

STATISTICAL SIGNAL PROCESSING



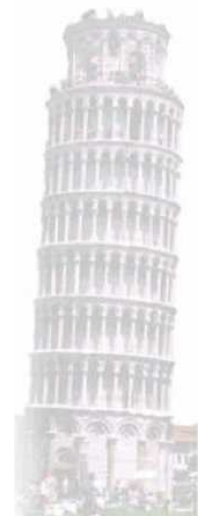
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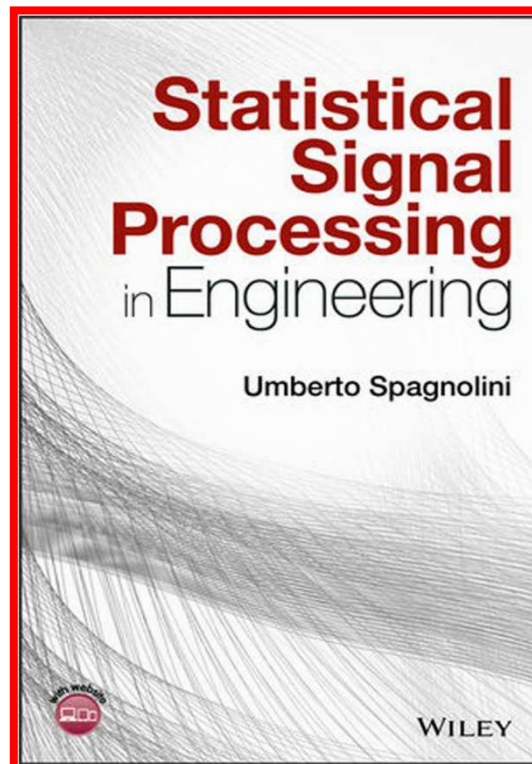
COURSE OUTLINE

- Discrete representation of continuous-time signals
- Introduction to discrete-time random processes
- Estimation of deterministic parameters
- Estimation of random parameters (Bayesian estimation)
- Modeling and identification of discrete-time random processes
- Smoothing, filtering, and prediction of random signals
- Power spectrum estimation (parametric and nonparametric)
- Basic concepts of Time-Frequency Analysis

EXAM OPERATING METHOD

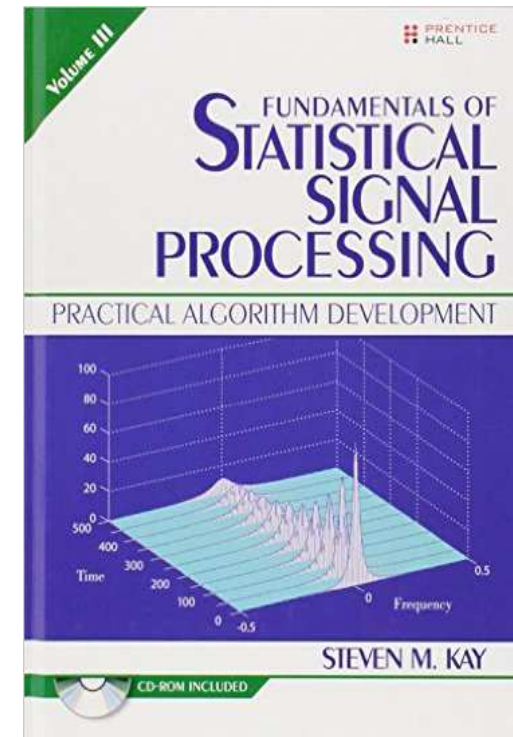
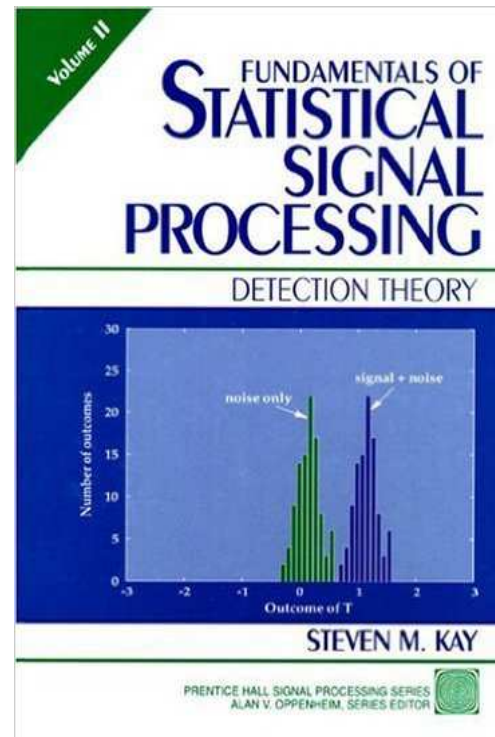
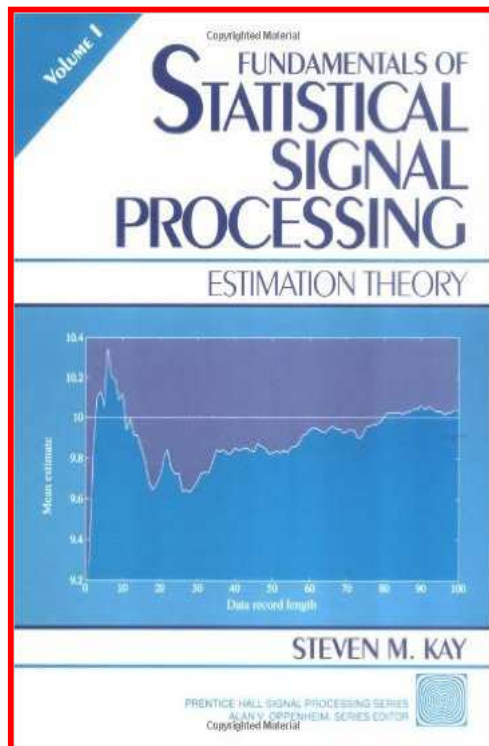
- Written and oral exam
- Exam enrollment: on-line via the website esami.unipi.it

- **COURSE TEXTBOOKS:**
- U. Spagnolini: *Statistical Signal Processing in Engineering*, Wiley. ISBN: 978-1-119-29397-2. First published: 15 December 2017.
- Material provided by the instructor (vugraphs + Matlab code)



■ COURSE TEXTBOOKS:

- **S. Kay: *Fundamentals of Statistical Signal Processing, Volume 1: Estimation Theory*, Prentice Hall, 1993.**

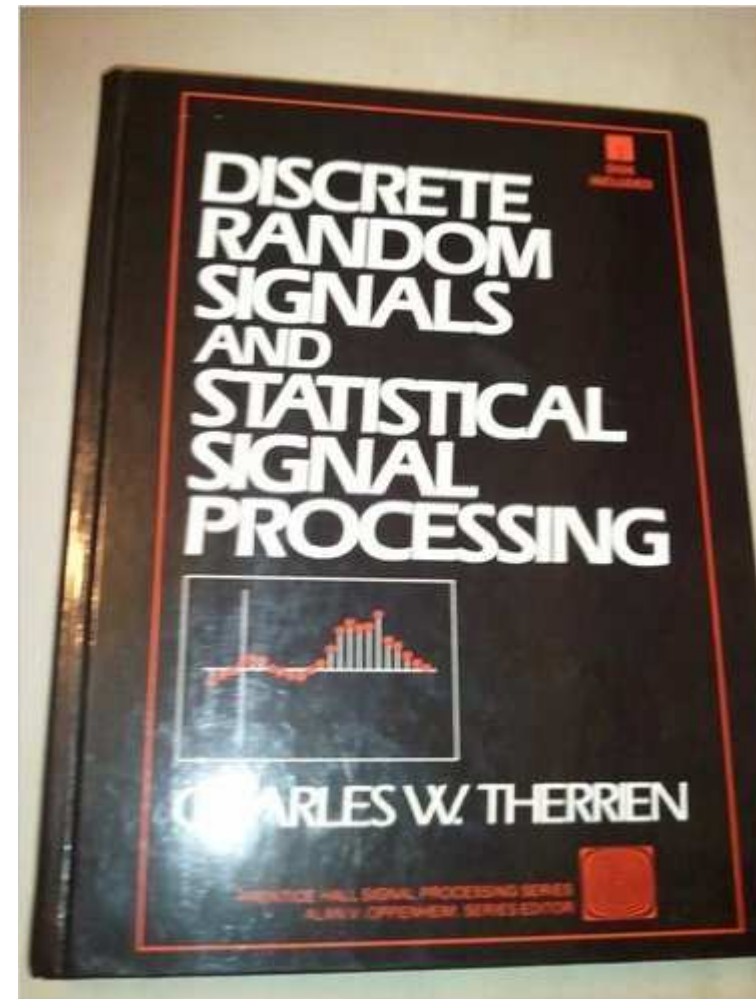


- ANOTHER EXCELLENT BOOK:

- Charles. W. Therrien:

*Discrete Random Signals and
Statistical Signal Processing*

Prentice Hall, 1992.



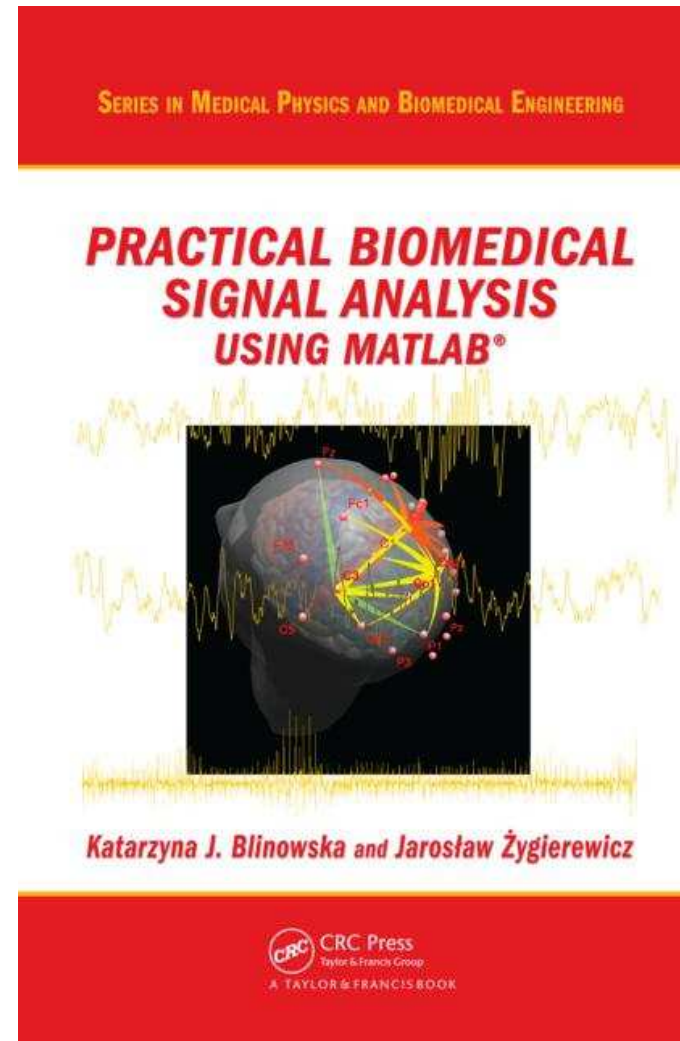
- **FOR FURTHER STUDIES:**

- Katarzyn J. Blinowska, Jaroslaw Zygiereicz

Practical Biomedical Signal Analysis Using MATLAB®

2011, CRC Press

Series: Series in Medical Physics and Biomedical Engineering





Discrete Representation of Continuous-Time Signals

- A **signal** is a physical quantity that can carry information.
- Physical and biological signals may be classified as deterministic or stochastic. The stochastic signal contrary to the deterministic one cannot be described by a mathematical function.
- Examples of **biomedical signals**:
 - electroencephalogram (EEG), electrocorticogram (ECoG), event-related potential (ERP), electrocardiogram (ECG), heart rate variability signal (HRV), electromyograms (EMG), electroenterograms (EEnG), and electrogastrograms (EGG).
 - Magnetic fields connected with the activity of brain (MEG) and heart (MCG).
 - Acoustic signals: phonocardiograms (PCG) and otoacoustic emissions (OAE).



Discrete Representation of Continuous-Time Signals

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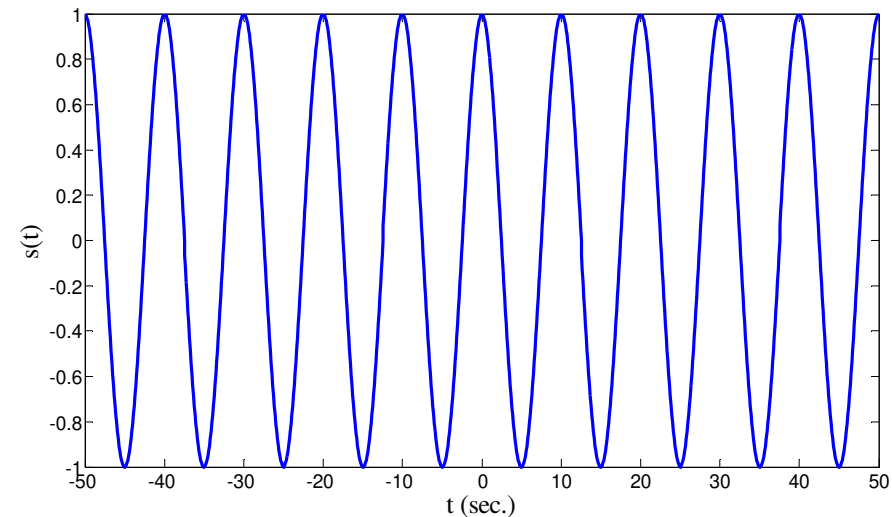
- In nature, most of the signals of interest are some physical values changing in time and/or space.
- These signals are continuous in time and in space.
- On the other hand, we use computers to store and analyze the data.
- To adapt the natural continuous data to the digital computer systems we need to digitize them.
- The most typical way to achieve this is to **sample** the physical values in certain moments in time and places in space and assign them a numeric value with a finite precision.
- **Analog-to-Digital Conversion (ADC)** is made of two processes:
 - **sampling** (selecting discrete moments in time), and
 - **quantization** (assigning a value of finite precision to an amplitude).

■ Example: sinusoidal signal

Graphical representation:

Analytical expression:

$$s(t) = A \cos(2\pi f_0 t + \theta), \quad t \in \mathbb{R}$$



■ **Fourier Analysis:** Discrete representation (in the frequency domain) of a finite-power periodic continuous-time signal is possible by resorting to the Fourier Series (FS):

$$s(t) \stackrel{FS}{\Leftrightarrow} S_k, \quad k \in \mathbb{Z}$$

- Discrete representation of a continuous-time signal, periodic of T_0 , having finite power P_s , is possible by resorting to the **Fourier Series**:

$$s(t) = s(t - T_0) \quad \forall t, \quad P_s \triangleq \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |s(t)|^2 dt < +\infty$$

Synthesis Equation

$$s(t) = \sum_{k=-\infty}^{+\infty} S_k \cdot e^{j2\pi \frac{k}{T_0} t} \quad \stackrel{FS}{\Leftrightarrow}$$

Analysis Equation

$$S_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t) \cdot e^{-j2\pi \frac{k}{T_0} t} dt$$

- The **Fourier Series (FS)** S_k provides a (frequency-domain) discrete representation of the continuous-time signal $s(t)$.

- We can use this discrete representation to calculate some signal parameters of interest (**Bessel-Parseval formula**):

- Power of $s(t)$:
$$P_s \triangleq \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |s(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |S_k|^2$$

- Cross-power between $s(t)$ and $x(t)$:
$$P_{sx} \triangleq \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t)x^*(t)dt = \sum_{k=-\infty}^{+\infty} S_k X_k^*$$

- Angle between $s(t)$ and $x(t)$:

$$\theta \triangleq \text{Angle}(s(t), x(t)) \rightarrow \cos \theta \triangleq \frac{\text{Re}\{P_{sx}\}}{\sqrt{P_s P_x}} \in [-1, 1]$$

- $\cos \theta = 0$ ($P_{sx} = 0$) $\rightarrow s(t)$ and $x(t)$ are **orthogonal**

- $\cos \theta = \pm 1 \rightarrow s(t)$ and $x(t)$ are **colinear**

- If the signal is **bandlimited**, the number of non-zero coefficients is finite.
- As an example:

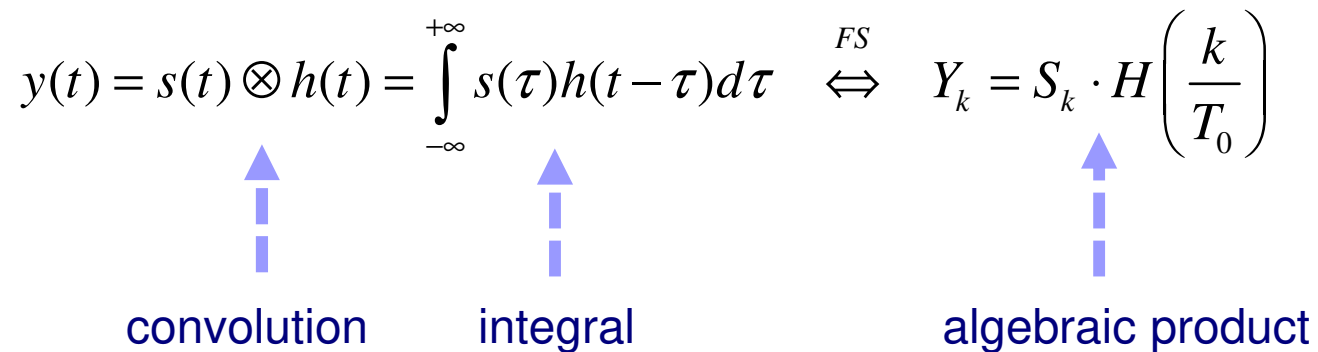
$$s(t) = A \cos(2\pi f_0 t + \theta) \stackrel{FS}{\Leftrightarrow} S_k = \begin{cases} \frac{A}{2} e^{j\theta}, & k = 1 \\ \frac{A}{2} e^{-j\theta}, & k = -1 \\ 0, & \text{otherwise} \end{cases}$$

■ Power of $s(t)$:
$$P_s = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |s(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |S_k|^2 = 2 \left(\frac{A}{2} \right)^2 = \frac{A^2}{2}$$

Bessel-Parseval equality

- The Fourier Series representation is also useful to find the output $y(t)$ of a **linear time-invariant (LTI) system** when the input is a finite-power periodic signal $s(t)$:

$$y(t) = s(t) \otimes h(t) = \int_{-\infty}^{+\infty} s(\tau) h(t - \tau) d\tau \quad \xLeftrightarrow{FS} \quad Y_k = S_k \cdot H\left(\frac{k}{T_0}\right)$$



convolution integral algebraic product

- $h(t)$ = impulse response of the LTI system
- $H(f) = \text{FT}[h(t)]$ frequency response of the LTI system

■ Another example of discrete representation: **bandlimited finite-energy aperiodic signals**.

■ We know from the **Sampling Theorem (Shannon Theorem, 1949)** that signals in this class can be represented in terms of its temporal samples via the following **Whittaker-Shannon interpolation formula**:

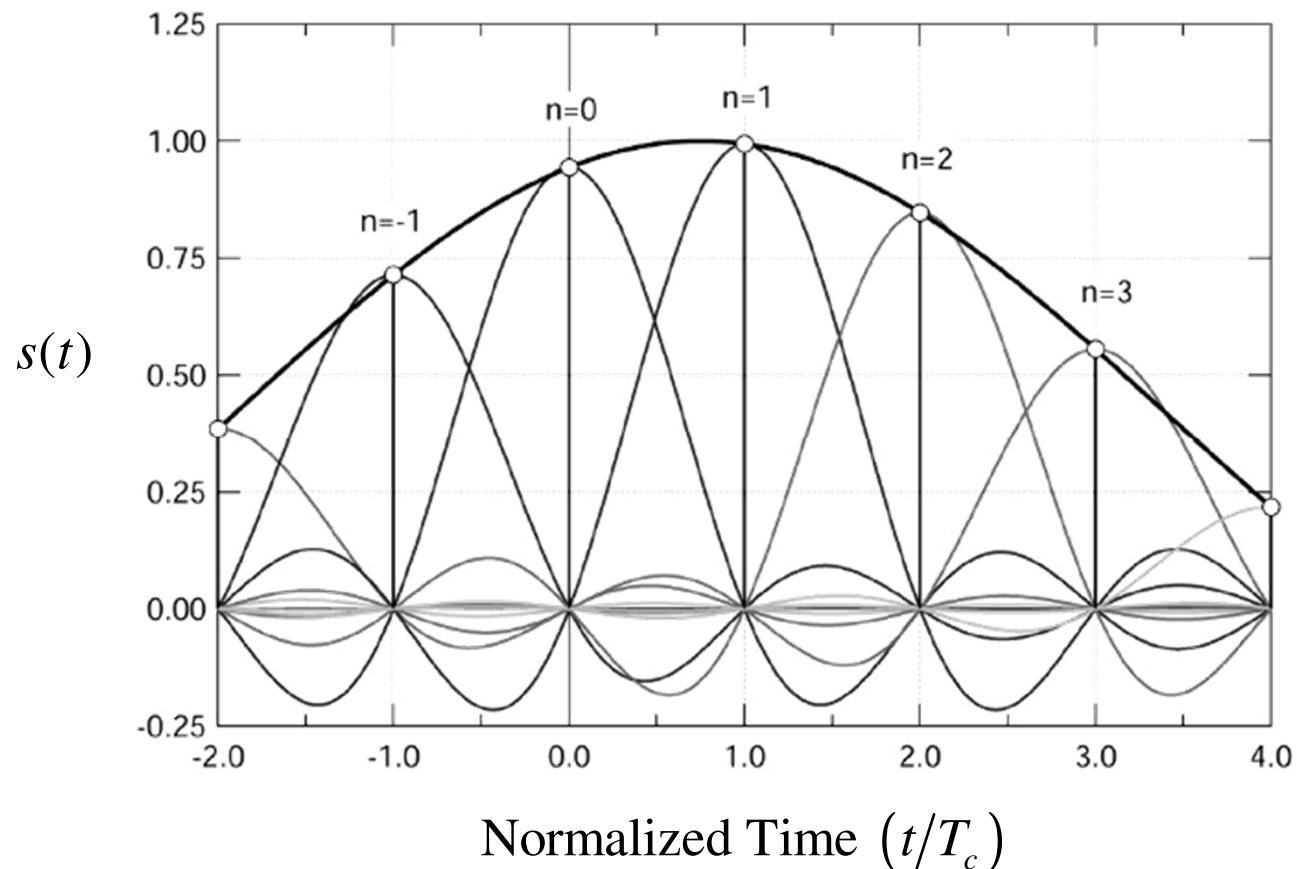
$$s(t) = 2BT_c \sum_{k=-\infty}^{+\infty} s(kT_c) \cdot \text{sinc}(2B(t - kT_c)), \quad \text{where } \text{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$$

■ B is the bandwidth of $s(t)$ and T_c is the sampling interval.

■ Perfect reconstruction of the signal $s(t)$ in terms of its samples is possible if and only if sampling satisfies the **Nyquist condition**: $T_c \leq 1/(2B)$.

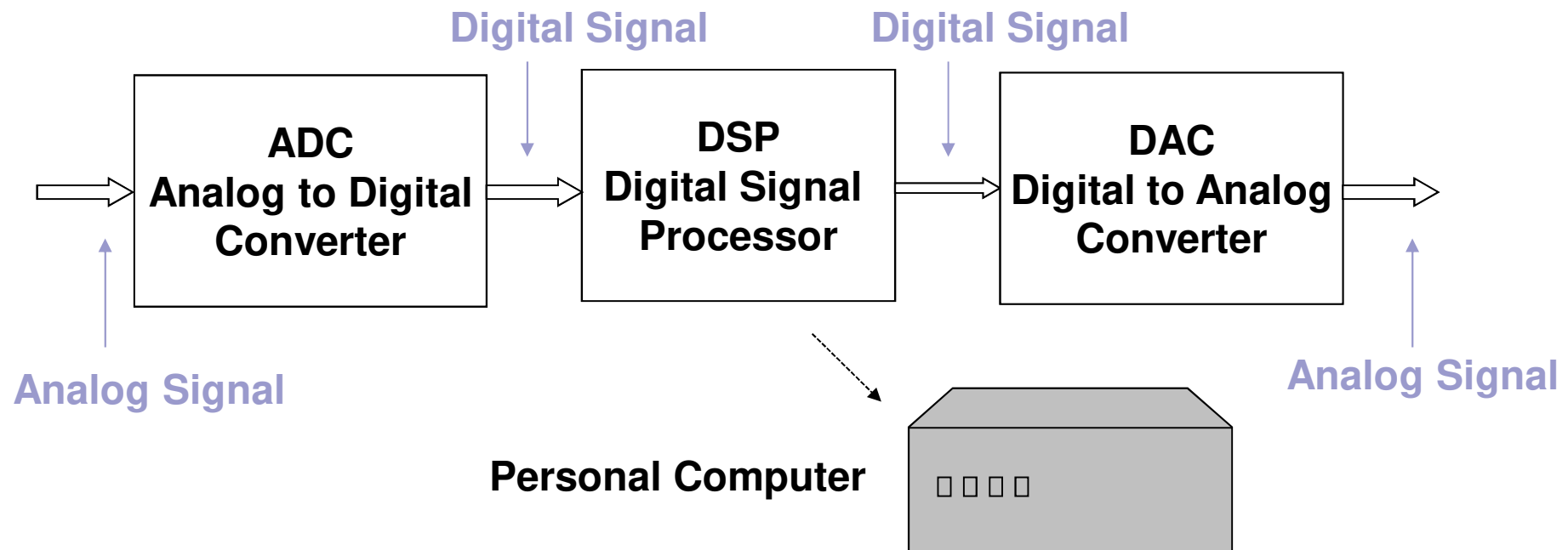
$$\begin{array}{ccc} s(t) & \overset{\text{sampling}}{\Leftrightarrow} & s(kT_c) \\ t \in \mathbb{R} & & k \in \mathbb{Z} \end{array}$$

$$s(t) = 2BT_c \sum_{k=-\infty}^{+\infty} s(kT_c) \cdot \text{sinc}(2B(t - kT_c)), \quad T_c = \frac{1}{2B}$$



■ Discrete representation of continuous-time signals.

■ Discrete representation is useful for storing and processing the signals of interest → processing can be done digitally, by using a dedicated microprocessor, i.e. a **Digital Signal Processor (DSP)**.



- Note that in both cases, the signal of interest is represented as a linear combination of some predefined continuous-time functions (**signal expansion**):

$$s(t) = \sum_{k=-\infty}^{+\infty} s_k \cdot \varphi_k(t)$$

- For finite-power periodic signals:**

$$s_k \triangleq S_k = FS\{s(t)\}, \quad \varphi_k(t) \triangleq e^{j2\pi \frac{k}{T_0} t}, \quad k \in \mathbb{Z}$$

- For bandlimited finite-energy aperiodic signals:**

$$s_k \triangleq \frac{2BT_c}{\sqrt{2B}} s(kT_c), \quad \varphi_k(t) \triangleq \sqrt{2B} \operatorname{sinc}(2B(t - kT_c)), \quad k \in \mathbb{Z}$$

- This approach can be generalized beyond the FS or the Sampling Theorem results: **Generalized Fourier Analysis** or **Basis Expansion**, sometimes called **Karhunen-Loève (KL) Expansion** or **Karhunen-Loève Transform (KLT)**:

$$\begin{array}{ccc} s(t) = \sum_{k=-\infty}^{+\infty} s_k \cdot \varphi_k(t) & \xLeftrightarrow{\text{Basis}} & s_k \\ t \in \mathbb{R} & & k \in \mathbb{Z} \end{array}$$

- The $\{\varphi_k(t)\}$'s are a set of predefined deterministic continuous-time signals called “**basis**” (or **dictionary**). Usually (but not always), they are **orthonormal functions**.
- The discrete representation of the continuous-time signal $s(t)$ is provided by the so-called **image** vector **s** (in general, infinite dimensional), whose components are the discrete values $\{s_k\}$, i.e. the coefficients of the linear combination:

$$s(t) \xLeftrightarrow{\text{Basis}} \mathbf{s} = \left[\cdots \quad s_{-1} \quad s_0 \quad s_1 \quad \cdots \right]^T \quad \text{image vector}$$

■ If the $\{\varphi_k(t)\}$'s are **orthonormal functions**, i.e.

$$\|\varphi_k(t)\|_2 = \sqrt{(\varphi_k(t), \varphi_k(t))} = 1, \quad (\varphi_k(t), \varphi_i(t)) = \delta_{k,i} = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases}$$

→ the elements of the **image** vector **s** can be derived as the **scalar product** (orthogonal projection) between the signal and the base functions:

$$s_k = (s(t), \varphi_k(t)), \quad \forall k$$

■ For example, for **finite energy signals**, the scalar product can be defined as the cross-energy between the two signals and the Euclidean norm induced by the scalar product is the square root of the energy:

$$(s(t), x(t)) \triangleq \int_{-\infty}^{+\infty} s(t)x^*(t)dt = E_{sx}, \quad \|s(t)\|_2 \triangleq \sqrt{(s(t), s(t))} = \sqrt{\int_{-\infty}^{+\infty} |s(t)|^2 dt} = \sqrt{E_s}$$

- Once we have defined the **scalar product** and the **norm** (induced by the scalar product), we can also define a **distance** between two signals:

$$d(s(t), x(t)) \triangleq \|s(t) - x(t)\|_2 = \sqrt{(s(t) - x(t), s(t) - x(t))}$$

- For **finite power periodic signals**, the scalar product can be defined as the cross-power between the two signals and the Euclidean norm induced by the scalar product is the square root of the power:

$$(s(t), x(t)) \triangleq \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t) x^*(t) dt = P_{sx}$$

$$\|s(t)\|_2 \triangleq \sqrt{(s(t), s(t))} = \sqrt{\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |s(t)|^2 dt} = \sqrt{P_s}$$

$$d(s(t), x(t)) \triangleq \|s(t) - x(t)\|_2 = \sqrt{(s(t) - x(t), s(t) - x(t))}$$

■ **Proof** of the result: $s_k = (s(t), \varphi_k(t))$, $\forall k$

■ **Assumption:** the $\{\varphi_k(t)\}$'s are **orthonormal** functions: $(\varphi_i(t), \varphi_k(t)) = \delta_{i,k}$

$$\begin{aligned}(s(t), \varphi_k(t)) &= \left(\sum_{i=-\infty}^{+\infty} s_i \cdot \varphi_i(t), \varphi_k(t) \right) \\ &= \sum_{i=-\infty}^{+\infty} s_i \cdot (\varphi_i(t), \varphi_k(t)) \\ &= \sum_{i=-\infty}^{+\infty} s_i \cdot \delta_{i,k} \\ &= s_k\end{aligned}$$

■ This proof makes use of the known property of the scalar product that:

$$(a \cdot s(t) + b \cdot q(t), \varphi_k(t)) = a \cdot (s(t), \varphi_k(t)) + b \cdot (q(t), \varphi_k(t))$$

■ **Important:** if the signals are expanded on an orthonormal basis, the **scalar product** between two signals and their **norms** can be obtained by processing the image vectors as well.

■ *Proof:*

$$s(t) = \sum_{k=-\infty}^{+\infty} s_k \cdot \varphi_k(t), \quad x(t) = \sum_{k=-\infty}^{+\infty} x_k \cdot \varphi_k(t)$$

$$\begin{aligned} (s(t), x(t)) &= \left(\sum_{i=-\infty}^{+\infty} s_i \cdot \varphi_i(t), \sum_{k=-\infty}^{+\infty} x_k \cdot \varphi_k(t) \right) \\ &= \sum_{i=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} s_i x_k^* \cdot (\varphi_i(t), \varphi_k(t)) \\ &= \sum_{i=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} s_i x_k^* \delta_{i,k} \\ &= \sum_{i=-\infty}^{+\infty} s_i x_i^* = \mathbf{x}^H \mathbf{s} \equiv \mathbf{s} \cdot \mathbf{x} \equiv (\mathbf{s}, \mathbf{x}) \end{aligned}$$

- Hence, whatever is the definition of scalar product we adopt, we have:

$$(s(t), x(t)) = (\mathbf{s}, \mathbf{x})$$

- If we assume $x(t)=s(t)$, we also get:

$$\|s(t)\|_2^2 = (s(t), s(t)) = (\mathbf{s}, \mathbf{s}) = \mathbf{s}^H \mathbf{s} = \|\mathbf{s}\|_2^2$$

- **In summary:** if the signals are expanded on an orthonormal basis, the scalar product between two signals and their norms can be obtained by processing the image vectors instead of processing the continuous-time signals.

- The Bessel-Parseval equality for the Fourier series is a particular case of this result:

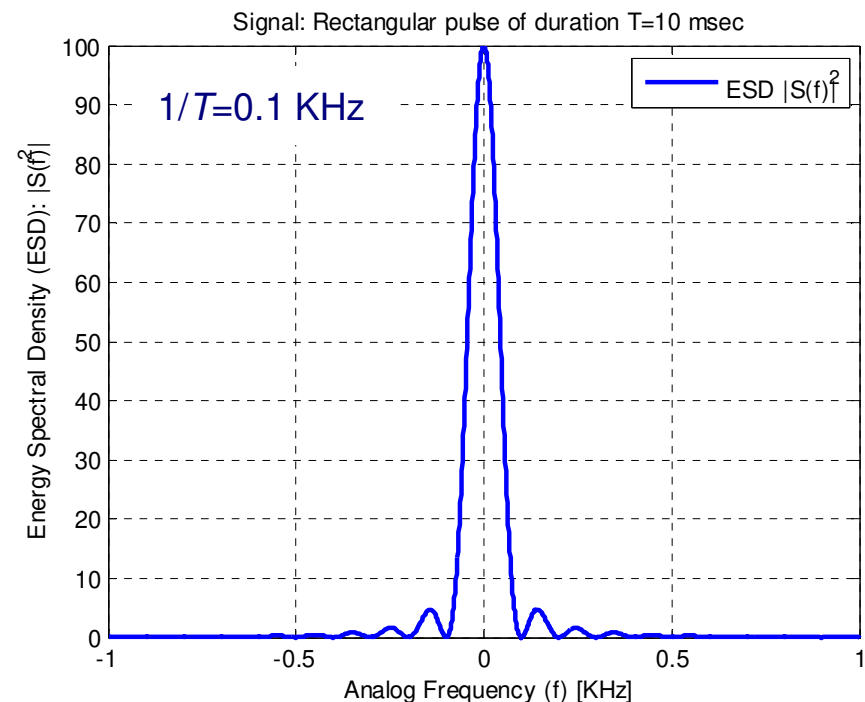
$$\|s(t)\|_2^2 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |s(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |S_k|^2 = \|\mathbf{s}\|_2^2$$

- In all practical applications, we observe the signal on a limited time interval.
- If we assume as $t=0$ the initial observation time and we denote by T the length of the observation interval, we have that the signal is of finite duration T and we are interested on the discrete representation of $s(t)$ for $t \in [0, T)$.
- Since the signal has finite duration T , its energy is finite, but the bandwidth is theoretically infinite, e.g.

$$s(t) = A \cdot \text{rect}\left(\frac{t - T/2}{T}\right)$$
$$\Updownarrow$$
$$S(f) = AT \cdot \text{sinc}(fT) e^{-j\pi fT}$$

- Energy Spectral Density (ESD):

$$ESD(f) = |S(f)|^2 = A^2 T^2 \cdot \text{sinc}^2(fT)$$



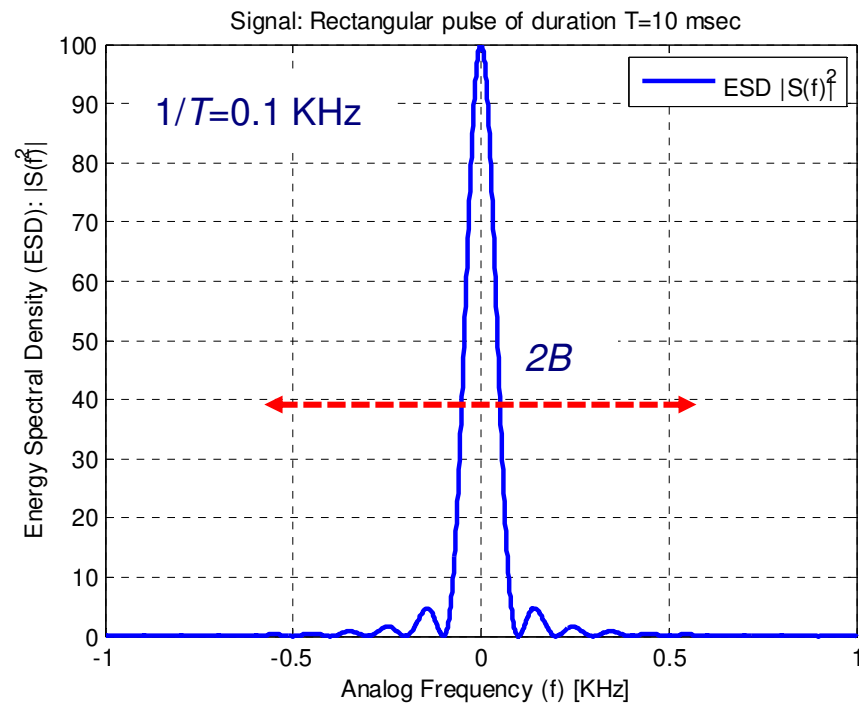
■ Energy of the signal:

$$E_s = \int_0^T |s(t)|^2 dt = \int_{-\infty}^{+\infty} |S(f)|^2 df$$

■ However, even if $s(t)$ is theoretically infinite-bandwidth, most of the energy is contained within a limited bandwidth B :

$$E_B = \int_{-B}^B |S(f)|^2 df = 2 \int_0^B |S(f)|^2 df$$

■ If B is large enough: $E_B \cong E_s$



■ To represent in discrete form a continuous-time signal observed in $[0, T)$ we can apply the following result: It can be proved that if we choose B large enough, such that $N=T/T_c=2BT \gg 1$, then:

$$\text{for } t \in [0, T) \rightarrow s(t) \cong \sum_{k=0}^{N-1} s\left(\frac{k}{2B}\right) \cdot \text{sinc}\left(2B\left(t - \frac{k}{2B}\right)\right) = \sum_{k=0}^{N-1} s_k \cdot \varphi_k(t)$$

$$\text{where: } \varphi_k(t) \triangleq \sqrt{2B} \text{sinc}\left(2B\left(t - \frac{k}{2B}\right)\right), \quad s_k \triangleq \frac{1}{\sqrt{2B}} s\left(\frac{k}{2B}\right), \quad 0 \leq k \leq N-1$$

$$\text{Basis: } \Psi \equiv \left\{ \varphi_k(t) = \sqrt{2B} \text{sinc}\left(2B\left(t - \frac{k}{2B}\right)\right) \right\}_{k=0}^{N-1}, \quad (\varphi_k(t), \varphi_i(t)) = \delta_{k,i}$$

$$s(t) \stackrel{\Psi}{\Leftrightarrow} \mathbf{s} = \begin{bmatrix} s_0 & s_1 & \cdots & s_{N-2} & s_{N-1} \end{bmatrix}^T$$

- The *approximation* should be interpreted as follows → if we define the “error” signal as:

$$s_{\Delta}(t) \triangleq s(t) - \hat{s}(t) = s(t) - \sum_{k=0}^{N-1} s\left(\frac{k}{2B}\right) \cdot \text{sinc}\left(2B\left(t - \frac{k}{2B}\right)\right)$$

$$E_s \triangleq \int_0^T |s(t)|^2 dt \quad [\text{Energy of } s(t)]$$

$$E_{\Delta} \triangleq \int_{-\infty}^{+\infty} |s_{\Delta}(t)|^2 dt \quad [\text{Energy of } s_{\Delta}(t)]$$

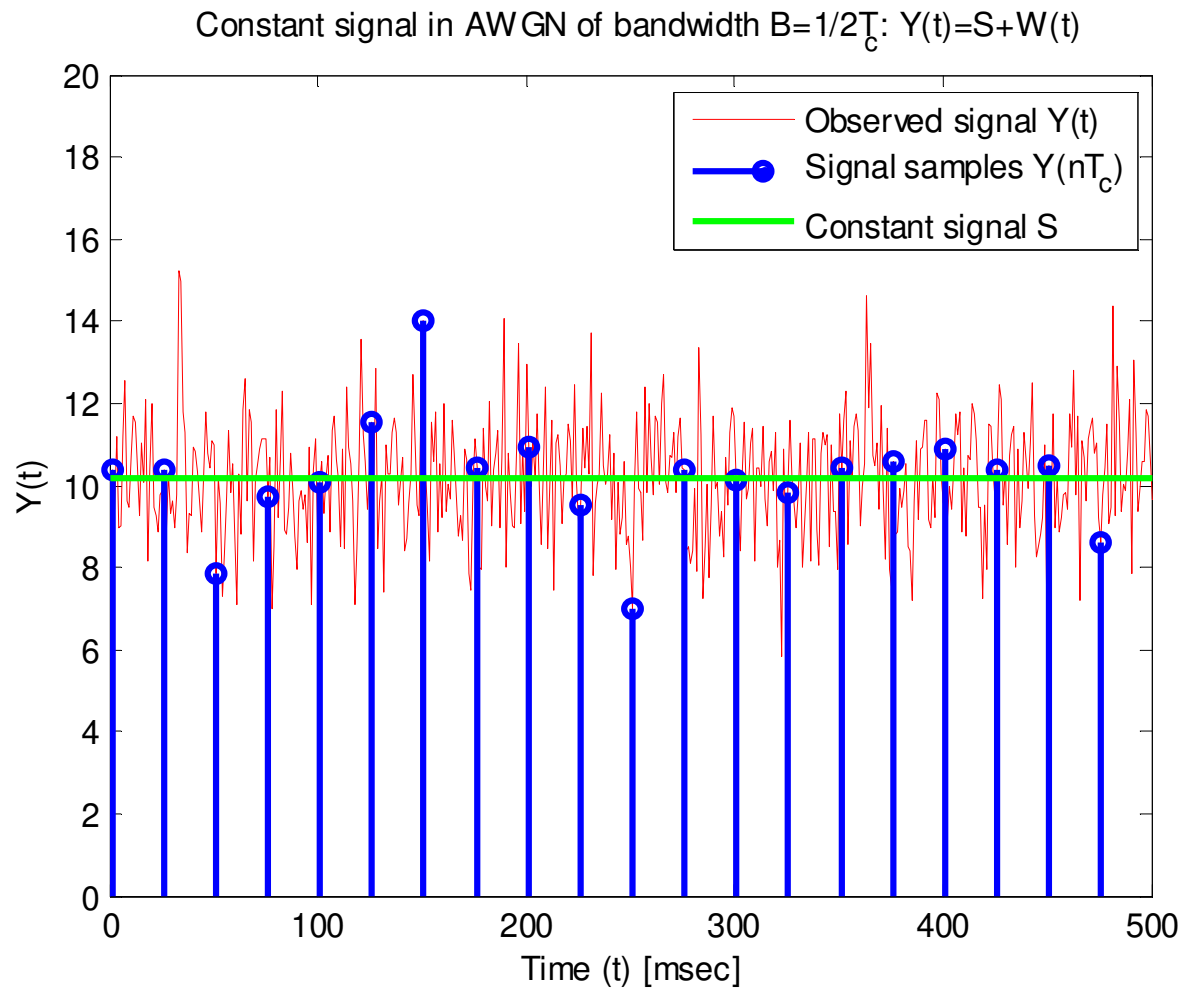
If B is large enough, such that $N = \frac{T}{T_c} = 2BT \gg 1$

then $E_{\Delta} \leq \alpha E_s$ with $\alpha \ll 1$

- In this case, we say that $X(t)$ has *essential* bandwidth B at level α and B is the essential bandwidth at level α of the signal $X(t)$.

Discrete Representation of Continuous-Time Signals

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$$T = 500 \text{ msec}$$

$$T_c = \frac{1}{2B} = 25 \text{ msec}$$

$$N = 2BT = 20$$



The Discrete Karhunen-Loève Transform (DKLT)

- The tendency nowadays is to sample the signal as soon as possible in the processing chain and perform all the processing digitally.
- If T is the observation interval and B the bandwidth of the signal (i.e of the anti-aliasing filter that is at the input of the Analog-to-Digital Converter (ADC)), we have seen that if the sampling interval is such that $T_c \leq 1/(2B)$ and $N = T/T_c \gg 1$, then we have:

$$X(t) \cong 2BT_c \sum_{k=0}^{N-1} X(kT_c) \cdot \text{sinc}(2B(t - kT_c)), \quad t \in [0, T)$$

$$\Psi \equiv \left\{ \varphi_k(t) = \sqrt{2B} \text{sinc}(2B(t - kT_c)) \right\}_{k=0}^{N-1}, \quad N = \frac{T}{T_c}$$

$$X(t) \stackrel{\Psi}{\Leftrightarrow} \mathbf{X} = \frac{2BT_c}{\sqrt{2B}} \left[X(0) \ X(T_c) \cdots X((N-1)T_c) \right]^T$$

- If we select the **Nyquist sampling interval** $T_c = 1/(2B)$, we have:

$$\mathbf{\Psi} \equiv \left\{ \varphi_k(t) = \sqrt{2B} \operatorname{sinc} \left(2B \left(t - \frac{k}{2B} \right) \right) \right\}_{k=0}^{N-1}, \quad N = \frac{T}{T_c} = 2BT$$


- The finite-energy (because of the finite duration N) discrete-time signal we obtain by sampling is:

$$\mathbf{X} = [X[0] \ X[1] \ \cdots \ X[N-1]]^T, \text{ where } X[n] \triangleq \frac{1}{\sqrt{2B}} X \left(\frac{n}{2B} \right)$$

- $N=2BT \gg 1$ is the dimension of the image vector \mathbf{X} and we also say that N is the dimension of the continuous-time signal $X(t)$.

- In many cases, it can be useful to expand the discrete-time signal obtained by sampling into a set of deterministic discrete-time orthonormal basis functions as follows:

$$X[n] = \sum_{i=0}^{K-1} \alpha_i \varphi_i[n], \quad n = 0, 1, \dots, N-1; \quad K \leq N$$

- The basis is orthonormal if : $(\varphi_i[n], \varphi_k[n]) = \delta_{i,k} = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases}$
 inner product between two basis functions

- Note that if $K < N$ the expansion represents a **data compression**:

$$\begin{array}{ccc} \mathbf{X} & \overset{\Psi}{\Leftrightarrow} & \mathbf{\alpha} \\ N \times 1 & & K \times 1 \end{array}$$

- In the case of finite duration N , i.e. finite-energy, discrete-time signals, the **inner product** and the induced **Euclidean norm** can be defined as follows:

$$(s[n], x[n]) \triangleq \sum_{n=0}^{N-1} s[n]x^*[n] = E_{sx} \quad (\text{cross-energy})$$

$$\|s[n]\|_2 \triangleq \sqrt{(s[n], s[n])} = \sqrt{\sum_{n=0}^{N-1} |s[n]|^2} = \sqrt{E_s} \quad (\sqrt{\text{energy}})$$

- The coefficients in the expansion are derived as the inner product between $X[n]$ and the basis functions:

$$\alpha_i = (X[n], \varphi_i[n]) = \sum_{n=0}^{N-1} X[n]\varphi_i^*[n] = \boldsymbol{\Phi}_i^H \mathbf{X}, \quad i = 0, 1, \dots, K-1.$$

■ **Proof** of the result: $\alpha_i = (X[n], \varphi_i[n]) \quad \forall k$

■ **Assumption:** the $\{\varphi_k[n]\}$'s are **orthonormal** functions: $(\varphi_k[n], \varphi_i[n]) = \delta_{k,i}$

$$\begin{aligned} (X[n], \varphi_i[n]) &= \sum_{n=0}^{N-1} X[n] \varphi_i^*[n] \\ &= \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \alpha_k \varphi_k[n] \varphi_i^*[n] \\ &= \sum_{k=0}^{K-1} \alpha_k \sum_{n=0}^{N-1} \varphi_k[n] \varphi_i^*[n] \\ &= \sum_{k=0}^{K-1} \alpha_k \cdot (\varphi_k(t), \varphi_i(t)) \\ &= \sum_{k=0}^{K-1} \alpha_k \cdot \delta_{k,i} \\ &= \alpha_i \end{aligned}$$

■ **Problem:** Which is the choice of the basis that makes the coefficients α_i uncorrelated? Remember that if $X(t)$ is a random process the coefficients are r.v.'s.

$$\mathbf{X} = [X[0] \ X[1] \ \cdots \ X[N-1]]^T$$

$N \times 1$

$$\mathbf{a} = [\alpha_0 \ \alpha_1 \ \cdots \ \alpha_{K-1}]^T$$

$K \times 1$

$$\boldsymbol{\varphi}_i = [\varphi_i[0] \ \varphi_i[1] \ \cdots \ \varphi_i[N-1]]^T$$

$N \times 1$

$$\rightarrow (\varphi_k[n], \varphi_i[n]) = (\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_i) = \boldsymbol{\varphi}_i^H \boldsymbol{\varphi}_k = \delta_{i,k}$$

$$\boldsymbol{\Phi} = [\boldsymbol{\varphi}_0 \ \boldsymbol{\varphi}_1 \ \cdots \ \boldsymbol{\varphi}_{K-1}], \quad \text{if } K = N \rightarrow \text{unitary matrix} \quad \boldsymbol{\Phi}^{-1} = \boldsymbol{\Phi}^H$$

$N \times K$

- Which is the choice of the basis that makes the coefficients α_i uncorrelated?
- Assume $K=N$ and, without loss of generality, also that $E\{X[n]\}=0 \rightarrow E\{\alpha_i\}=0$.

$$X[n] = \sum_{i=0}^{N-1} \alpha_i \varphi_i[n], \quad n = 0, 1, \dots, N-1$$

$$\mathbf{X} = \sum_{i=0}^{N-1} \alpha_i \boldsymbol{\varphi}_i = \boldsymbol{\Phi} \boldsymbol{\alpha} \quad \rightarrow \quad \boldsymbol{\Phi}^H \mathbf{X} = \boldsymbol{\Phi}^H \boldsymbol{\Phi} \boldsymbol{\alpha} = \boldsymbol{\alpha}$$

$$\boldsymbol{\Phi}: \quad E\{\boldsymbol{\alpha} \boldsymbol{\alpha}^H\} = \begin{bmatrix} \sigma_0^2 & 0 & \dots & 0 \\ 0 & \sigma_1^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{N-1}^2 \end{bmatrix} = \boldsymbol{\Lambda}, \quad \text{where } \sigma_i^2 \triangleq E\{|\alpha_i|^2\}$$

and $E\{\alpha_i \alpha_k^*\} = 0$ if $i \neq k$

$$\Phi: E\{\alpha\alpha^H\} = E\{\Phi^H \mathbf{X}\mathbf{X}^H \Phi\} = \Phi^H E\{\mathbf{X}\mathbf{X}^H\} \Phi = \Phi^H \mathbf{R}_X \Phi = \Lambda$$


$$\Lambda = \Phi^H \mathbf{R}_X \Phi \Rightarrow \Phi \Lambda \Phi^H = \Phi \Phi^H \mathbf{R}_X \Phi \Phi^H \Rightarrow \mathbf{R}_X = \Phi \Lambda \Phi^H$$

■ Matrix Φ diagonalizes the correlation matrix $\mathbf{R}_X \rightarrow$ the basis that makes the coefficients uncorrelated is obtained from the **Eigenvector-Eigenvalue Decomposition (EVD)** of the correlation matrix \mathbf{R}_X of the random vector $\mathbf{X} \rightarrow \Phi$ is formed by the eigenvectors of \mathbf{R}_X .

■ The expansion on this basis is called **Karhunen-Loève expansion** and the vector of coefficients α is called **discrete Karhunen-Loève transform (DKLT)**.

- Definition of the correlation matrix \mathbf{R}_X of a random vector \mathbf{X} :

$$\mathbf{R}_X \triangleq E\{\mathbf{X}\mathbf{X}^H\} = E\left\{\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \begin{bmatrix} X_1^* & X_2^* & \cdots & X_N^* \end{bmatrix}\right\}$$

$$= E\left\{\begin{bmatrix} |X_1|^2 & X_1 X_2^* & \cdots & X_1 X_N^* \\ X_2 X_1^* & |X_2|^2 & \cdots & X_2 X_N^* \\ \vdots & \vdots & \ddots & \vdots \\ X_N X_1^* & X_N X_2^* & \cdots & |X_N|^2 \end{bmatrix}\right\}$$

$$\mathbf{R}_X = \begin{bmatrix} E\{|X_1|^2\} & E\{X_1 X_2^*\} & \cdots & E\{X_1 X_N^*\} \\ E\{X_2 X_1^*\} & E\{|X_2|^2\} & \cdots & E\{X_2 X_N^*\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{X_N X_1^*\} & E\{X_N X_2^*\} & \cdots & E\{|X_N|^2\} \end{bmatrix}$$

$$= \begin{bmatrix} E\{|X_1|^2\} & E\{X_1 X_2^*\} & \cdots & E\{X_1 X_N^*\} \\ \left(E\{X_1 X_2^*\}\right)^* & E\{|X_2|^2\} & \cdots & E\{X_2 X_N^*\} \\ \vdots & \vdots & \ddots & \vdots \\ \left(E\{X_1 X_N^*\}\right)^* & \left(E\{X_2 X_N^*\}\right)^* & \cdots & E\{|X_N|^2\} \end{bmatrix} = \mathbf{R}_X^H$$

- If the data vector \mathbf{X} is obtained by sampling a continuous-time wide-sense stationary (w.s.s.) process $X(t)$:

$$X_{i+1} \triangleq [\mathbf{X}]_{i+1} = X[i] = \frac{1}{\sqrt{2B}} X\left(\frac{i}{2B}\right), \quad i = 0, 1, 2, \dots, N-1$$

$$[\mathbf{R}_X]_{i+1, j+1} = E\{X_{i+1} X_{j+1}^*\} = E\{X[i] X^*[j]\} = R_X[j-i]$$

■ ACF of the w.s.s. discrete-time process $X[n]$

$$= \frac{1}{2B} E\left\{X\left(\frac{i}{2B}\right) X^*\left(\frac{j}{2B}\right)\right\} = \frac{1}{2B} R_X\left(\frac{j-i}{2B}\right)$$

$$i, j = 0, 1, 2, \dots, N-1$$

■ ACF of the w.s.s. continuous-time process $X(t)$

■ Hence, if the data vector \mathbf{X} is obtained by sampling a continuous-time w.s.s. process $X(t)$ with Autocorrelation Function (ACF) $R_X(\tau) = E\{X(t)X^*(t+\tau)\}$, in addition to be Hermitian, the correlation matrix of \mathbf{X} has a Toeplitz structure:

$$\mathbf{R}_X = \frac{1}{2B} \begin{bmatrix} R_X(0) & R_X\left(\frac{1}{2B}\right) & R_X\left(\frac{2}{2B}\right) & \cdots & R_X\left(\frac{N-2}{2B}\right) & R_X\left(\frac{N-1}{2B}\right) \\ R_X^*\left(\frac{1}{2B}\right) & R_X(0) & R_X\left(\frac{1}{2B}\right) & \cdots & R_X\left(\frac{N-3}{2B}\right) & R_X\left(\frac{N-2}{2B}\right) \\ R_X^*\left(\frac{2}{2B}\right) & R_X^*\left(\frac{1}{2B}\right) & R_X(0) & \cdots & R_X\left(\frac{N-4}{2B}\right) & R_X\left(\frac{N-3}{2B}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R_X^*\left(\frac{N-2}{2B}\right) & R_X^*\left(\frac{N-3}{2B}\right) & R_X^*\left(\frac{N-4}{2B}\right) & \cdots & R_X(0) & R_X\left(\frac{1}{2B}\right) \\ R_X^*\left(\frac{N-1}{2B}\right) & R_X^*\left(\frac{N-2}{2B}\right) & R_X^*\left(\frac{N-3}{2B}\right) & \cdots & R_X^*\left(\frac{1}{2B}\right) & R_X(0) \end{bmatrix}$$

- The **discrete Karhunen-Loève transform (DKLT)** is a **linear whitening (decorrelating) transformation** of the original data vector \mathbf{X} :

$$\underset{N \times 1}{\mathbf{X}} \xleftrightarrow{DKLT} \underset{N \times 1}{\boldsymbol{\alpha}}$$

$$\boldsymbol{\alpha} = \boldsymbol{\Phi}^H \mathbf{X}$$

$$\boldsymbol{\Phi} : E\{\boldsymbol{\alpha}\boldsymbol{\alpha}^H\} = \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{N-1}^2 \end{bmatrix}, \quad \text{where } \sigma_i^2 \triangleq E\{|\alpha_i|^2\}$$

- In many receivers, the first operation that is performed is a **linear whitening transformation** of the data vector \mathbf{X} .

- The **DKLT** of the original data vector \mathbf{X} requires knowledge of the correlation matrix \mathbf{R}_X of the random vector \mathbf{X} , i.e. of the ACF of the w.s.s. process $X(t)$.
- Φ is obtained from the **Eigenvector-Eigenvalue Decomposition (EVD)** of the correlation matrix $\mathbf{R}_X \rightarrow \Phi$ is formed by the eigenvectors of \mathbf{R}_X :

$$\mathbf{R}_X = \Phi \Lambda \Phi^H$$

- In many applications, we do not know a priori the correlation matrix \mathbf{R}_X , so we have to estimate it from the data:

$$\hat{\mathbf{R}}_X \Rightarrow (\hat{\Phi}, \hat{\Lambda}) \text{ from the EVD of } \hat{\mathbf{R}}_X : \hat{\mathbf{R}}_X = \hat{\Phi} \hat{\Lambda} \hat{\Phi}^H$$

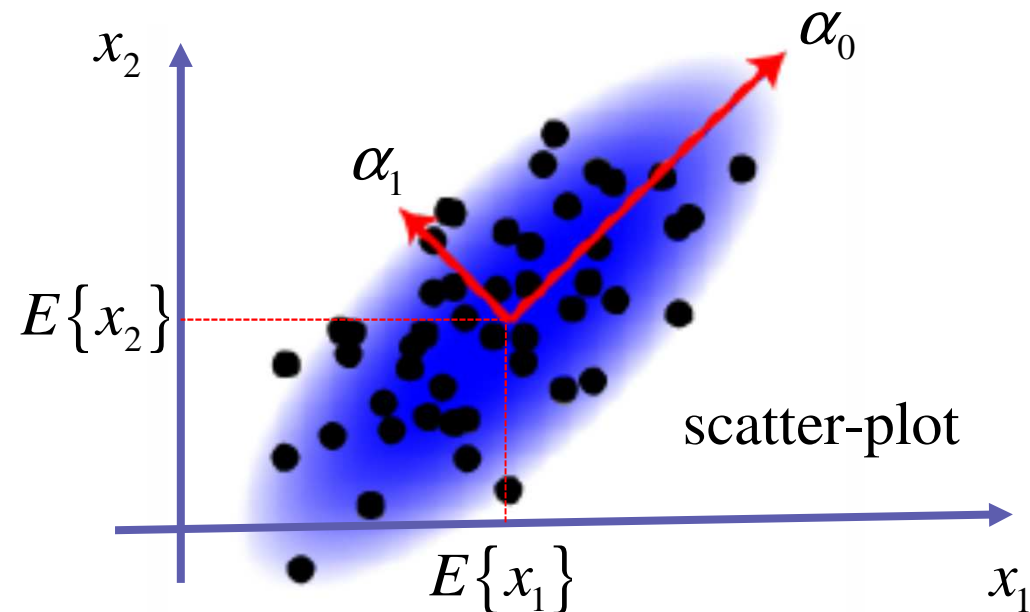
- We will see later on how to estimate the correlation matrix of an N -dimensional random vector \mathbf{X} given a number $M > N$ of independent realizations of \mathbf{X} .

- Φ is a unitary matrix and $\alpha = \Phi^H \mathbf{X}$ is a linear transformation of the data.
- It is basically a rotation of the reference system in such a way that the components in the new reference system are uncorrelated:
- If \mathbf{X} is non-zero mean, we can remove the mean, and Φ is obtained by the EVD of the covariance matrix of vector \mathbf{X} .

$$\alpha = \Phi^H (\mathbf{X} - \mathbf{m}_X)$$

where $\mathbf{m}_X \triangleq E\{\mathbf{X}\}$,

$$\Phi: \quad \mathbf{C}_X \triangleq E\{(\mathbf{X} - \mathbf{m}_X)(\mathbf{X} - \mathbf{m}_X)^H\} = \Phi \Lambda \Phi^H$$



■ **Important:** It is well-known that if the correlation time L of the discrete-time sequence $X[n]$ is much lower than the vector size N , then the matrix Φ that diagonalizes the correlation matrix \mathbf{R}_X is (approximately) the DFT matrix and the DKLT coincides with the (unitary) **Discrete Fourier Transform (DFT)**:

$$L \ll N \rightarrow \varphi_i[n] \cong \frac{1}{\sqrt{N}} e^{j2\pi \frac{i}{N}n}; \quad i, n = 0, 1, \dots, N-1$$

$$\mathbf{X} = \Phi \boldsymbol{\alpha} \cong IDFT\{\boldsymbol{\alpha}\} \Leftrightarrow \boldsymbol{\alpha} = \Phi^H \mathbf{X} \cong DFT\{\mathbf{X}\}$$

$$X[n] \cong \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \alpha_i e^{j2\pi \frac{i}{N}n}$$

$$\alpha_i = (X[n], \varphi_i[n]) \cong \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X[n] e^{-j2\pi \frac{i}{N}n}$$

$$\Phi = \frac{1}{\sqrt{N}} \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{j\frac{2\pi}{N}} & \dots & e^{j\frac{2\pi}{N}(N-1)} \\ 1 & e^{j\frac{2\pi}{N}2} & \dots & e^{j\frac{2\pi}{N}2(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & e^{j\frac{2\pi}{N}(N-1)} & \dots & e^{j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}}_{DFT \text{ matrix } N \times N}$$

$$\Phi^{-1} = \Phi^H$$

Unitary matrix

■ Moreover, if $N \gg L$, the eigenvalues of \mathbf{R}_x are the samples of the Power Spectral Density (PSD):

$$\sigma_i^2 = E\{|\alpha_i|^2\} \stackrel{N \gg L}{\cong} S_X \left(e^{j2\pi \frac{i}{N}} \right) = DFT\{R_X[m]\}, \quad i = 0, 1, 2, \dots, N-1$$

■ *Proof.*

$$\begin{aligned}
 \sigma_i^2 &= E\left\{|\alpha_i|^2\right\} = E\left\{\alpha_i \alpha_i^*\right\} \\
 &= E\left\{\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X[n] e^{-j2\pi \frac{i}{N} n} \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X^*[k] e^{j2\pi \frac{i}{N} k}\right\} \\
 &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} E\left\{X[n] X^*[k]\right\} e^{-j2\pi \frac{i}{N} (n-k)} \\
 &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} R_X[n-k] e^{-j2\pi \frac{i}{N} (n-k)} = \frac{1}{N} \sum_{m=-(N-1)}^{N-1} (N-|m|) R_X[m] e^{-j2\pi \frac{i}{N} m} \\
 &= \sum_{m=-(N-1)}^{N-1} \left(1 - \frac{|m|}{N}\right) R_X[m] e^{-j2\pi \frac{i}{N} m} \stackrel{N \gg L}{\cong} \sum_{m=-L}^L R_X[m] e^{-j2\pi \frac{i}{N} m} \\
 &= DFT\left\{R_X[m]\right\} = S_X\left(e^{j2\pi \frac{i}{N}}\right), \quad i = 0, 1, 2, \dots, N-1
 \end{aligned}$$

- Why is it important to come up with a discrete-representation where the components of the image vector are uncorrelated?
- This can be easily understood if for example the process $X(t)$ is Gaussian. In this case, also the discrete-time process $X[n]$ obtained by sampling is **Gaussian** and, as a consequence, also the image vector α obtained as a linear transformation of the image vector \mathbf{X} .
- We know that for Gaussian random vectors, uncorrelation implies independence (remember that independence always implies uncorrelation, but the reverse is usually not true). Hence, we can easily derive the joint **probability density function (pdf)** of the image vector α as the product of the marginal pdf's of the components of α :

$$f_{\alpha}(\alpha) = \prod_{i=0}^{N-1} f_{\alpha_i}(\alpha_i), \quad \text{where } N = 2BT$$

- Sometimes the **DKLT** is called **Principal Component Analysis (PCA)** and is used for dimensionality reduction (or redundancy reduction):

$$X[n] = \sum_{i=0}^{N-1} \alpha_i \varphi_i[n] \cong \sum_{i=0}^{K-1} \alpha_i \varphi_i[n]$$

$$\begin{matrix} \mathbf{X} \\ N \times 1 \end{matrix} \xrightleftharpoons{PCA} \begin{matrix} \mathbf{a} \\ K \times 1 \end{matrix}$$

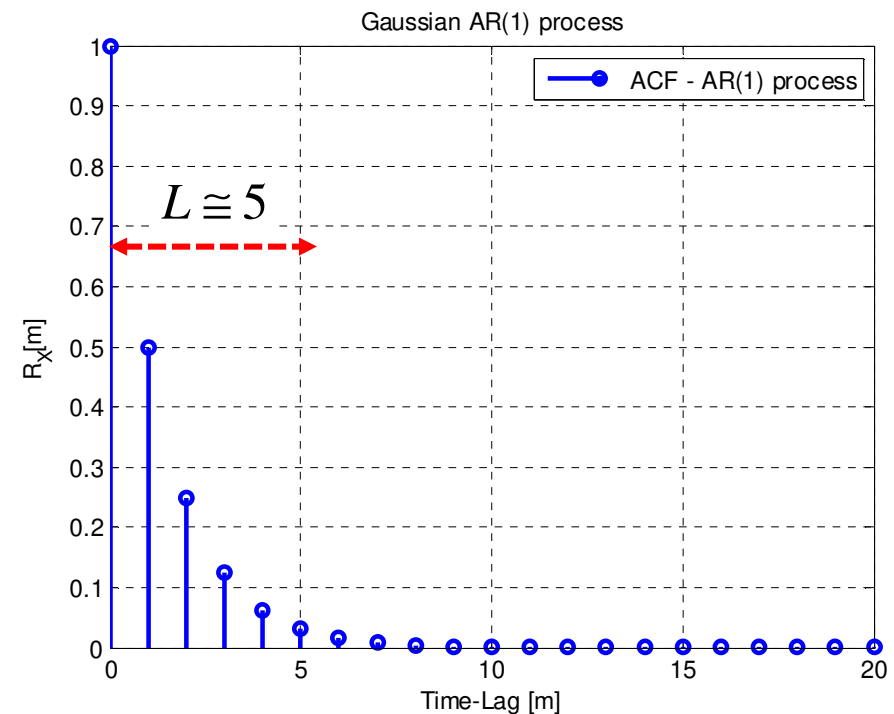
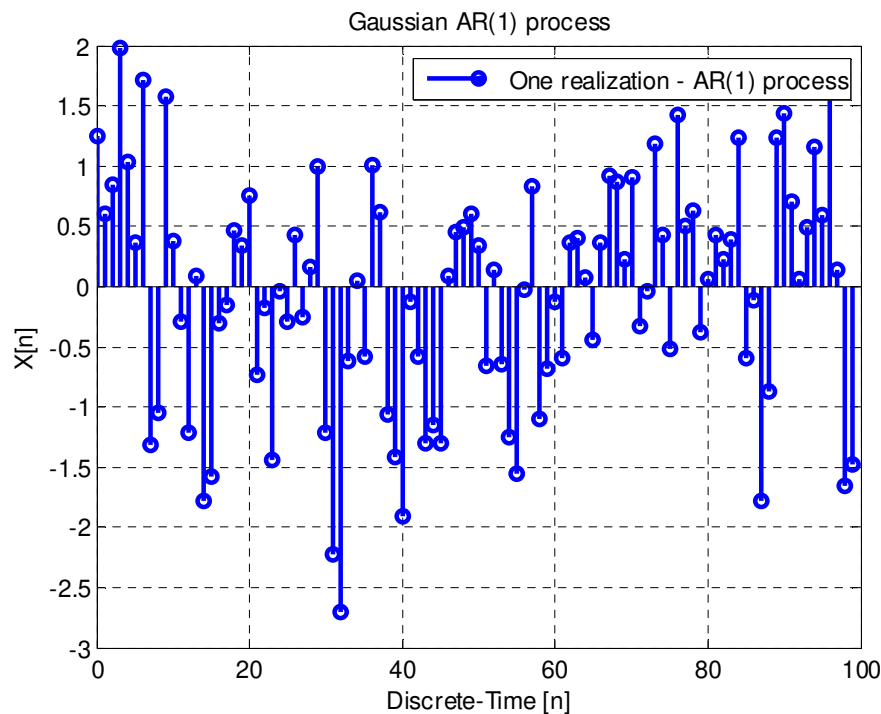
where :

- 1) $\sigma_i^2 \triangleq E\{|\alpha_i|^2\}$, $\sigma_i^2 \geq \sigma_{i+1}^2$ (decreasing order)
- 2) $E\{\alpha_i \alpha_j^*\} = 0$ for $i \neq j$
- 3) $K < N$ such that $\sum_{i=0}^{K-1} \sigma_i^2 \gg \sum_{i=K}^{N-1} \sigma_i^2$

- The best choice of K basis functions (dimensionality reduction) corresponds to the K eigenvectors of \mathbf{R}_X with the largest eigenvalues (the principal components).

■ Example: $X[n]$ random process with autocorrelation function (ACF):

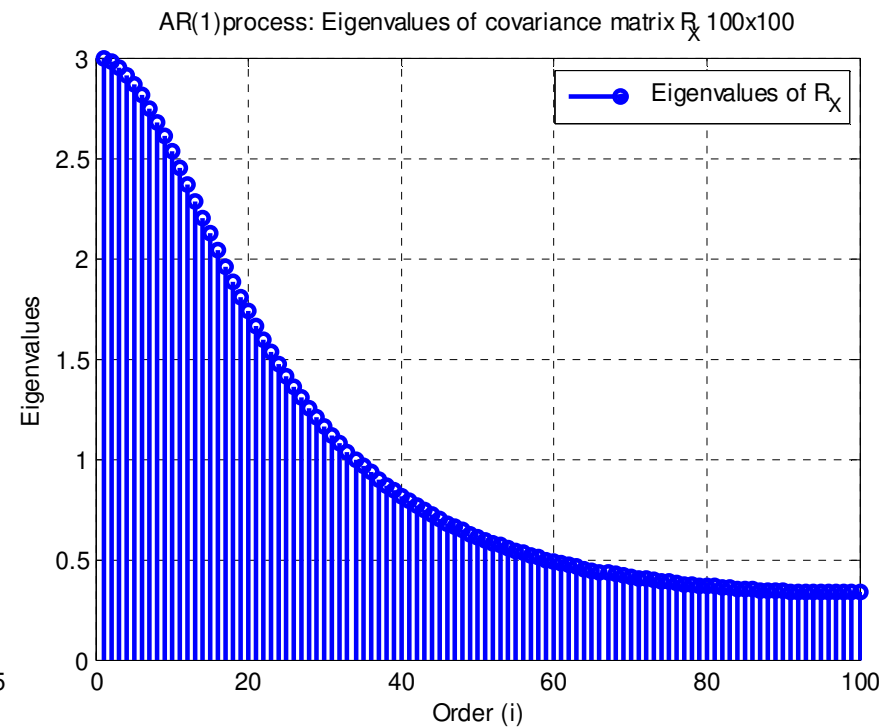
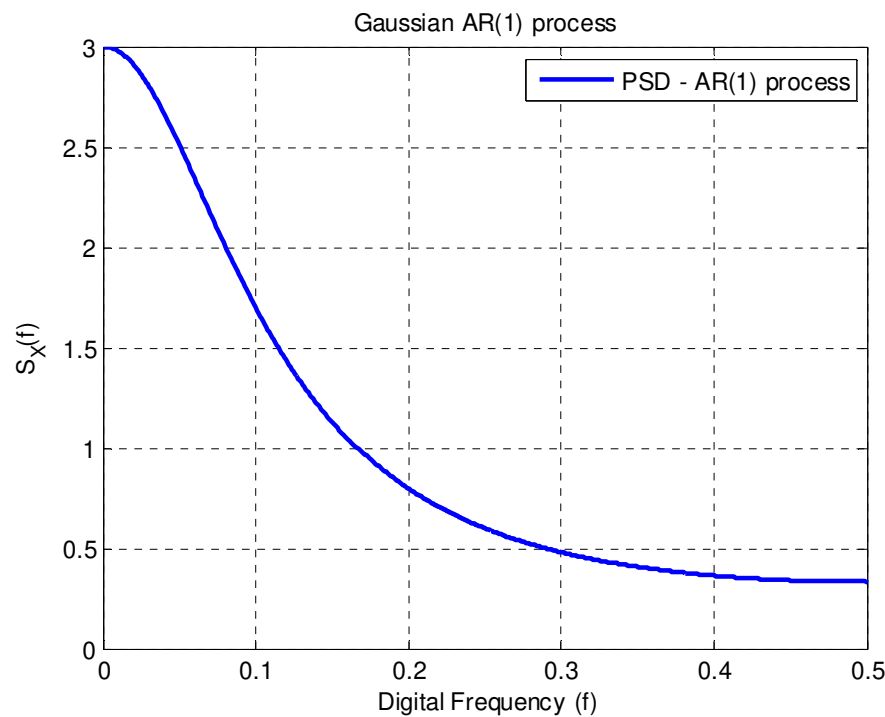
$$R_X[m] = E\{X[n]X^*[n+m]\} = (0.5)^{|m|} \rightarrow L \cong 5 \ll N = 100$$



$$S_X(e^{j2\pi f}) = DTFT\{R_X[m]\}$$

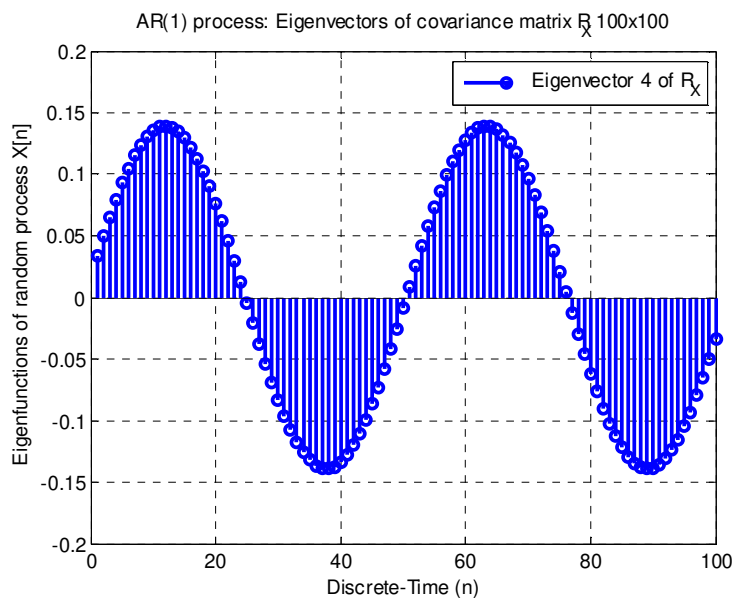
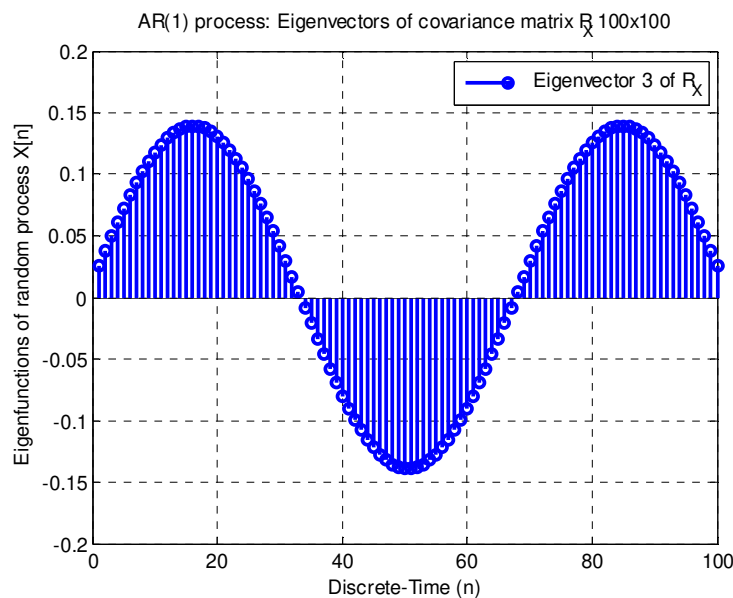
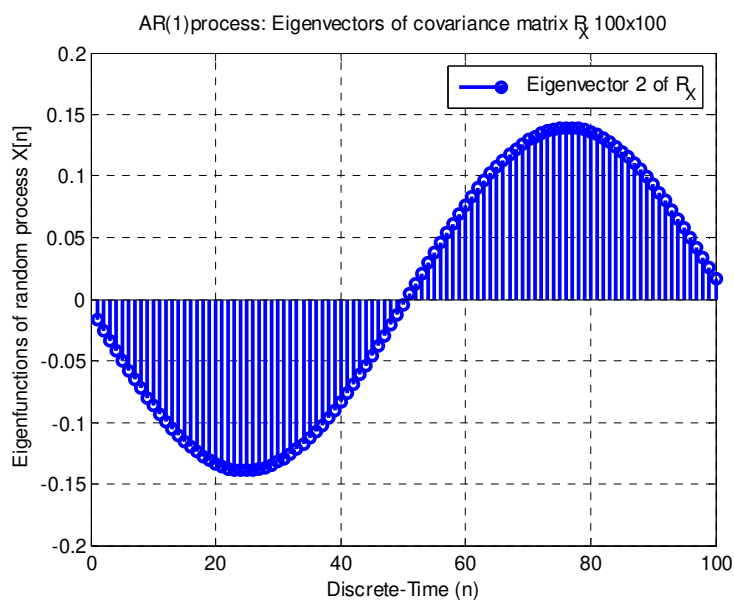
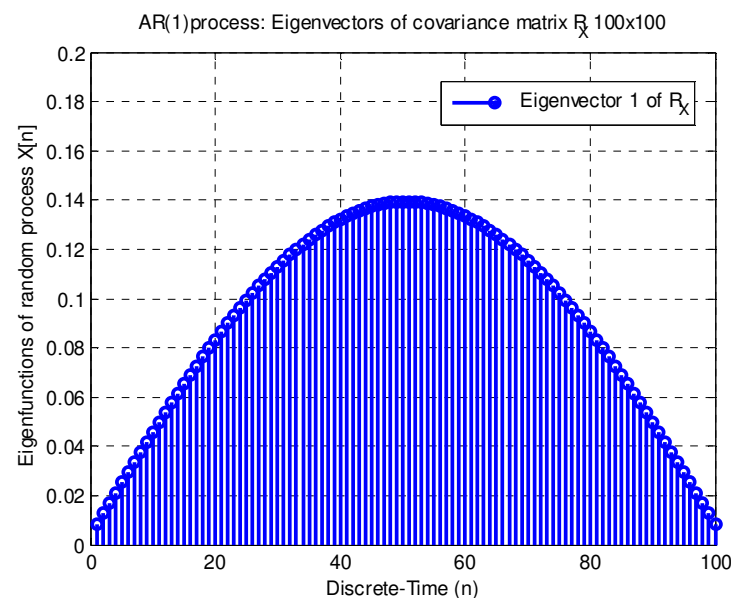
$$\text{Eigenvalues: } \sigma_i^2 \triangleq E\{|\alpha_i|^2\}$$

in decreasing order



Basis Expansion of Discrete-Time Signals

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■ **Example:** $X(t) = A \cos(2\pi f_0 t + \theta) + W(t), \quad t \in [0, T), \quad S_w(f) = \frac{N_0}{2}$

signal of interest $s(t)$

Additive White Gaussian Noise (AWGN)

■ A typical problem is to estimate the signal parameters (amplitude and phase) based on the observation of a noise corrupted version of the signal in the interval $[0, T)$.

■ The signal frequency f_0 may be a priori known or unknown. Let's assume first that it is a-priori known.

■ After anti-aliasing filtering with bandwidth $B=1/(2T_c)$ and ADC:

$$X[n] = A \cos(2\pi F_0 n + \theta) + W[n], \quad n = 0, 1, \dots, N-1$$

$$\text{where } F_0 \triangleq f_0 T_c \in (0, 1/2), \quad N = T/T_c = 2BT, \quad W[n] \in \mathcal{N}(0, \sigma_w^2) \quad \text{IID}$$

■ The noise samples are Independent and Identically distributed (IID).

- The discrete-time signal obtained by sampling can be expanded on the orthonormal basis functions as follows:

$$\begin{aligned} X[n] &= A \cos(2\pi F_0 n + \theta) + W[n] \\ &= A \cos(\theta) \cos(2\pi F_0 n) - A \sin(\theta) \sin(2\pi F_0 n) + W[n] \\ &= \sqrt{\frac{N}{2}} A \cos(\theta) \varphi_0[n] - \sqrt{\frac{N}{2}} A \sin(\theta) \varphi_1[n] + W[n] \end{aligned}$$

- where:

$$\varphi_0[n] \triangleq \sqrt{\frac{2}{N}} \cos(2\pi F_0 n), \quad \varphi_1[n] \triangleq \sqrt{\frac{2}{N}} \sin(2\pi F_0 n), \quad n = 0, 1, \dots, N-1$$

$$(\varphi_i, \varphi_k) \triangleq \sum_{n=0}^{N-1} \varphi_i[n] \varphi_k^*[n] = \delta_{i,k} \quad (\text{Kronecker's delta}) \quad \Leftrightarrow \quad N \gg 1$$

■ In fact, if $N \gg 1$:

$$\begin{aligned} (\varphi_0, \varphi_0) &= \sum_{n=0}^{N-1} \sqrt{\frac{2}{N}} \cos(2\pi F_0 n) \sqrt{\frac{2}{N}} \cos(2\pi F_0 n) = \frac{2}{N} \sum_{n=0}^{N-1} \cos^2(2\pi F_0 n) \\ &= \frac{2}{N} \sum_{n=0}^{N-1} \left(\frac{1}{2} + \frac{1}{2} \cos(4\pi F_0 n) \right) = 1 + \frac{1}{N} \sum_{n=0}^{N-1} \cos(4\pi F_0 n) \cong 1 \end{aligned}$$

$$\begin{aligned} (\varphi_1, \varphi_1) &= \sum_{n=0}^{N-1} \sqrt{\frac{2}{N}} \sin(2\pi F_0 n) \sqrt{\frac{2}{N}} \sin(2\pi F_0 n) = \frac{2}{N} \sum_{n=0}^{N-1} \sin^2(2\pi F_0 n) \\ &= \frac{2}{N} \sum_{n=0}^{N-1} \left(\frac{1}{2} - \frac{1}{2} \cos(4\pi F_0 n) \right) = 1 - \frac{1}{N} \sum_{n=0}^{N-1} \cos(4\pi F_0 n) \cong 1 \end{aligned}$$

$$\begin{aligned} (\varphi_0, \varphi_1) &= \sum_{n=0}^{N-1} \sqrt{\frac{2}{N}} \cos(2\pi F_0 n) \sqrt{\frac{2}{N}} \sin(2\pi F_0 n) = \frac{2}{N} \sum_{n=0}^{N-1} \frac{1}{2} \sin(4\pi F_0 n) \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \sin(4\pi F_0 n) \cong 0 \end{aligned}$$

- The other $N-2$ orthonormal basis functions should be chosen to be unit-norm and orthogonal to the first two (we do not need to specify them!).
- The discrete-time signal can be expanded by using an orthonormal basis set and the coefficients of the expansion are derived as follows:

$$\alpha_k = (X[n], \varphi_k[n]) = \sum_{n=0}^{N-1} X[n] \varphi_k^*[n]$$

$$\mathbf{a} = [\alpha_0 \quad \alpha_1 \quad \cdots \quad \alpha_{N-1}]^T$$

$$\begin{aligned} \alpha_k &= (X[n], \varphi_k[n]) = \sum_{n=0}^{N-1} X[n] \varphi_k^*[n] \\ &= \sum_{n=0}^{N-1} \left(\sqrt{\frac{N}{2}} A \cos(\theta) \varphi_0[n] - \sqrt{\frac{N}{2}} A \sin(\theta) \varphi_1[n] + W[n] \right) \varphi_k^*[n] \\ &= \sqrt{\frac{N}{2}} A \cos(\theta) \sum_{n=0}^{N-1} \varphi_0[n] \varphi_k^*[n] - \sqrt{\frac{N}{2}} A \sin(\theta) \sum_{n=0}^{N-1} \varphi_1[n] \varphi_k^*[n] + \sum_{n=0}^{N-1} W[n] \varphi_k^*[n] \end{aligned}$$

$$\alpha_k = \sqrt{\frac{N}{2}} A \cos(\theta) \delta_{0,k} - \sqrt{\frac{N}{2}} A \sin(\theta) \delta_{1,k} + W_k, \quad k = 0, 1, \dots, N-1$$

■ where: $W_k \triangleq (W[n], \varphi_k[n]) = \sum_{n=0}^{N-1} W[n] \varphi_k^*[n]$

$$\mathbf{\alpha} = \mathbf{s} + \mathbf{W} = \underbrace{\begin{bmatrix} \sqrt{\frac{N}{2}} A \cos(\theta) \\ -\sqrt{\frac{N}{2}} A \sin(\theta) \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\mathbf{s}} + \underbrace{\begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ \vdots \\ W_{N-1} \end{bmatrix}}_{\mathbf{W}}$$

$$E\{W_k\} = \sum_{n=0}^{N-1} E\{W[n]\} \varphi_k^*[n] = 0$$



$$E\{W_k W_i^*\} = E\left\{\sum_{n=0}^{N-1} W[n] \varphi_k^*[n] \sum_{l=0}^{N-1} W^*[l] \varphi_i[l]\right\}$$

$$= \sum_{n=0}^{N-1} \sum_{l=0}^{N-1} E\{W[n] W^*[l]\} \varphi_i[l] \varphi_k^*[n]$$

$$= \sum_{n=0}^{N-1} \sum_{l=0}^{N-1} \sigma_W^2 \delta[n-l] \varphi_i[l] \varphi_k^*[n]$$

$$= \sigma_W^2 \sum_{n=0}^{N-1} \varphi_i[n] \varphi_k^*[n] = \sigma_W^2 \delta_{i,k}$$

$$\Rightarrow W_k \in \mathcal{N}(0, \sigma_W^2) \quad \text{IID,} \quad \text{where} \quad \sigma_W^2 = \frac{N_0}{2T_c} = 2B \frac{N_0}{2} = N_0 B$$



- Only the first two components bring information on the amplitude and phase of the useful signal. The other components are **irrelevant**, so we do not need to compute them:

$$\alpha_0 = (X[n], \varphi_0[n]) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} X[n] \cos(2\pi F_0 n)$$

$$\alpha_1 = (X[n], \varphi_1[n]) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} X[n] \sin(2\pi F_0 n)$$

- This operation is equivalent to the extraction of the **In-phase (I)** and **Quadrature (Q) components**, typically called “**demodulation**”, but in digital fashion, after the ADC.

- Note that the components of the transformed vector α are uncorrelated:

$$\alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{N}{2}} A \cos(\theta) \\ -\sqrt{\frac{N}{2}} A \sin(\theta) \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ \vdots \\ W_{N-1} \end{bmatrix} \rightarrow E\{\alpha\} = E\{s + W\} = s$$


$$\text{cov}(\alpha_k, \alpha_i) = E\{(\alpha_k - s_k)(\alpha_i - s_i)^*\} = E\{W_k W_i^*\} = \sigma_W^2 \delta_{i,k}$$

$$\rightarrow \text{cov}(\alpha_k, \alpha_i) = 0 \quad \text{for } i \neq k$$

- The correlation matrix of vector α is given by:

$$\mathbf{R}_\alpha = E\{\alpha\alpha^H\} = \begin{bmatrix} \sigma_0^2 & r_{01} & \cdots & 0 \\ r_{10} & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{N-1}^2 \end{bmatrix} = \mathbf{C}_\alpha + \mathbf{n}_\alpha \mathbf{n}_\alpha^H = \sigma_w^2 \mathbf{I} + \mathbf{s} \mathbf{s}^H$$

$$r_{01} = r_{10} = E\{\alpha_1 \alpha_0^*\} = E\{\alpha_1\} E\{\alpha_0^*\} = s_1 s_0^* = -\frac{NA^2}{2} \cos(\theta) \sin(\theta) = -\frac{NA^2}{4} \sin(2\theta)$$

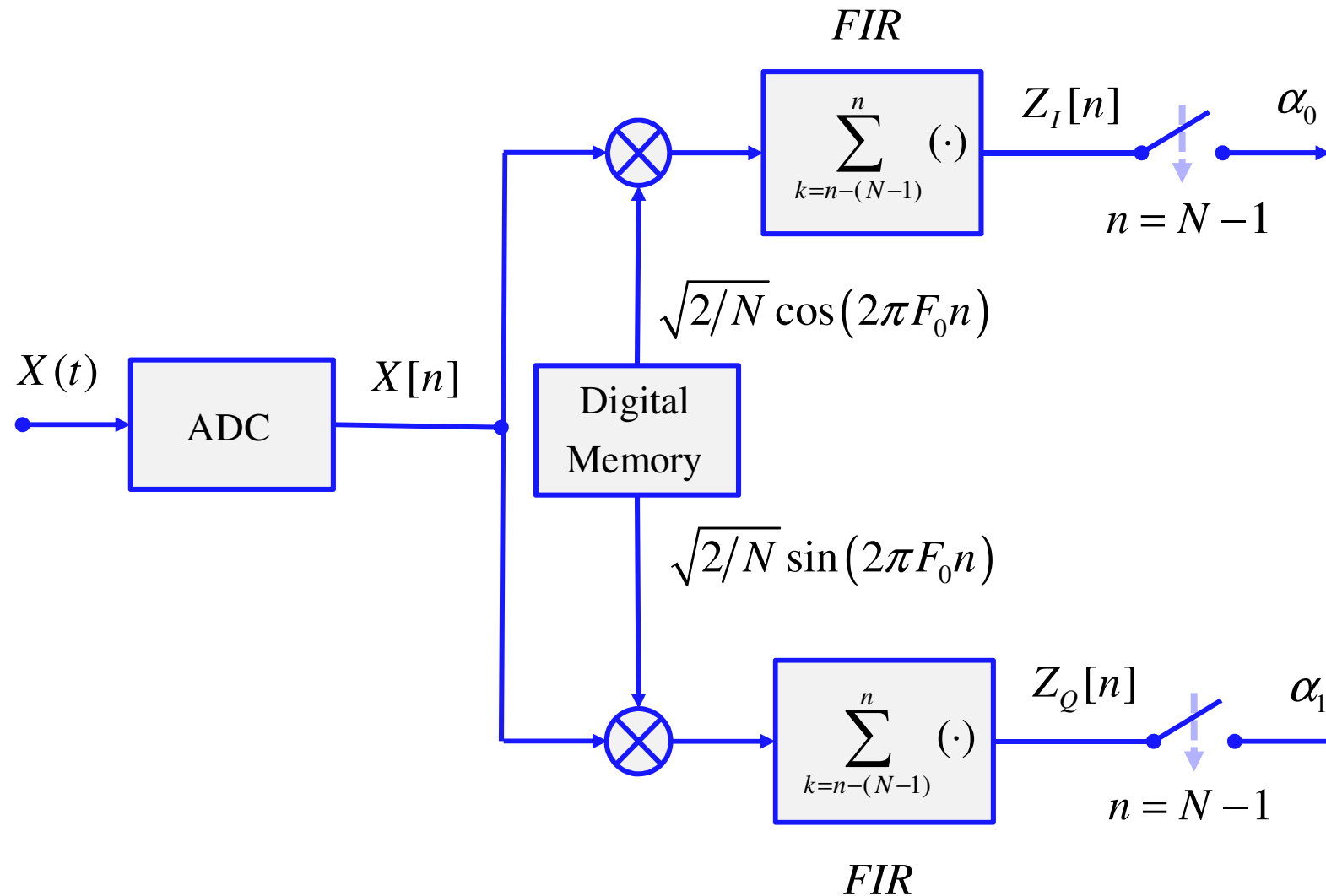


$$\sigma_i^2 = E\{|\alpha_i|^2\} = E\{|s_i + W_i|^2\} = |s_i|^2 + E\{W_i^2\} = |s_i|^2 + \sigma_w^2$$

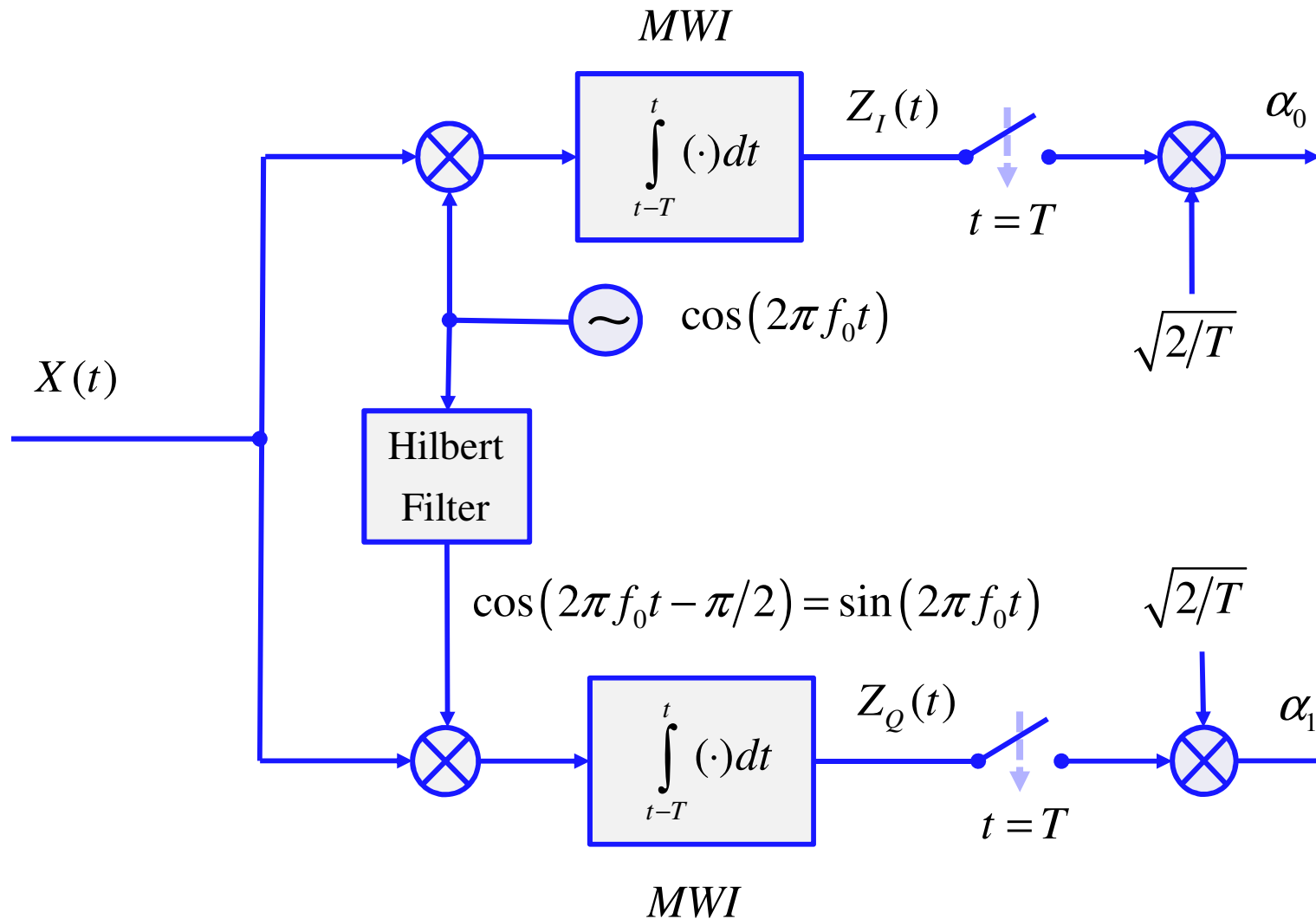
$$\sigma_0^2 = \frac{NA^2}{2} \cos^2(\theta) + \sigma_w^2, \quad \sigma_1^2 = \frac{NA^2}{2} \sin^2(\theta) + \sigma_w^2, \quad \sigma_i^2 = \sigma_w^2 \quad \text{for } 2 \leq i \leq N-1$$

Principal components

- Block diagram where the **inner product (correlation)** is implemented digitally:



- Alternative approach: **correlator** structure that uses an analog **moving window integrator (MWI)**:



- **Typical approach** → first sample (ADC conversion) to get a discrete-time signal and then expand it by using the discrete-time orthonormal basis functions:

$$\varphi_0[n] = \sqrt{\frac{2}{N}} \cos(2\pi F_0 n), \quad \varphi_1[n] = \sqrt{\frac{2}{N}} \sin(2\pi F_0 n), \quad n = 0, 1, \dots, N-1$$

- Only the first two components of the new image vector are **relevant** (i.e. they are the two **principal components**), so the 2D image vector is:

$$\mathbf{a} \triangleq \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{N}{2}} A \cos(\theta) \\ -\sqrt{\frac{N}{2}} A \sin(\theta) \end{bmatrix} + \begin{bmatrix} W_0 \\ W_1 \end{bmatrix} = \mathbf{s} + \mathbf{W}$$



- The components α_0 and α_1 are typically called the **In-phase (I)** and **Quadrature (Q) components** with respect to the frequency f_0 .
- If $N=2BT \gg 1$ the two approaches to extract the I & Q components are equivalent from the point of view of **Signal to Noise power Ratio (SNR)**.
- The first approach is usually preferred because more efficient in terms of computational complexity (down-conversion to baseband, i.e. extraction of the I & Q components, is done digitally).
- Let us calculate the *Signal to Noise Ratio* (SNR) for the image vector \mathbf{X} (*Input SNR*) and for the transformed image vector α , after the discrete-time base expansion (*Output SNR*).

- *Signal to Noise Ratio (SNR) for the image vector \mathbf{X} (Input SNR):*

$$X[n] = \sqrt{\frac{N}{2}} A \cos(\theta) \varphi_0[n] - \sqrt{\frac{N}{2}} A \sin(\theta) \varphi_1[n] + W[n]; \quad n = 0, 1, \dots, N-1$$

where $W[n] \in \mathcal{N}(0, \sigma_w^2)$ IID, $\sigma_w^2 = N_0 B$, $N = T/T_c = 2BT$

$$\mathbf{X} = \underbrace{\sqrt{\frac{N}{2}} A \cos(\theta) \boldsymbol{\varphi}_0 - \sqrt{\frac{N}{2}} A \sin(\theta) \boldsymbol{\varphi}_1}_{\mathbf{s}_X} + \mathbf{W}_X$$

where $\mathbf{W}_X \triangleq [W[0] \ W[1] \ \dots \ W[N-1]]^T$

$$\begin{aligned}
 SNR_{IN} &= \frac{E\{\|\mathbf{s}_X\|_2^2\}}{E\{\|\mathbf{W}_X\|_2^2\}} = \frac{\|\mathbf{s}_X\|_2^2}{E\left\{\sum_{n=0}^{N-1} W^2[n]\right\}} = \frac{\left\|\sqrt{\frac{N}{2}}A\cos(\theta)\boldsymbol{\phi}_0 - \sqrt{\frac{N}{2}}A\sin(\theta)\boldsymbol{\phi}_1\right\|_2^2}{\sum_{n=0}^{N-1} E\{W^2[n]\}} \\
 &= \frac{\frac{NA^2}{2}\cos^2(\theta)\|\boldsymbol{\phi}_0\|_2^2 + \frac{NA^2}{2}\sin^2(\theta)\|\boldsymbol{\phi}_1\|_2^2 - 2 \cdot \frac{NA^2}{2}\cos(\theta)\sin(\theta)(\boldsymbol{\phi}_0, \boldsymbol{\phi}_1)}{N\sigma_W^2} \\
 &= \frac{\frac{NA^2}{2}\cos^2(\theta) + \frac{NA^2}{2}\sin^2(\theta)}{N\sigma_W^2} = \frac{A^2}{2\sigma_W^2}
 \end{aligned}$$

$$SNR_{IN} = \frac{A^2}{2\sigma_W^2}$$

■ *Signal to Noise Ratio (SNR) for the transformed image vector α (Output SNR):*

$$\mathbf{\alpha} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{N}{2}} A \cos(\theta) \\ -\sqrt{\frac{N}{2}} A \sin(\theta) \end{bmatrix} + \begin{bmatrix} W_0 \\ W_1 \end{bmatrix} = \mathbf{s} + \mathbf{W}, \quad \text{where } W_k \in \mathcal{N}(0, \sigma_w^2), \quad IID$$

$$SNR_{OUT} = \frac{E\{\|\mathbf{s}\|_2^2\}}{E\{\|\mathbf{W}\|_2^2\}} = \frac{s_0^2 + s_1^2}{E\{W_0^2 + W_1^2\}} = \frac{A^2 N/2}{2\sigma_w^2} = \frac{A^2 N/2}{2 \cdot N_0 B} = \frac{A^2 T}{2N_0} = \frac{E_s}{N_0}$$

■ The **Processing Gain (PG)** is the gain in SNR that we achieve thanks to the linear transformation (i.e. the discrete-time base expansion):

$$PG = \frac{SNR_{OUT}}{SNR_{IN}} = \frac{A^2 N/2}{2\sigma_w^2} \bigg/ \frac{A^2}{2\sigma_w^2} = \frac{N}{2}$$

- Calculation of these two principal components is equivalent to calculating the **Discrete-Time Fourier Transform (DTFT)** of the observed signal $X[n]$ at the digital frequency F_0 , which is the most powerful frequency component present in the *Signal of Interest* (Sol) $s[n]$ observed in $[0, T)$:

$$\begin{aligned}\alpha_0 &= \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} X[n] \cos(2\pi F_0 n) = \sqrt{\frac{2}{N}} \Re e \left\{ \sum_{n=0}^{N-1} X[n] e^{-j2\pi F_0 n} \right\} \\ &= \sqrt{\frac{2}{N}} \Re e \left\{ DTFT \{ \mathbf{X} \} \Big|_{F=F_0} \right\}\end{aligned}$$

$$\begin{aligned}\alpha_1 &= \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} X[n] \sin(2\pi F_0 n) = -\sqrt{\frac{2}{N}} \Im m \left\{ \sum_{n=0}^{N-1} X[n] e^{-j2\pi F_0 n} \right\} \\ &= -\sqrt{\frac{2}{N}} \Im m \left\{ DTFT \{ \mathbf{X} \} \Big|_{F=F_0} \right\}\end{aligned}$$

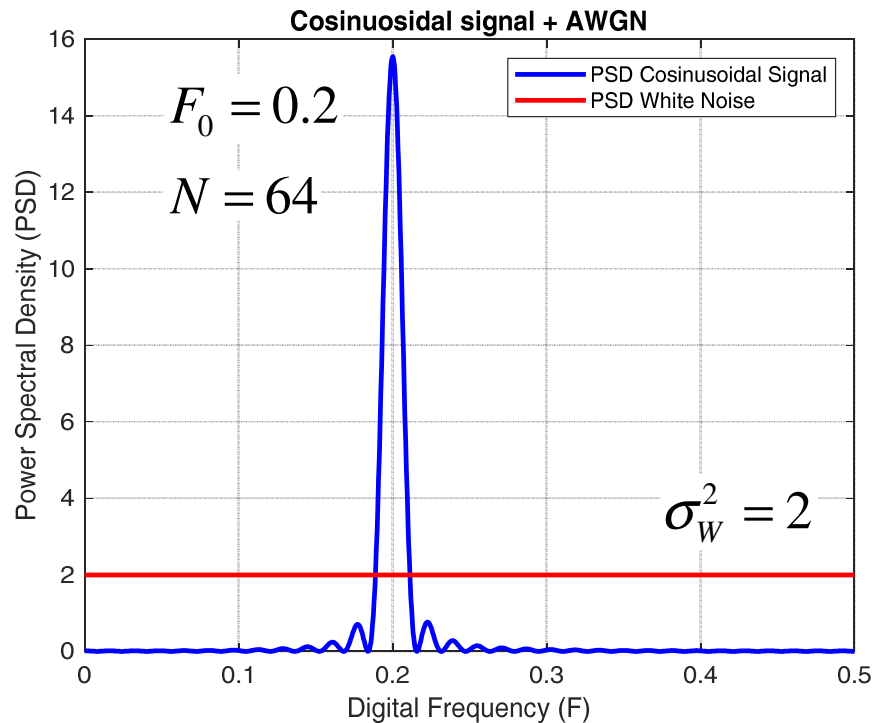
Basis Expansion of Discrete-Time Signals

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$$X[n] = s[n] + W[n] = A \cos(2\pi F_0 n + \theta) + W[n], \quad n = 0, 1, \dots, N-1$$

$$S(e^{j2\pi F}) = DTFT\{s[n]\} = \sum_{n=0}^{N-1} s[n] e^{-j2\pi F n}$$

$$PSD_S(e^{j2\pi F}) = \lim_{N \rightarrow \infty} \frac{1}{N} |S(e^{j2\pi F})|^2$$



$$SNR(F) = \frac{PSD_S(e^{j2\pi F})}{PSD_W(e^{j2\pi F})}$$

$$F_0 = \arg \max_F \{SNR(F)\}$$



$$\alpha_0 = \sqrt{\frac{2}{N}} \Re \left\{ DTFT \{ \mathbf{X} \} \Big|_{F=F_0} \right\}$$

$$\alpha_1 = -\sqrt{\frac{2}{N}} \Im \left\{ DTFT \{ \mathbf{X} \} \Big|_{F=F_0} \right\}$$

- The values of the DTFT at frequencies different from F_0 are not relevant, i.e. they do not bring additional useful information about the amplitude and phase of the signal of interest $s[n]$.
- The two principal components α_0 and α_1 contains all the relevant information about the amplitude A and the phase θ of the continuous-time signal of interest $s(t)$. Hence, A and θ can be estimated from α_0 and α_1 .

- This is possible if we know *a priori* the signal frequency F_0 (remember that the basis functions should be deterministic perfectly known).
- What about if the signal frequency F_0 is also unknown?



We use the FFT to calculate the DTFT of the N -dimensional vector \mathbf{X} at all the discrete frequencies $F_k = k/N$ (or $F_k = k/N_{zp}$ if we use zero-padding) and then we select the two principal components, i.e. the real and imaginary parts of the FFT at the frequency for which the $|\text{FFT}|^2$ assumes the greatest value [*we will investigate this problem in detail later on*].