributes are dependent on $\{a-,c+,g+\}$, i.e. contained in $\{a-,c+,g+\}''$ (take the meet of the elements labelled by $a-$, $c+$, and $g+$, and list all labelled elements above this meet: the result will be $\{a-,c+,g+\}'' = \{a-,c+,d-,e+,f+,g+\}$). The development of the Hasse diagram representing 286 elements has followed the scheme: Search for a suitable chain $L\times L =: R_0 \supset R_1 \supset R_2 \supset \ldots \supset R_n := id_L$ of equivalence relations with convex equivalence classes on the lattice L such that $[a]R_{i-1}/R_i$ has dimension ≤ 3 for all $a \in L$ and $i = 1,\ldots,n$. In our example we have $n = 3$; L/R_1 and $[1]R_1/R_2$ may be represented by the following Hasse diagrams:





Besides the description by Hasse diagrams, there are other useful methods for describing concept lattices. For instance, a concept lattice might be completely determined by some relative sublattice (e.g. its scaffolding) which can be diagrammed (cf. Wille [39]). A rich multitude of descriptions arises from the general idea of measurement which especially leads to numerical representations; this will be discussed further in Section 7.

6. FROM STRUCTURES TO CONCEPT LATTICES

Data models are often not contexts as defined in Section 2, i.e. relational structures with only unary relations; they frequently use many-placed relations and operations. Nevertheless, it is also desirable to have hierarchies of concepts for arbitrary relational structures available. For this, the general scheme is to derive (unary) attributes from the given structure and to form the concept lattice of the context consisting of the elements of the structure and these derived attributes. For instance, an n-ary relation R on a set G may be resolved into unary relations defined by $R_{(g_1,\ldots,g_{i-1},g_{i+1},\ldots,g_n)}(x) :\Leftrightarrow R(g_1,\ldots,g_{i-1},x,g_{i+1},\ldots,g_n)$ where $(g_1,\ldots,g_{i-1},g_{i+1},\ldots,g_n) \in G^{n-1}$ and $i \in \{1,\ldots,n\}$. The derivation always depends on which concepts will be of interest with respect to the data. In this section we restrict our considerations to structures of "nominal character"; further structures are discussed in Section 7.

In many situations, the term "attribute" has an even more general meaning, i.e. an attribute may allow several values instead of one. A person may have any one of several values for the attribute "profession", for instance, "artist", "bus driver", "carpenter", etc.; there are also people with no profession (no value). We capture the general case by the definition of a *many-valued context* which is a quadruple (G,M,W,I) such that G, M, and W are sets, and I is a binary relation between G and $M\times W$ where $gI(m,w_1)$ and $gI(m,w_2)$ always imply $w_1 = w_2$; the elements of G, M, and W are called *objects*, *(many-valued) attributes*, and *attribute values*, respectively. If $gI(m,w)$ for $g \in G$, $m \in M$, and $w \in W$ we say: the object g *has* value w for the attribute m. If $|W| = n$ then (G,M,W,I) is called an *n-valued context*. A one-valued context can be considered as a context as defined in Section 2. Now, in the case where the attribute values are understood as nominal data, we suggest the derivation to "one-valued" attributes by just taking $(G,M\times W,I)$ as the derived context; let us indicate this case by calling (G,M,W,I) a *nominal context*. Hence, for a nominal context (G,M,W,I), its concept lattice is just $\mathfrak{L}(G,M\times W,I)$. In this setting, the context of our planets in Section 3 can be understood as a nominal context with $M :=$ {size, distance from sun, moon} and $W :=$ {small, medium, large, near, far, yes, no}. The dichotomic contexts turn out to be always 2-valued nominal contexts.

It is common to assume that an n-valued context (G,M,W,I) is *complete*, i.e. for every $g \in G$ and $m \in M$ there exists a $w \in W$ with $gI(m,w)$; complete n-valued contexts are just a translated version of "relational data models" (cf. Date [9]). For attacking the determination problem and the description problem, the knowledge of special properties satisfied by the concept lattice of a complete nominal context is helpful. For a characterization of the concept lattices of complete n-valued nominal contexts the following definition is used. An element d of a complete lattice L has *valence* n if n is the smallest cardinality of a subset D of $L\setminus\{0\}$ containing d which is maximal with respect to the property that $x\wedge y = 0$ for all $x,y \in D$ with $x \neq y$. Recall that a lattice is *atomistic* if each lattice element is a supremum of atoms.

THEOREM. *A complete lattice L is isomorphic to a concept lattice of a complete n-valued nominal context if and only if L is atomistic and has an infimum-dense subset of elements of valence $\leq n$.*

PROOF. We assume that L has the desired properties. Let A be the set of all atoms of L and let $A(b) := \{a \in A \mid a \leq b\}$ for $b \in L$; furthermore, let M be the set of all elements of valence $\leq n$ in L and let W be a set with $|W| = n$ and $1 \in W$. For each $m \in M$ there must be a subset W_m of W with $1 \in W_m$ and an injective map $\iota_m : W_m \to L$ such that $\iota_m 1 = m$ and $\{A(\iota_m w) \mid w \in W_m\}$ is a partition of A. For $a \in A$, $m \in M$, and $w \in W$, we define $aI(m,w)$ if and only if $w \in W_m$

and $a \leq \iota_m w$. Since $\{\iota_m w \mid m \in M$ and $w \in W_m\}$ is infimum-dense in L and A is supremum-dense in L, L is isomorphic to $\underline{\mathfrak{L}}(A,M{\times}W,I)$ by the theorem of Section 2 where (A,M,W,I) is a complete n-valued nominal context. Conversely, let (G,M,W,I) be a complete n-valued nominal context. If $\{g_1\}'' \neq \{g_2\}''$ for $g_1,g_2 \in G$, then w.o.l.g. there exists an $(m,w) \in M{\times}W$ with $g_1I(m,w)$ but not $g_2I(m,w)$. By the completeness, there is a $\bar{w} \in W$ with $g_2I(m,\bar{w})$. Now, $\{(m,w)\}' \cap \{(m,\bar{w})\}' = \emptyset$ implies $\{g_1\}'' \cap \{g_2\}'' = \emptyset$. Hence $\{\{g\}'' \mid g \in G\}$ is a partition of G. Thus, $\underline{\mathfrak{L}}(G,M{\times}W,I)$ is atomistic. Obviously, a concept $(\{(m,w)\}',\{(m,w)\}'')$ has valence $\leq n$ in $\underline{\mathfrak{L}}(G,M{\times}W,I)$ for all $(m,w) \in M{\times}W$. Since the set of all these concepts is infimum-dense in the concept lattice, the theorem is proved.

COROLLARY. *A finite lattice L is isomorphic to a concept lattice of a finite complete n-valued nominal context if and only if L is atomistic and every $\wedge$-irreducible element of L has valence $\leq n$.*

COROLLARY. *A finite lattice L is isomorphic to a concept lattice of a finite complete 2-valued nominal context if and only if L is atomistic and every $\wedge$-irreducible element of L has a pseudo-complement.*

The concept lattice of a nominal context (G,M,W,I) only clarifies the dependencies between the "one-valued" attributes of $M{\times}W$ but not the dependencies between the many-valued attributes of M which are in general defined as follows (since the elements of M can be understood as partial maps from G into W, we write $mg = w$ instead of $gI(m,w)$): For $m \in M$ and $B \subseteq M$, the attribute m is *dependent* on B if $mg_1 = mg_2$ for all $g_1,g_2 \in G$ whenever $ng_1 = ng_2$ for all $n \in B$ or equivalently, if there exists a mapping $\alpha : W^B \to B$ such that $\alpha(ng)_{n \in B} = mg$ for all objects g for which ng is defined for all $n \in B$ (special dependencies as monotone, linear, quadratic, etc. usually refer to restrictions on the mappings α). The closed subsets of M with respect to the defined dependence are exactly the intents of the context $(G{\times}G,M,\sim)$ where $(g_1,g_2) \sim m :\Leftrightarrow mg_1 = mg_2$. Here lattices of (partial) equivalence relations arise.

7. MEASUREMENT

Numerical descriptions of concept lattices may be viewed as a part of that area concerned with the measurements of objects. Concept lattices may even arise from many-valued contexts whose attribute values are numerical; there the attributes can be understood as some kind of measure of objects. The common theory of measurement defines a scale (measure) to be a (relatively closed) homomorphism from an empirical relational structure into a numerical relational structure (cf. Pflanzagl [26], Krantz, Luce, Suppes, Tversky [20], Roberts [31]). In our approach, it is desirable to have a type-free notion of measure (instead of the type-fixed notion homomorphism) wherefore we separate the notions of scale and measure. A *scale* is defined as a *numerical context* $\Sigma := (G_\Sigma, M_\Sigma, I_\Sigma)$ where G_Σ is a set of numbers or number-valued functions. Then a Σ-*measure* of a context (G,M,I) is a mapping μ from G into G_Σ such that $\mu^{-1}A$ is an extent of (G,M,I) for every extent A of Σ; μ is called *full* if μ^{-1} induces an isomorphism from the concept lattice of $(\mu G, M_\Sigma, I_\Sigma \cap \mu G \times M_\Sigma)$ onto $\underline{\mathfrak{B}}(G,M,I)$. The description problem appears now within the *representation problem* which asks for (full) Σ-measures of a given context into an appropriate scale Σ. Paradigms for scales are the *real ordinal scale* $\Sigma_O := (\mathbb{R}, M_O, \in)$ with $M_O := \{(-\infty, r] \mid r \in \mathbb{R}\}$, the *real interval scale* $\Sigma_I := (\mathbb{R}, M_I, \in)$ with $M_I := M_O \cup \{\{r-s, r, r+s\} \mid r \in \mathbb{R},\ s \in \mathbb{R}^+\}$, the *real ratio scale* $\Sigma_R := (\mathbb{R}, M_R, \in)$ with $M_R := M_I \cup \{\{r:s, r, r\cdot s\} \mid r \in \mathbb{R},\ s \in \mathbb{R}^+\}$, and their multidimensional analogues.

Our setting of measurement provides a general scheme for deriving a context from a many-valued context (G,M,W,I). If an attribute $m \in M$ is understood as a map into a scale Σ, then we propose to resolve m into the set $\{m\} \times M_\Sigma$ where an object $g \in G$ has the attribute (m,n) for $n \in M_\Sigma$ if $g \in m^{-1}\{n\}'$; this translation guarantees that m will be a Σ-measure of the derived context. For instance, if all attributes in M are maps into the real ordinal scale Σ_O, we obtain the context $(G, M \times \mathbb{R}, I_\leq)$ where $gI_\leq(m,r) :\Leftrightarrow mg \leq r$; in such a case, we call (G,M,W,I) an *ordinal context*. The embedding of $\underline{\mathfrak{B}}(G, M\times\mathbb{R}, I_\leq)$ into $\mathbb{R}^M$ will be discussed in a more general frame below.

A main question in the theory of measurement is whether a Σ-measure μ_0 of a context (G,M,I) is *unique* in the sense that for every Σ-measure μ of (G,M,I) which admits a bijection $\beta: \mu_0 G \to \mu G$ inducing an isomorphism between the corresponding concept lattices and satisfying $\beta_0\mu_0 = \mu$, there exists a bijection $\sigma: G_\Sigma \to G_\Sigma$ which induces an automorphism of $\underline{\mathfrak{B}}(\Sigma)$ and extends β. This indicates that the representation problem also leads to a study of automorphisms of concept lattices. For instance, the automorphisms of the concept lattice of Σ_O, Σ_I, and Σ_R are induced by the order automorphisms, the positive affine mappings, and the positive linear mappings of $\mathbb{R}$, respectively (the set of all permutations of G_Σ inducing automorphisms on $\underline{\mathfrak{B}}(\Sigma)$ is usually considered as the *scale type* of Σ).

How lattice-theoretic developments are connected with a systematic study of the representation problem may be demonstrated in the case of order measurement. Let $(\Omega_t)_{t\in T}$ be a family of complete chains (of numbers) and let Ω be the direct product of the Ω_t $(t \in T)$; then $\Omega_\leq := (\Omega, \Omega, \leq)$ is called an *order scale of dimension* $|T|$ *and of length* $l(\Omega)$. Since Ω is a complete lattice, the mapping given by $x \mapsto ((x],[x))$ is an isomorphism from Ω onto $\mathfrak{B}(\Omega_{\leq})$ by the theorem of Section 2 (we recall that $(x] := \{y | y \leq x\}$ and $[x) := \{z | x \leq z\}$).

PROPOSITION. *For a full $\Omega_{\leq}$-measure μ of a context (G,M,I), let $\overline{\mu}(A,B) := \bigvee \mu A$ for all $(A,B) \in \mathfrak{B}(G,M,I)$. Then the mapping given by $\mu \mapsto \overline{\mu}$ is a bijection from the set of all full $\Omega_{\leq}$-measures of (G,M,I) onto the set of all $\bigvee$-embeddings of $\mathfrak{B}(G,M,I)$ into Ω; in particular, $\mu g = \overline{\mu}(\{g\}'',\{g\}')$ for all $g \in G$.*

PROOF. If μ is a full $\Omega_{\leq}$-measure of (G,M,I) then an isomorphism from $\mathfrak{B}(G,M,I)$ onto $\mathfrak{B}(\mu G,\Omega,\leq)$ is given by $(A,B) \mapsto (\mu A,(\mu A)')$; in particular, μA is an extent of $(\mu G,\Omega,\leq)$ for each extent A of (G,M,I) and therefore $(\bigvee\mu A] \cap \mu G = \mu A$. Thus, $\overline{\mu}$ is injective. Let $(A_j,B_j) \in \mathfrak{B}(G,M,I)$ for $j \in J$ and let $a := \bigvee_{j \in J} \bigvee \mu A_j$. There exists a concept (A,B) of (G,M,I) with $\mu A = (a] \cap \mu G$. Since $a \leq \bigvee \mu C$ for every concept (C,D) of (G,M,I) while $\bigvee_{j \in J} (A_j,B_j) \leq (C,D)$, it follows $(A,B) = \bigvee_{j \in J} (A_j,B_j)$. Hence $\overline{\mu}$ is a $\bigvee$-embedding of $\mathfrak{B}(G,M,I)$ into Ω. Since $(\mu g] \cap \mu G$ is the smallest extent of $(\mu G,\Omega,\leq)$ containing g $(g \in G)$, we have $(\mu g] \cap \mu G = \mu\{g\}''$ and hence $\mu g = \overline{\mu}(\{g\}'',\{g\}')$. Now, let ι be a $\bigvee$-embedding of $\mathfrak{B}(G,M,I)$ into Ω. We define $\underline{\iota} g := \iota(\{g\}'',\{g\}')$ for all $g \in G$. Let A be an extent of (G,M,I). Obviously, $(\bigvee\underline{\iota}A] \cap \underline{\iota}G \supseteq \underline{\iota}A$. Let $\underline{\iota}g \leq \bigvee\underline{\iota}A$. Because of $\bigvee\underline{\iota}A \leq \iota(A,A')$, it follows $\iota(\{g\}'',\{g\}') \leq \iota(A,A')$ and hence $(\{g\}'',\{g\}') \leq (A,A')$ wherefore $g \in A$ and $\underline{\iota}g \in \underline{\iota}A$. Thus, $(\bigvee\underline{\iota}A] \cap \underline{\iota}G = \underline{\iota}A$. This means that $\underline{\iota}$ induces an injective map from $\mathfrak{B}(G,M,I)$ into $\mathfrak{B}(\underline{\iota}G,\Omega,\leq)$. Let $a \in \Omega$ and let $A := \underline{\iota}^{-1}(a] = \{g \in G | \iota(\{g\}'',\{g\}') \leq a\}$. Then $\iota(A'',A') = \iota\bigvee\{(\{g\}'',\{g\}') | g \in A\} = \bigvee\{\iota(\{g\}'',\{g\}') | g \in A\} \leq a$ which implies $A = A''$. Therefore $\underline{\iota}$ is an $\Omega_{\leq}$-measure of (G,M,I) and induces a bijective map from $\mathfrak{B}(G,M,I)$ onto $\mathfrak{B}(\underline{\iota}G,\Omega,\leq)$. Obviously, $(A,B) \leq (C,D)$ implies $\underline{\iota}A \subseteq \underline{\iota}C$. Conversely, if $\underline{\iota}A \subseteq \underline{\iota}C$ then $\iota(A,B) \leq \iota(C,D)$ and hence $(A,B) \leq (C,D)$. Now we can summarize that $\underline{\iota}$ induces an isomorphism from $\mathfrak{B}(G,M,I)$ onto $\mathfrak{B}(\underline{\iota}G,\Omega,\leq)$, i.e. $\underline{\iota}$ is a full $\Omega_{\leq}$-measure of (G,M,I). Finally, $\underline{\overline{\mu}}g = \overline{\mu}(\{g\}'',\{g\}') = \bigvee\mu\{g\}'' = \mu g$ for each full $\Omega_{\leq}$-measure μ of (G,M,I), and $\overline{\underline{\iota}}(A,B) = \bigvee\underline{\iota}A = \bigvee\{\iota(\{g\}'',\{g\}') | g \in A\} = \iota\bigvee\{(\{g\}'',\{g\}') | g \in A\} = \iota(A,B)$ for each $\bigvee$-embedding ι of $\mathfrak{B}(G,M,I)$ into Ω. This finishes the proof of the proposition.

By the preceding proposition, full order scaling can be analysed purely lattice-theoretically. For instance, the smallest dimension of an order scale $\Omega_{\leq}$ which admits a full $\Omega_{\leq}$-measure of (G,M,I) equals the $\bigvee$-*dimension* of $\mathfrak{B}(G,M,I)$ which is defined as the smallest number of complete chains whose direct product admits a $\bigvee$-embedding of the lattice. The smallest length of such an order scale is given by the $\bigvee$-*rank* of $\mathfrak{B}(G,M,I)$ which is defined as the smallest length of a direct product of complete chains which admits a $\bigvee$-embedding of the lattice. The following theorem gives a basic analysis of $\vee$-embeddings into direct products of complete chains in the case of finite lattices (cf. Lea [22], Ritzert [29]).

THEOREM. *Let L be a finite lattice. If $\chi := \{C_t \mid t \in T\}$ is a partition of $M(L)$ into chains then $\hat{\chi}: L \to \mathop{\times}_{t \in T}(C_t \cup \{1_L\})$ defined by $\hat{\chi}a := (a_t)_{t \in T}$ with $a_t := \min\{c \in C_t \cup \{1_L\} \mid a \leq c\}$ is a $\vee$-embedding. If $\iota: L \to \Omega$ is any $\vee$-embedding in a direct product of complete chains then there exists always a partition $\chi := \{C_t \mid t \in T\}$ of $M(L)$ into chains and a $\vee$-homomorphism κ from Ω onto $\mathop{\times}_{t \in T}(C_t \cup \{1_L\})$ such that κ maps $M(\Omega) \cup \{1_\Omega\}$ onto $M(\mathop{\times}_{t \in T}(C_t \cup \{1_L\})) \cup \{1_{\times C_t}\}$ and $\hat{\chi} = \kappa \circ \iota$.*

PROOF. Since every element of L is a meet of $\wedge$-irreducible elements, $\hat{\chi}$ is injective. Obviously, $a_t \vee b_t = (a \vee b)_t$ for all $a,b \in L$ and $t \in T$. Thus, $\hat{\chi}$ is a $\vee$-embedding of L into $\mathop{\times}_{t \in T}(C_t \cup \{1_L\})$.
Let $\Omega := \mathop{\times}_{s \in S} \Omega_s$ and let $\pi_s: \Omega \to \Omega_s$ the canonical projection for $s \in S$. If $\pi_s \iota a = \pi_s \iota b^s$ and $a \leq b^s$ for $a \in M(L)$, $s \in S$, and $b^s \in L$, then $\iota a = \bigwedge_{s \in S} \iota b^s$ and hence $a = \bigwedge_{s \in S} b^s$ wherefore $a = b^s$ for some $s \in S$. Thus, there exists an injective map $\sigma: M(L) \to S$ such that $\pi_{\sigma a} \iota a = \pi_{\sigma a} \iota b$ implies $a \geq b$ for all $a \in M(L)$ and $b \in L$. $M(\Omega)$ contains exactly one element $m_{\sigma a}$ with $\pi_{\sigma a} m_{\sigma a} = \pi_{\sigma a} \iota a$ for each $a \in M(L)$. If $\sigma a = \sigma b$ and $m_{\sigma a} < m_{\sigma b}$ for $a,b \in M(L)$ then $a < b$, because otherwise we would have $b < a \vee b$ but $\pi_{\sigma b} \iota b = \pi_{\sigma b} \iota(a \vee b)$. Now, let $C_{\sigma a}$ be the chain consisting of all elements $b \in M(L)$ such that $m_{\sigma a}$ and $m_{\sigma b}$ are comparable in Ω; furthermore, let $T := \sigma M(L)$. Then $\chi := \{C_t \mid t \in T\}$ is a partition of $M(L)$ into chains. For $x \in \Omega$ we define $\kappa x := (x_t)_{t \in T}$ with $x_t := \min\{b \in C_t \cup \{1_L\} \mid x \leq m_{\sigma b}\}$ where $m_{\sigma 1_L} = 1_\Omega$. Again, $x_t \vee y_t \vee z_t \vee \ldots = (x \vee y \vee z \vee \ldots)_t$ for all $x,y,z,\ldots \in \Omega$ and $t \in T$; hence κ is a $\vee$-homomorphism. Obviously, $\kappa(M(\Omega) \cup \{1_\Omega\}) = M(\mathop{\times}_{t \in T}(C_t \cup \{1_L\})) \cup \{1_{\times C_t}\}$ and κ is surjective. If $\iota a \leq m_{\sigma b}$ for $a \in L$ and $b \in M(L)$ then $a \leq b$, because otherwise we would have $b < a \vee b$ but $\pi_{\sigma b} \iota b = \pi_{\sigma b} \iota(a \vee b)$. Therefore $(\iota a)_t = a_t$ for all $a \in L$ and $t \in T$. Hence $\hat{\chi} = \kappa \circ \iota$.

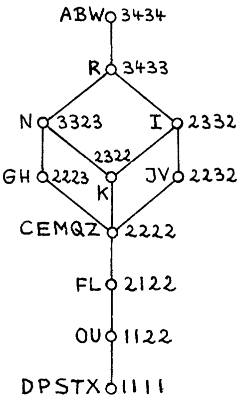
For the first assertion of the following corollary we used, of course, the theorem of R.P. Dilworth on the width of ordered sets [14].

COROLLARY. *For a finite lattice L, the $\vee$-dimension of L is equal to the width of $M(L)$ and the $\vee$-rank of L is equal to the cardinality of $M(L)$.*

COROLLARY. *A finite context (G,M,I) admits a unique full $\Omega_\leq$-measure with $\Omega = \mathop{\times}_{t \in T} \Omega_t$ if and only if there is exactly one* *partition* $\{C_t \mid t \in T\}$ *of* $M(L)$ *into chains such that the length of* Ω_t *is equal to the cardinality of* C_t *for all* $t \in T$*; then, in particular, the dimension and the length of the order scale* $\Omega_\leq$ *will be equal to the width and the cardinality of* $M(\underline{\mathfrak{B}}(G,M,I))$*, respectively.*

To end this section, we apply the results of our lattice-theoretical analysis to a simple example of an ordinal context. The context below is the list of the general grades on conduct (c), diligence (d), attentiveness (a), and orderliness (o) given to a class of the Ludwig-Georgs-Gymnasium (Darmstadt) for the winter term 1980/81 (the best grade is 1).

	c	d	a	o
Anna	3	4	3	4
Berend	3	4	3	4
Christa	2	2	2	2
Dieter	1	1	1	1
Ernst	2	2	2	2
Fritz	2	1	2	2
Gerda	2	2	2	3
Horst	2	2	2	3
Ingolf	2	3	3	2
Jürgen	2	2	3	2
Karl	2	3	2	2
Linda	2	1	2	2
Manfred	2	2	2	2
Norbert	3	3	2	3
Olga	1	1	2	2
Paul	1	1	1	1
Quax	2	2	2	2
Rudolf	3	4	3	3
Stefan	1	1	1	1
Till	1	1	1	1
Uta	1	1	2	2
Volker	2	2	3	2
Walter	3	4	3	4
Xaver	1	1	1	1
Zora	2	2	2	2



The concept lattice has eight $\wedge$-irreducible elements, namely γR, γN, γI, γG, γJ, γF, γO, and γD; its width is two. Thus, the smallest dimension (length) of an order scale $\Omega_\leq$ allowing a full $\Omega_\leq$-measure is two (eight); for instance, one may take as chains $\{\gamma A, \gamma R, \gamma N, \gamma G, \gamma F, \gamma O, \gamma D\}$ and $\{\gamma A, \gamma I, \gamma J\}$. Since the concept lattice has an automorphism interchanging $\wedge$-irreducible elements, the context admits no unique full $\Omega_\leq$-measure of dimension two. Surprisingly, only the order scale built out of eight chains of length one admits a unique full $\Omega_\leq$-measure. In spite of the missing uniqueness, the conclusion seems reasonable that the given data are two-dimensional with even one main dimension. In general, it may be fruitful to explore whether the described order measurement can be used as basis for a factor analysis of ordinal data (cf. Rummel [32]).

8. COMPLETIONS OF PARTIAL CONCEPT LATTICES

Up to now, it was always assumed that the concept lattices are derived from known contexts. But in many situations one has only a vague knowledge of a context although many of its concepts are fairly clear. For instance, it seems to be impossible to write down a comprehensive context of musical instruments, but everyone uses concepts as violin, trumpet, guitar, string instrument, etc. and meaningful sentences such as "a violin is a string instrument". In such cases of concepts of a vague context we have a modified determination problem: *How can one extend a partial lattice of concepts to a complete concept lattice of some context*? The following theorem (see MacNeille [23], Schmidt [34]) describes the smallest solution, usually called the *Dedekind-MacNeille completion.*

THEOREM. *For an ordered set* $(P,\leq)$ *an embedding* ι *of* $(P,\leq)$ *into* $\mathfrak{B}(P,P,\leq)$ *is defined by* $\iota x := ((x],[x))$ $(x \in P)$; *especially,* $\iota\bigwedge X = \bigwedge\iota X$ *and* $\iota\bigvee X = \bigvee\iota X$ *if* $\bigwedge X$ *and* $\bigvee X$ *exist in* $(P,\leq)$, *respectively. If* λ *is any embedding of* $(P,\leq)$ *into a complete lattice* L *then there exists an order embedding* κ *of* $\mathfrak{B}(P,P,\leq)$ *into* L *such that* $\lambda = \kappa\circ\iota$.

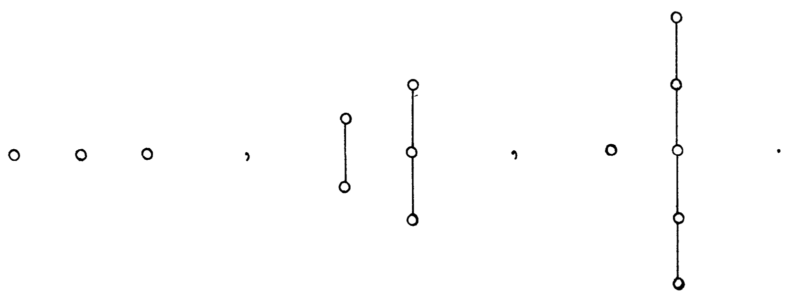
PROOF. By definition, the concepts of $(P,P,\leq)$ are exactly the pairs (A,B) such that $A,B \subseteq P$ and $A' = \{x \in P \mid x \leq y$ for all $y \in B\}$, $B' = \{y \in P \mid x \leq y$ for all $x \in A\}$; in particular, $((x],[x))$ with $x \in P$ are concepts of $(P,P,\leq)$ what confirms ι as an embedding. If $\bigwedge X$ exists in $(P,\leq)$, then $(\bigwedge X] = \bigcap_{x\in X} (x]$ and hence $\iota\bigwedge X =$ $((\bigwedge X],[\bigwedge X)) = (\bigcap_{x\in X} (x],(\bigcap_{x\in X} (x])') = \bigwedge\iota X$ by the theorem of Section 2; the assertion for $\bigvee X$ is proved dually. Now, let λ be an embedding of $(P,\leq)$ into a complete lattice L. We define $\kappa(A,B) := \bigvee\lambda A$ for $(A,B) \in \mathfrak{B}(P,P,\leq)$. Obviously, $\lambda = \kappa\circ\iota$. If $\kappa(A_1,B_1) \leq \kappa(A_2,B_2)$ then $\bigvee\lambda A_1 \leq \bigvee\lambda A_2 \leq \lambda b$ for all $b \in B_2$ and hence $b \in A_1' = B_1$ for all $b \in B_2$ because λ is an embedding. Thus, $B_1 \supseteq B_2$ and therefore $(A_1,B_1) \leq (A_2,B_2)$; in particular, this includes the injectivity of κ. Since κ is order-preserving, κ is the desired order embedding.

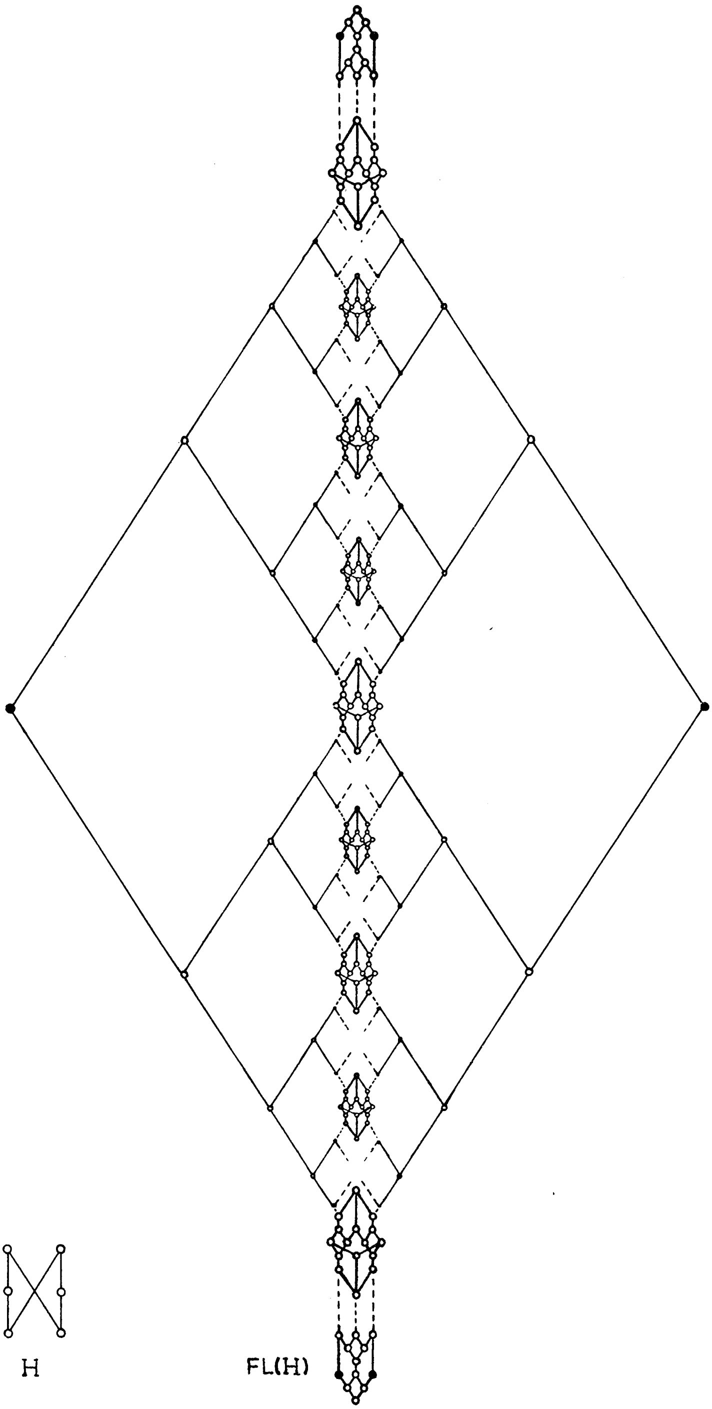
Instead of closing a partial concept lattice to its smallest completion, one is usually more interested to unfold the concepts as far as possible to reach a rich context. Since meet and join are the fundamental operations for concepts which may produce new concepts from known ones, they form a natural tool for the desired extensions. For instance, we may start with some known concepts of musical instruments; then we try to produce new concepts by forming various meets and joins and to separate them by listing more and more objects and attributes. Such a procedure would follow the free generating process in a lattice. Unfortunately, the common set-theoretical language does not allow us to speak of a "free complete lattice"; hence, there is in this sense no largest solution of the modified determination problem available. But if we strictly bound the cardinality of subsets of which meets and joins are taken by some fixed cardinal number α, then there exists a lattice freely α-generated by a given partial concept lattice (see Crawley, Dean [7]); for the final completion one may use the preceding theorem. Now, these large solutions of the modified determination problem give rise to a modified description problem: *How can one describe a lattice freely α-generated by a partial lattice?* The common answer uses lattice terms to describe the lattice elements and gives a procedure how the order of the lattice can be derived from the describing terms (see Whitman [37], [38], Dean [10], Crawley, Dean [7], Jónsson [19], Lakser [21], Grätzer, Lakser, Platt [16]).

Again, descriptions by Hasse diagrams are desirable. But it seems that already a lattice freely ($\aleph_0$-) generated by three unordered elements cannot be described by a readable Hasse diagram. Nevertheless, many lattices freely generated by a partial lattice can be "drawn". So far, only the case of finite ordered sets has been completely analysed (if one agrees that a lattice having a sublattice freely generated by three unordered elements does not have a readable Hasse diagram). Those finite ordered sets whose free completions can be "drawn" are characterized by the theorem below (see Rival, Wille [30]). It is surprising that all these free completions can be embedded into a lattice freely generated by the six-element ordered set H; the Hasse diagrams of H and of its free completion are on the next page. The Hasse diagrams of all these free completions can be obtained by combining some of the ten diagrams given in [30]. In the case where a free completion of a partial lattice cannot be "drawn", it is still interesting to explore how far, following the generating process, the Hasse diagram can be developed.

THEOREM. *For a finite ordered set P the following conditions are equivalent*:

- (*i*) *A lattice freely generated by P does not contain a sublattice freely generated by three unordered elements*;
- (*ii*) *A lattice freely generated by H has a subset isomorphic to P which freely generates a sublattice*;
- (*iii*) *P contains no subset having one of the following Hasse diagrams*:





H FL(H)

9. FURTHER REMARKS

This restructuring of lattice theory does not pretend to be complete in any sense. Many ties of lattice theory to its surroundings have not been mentioned. A comprehensive development of lattice theory should integrate and elaborate much more: for instance, origins as in logic (cf. Rasiowa [28]), connections as in geometry (cf. Birkhoff [5]), interpretations as in computer science (cf. Scott [33]), and applications as in quantum mechanics (cf. Hooker [18]). Especially, its significance to other mathematical disciplines which is still considered to be the main source has to be clarified. Besides the interpretation by hierarchies of concepts, other basic interpretations of lattices should be introduced; an important rôle is already played by the interpretation of lattices as closure systems. Feedback will always be given by the communication with potential users of lattice theory.

Coming back to the initial question: Why develop lattice theory? we may conclude that there is no short answer. The justification of lattice theory arises from its place in the landscape of our culture in general.

ACKNOWLEDGEMENTS. Thoughts and ideas in this paper have been matured to a great extent within the "Arbeitsgemeinschaft Mathematisierung" at the TH Darmstadt in which we have worked on the theme of restructuring mathematics now for more than three years. I would like to thank all members of the Arbeitsgemeinschaft for their stimulation, contributions, and continuous support. I would also like to thank all those who have joined discussions at the several lectures which I gave on the subject during the last two years. I am especially grateful to Ivan Rival for his encouragement and help during the writing of the final manuscript. This work was partially supported by the N.S.E.R.C. of Canada under the International Scientific Exchange Programme.

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