INTRODUCTION TO DISCRETE-TIME RANDOM PROCESSES

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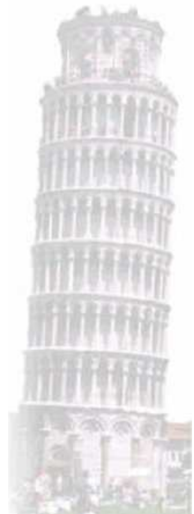
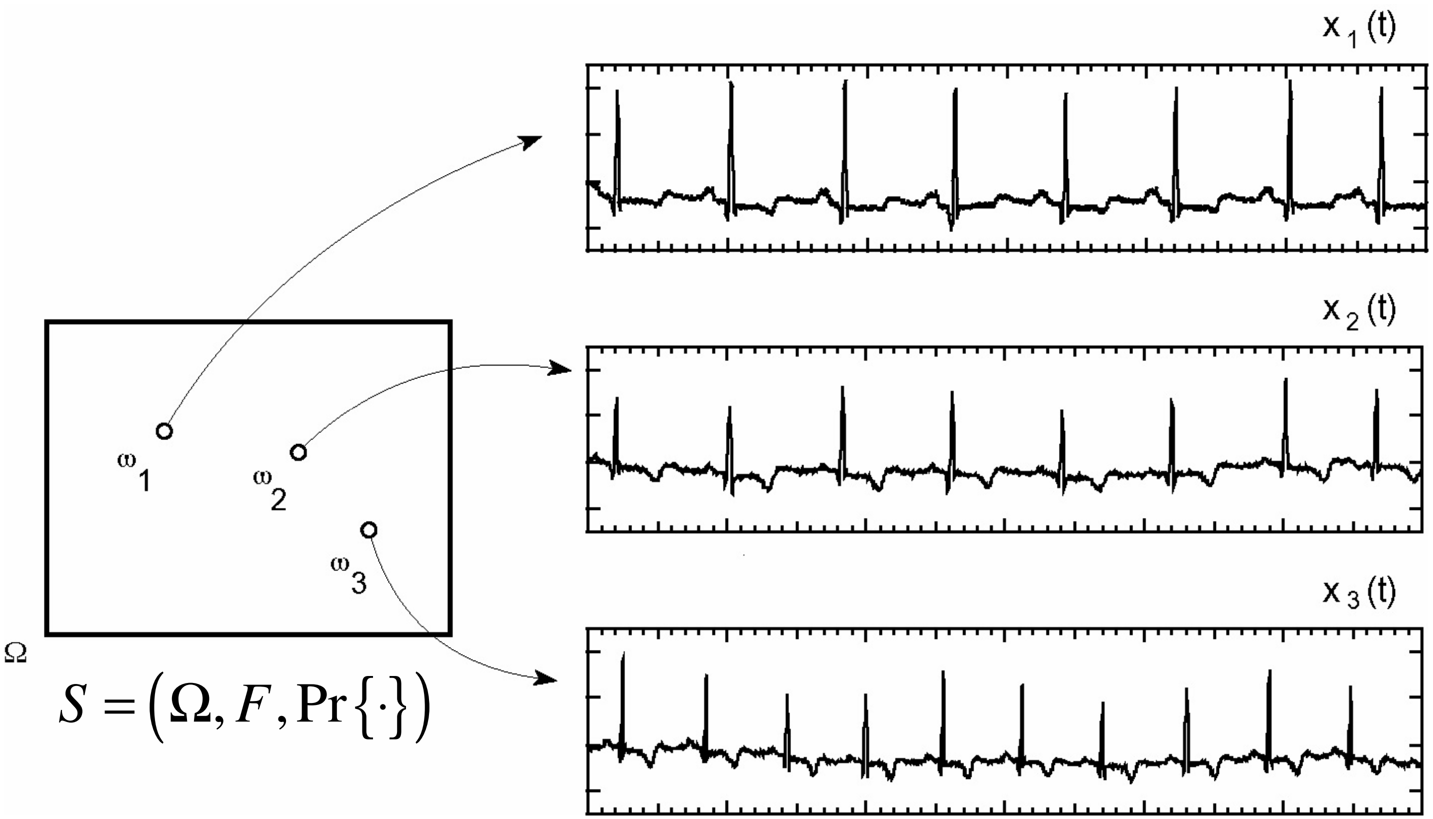


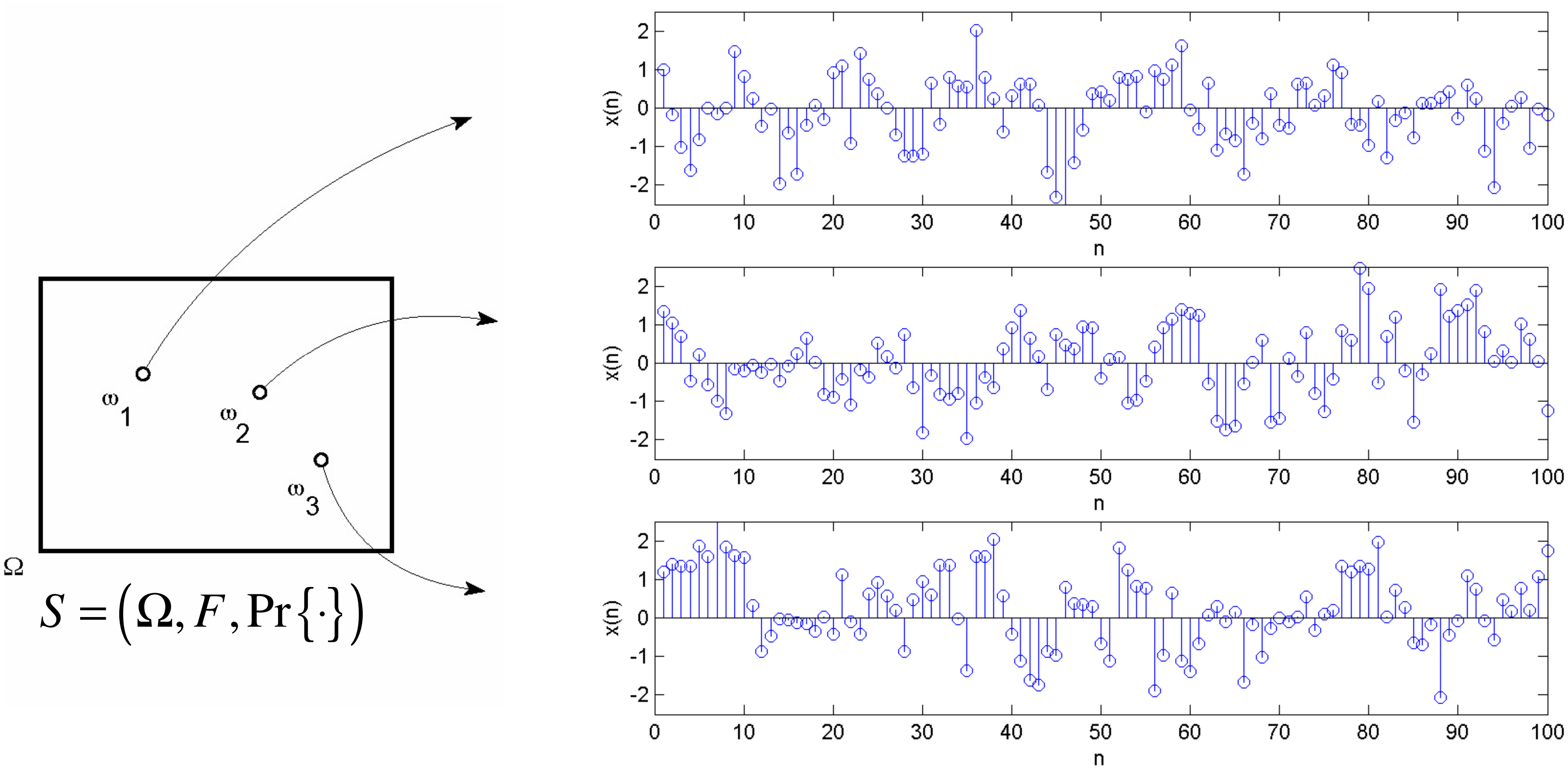
Random Processes

- Examples of di electrocardiograms of patients suffering for cardiac arrhythmia.



- Signals that bring information are random signals.
- The realizations of a random process (r.p.) are deterministic signals.
- Sampling a continuous-time r.p. we get a **discrete-time random process**.

Discrete-Time Random Processes (1/3)



- The realizations of a **discrete-time random process** are deterministic discrete-time signals.

Discrete-Time Random Processes (2/3)

- Consider a sequence *x*[*n*] such that its value for any choice of the parameter *n* is a *random variable* (r.v.).

  Then, *x*[*n*] is a *discrete-time random process* or a *random sequence*.

- If the parameter *n* represents a time index (i.e. the discrete-time), the random sequence is sometime referred to as a *time series*.

- **Example**: The *Bernoulli process* is a sequence of Independent and Identically Distributed (IID) binary random variables:

$$x[n] = \{ x[n] \mid x[n] = \pm 1, \forall n \in \mathbb{N} \}$$

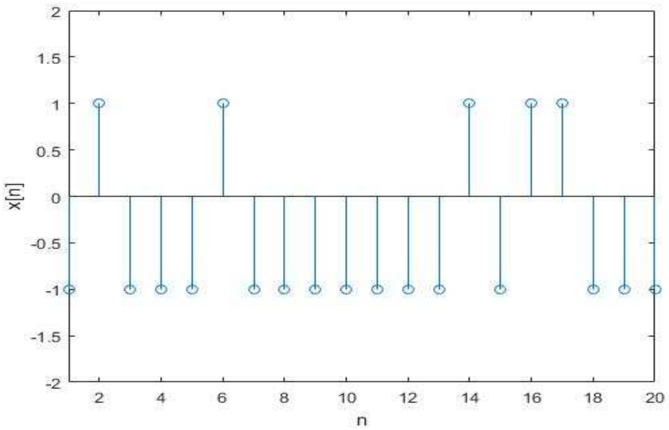
where, each r.v. sampled from *x*[*n*] follows a Bernoulli distribution:

$$\Pr\{x[n] = +1\} = p, \qquad \Pr\{x[n] = -1\} = 1 - p$$

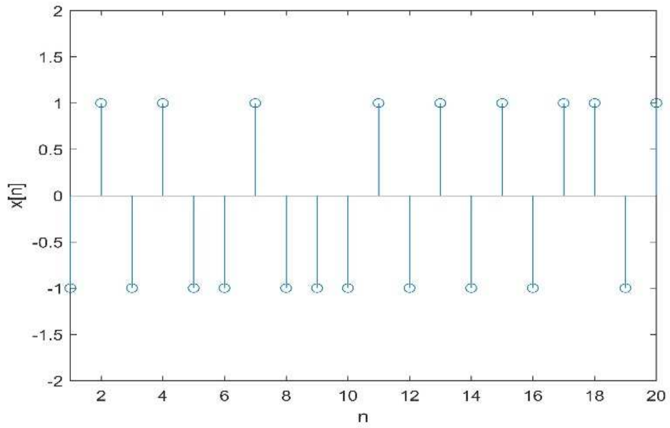
Discrete-Time Random Processes (3/3)

- Examples of realizations of a Bernoulli process for three different values of *p*:

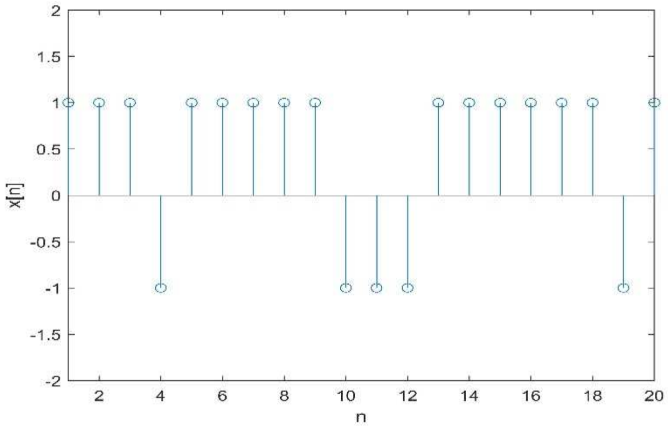
p=0.2



p=0.5



p=0.8



Parametric Discrete-Time Random Processes (1/2)

- If a discrete random process is of the form

$$x[n] = f\left(\mathbf{a}, n\right);$$

where **a** is a vector of random parameters, then it is called *parametric* process.

- **Example:**

$$x[n] = A\cos\left(2\pi F_0 n + \theta_0\right), \quad 0 \leq F_0 < \frac{1}{2}$$

- The amplitude *A* can be modelled as a Rayleigh r.v., the phase θ_0 is typically modelled as a r.v. uniformly distributed in $[-\pi,\pi)$.

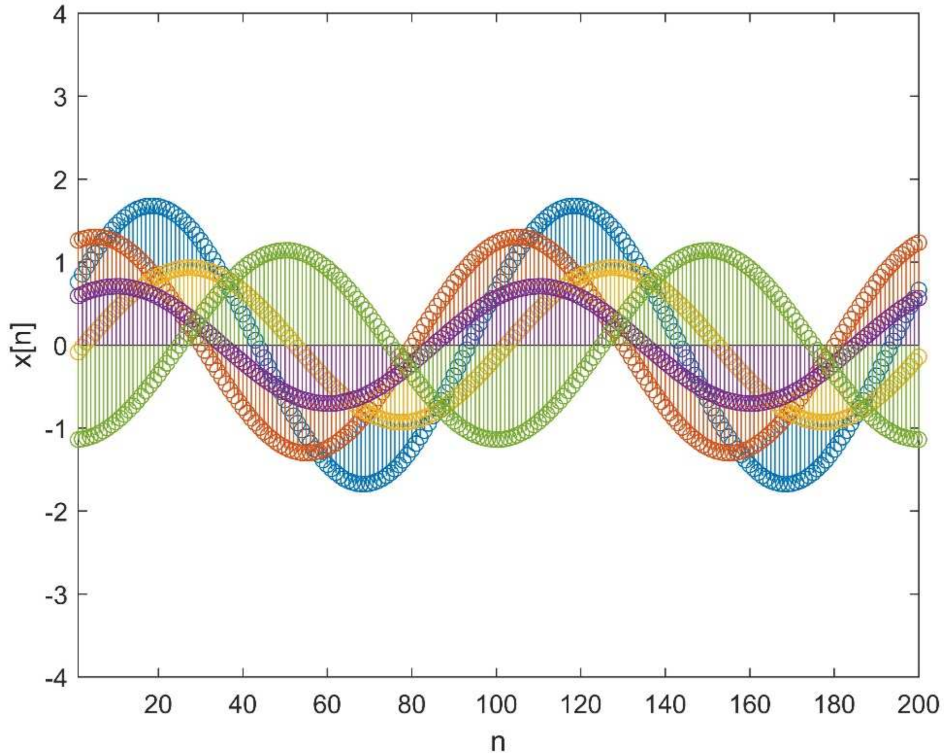
$$A \in \mathcal{R}(\sigma_A^2) \qquad \theta_0 \in \mathcal{U}(-\pi,\pi) \qquad A \text{ and } \theta_0 \text{ indipendent}$$

$$f_A(a) = \frac{a}{\sigma_A^2} e^{-\frac{a^2}{2\sigma_A^2}} u(a), \quad E\{A^2\} = 2\sigma_A^2$$

Parametric Discrete-Time Random Processes (2/2)

- Five realizations of the parametric process $x[n]$:

  - Scale parameter of the Rayleigh pdf: $\sigma_A^2 = 1/2$
  - Normalized frequency: F_0=0.01
  - Period: T_0=1/F_0=100



- A parametric process is fully characterized by the function $f(\mathbf{a},n)$ and by the joint pdf of the random parameters.

Characterization of a discrete-time random process (1/2)

- In general, a discrete random process is fully statistically characterized by the knowledge of the joint **cumulative distribution function (cdf)** of any set of *N* random variables sampled from *x*[*n*]:

$$\left\{x[n_1], x[n_2], \cdots, x[n_N]\right\}, \quad \forall N \in \mathbb{N} \text{ and } \forall \mathcal{S}_N = \left\{n_1, \ldots, n_N\right\}$$

- The cdf is defined as:

$$P_X\left(x_1, x_2, \ldots, x_N; n_1, n_2, \ldots, n_N\right) \triangleq \Pr\left\{x[n_1] \leq x_1, x[n_2] \leq x_2, \ldots, x[n_N] \leq x_N\right\}$$

- Alternatively, we can characterize *x*[*n*] by means of the joint **probability density function (pdf)**:

$$p_X\left(x_1, x_2, \ldots, x_N; n_1, n_2, \ldots, n_N\right) \triangleq \frac{\partial^N P_X\left(x_1, x_2, \ldots, x_N; n_1, n_2, \ldots, n_N\right)}{\partial x_1 \partial x_2 \cdots \partial x_N}$$

Characterization of a discrete-time random process (2/2)

- In practical applications, it is quite impossible to fully characterize a random process by means of the joint pdf for any order *N*:

$$p_X\left(x_1, x_2, \ldots, x_N; n_1, n_2, \ldots, n_N\right), \forall N \in \mathbb{N} \text{ and } \forall \mathcal{S}_N = \left\{n_1, \ldots, n_N\right\}$$

- In particular, the knowledge of the joint pdf (or of the joint cdf) of any *order N,* and for any possible set with cardinality *N* of random variables is often not achievable.

- For this reason, we usually have to rely only on some **statistical indices** that can be easily evaluated or, *at least estimated* from the available data.

- The two main sets of indices are the *first order statistics* (e.g. mean value, power, variance, skewness, kurtosis, etc.) and the *second order statistics* (e.g. the autocorrelation function (ACF), the autocovariance function, the power spectral density (PSD), etc.) of the process.

First order statistics

- **First order statistics**: they concern one single variable extracted by the process, so they require knowledge of only the first order pdf.

- **Moment of order *k*:**

$$E\left\{x^k[n]\right\} \triangleq \int_{-\infty}^{+\infty} x^k p_X(x;n)dx$$

In general, they are function of *n*!

- *Mean* (ordinary moment of order 1):

$$\eta_X[n] \triangleq E\left\{x[n]\right\} \triangleq \int_{-\infty}^{+\infty} x p_X(x;n)dx$$

- *Statistical power* (ordinary moment of order 2):

$$\mathrm{P}_X[n] \triangleq E\left\{x^2[n]\right\} \triangleq \int_{-\infty}^{+\infty} x^2 p_X(x;n)dx$$

- *Variance* (central moment of order 2):

$$\sigma_X^2[n] = E\left\{\left(x[n]-\eta_X[n]\right)^2\right\} \triangleq \int_{-\infty}^{+\infty} \left(x-\eta_X[n]\right)^2 p_X(x;n)dx = \mathrm{P}_X[n]-\eta_X^2[n]$$

Second order statistics

- **Second order statistics**: they concern two variables extracted from the process, so they require knowledge of the second order pdf.

- **Autocorrelation function (ACF):**

$$R_X[n_1,n_2] \triangleq E\{x[n_1]x[n_2]\} \triangleq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 p_X(x_1,x_2;n_1,n_2)dx_1 dx_2$$

In general, they are function of n_1 *and* n_2

- *Covariance function*:

$$C_X[n_1,n_2] \triangleq E\{(x[n_1]-\eta_X[n_1])(x[n_2]-\eta_X[n_2])\} = R_X[n_1,n_2]-\eta_X[n_1]\eta_X[n_2]$$

- Two random variables, $x[n_1]$ and $x[n_2]$, are said to be

  - *Orthogonal if:* $R_X[n_1,n_2] = E\{x[n_1]x[n_2]\} = 0$

  - *Uncorrelated if:* $C_X[n_1,n_2] = 0 \quad \rightarrow \quad R_X[n_1,n_2] = \eta_X[n_1]\eta_X[n_2]$

- Note that: $P_X[n] = R_X[n,n], \quad \sigma_X^2[n] = C_X[n,n]$

Stationary discrete-time random processes

- **Definition**: A discrete random process $x[n]$ is said to be *stationary in the strict sense* if its joint pdf of any order is invariant to any shift L, i.e.:

$$p_X\left(x_1, x_2, \ldots, x_N; n_1, n_2, \ldots, n_N\right) = p_X\left(x_1, x_2, \ldots, x_N; n_1 - L, n_2 - L, \ldots, n_N - L\right)$$

$$\forall N \in \mathbb{N},\ \ \forall \mathcal{S}_N = \left\{n_1, \ldots, n_N\right\},\ \ \forall L \in \mathbb{Z}$$

- The strict-sense stationarity implies that the process $x[n]$ and its shifted version $x[n-L]$ have exactly that same statistics of any order, or in other words they are statistically equivalent.

> The statistical equivalence does not imply that each possible sample functions of $x[n]$ is equal to the ones of $x[n-L]$.

- The strict-sense stationarity is very difficult to check in practice.

Wide-sense stationarity

- **Definition**: A discrete random process $x[n]$ is said to be *wide-sense stationary* (w.s.s.) if its mean is constant and its ACF if a function only of the spacing between the sampled random variables (the so-called *lag*):
  - $\eta_X[n] \equiv \eta_X$
  - $R_X[n_1, n_2] \equiv R_X[0, n_2 - n_1] \equiv R_X[m]$

and consequently

  - $C_X[n_1, n_2] \equiv C_X[0, n_2 - n_1] \equiv C_X[m]$
  - $\sigma_X^2[n] = C_X[n,n] \equiv C_X[0] \triangleq \sigma_X^2$
  - $P_X[n] = R_X[n,n] \equiv R_X[0] \triangleq P_X = \sigma_X^2 + \eta_X^2$

- Strict-sense stationarity implies wide-sense stationarity.
- The reverse is not true, unless the process is Gaussian.

Examples of ACFs

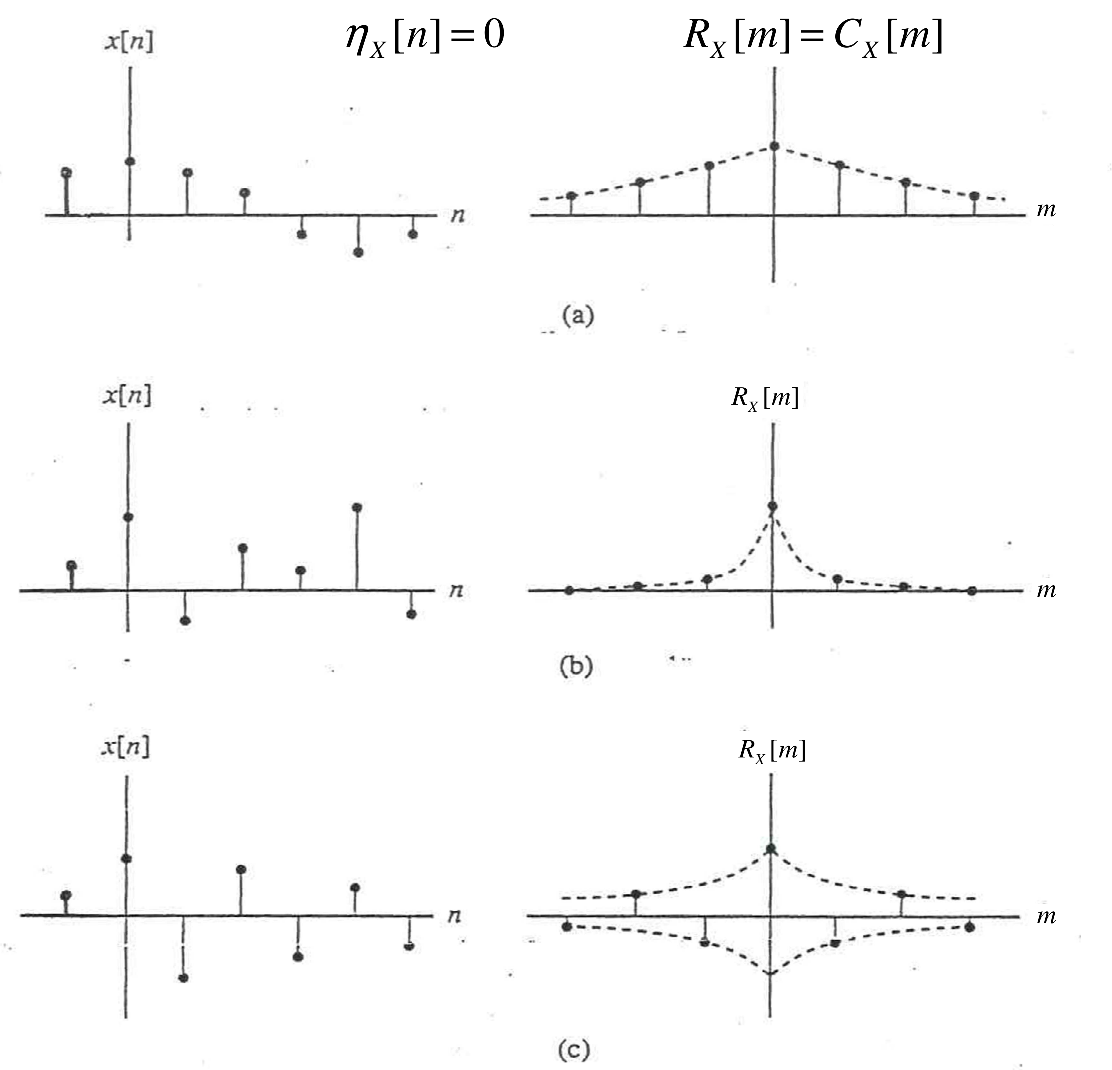


Figure Examples of random sequences and their correlation functions. (a) High correlation. (b) Low correlation. (c) High negative correlation.

Properties of the ACF (1/3)

- Let $x[n]$ be a w.s.s. discrete-time **real-valued** process. Its ACF satisfies the following properties:

- **1. Simmetry**

$$R_X[m] = E\{x[n]x[n+m]\} = E\{x[n-m]x[n]\} = E\{x[n]x[n-m]\} = R_X[-m]$$

The same property holds for the covariance function: $C_X[m] = C_X[-m]$

- **2.** The ACF has is maximum value in 0:

$$R_X[0] \geq |R_X[m]|, \quad \forall m \in \mathbb{Z}$$

$$C_X[0] \geq |C_X[m]|, \quad \forall m \in \mathbb{Z}$$

Properties of the ACF (2/3)

- **Simmetry**: If $x[n]$ is a w.s.s. discrete-time **complex-valued** process, its ACF satisfies the **Hermitian symmetry** property:

$$R_X[m] \triangleq E\left\{x[n]x^*[n+m]\right\} = E\left\{x[n-m]x^*[n]\right\}$$
$$= \left(E\left\{x[n]x^*[n-m]\right\}\right)^* = R_X^*[-m]$$

- From now on, if not otherwise explicitly stated, we will consider the case of real-valued processes.

Properties of the ACF (3/3)

- **3.** If $x[n]$ contains a periodic component with period N_0, then the ACF and the correlation function (as well as all of its moments) contain a periodic component of the same period N_0.

$$R_X[m] = E\{x[n]x[n+m]\} = E\{x[n]x[n+m-N_0]\} = R_X[m-N_0]$$

- **4.** If the ACF of a stationary (at least in wide-sense) process does not contain periodic components, then

$$\lim_{m\to\infty} R_X[m] = \lim_{m\to\infty}\left[C_X[m] + \eta_X^2\right] = \lim_{m\to\infty} C_X[m] + \eta_X^2 = \eta_X^2$$

- This is because, if the process does not contain periodic components, when the lag m increases the two random variables $x[n]$ and $x[n+m]$ tend to be uncorrelated.

Discrete processes and random vectors

- Let **x** be a random vector consisting of *N* random variables sampled from a discrete-time random process *x*[*n*]:

$$\mathbf{x} = \begin{bmatrix} x[0] & x[1] & \cdots & x[N-1] \end{bmatrix}^T \in \mathbb{R}^N,$$

- The *mean vector* is defined as:

$$\boldsymbol{\eta}_X \triangleq E\{\mathbf{x}\} = \begin{bmatrix} E\{x[0]\} & E\{x[1]\} & \cdots & E\{x[N-1]\} \end{bmatrix}^T = \begin{bmatrix} \eta_X[0] & \eta_X[1] & \cdots & \eta_X[N-1] \end{bmatrix}^T$$

- The mean vector of **x** is completely determined by the mean function $\eta_X[n]$ of the random process *x*[*n*].

- For a w.s.s random process, the mean vector has all its components equal to the same value, which is the mean value of the process:

$$\left[\boldsymbol{\eta}_X\right]_{n+1} = E\{x[n]\} = \eta_X, \quad n = 0,1,\ldots,N-1 \quad \Rightarrow \quad \boldsymbol{\eta}_X = \eta_X \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T = \eta_X \mathbf{1}$$

Correlation matrix

- The **correlation matrix** is defined as:

$$\mathbf{R}_X \triangleq E\left\{\mathbf{x}\mathbf{x}^T\right\} \in \mathbb{R}^{N\times N}$$

- $\mathbf{R}_X$ is symmetric and positive semidefinite:

$$\mathbf{R}_X = E\left\{\mathbf{x}\mathbf{x}^T\right\} = E\left\{\left(\mathbf{x}\mathbf{x}^T\right)^T\right\} = \left(E\left\{\mathbf{x}\mathbf{x}^T\right\}\right)^T = \mathbf{R}_X^T$$

$$\text{EVD}: \quad \mathbf{R}_X = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T, \quad \lambda_k = \left[\mathbf{\Lambda}\right]_k \geq 0 \quad \forall k$$

$$\text{Eigenvalue-Eigenvector Decompostion (EVD)}$$

- The correlation matrix of **x** is completely specified by the correlation function of the process *x*[*n*]:

$$\left[\mathbf{R}_X\right]_{i+1,\,j+1} = E\left\{x[i]x[j]\right\} = R_X[i,j], \quad i,j = 0,1,\ldots,N-1$$

Covariance matrix

- If $x[n]$ is w.s.s., $\mathbf{R}_X$ is not only symmetric, but it is also a **Toeplitz** matrix, i.e.

$$[\mathbf{R}_X]_{i+1,j+1} = E\{x[i]x[j]\} = R_X[i,j] = R_X[j-i], \quad i,j = 0,1,\ldots,N-1$$

$$\Downarrow$$

$$\mathbf{R}_X = \begin{bmatrix} R_X[0] & R_X[1] & \cdots & R_X[N-1] \\ R_X[-1] & R_X[0] & \ddots & \vdots \\ \vdots & \ddots & \ddots & R_X[1] \\ R_X[-(N-1)] & \cdots & R_X[-1] & R_X[0] \end{bmatrix} = \mathbf{R}_X^T$$

- The fact that a matrix has the **Toeplitz** structure has some important implications for the calculation of its inverse.

Covariance matrix

- The **covariance matrix** of **x** is defined as:

$$\mathbf{C}_X \triangleq E\left\{\left(\mathbf{x}-\boldsymbol{\eta}_X\right)\left(\mathbf{x}-\boldsymbol{\eta}_X\right)^T\right\} = \mathbf{R}_X - \boldsymbol{\eta}_X \boldsymbol{\eta}_X^T \in \mathbb{R}^{N\times N}$$

- $\mathbf{C}_X$ is symmetric and positive semidefinite.
- The covariance matrix of **x** is completely specified by the covariance function of $x[n]$

$$\left[\mathbf{C}_X\right]_{i+1,\,j+1} = E\left\{\left(x[i]-\eta_X[i]\right)\left(x[j]-\eta_X[j]\right)\right\} = C_X[i,j], \quad i,j = 0,1,\ldots,N-1$$

- If $x[n]$ is w.s.s., $\mathbf{C}_X$ is also a Toeplitz matrix, i.e.

$$\left[\mathbf{C}_X\right]_{i+1,\,j+1} = C_X[j-i] \quad \rightarrow \quad \mathbf{C}_X = \begin{bmatrix} C_X[0] & C_X[1] & \cdots & C_X[N-1] \\ C_X[-1] & C_X[0] & \ddots & \vdots \\ \vdots & \ddots & \ddots & C_X[1] \\ C_X[-(N-1)] & \cdots & C_X[-1] & C_X[0] \end{bmatrix} = \mathbf{C}_X^T$$

Gaussian discrete random processes

- A discrete-time random process $x[n]$ is said to be ***Gaussian*** if every set of N random variables $\{x[n_1], x[n_2],\ldots, x[n_N]\}$ are jointly Gaussian distributed for every N and for every N-tuple $\{n_1,\ldots,n_N\}$.

- Alternatively, $x[n]$ is said to be *Gaussian* if the vector of random variables

$$\mathbf{x} = \left[x[n_0] \quad x[n_0+1] \quad \cdots \quad x[n_0+N-1] \right]^T, \quad \forall n_0, \forall N$$

has a Gaussian pdf with mean value η_X and covariance matrix $\mathbf{C}_X$:

$$f_X(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^N |\mathbf{C}_X|}} \exp\left[-\frac{1}{2} (\mathbf{x} - \boldsymbol{\eta}_X)^T \mathbf{C}_X^{-1} (\mathbf{x} - \boldsymbol{\eta}_X) \right]$$

$$\mathbf{x} \sim \mathcal{N}(\boldsymbol{\eta}_X, \mathbf{C}_X)$$

Properties of a Gaussian random process

The pdf of any order N of a Gaussian vector $\mathbf{x}$ is fully characterized by the knowledge of the mean value vector η_X and the covariance matrix $\mathbf{C}_X$.

The Gaussianity property is invariant under affine transformations.

$$\mathbf{x} \sim \mathcal{N}\left(\boldsymbol{\eta}_X, \mathbf{C}_X\right) \Longrightarrow \mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b} \sim \mathcal{N}\left(\mathbf{A}\boldsymbol{\eta}_X + \mathbf{b}, \mathbf{A}\mathbf{C}_X\mathbf{A}^T\right)$$

$$\boldsymbol{\eta}_Y = E\{\mathbf{y}\} = E\{\mathbf{A}\mathbf{x} + \mathbf{b}\} = \mathbf{A}E\{\mathbf{x}\} + \mathbf{b} = \mathbf{A}\boldsymbol{\eta}_X + \mathbf{b}$$

$$\mathbf{C}_Y = E\left\{(\mathbf{y} - \boldsymbol{\eta}_Y)(\mathbf{y} - \boldsymbol{\eta}_Y)^T\right\} = E\left\{(\mathbf{A}\mathbf{x} - \mathbf{A}\boldsymbol{\eta}_X)(\mathbf{A}\mathbf{x} - \mathbf{A}\boldsymbol{\eta}_X)^T\right\}$$

$$= E\left\{\mathbf{A}(\mathbf{x} - \boldsymbol{\eta}_X)(\mathbf{x} - \boldsymbol{\eta}_X)^T\mathbf{A}^T\right\} = \mathbf{A}E\left\{(\mathbf{x} - \boldsymbol{\eta}_X)(\mathbf{x} - \boldsymbol{\eta}_X)^T\right\}\mathbf{A}^T = \mathbf{A}\mathbf{C}_X\mathbf{A}^T$$

Properties of a Gaussian random process

Every r-tuple of random variables taken from a Gaussian vector **x** of dimension N (with $r<N$) forms a Gaussian vector of dimension r.

In particular, each entry of a Gaussian random vector is a Gaussian r.v.:

$$[\mathbf{x}]_k \sim \mathcal{N}\left([\boldsymbol{\eta}_X]_k, [\mathbf{C}_X]_{k,k}\right)$$

Every r-tuple of random variables taken from a Gaussian vector **x** conditioned to another k-tuple, also taken from the same vector **x**, are (conditionally) Gaussian distributed.

Properties of a Gaussian random process

If *N* jointly Gaussian random variables {*x*[*n*]}'s are mutually uncorrelated, they are also mutually independent.

$$\text{If: } c_{X_iX_j} = \operatorname{cov}\{x[i], x[j]\} = 0 \quad \text{for } i \neq j \;\Rightarrow\; \mathbf{C}_X = \begin{bmatrix} \sigma_{X_1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{X_2}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_{X_N}^2 \end{bmatrix},$$

where $\sigma_{X_i}^2 \triangleq \operatorname{var}\{x[i]\}$

$$\text{Then: } (\mathbf{x} - \boldsymbol{\eta}_X)^T \mathbf{C}_X^{-1} (\mathbf{x} - \boldsymbol{\eta}_X) = \sum_{i=1}^{N} \frac{(x_i - \eta_{X_i})^2}{\sigma_{X_i}^2} \quad \text{and} \quad |\mathbf{C}_X| = \prod_{i=1}^{N} \sigma_{X_i}^2$$

$$\text{Hence: } f_X(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^N \prod_{i=1}^{N} \sigma_{X_i}^2}} \exp\left[-\frac{1}{2} \sum_{i=1}^{N} \frac{(x_i - \eta_{X_i})^2}{\sigma_{X_i}^2}\right] = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_{X_i}^2}} e^{-\frac{(x_i - \eta_{X_i})^2}{2\sigma_{X_i}^2}} = \prod_{i=1}^{N} f_{X_i}(x_i)$$

Stationary discrete-time White process

- A *zero-mean*, stationary (at least in wide-sense), discrete-time random process $x[n]$ is called a ***white process***, or a *white noise process,* if the random variables sampled from it are pairwise uncorrelated.

- The mean value and ACF of a discrete-time white process can be expressed as:

$$\eta_X = 0$$

$$R_X[m] = C_X[m] = \sigma_X^2 \delta[m] = \begin{cases} \sigma_X^2 & m = 0 \\ 0 & m \neq 0 \end{cases}$$

- We say that the process is δ–correlated.

> A discrete-time white process need not to be Gaussian. The only requirement is that its mean is zero and the sampled r.v.'s are uncorrelated.

Stationary discrete-time White Gaussian process

If the N random variables $\{x[n]\}$'s have been obtained by sampling a white Gaussian process, they are not only *uncorrelated*, but also *independent*.

In particular, if the process is a w.s.s. white Gaussian process, the $\{x[n]\}$'s are <u>independent and identically distributed</u> (IID):

$$\eta_{X_i} \triangleq E\{x[n]\} = \eta_X, \quad \sigma_{X_i}^2 \triangleq \mathrm{var}\{x[i]\} = \sigma_X^2 \quad \forall i$$

$$\Downarrow$$

$$\boldsymbol{\eta}_X = \eta_X \mathbf{1}, \quad \mathbf{C}_X = \sigma_X^2 \mathbf{I}$$

$$\Downarrow$$

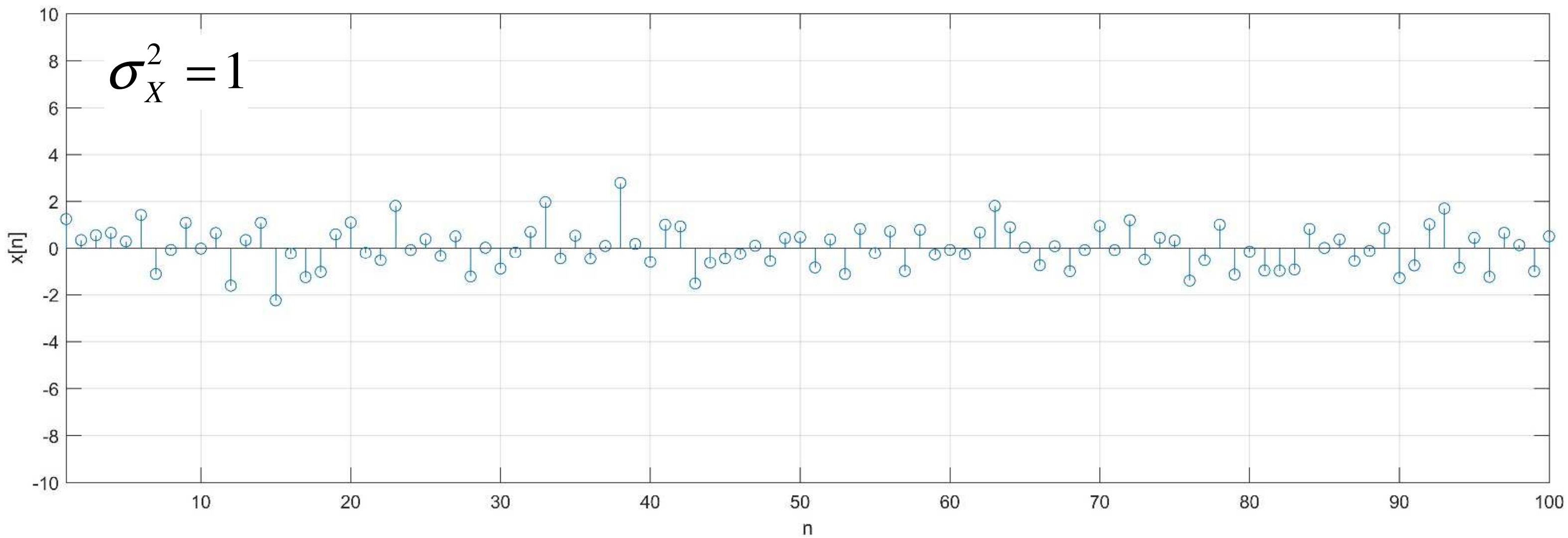
$$f_X(\mathbf{x}) = \prod_{i=1}^{N} f_{X_i}(x_i) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x_i-\eta_X)^2}{2\sigma_X^2}}$$

White Gaussian process: realizations (1/2)

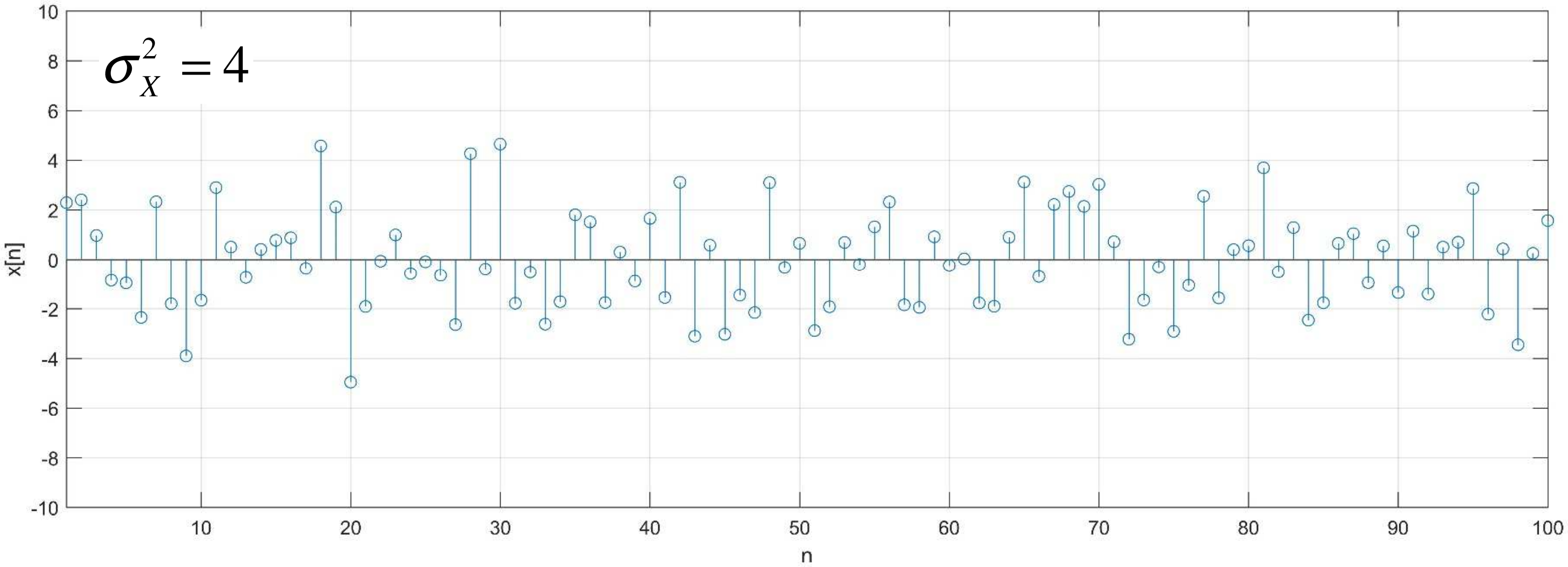
- Examples of possible realizations of a w.s.s. white Gaussian process

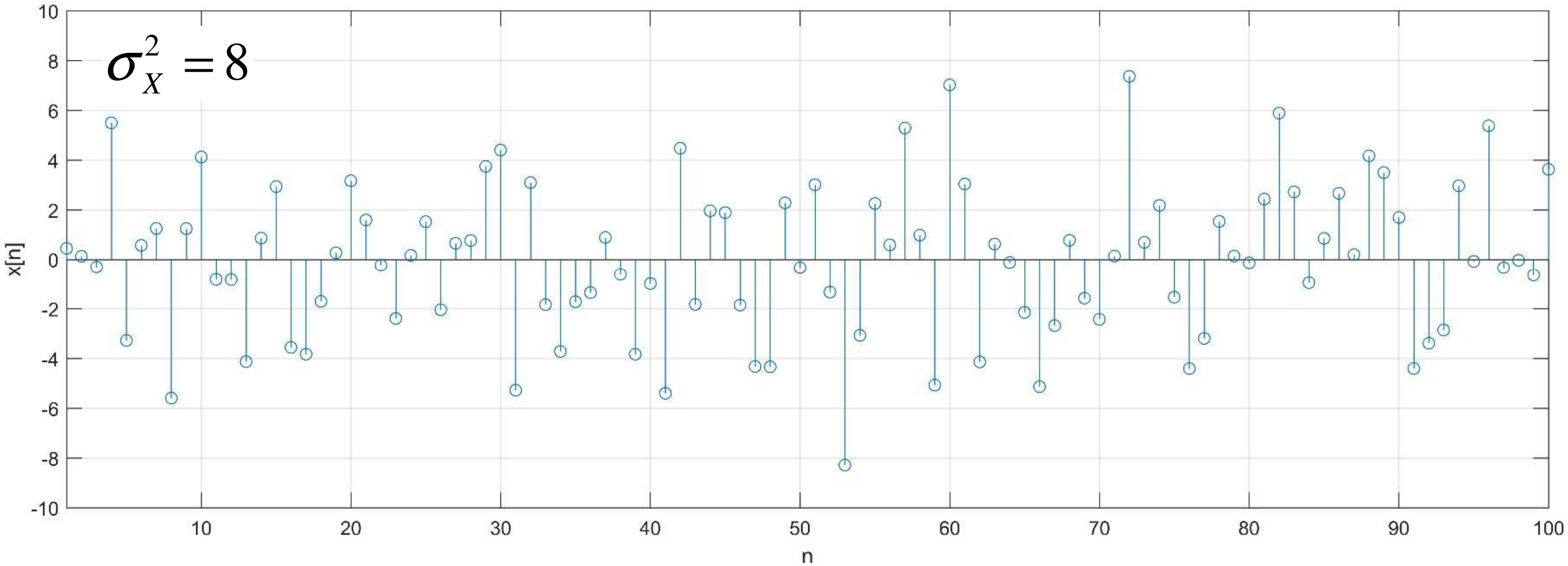
$$x[n] \sim \mathcal{N}\left(0, \sigma_X^2\right), \quad n \in \mathbb{Z} \qquad \text{and} \qquad R_X[m] = \sigma_X^2 \delta[m]$$

for different levels of the power:



White Gaussian process: realizations (2/2)





Stationarity: Example 1

- ***Example***: Let $v[n]$ be a discrete process of IID random variables with mean η and variance σ^2. The ACF of $v[n]$ is:

$$R_V[n_1,n_2] = E\{v[n_1]v[n_2]\} = \begin{cases} E\{v[n_1]\}E\{v[n_2]\} = \eta^2, & n_1 \neq n_2 \\ E\{v^2[n_1]\} = \sigma^2 + \eta^2, & n_1 = n_2 \end{cases}$$

- The ACF can be rewritten as:

$$R_V[n_1,n_2] = \sigma^2\delta[n_2 - n_1] + \eta^2 \equiv \sigma^2\delta[m] + \eta^2 = R_V[m]$$

- Since the mean is constant and the ACF is a function only of the difference $m=n_2-n_1$, $v[n]$ is a w.s.s. discrete-time random process.
- If the mean value is zero, i.e. $\eta=0$, $v[n]$ is a w.s.s. discrete-time white random process.

Stationarity: Example 2

- ***Example***: With reference to *Example* 1, let $x[n]$ be a discrete process such that:

$$x[n] = n \cdot v[n-1]$$

- Its ACF can be expressed as:

$$\begin{aligned} R_X[n_1,n_2] &= E\{x[n_1]x[n_2]\} = E\{n_1 v[n_1-1] n_2 v[n_2-1]\} \\ &= n_1 n_2 E\{v[n_1-1]v[n_2-1]\} = n_1 n_2 \left(\sigma_X^2 \delta[n_2-n_1] + \eta^2\right) \end{aligned}$$

- Since the ACF in *not* purely a function of $m=n_2-n_1$, $x[n]$ is not wide-sense stationary.

- The covariance function of $x[n]$ can be found to be

$$C_X[n_1,n_2] = R_X[n_1,n_2] - \eta_X[n_1]\eta_X[n_2] = R_X[n_1,n_2] - n_1 n_2 \eta^2 = n_1 n_2 \sigma_X^2 \delta[n_2-n_1]$$

which is not a function of only $m=n_2-n_1$.

Stationarity: Example 3

- ***Example***: With reference to *Example* 1, let $x[n]$ be a discrete process such that:

$$x[n] = v[n] + \frac{1}{2}v[n-1]$$

where, for simplicity, we assume that $\eta \equiv 0$ and $\sigma^2 = 1$: $R_V[m] = \sigma^2\delta[m] + \eta^2 = \delta[m]$

- The ACF can be expressed as:

$$R_X[n_1, n_2] = E\{x[n_1]x[n_2]\} = E\left\{\left(v[n_1] + \frac{1}{2}v[n_1 - 1]\right)\left(v[n_2] + \frac{1}{2}v[n_2 - 1]\right)\right\}$$

$$= E\{v[n_1]v[n_2]\} + \frac{1}{2}E\{v[n_1 - 1]v[n_2]\} + \frac{1}{2}E\{v[n_1]v[n_2 - 1]\} + \frac{1}{4}E\{v[n_1 - 1]v[n_2 - 1]\}$$

$$= \frac{5}{4}\delta[n_2 - n_1] + \frac{1}{2}\delta[n_2 - n_1 - 1] + \frac{1}{2}\delta[n_2 - n_1 + 1]$$

$$= \frac{5}{4}\delta[m] + \frac{1}{2}\delta[m-1] + \frac{1}{2}\delta[m+1] = R_X[m]$$

- Since the ACF is a function of only $m = n_2 - n_1$, $x[n]$ is wide-sense stationary.

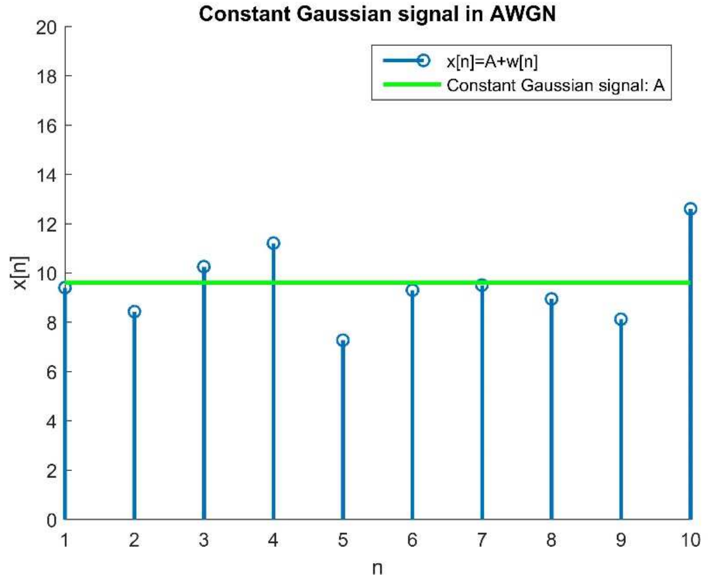
Stationarity: Example 4 (1/2)

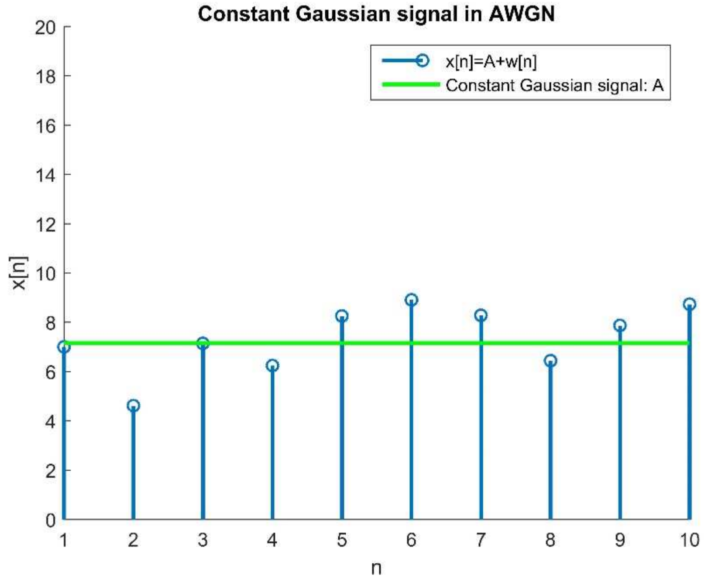
- ***Example***: Let $x[n]$ be a discrete process defined as:

$$x[n] = A + v[n]$$

where $v[n]$ is a stationary white Gaussian process with variance σ^2, while A is a Gaussian random variable such that $A \sim \mathcal{N}(\eta_A, \sigma_A^2)$ and A independent of $v[n]$.

- Two possible realizations: $\eta_A = 10, \sigma_A^2 = 4, \sigma_v^2 = 2$





Stationarity: Example 4 (2/2)

- Evaluation of the ACF of the process *x*[*n*]:

$$x[n] = A + v[n]$$

- *v*[*n*] is a stationary white process with variance σ^2,
- $A \sim \mathcal{N}(\eta_A, \sigma_A^2)$ is independent of *v*[*n*].

- The mean of the process: $\eta_X = E\{x[n]\} = E\{A\} + E\{v[n]\} = \eta_A$

- The ACF can be expressed as:

$$\begin{aligned} R_X[n_1, n_2] &= E\{x[n_1]x[n_2]\} = E\{(A + v[n_1])(A + v[n_2])\} \\ &= E\{A^2\} + E\{Av[n_1]\} + E\{Av[n_2]\} + E\{v[n_1]v[n_2]\} \\ &= \eta_A^2 + \sigma_A^2 + \sigma^2\delta[m] = R_X[m] \end{aligned}$$

$$C_X[m] = R_X[m] - \eta_A^2 = \sigma_A^2 + \sigma^2\delta[m] \quad \rightarrow \quad \lim_{m \to \infty} C_X[m] = \sigma_A^2 \neq 0$$

- Since the ACF is a function of only $m=n_2-n_1$, then *x*[*n*] is wide-sense stationary.

Cross-correlation and cross-covariance functions

- It is often necessary to deal with correlations (or covariances) between two or more random processes.
- To this purpose, the **cross-correlation function** between two discrete-time random processes, *x*[*n*] and *y*[*n*], can be defined as:

$$R_{XY}[n_1,n_2] \triangleq E\{x[n_1]y[n_2]\}$$

- Similarly, the **cross-covariance function** is defined as:

$$C_{XY}[n_1,n_2] \triangleq E\{(x[n_1]-\eta_X[n_1])(y[n_2]-\eta_Y[n_2])\} = R_{XY}[n_1,n_2]-\eta_X[n_1]\eta_Y[n_2]$$

- Two random processes, *x*[*n*] and *y*[*n*], are said to be *orthogonal* if their cross-correlation function is identical to zero.
- *x*[*n*] and *y*[*n*] are said to be *uncorrelated* if their cross-covariance function is zero.

Cross-correlation and cross-covariance functions

- Two discrete random processes, $x[n]$ and $y[n]$, are said to be wide sense *jointly stationary* if:

  1. $x[n]$ and $y[n]$ are wide-sense stationary;
  2. The cross-correlation function is a function of only the *lag* $m=n_2-n_1$:

$$R_{XY}[n_1,n_2]=R_{XY}[0,n_2-n_1]\equiv R_{XY}[n_2-n_1]\equiv R_{XY}[m]$$

- If $x[n]$ and $y[n]$ are *jointly stationary*, then the cross-correlation and the cross-covariance functions are functions only of the *lag* $m=n_2-n_1$:

$$R_{XY}[n_2-n_1]\equiv R_{XY}[m]\triangleq E\left\{x[n]y[n+m]\right\}$$

$$C_{XY}[n_2-n_1]\equiv C_{XY}[m]\triangleq E\left\{\left(x[n]-\eta_X\right)\left(y[n+m]-\eta_Y\right)\right\}=R_{XY}[m]-\eta_X\eta_Y$$

Cross-correlation and cross-covariance functions

- The cross-correlation and the cross-covariance functions have no special symmetry (they can even be one-sided).

- The correlation between $x[n_1]$ and $y[n_2]$ is obviously the same as the correlation between $y[n_2]$ and $x[n_1]$:

$$R_{XY}[n_1,n_2] \triangleq E\{x[n_1]y[n_2]\} = E\{y[n_2]x[n_1]\} = R_{YX}[n_2,n_1]$$

However, the correlation between $x[n_1]$ and $y[n_2]$ is *NOT* the same as the correlation between $y[n_1]$ and $x[n_2]$.

$$R_{XY}[n_1,n_2] \triangleq E\{x[n_1]y[n_2]\} \neq E\{y[n_1]x[n_2]\} = E\{x[n_2]y[n_1]\} = R_{XY}[n_2,n_1]$$

Cross-correlation and cross-covariance functions

- If $x[n]$ and $y[n]$ are *jointly stationary*, then the cross-correlation is a function only of the *lag* $m=n_2-n_1$ and we have:

$$R_{XY}[n_1,n_2]=R_{YX}[n_2,n_1] \quad \Rightarrow \quad R_{XY}[m]=R_{YX}[-m]$$

$$R_{XY}[n_1,n_2]\neq R_{XY}[n_2,n_1] \quad \Rightarrow \quad R_{XY}[m]\neq R_{XY}[-m]$$

- Exactly the same results are valid for the cross-covariance function:

$$C_{XY}[m]=C_{YX}[-m], \qquad C_{XY}[m]\neq C_{XY}[-m]$$

Cross-correlation and cross-covariance functions

Example: **IIR causal filter**

$$x[n] \longrightarrow \boxed{l[n]} \longrightarrow y[n]$$

- *x*[*n*] is a white random process, then $R_X[m] = \sigma_X^2 \delta[m]$
- The output *y*[*n*] can be expressed as: $y[n] = \sum_{k=0}^{+\infty} l[k]x[n-k]$
- Then, the cross-correlation function between *x*[*n*] and *y*[*n*] is

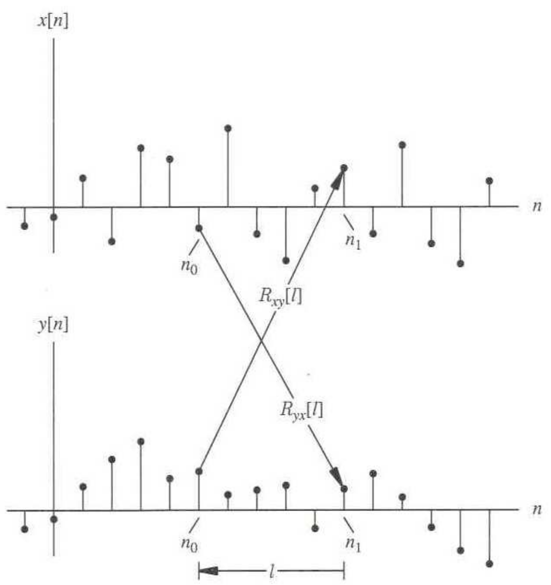
$$R_{XY}[m] \triangleq E\{x[n]y[n+m]\} = E\left\{ x[n]\sum_{k=0}^{+\infty} l[k]x[n+m-k] \right\}$$

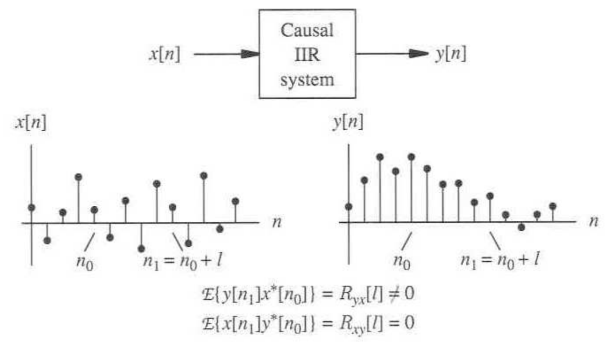
$$= \sum_{k=0}^{+\infty} l[k]E\{x[n]x[n+m-k]\} = \sum_{k=0}^{+\infty} l[k]R_X[m-k] = \sigma_X^2 l[m]$$

- Since *l*[*n*] is causal, we have: $R_{XY}[m] \neq 0$ only if $m \geq 0 \quad \Rightarrow \quad R_{XY}[m] \neq R_{XY}[-m]$

Cross-correlation and cross-covariance functions

- In particular, let $x[n_0]$ and $y[n_1]$ be two samples from the input and from the output signals, respectively.
- If $n_0<n_1$, i.e. $m=n_1-n_0>0$, due to the filter memory, the input at n_0 affects the output at n_1, so $x[n_0]$ and $y[n_1]$ are correlated.
- On the contrary, the input at n_1 does *NOT* affects the output at n_0, since the filter is causal. Consequently, $y[n_0]$ and $x[n_1]$ are *NOT* correlated.





(a)

(b)

Power Spectral Density (PSD)

- There are two possible definitions of the **Power Spectral Density (PSD)** of w.s.s. discrete-time random process:

Definition 1: $$S_X(e^{j2\pi f}) \triangleq \lim_{N\to\infty} \frac{1}{N} E\left\{\left|X_N(e^{j2\pi f})\right|^2\right\}$$

where $$X_N(e^{j2\pi f}) \triangleq DTFT\left\{x[n]\right\} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi fn}$$

Definition 2: $$S_X\left(e^{j2\pi f}\right) \triangleq DTFT\left\{R_X[m]\right\} = \sum_{m=-\infty}^{\infty} R_X[m]e^{-j2\pi fm}$$

where $R_X[m] \triangleq E\left\{x[n]x^*[n+m]\right\} = R_X^*[-m]$ is the ACF

[The conjugate is present only for complex-valued processes]

Power Spectral Density (PSD)

- The second definition of the PSD is also known as the **Einstein–Wiener–Khintchine Theorem:**

$$S_X\left(e^{j2\pi f}\right) \triangleq \sum_{m=-\infty}^{+\infty} R_X[m]e^{-j2\pi fm}, \quad |f| \leq 1/2$$

- Note that the PSD is *periodic* with period equal to 1.
- The inverse relation is given by:

$$R_X[m] = \int_{-1/2}^{1/2} S_X\left(e^{j2\pi f}\right)e^{j2\pi fm}df$$

- Why the term "Power Spectral Density"? The reason is clear from:

$$P_X \triangleq E\left\{|x[n]|^2\right\} = R_X[0] = \int_{-1/2}^{1/2} S_X\left(e^{j2\pi f}\right)df$$

Properties of the PSD

- Let $x[n]$ be a **real** discrete-time random process, then its PSD satisfies two fundamental properties:

1. The PSD is a *real* and *even* function of f

$$S_X\left(e^{j2\pi f}\right) \in \mathbb{R}, \quad S_X\left(e^{j2\pi f}\right) = S_X\left(e^{-j2\pi f}\right), \quad |f| \le 1/2$$

2. The PSD is non-negative

$$S_X\left(e^{j2\pi f}\right) \ge 0, \quad |f| \le 1/2$$

- It is easy to show that Properties 1 and 2 derive directly from the two corresponding properties of the ACF and from the properties of the FT for discrete-time signals.

The link between the ACF and the PSD

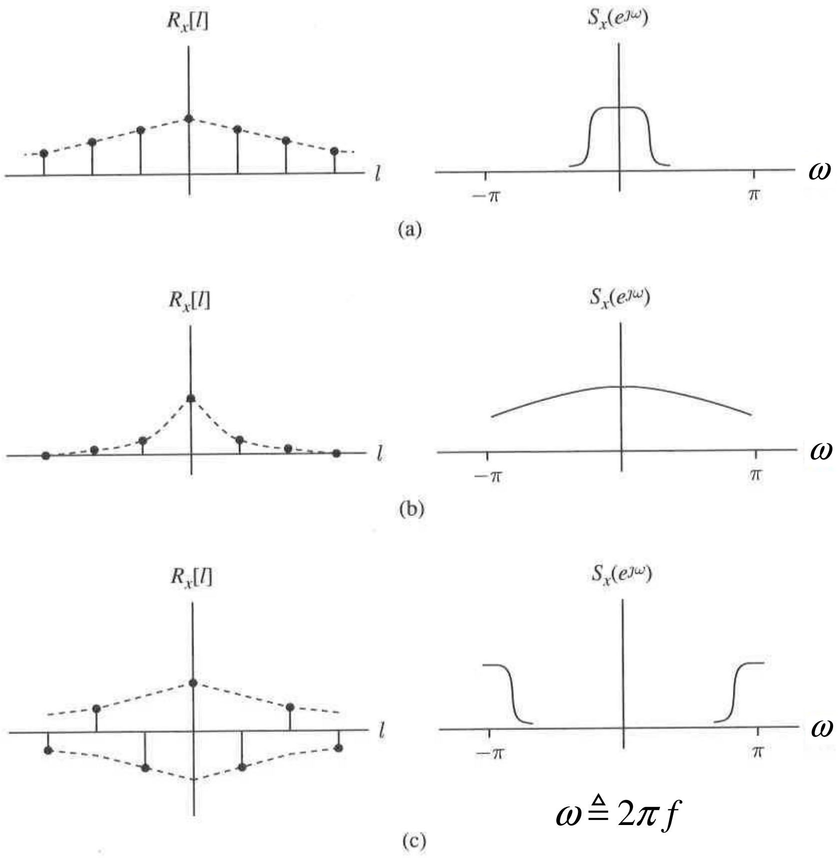


Figure Comparison of correlation functions and power density spectra. (a) High correlation. (b) Low correlation. (c) High negative correlation.

- When the ACF is broad, indicating small changes between samples of $x[n]$, the PSD is narrow, i.e. $x[n]$ has only low-frequency components.

- When the ACF is narrow, indicating low correlation between samples that are not too far, the PSD is broad, i.e. $x[n]$ can have high-frequency components which allows it to change significantly (and irregularly) between successive samples.

- When the ACF shows rapidly varying characteristic, corresponding to negative correlation, the PSD contains *only* high frequencies (i.e. close to 1/2).

The PSD of a white discrete-time random process

- As previously discussed, any discrete-time zero-mean random process whose samples are uncorrelated is called a **discrete-time white process**.
- Consequently, the ACF of a w.s.s. discrete-time white random process is

$$R_W[m] = \sigma_W^2 \delta[m]$$

- Then, the PSD is given by:

$$S_W\left(e^{j2\pi f}\right) = \sigma_W^2 \sum_{m=-\infty}^{+\infty} \delta[m] e^{-j2\pi fm} = \sigma_W^2$$

- The PSD is constant over all the frequency band, and this fact justifies the name "white process".
- Unlike a continuous-time white process, a discrete-time white process has finite power equal to:

$$R_W[0] = \int_{-1/2}^{1/2} S_W\left(e^{j2\pi f}\right) df = \int_{-1/2}^{1/2} \sigma_W^2 df = \sigma_W^2$$

PSD: continuous and discrete parts

- In the most general case, the PSD can be written in the form:

$$S_X\left(e^{j2\pi f}\right)=S_X^c\left(e^{j2\pi f}\right)+2\pi\sum_{i=1}^{L}P_i\delta\left(e^{j2\pi f}-e^{j2\pi f_i}\right)$$

- The first term represents the continuous part of the spectrum, the second term is the discrete part, i.e. the Dirac's delta functions present in the spectrum.

- The Dirac's delta functions arise from *periodic* or *almost periodic* components of the discrete random process. The continuous part in the PSD is due to the aperiodic (or transient) components in the data.

- **Note:** Let $s[n]=\cos\left(2\pi F_0 n+\theta_0\right)$

  1. If $F_0=K/N$ is the ratio between two integers, even if N/K is not an integer, then $s[n]$ is said to be periodic (of period N).

  2. If F_0 is not a rational number, then $s[n]$ is said to be quasi-periodic.

Example: cosinusoidal process in AWGN

- **Example:** cosinusoidal process in Additive White Gaussian Noise (AWGN)

$$x[n] = s[n] + w[n] = A\cos\left(2\pi F_0 n + \theta_0\right) + w[n], \quad 0 \le F_0 < \frac{1}{2}$$

- The amplitude A can be modelled as a Rayleigh r.v., the phase θ_0 is typically modelled as a r.v. uniformly distributed in $[-\pi,\pi)$, both are independent of $w[n]$.

$$A \in \mathcal{R}(\sigma_A^2) \qquad \theta_0 \in \mathcal{U}(-\pi,\pi) \qquad A \text{ and } \theta_0 \text{ indipendent}$$

$$f_A(a) = \frac{a}{\sigma_A^2} e^{-\frac{a^2}{2\sigma_A^2}} u(a), \quad E\{A^2\} = 2\sigma_A^2$$

- Autocorrelation function (ACF):

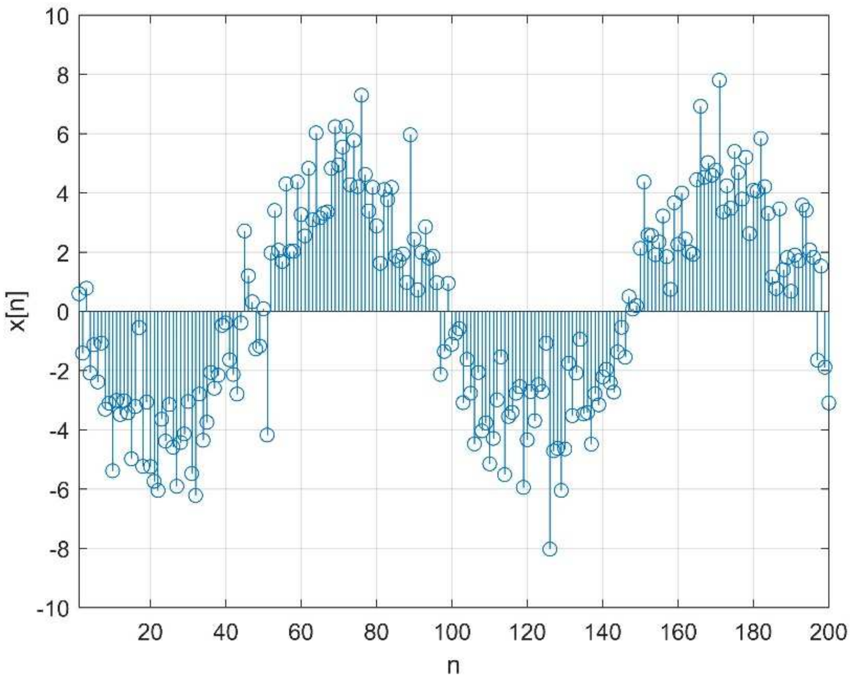
$$\begin{aligned} R_X[m] &= E\{x[n]x[n+m]\} = E\left\{\left(s[n]+w[n]\right)\left(s[n+m]+w[n+m]\right)\right\} \\ &= R_S[m] + R_{SW}[m] + R_{WS}[m] + R_W[m] = R_S[m] + R_W[m] \end{aligned}$$

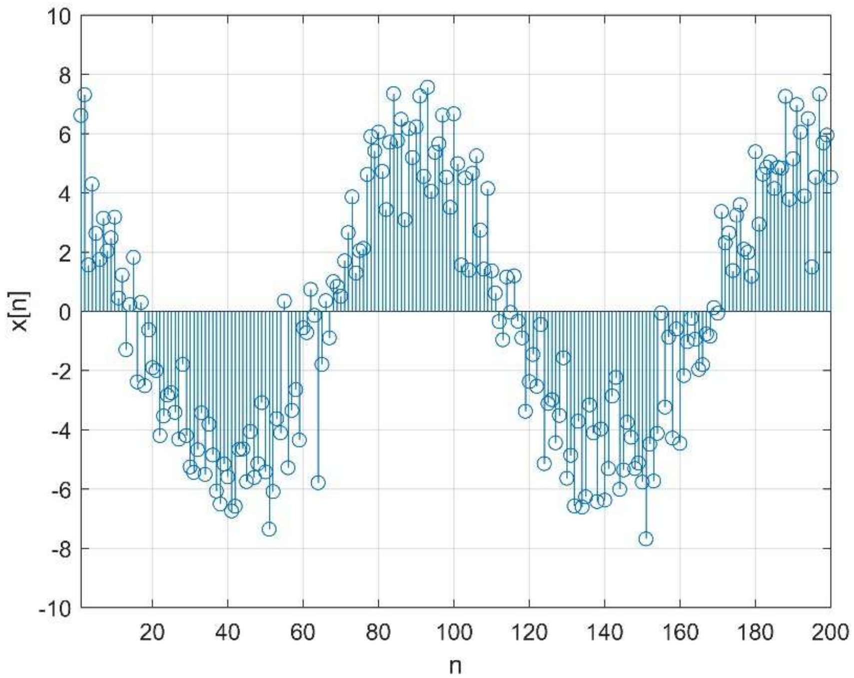
Example: cosinusoidal process in AWGN

- Examples of realizations of a cosinusoidal process in AWGN:

$$x[n] = A\cos\left(2\pi F_0 n + \theta_0\right) + w[n], \quad F_0 = 0.1$$

$$A \in \mathcal{R}(\sigma_A^2), \quad \sigma_A^2 = 2 \qquad \theta_0 \in \mathcal{U}(-\pi,\pi) \qquad w[n] \sim \mathcal{N}\left(0,\sigma_W^2\right), \sigma_W^2 = 2$$





Example: cosinusoidal process in AWGN

- Autocorrelation function of the cosinusoidal process:

$$R_S[m] = E\{s[n]s[n+m]\} = E\{A^2\}E\{\cos(2\pi F_0 n + \theta_0)\cos(2\pi F_0(n+m) + \theta_0)\}$$

$$= \frac{E\{A^2\}}{2}E\{\cos(2\pi F_0 m) + \cos(2\pi F_0(2n+m) + 2\theta_0)\} = \sigma_A^2 \cos(2\pi F_0 m)$$

where the last equality follows from the Werner formula and from the fact that

$$E\{\cos(2\pi F_0(2n+m) + 2\theta_0)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\pi F_0(2n+m) + 2\theta_0)\, d\theta_0 = 0, \quad \forall n \in \mathbb{Z}$$

- Autocorrelation function of the AWGN:

$$R_W[m] = \sigma_W^2 \delta[m]$$

- Finally, the autocorrelation function of *x*[*n*] can be expressed as:

$$R_X[m] = R_S[m] + R_W[m] = \sigma_A^2 \cos(2\pi F_0 m) + \sigma_W^2 \delta[m]$$

Example: cosinusoidal process in AWGN

- The mean value of $x[n]$ is:

$$
\begin{aligned}
E\{x[n]\} &= E\{s[n]\} + E\{w[n]\} \\
&= E\{A\cos(2\pi F_0 n + \theta_0)\} \\
&= E\{A\}E\{\cos(2\pi F_0 n + \theta_0)\} = 0
\end{aligned}
$$

where the last equality follows from the fact that

$$
E\{\cos(2\pi F_0 n + \theta_0)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\pi F_0 n + \theta_0)\, d\theta_0 = 0, \quad \forall n \in \mathbb{Z}
$$

- Since the mean is constant and the ACF is a function of only the lag m, the process $x[n]$ is a wide sense stationary (w.s.s.) process. Hence, we can calculate the Power spectral Density (PSD).

Example: cosinusoidal process in AWGN

- The Power Spectral Density (PSD) is given by the FT of the ACF:

$$S_X\left(e^{j2\pi f}\right) \triangleq \sum_{m=-\infty}^{+\infty} R_X[m]e^{-j2\pi fm}$$

$$= FT\left\{\sigma_A^2 \cos\left(2\pi F_0 m\right) + \sigma_W^2 \delta[m]\right\}$$

$$= \sigma_A^2 FT\left\{\cos\left(2\pi F_0 m\right)\right\} + \sigma_W^2 FT\left\{\delta[m]\right\}$$

$$= \underbrace{2\pi \cdot \frac{\sigma_A^2}{2}\left[\delta\left(e^{j2\pi f} - e^{j2\pi F_0}\right) + \delta\left(e^{j2\pi f} - e^{-j2\pi F_0}\right)\right]}_{\text{Discrete part of the spectrum}} + \sigma_W^2$$

Continuous part of the spectrum

Cosinusoidal signal (periodic) in AWGN (aperiodic)

$$x[n] = s[n] + w[n] = \underbrace{A\cos\left(2\pi F_0 n + \theta_0\right)}_{\text{Periodic component of the process}} + w[n]$$

Aperiodic component

$$R_X[m] = R_S[m] + R_W[m] = \underbrace{\sigma_A^2 \cos\left(2\pi F_0 m\right)}_{\text{Periodic component of the ACF}} + \sigma_W^2 \delta[m]$$

Aperiodic component of the ACF

$$S_X\left(e^{j2\pi f}\right) = S_S\left(e^{j2\pi f}\right) + S_W\left(e^{j2\pi f}\right)$$

$$= \underbrace{2\pi \cdot \frac{\sigma_A^2}{2}\left[\delta\left(e^{j2\pi f} - e^{j2\pi F_0}\right) + \delta\left(e^{j2\pi f} - e^{-j2\pi F_0}\right)\right]}_{\text{Discrete part of the spectrum}} + \sigma_W^2$$

Continuous part of the spectrum

Cosinusoidal signal (periodic) in AWGN (aperiodic)

- The statistical power of the cosinusoidal process *s*[*n*]:

$$P_S = E\left\{s^2[n]\right\} = R_S[0] = \frac{E\left\{A^2\right\}}{2} = \sigma_A^2$$

$$\begin{aligned}
P_S &= \int_{-1/2}^{1/2} S_S\left(e^{j2\pi f}\right) df \\
&= \int_{-1/2}^{1/2} 2\pi \cdot \frac{\sigma_A^2}{2}\left[\delta\left(e^{j2\pi f} - e^{j2\pi F_0}\right) + \delta\left(e^{j2\pi f} - e^{-j2\pi F_0}\right)\right] df \\
&= \frac{\sigma_A^2}{2} \int_{-1/2}^{1/2} 2\pi \cdot \delta\left(e^{j2\pi f} - e^{j2\pi F_0}\right) df + \frac{\sigma_A^2}{2} \int_{-1/2}^{1/2} 2\pi \cdot \delta\left(e^{j2\pi f} - e^{-j2\pi F_0}\right) df \\
&= \frac{\sigma_A^2}{2} \int_{-\pi}^{\pi} \delta\left(e^{j\omega} - e^{j2\pi F_0}\right) d\omega + \frac{\sigma_A^2}{2} \int_{-\pi}^{\pi} \delta\left(e^{j\omega} - e^{-j2\pi F_0}\right) d\omega \\
&= \frac{\sigma_A^2}{2} + \frac{\sigma_A^2}{2} = \sigma_A^2
\end{aligned}$$

Example: Costant signal (periodic) in AWGN (aperiodic)

- **Example**: $x[n] = A + v[n]$

- $v[n]$ is a stationary white process with variance σ^2
- $A \sim \mathcal{N}(\eta_A, \sigma_A^2)$ is independent of $v[n]$. A represents a periodic component at zero frequency.

- The mean of the process is: $\eta_X \triangleq E\{x[n]\} = E\{A\} + E\{v[n]\} = \eta_A$

- We already derived the ACF: $R_X[m] = \sigma_A^2 + \eta_A^2 + \sigma^2 \delta[m]$

- The PSD is given by: $S_X\left(e^{j2\pi f}\right) = FT\{R_X[m]\} = FT\left\{\sigma_A^2 + \eta_A^2\right\} + FT\left\{\sigma^2 \delta[m]\right\}$

$$= 2\pi \cdot \left(\sigma_A^2 + \eta_A^2\right) \delta\left(e^{j2\pi f} - 1\right) + \sigma^2$$

Example: Costant signal (periodic) in AWGN (aperiodic)

- PSD: $S_X\left(e^{j2\pi f}\right) = 2\pi \cdot \left(\sigma_A^2 + \eta_A^2\right)\delta\left(e^{j2\pi f} - 1\right) + \sigma^2$
- **Comment**: A random process *x*[*n*] with a non-zero mean value η_A has a Dirac's delta function of area $(\eta_A)^2$ in the PSD at *f*=0.

- If the process is zero-mean, i.e. $\eta_X \triangleq E\{x[n]\} = \eta_A = 0$
- ACF: $R_X[m] = \sigma_A^2 + \sigma^2\delta[m]$
- PSD: $S_X\left(e^{j2\pi f}\right) = 2\pi \cdot \sigma_A^2\delta\left(e^{j2\pi f} - 1\right) + \sigma^2$
- A random process *x*[*n*] that contains a random constant component *A* (i.e. periodic at zero frequency) has a Dirac's delta function at *f*=0 of area *var*{*A*} in the PSD.

Some considerations

- **Case 1**: A random process $x[n]$ with a non-zero mean value η_x has a Dirac's delta function at f=0 of area $(\eta_x)^2$ in the PSD.

- **Case 2**: A random process $x[n]$ with a zero mean random constant component (i.e. periodic at zero frequency) has a Dirac's delta function at f=0 of area $var\{A\}$ in the PSD.

⚠ It is not possible to discriminate between Case 1 and Case 2 through the knowledge of the ACF or the PSD. We need some a priori information.

Complex Power Spectral Density (1/3)

- The use of the **z-transform** in the analysis of stationary discrete-time random processes is of fundamental importance.

- A quantity related to the second order moments of a stationary discrete-time random process in the domain of the *z*-transform is the **complex PSD (CPSD)**:

$$S_X(z) \triangleq ZT\{R_X[m]\} = \sum_{m=-\infty}^{+\infty} R_X[m] z^{-m}, \quad z \in ROC \subseteq \mathbb{C}$$

- **ROC** = **Region of convergence**, i.e. those values of z for which the sum above converges to a finite value.

- The ROC of a complex PSD is always an annular region, i.e. of the form:

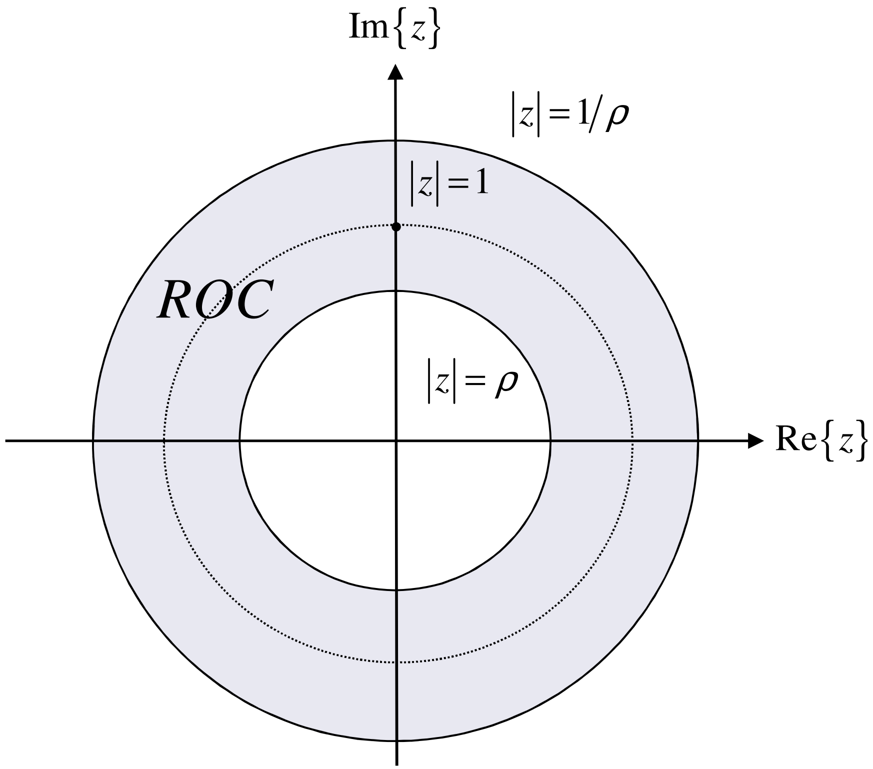
$$ROC \equiv \{z \mid \rho < |z| < 1/\rho\}, \quad \text{with } 0 < \rho < 1$$

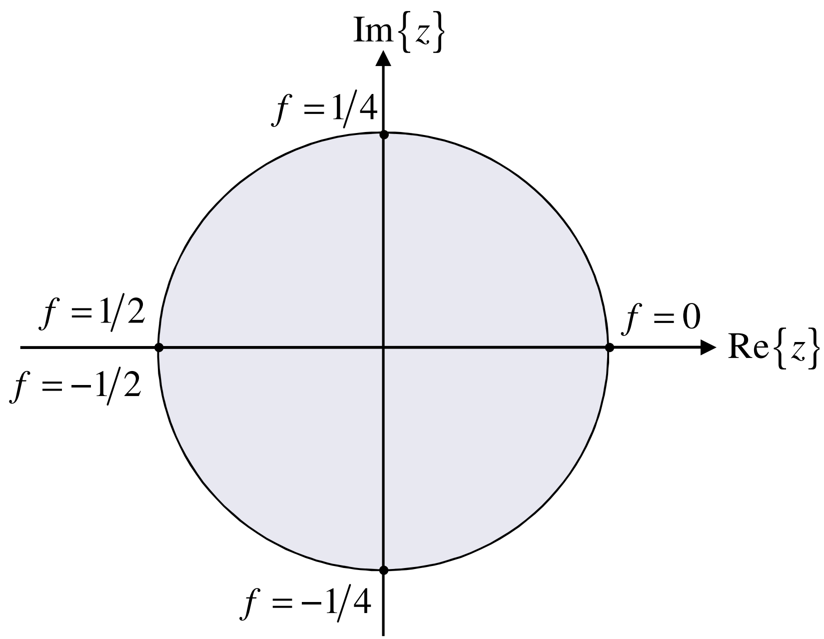
Complex Power Spectral Density (2/3)

- The PSD introduced before can be regarded as a special case of the CPSD when the latter is evaluated on the unit circle (the unit circle belongs to the ROC):

$$S_X\left(e^{j2\pi f}\right) = S_X(z)\Big|_{z=e^{j2\pi f}}$$

$$z = e^{j2\pi f} \rightarrow |z| = 1, \quad \angle z = 2\pi f$$





Complex Power Spectral Density (3/3)

- The inverse formula is:

$$R_X[m] = \frac{1}{j2\pi}\oint_{\partial C} S_X(z)\, z^{m-1} dz$$

where the contour of the integration region *C, i.e.* ∂C, is taken to be counterclockwise and in the convergence region of the z-transform.

- From the property of symmetry of the ACF, $R_X[-m] = R_X^*[m]$, we have that:

$$S_X(z) = \sum_{m=-\infty}^{+\infty} R_X[m] z^{-m} = \sum_{m=-\infty}^{+\infty} R_X[-m] z^{m} = \left(\sum_{m=-\infty}^{+\infty} R_X[m] \left(1/z^*\right)^{-m} \right)^* = S_X^*\left(1/z^*\right)$$

- For <u>real-valued</u> random processes, i.e. for real ACFs, we have:

$$S_X(z) = S_X^*\left(1/z^*\right) = S_X(1/z) = S_X\left(z^{-1}\right)$$

Complex PSD: continuous and discrete parts

- As for the PSD, the complex PSD can be written in the form:

$$S_X(z) = S_X^c(z) + 2\pi \sum_{i=1}^{L} P_i \delta\left(z - e^{j2\pi f_i}\right)$$

- The first term represents the continuous part of the spectrum, the second term is the discrete part, i.e. the Dirac's delta functions present in the spectrum.

- The Dirac's delta functions arise from *periodic* or *almost periodic* components of the random process. The continuous part is due to the *aperiodic* components of the random process.

Note that, due to mathematical problems related to the Dirac's delta function, the direct and the inverse formulas that link the complex PSD and the ACF are valid only for the continuous part of the complex PSD.

Poles and zeros of the Complex PSD

- The continuous part of the complex PSD can be usually expressed as the ratio of two polynomials in z (we will investigate this in details later on).

- Due to the following relations for *complex-valued* and *real-valued* random processes:

$$S_X(z) = S_X^*\left(1/z^*\right) \qquad\qquad S_X(z) = S_X\left(z^{-1}\right)$$

the polynomials at the numerators and at the denominator have to be factorized as:

$$\left(1 - z_0 z^{-1}\right) \quad \rightarrow \quad \left(1 - z_0 z^*\right)^* = \left(1 - z_0^* z\right) \quad \Rightarrow \quad \left(1 - z_0 z^{-1}\right)\left(1 - z_0^* z\right)$$

- This implies that for every pole (and zero) that occurs at

$$z_0 = \left|z_0\right| e^{j\phi_0}$$

there is a corresponding pole (and zero) at

$$\frac{1}{z_0^*} = \left(\frac{1}{\left|z_0\right|} e^{-j\phi_0}\right)^* = \frac{1}{\left|z_0\right|} e^{j\phi_0}$$

Poles and zeros of the Complex PSD: Properties

- Hence, poles and zeros of the CPSD occur at conjugate reciprocal positions.
- For real-valued processes the CPSD can be expressed in terms of real polynomials in *z*, as a consequence the roots must occur in conjugate pairs.
- This implies that for real-valued random processes, poles and the zeros of the complex PSD occur in group of four at locations:

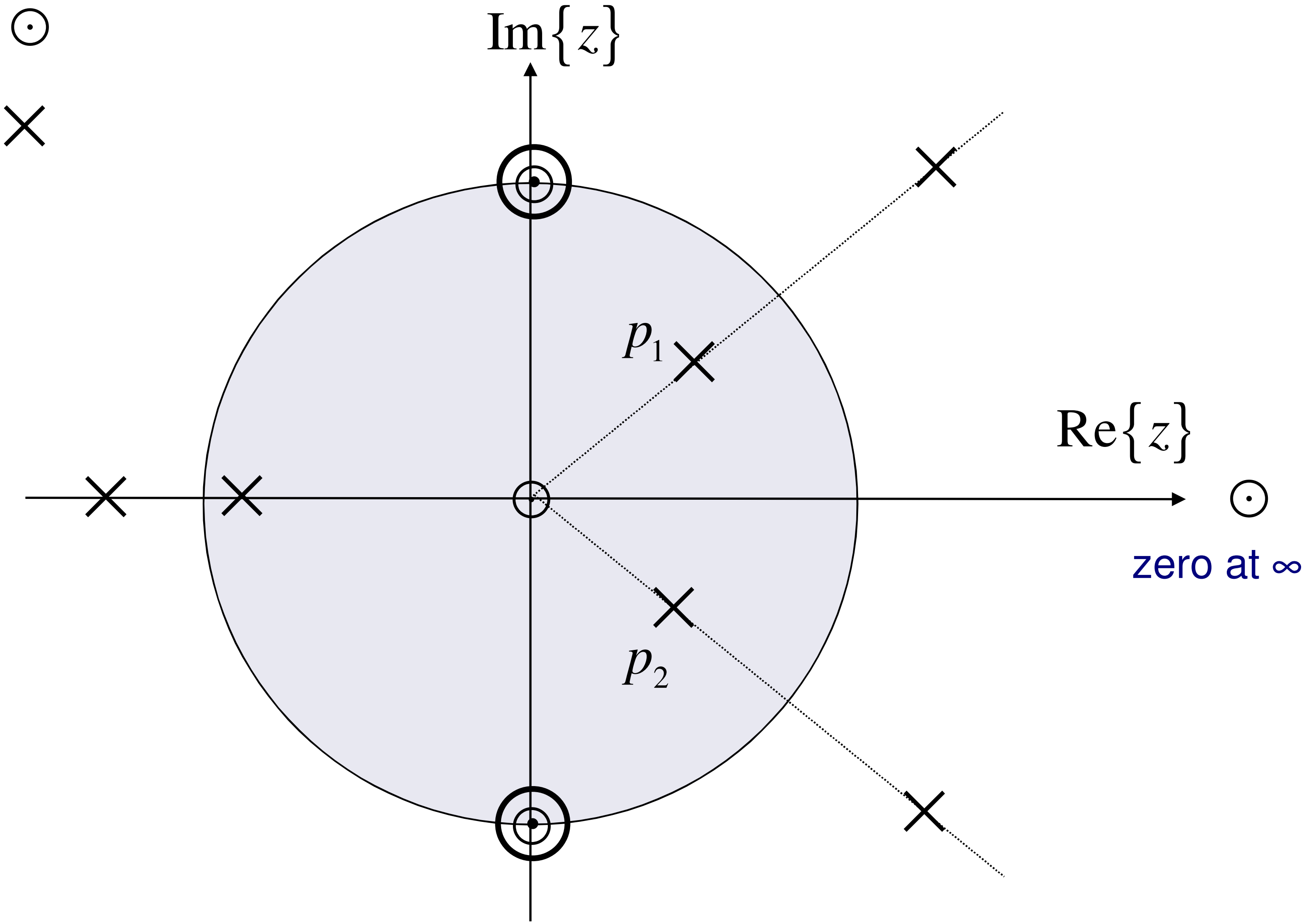
$$z_0, \quad 1/z_0^*, \quad z_0^*, \quad 1/z_0$$

- Roots on real axis occur in pair at locations: $z_0, 1/z_0$.
- When zeros occur on the unit circle, they must occur in even multiplicities.
- Poles on the unit circle are not allowed to exist; they could eventually contribute to the discrete part of the complex PSD.
- A pole (zero) at the origin implies a pole (zero) at ∞.

Poles and zeros for real random processes

Zeros: $\odot$

Poles: $\times$



- The ROC cannot contain any pole.

Region of convergence

- The symmetry property of the complex PSD of real-valued processes, i.e.

$$S_X(z) = S_X(z^{-1}) \qquad (*)$$

has specific implications in the shape of the region of convergence.

- In particular, if $S_X(z)$ converges only in the region $|z| > \rho$ where ρ is a positive real number, because of property (*), $S_X(z)$ should also converge for z such that $|z^{-1}| > \rho$ i.e. $|z| < 1/\rho$

- Then, the complex PSD always converges on an annular region of the form:

$$\rho < |z| < 1/\rho, \quad 0 < \rho < 1$$

- ρ must be strictly less then 1, otherwise the convergence region is an empty set.

The unitary circle belongs to the convergence region.

Linear Time-Invariant systems (1/7)

Linear shift-invariant system

$$w[n] \longrightarrow \boxed{\text{LTI system}} \longrightarrow x[n]$$

- A linear time-invariant (LTI) system can be characterized by its *impulse response*, say $l[n]$.

- The impulse response is defined as the output of the LTI system when the input is the Kronecker delta function with unit amplitude.

$$w[n] = \delta[n] \longrightarrow \boxed{l[n]} \longrightarrow x[n] = l[n]$$

- The system input and output are related by the linear convolution operator:

$$x[n] = w[n] \otimes l[n] = \sum_{k=-\infty}^{+\infty} l[k] w[n-k] = \sum_{k=-\infty}^{+\infty} w[k] l[n-k]$$

Linear Time-Invariant systems (2/7)

- Along with the *impulse response* $l[n]$, an LTI system can be equivalently characterized by the *frequency response* and by the *transfer function.*

- The **transfer function** is defined as the Z-transform of $l[n]$:

$$L(z) = Z\{l[n]\} = \sum_{n=-\infty}^{+\infty} l[n] z^{-n}$$

- The **frequency response** is defined as the Fourier Transform of $l[n]$:

$$L\left(e^{j2\pi f}\right) = FT\{l[n]\} = \sum_{n=-\infty}^{+\infty} l[n] e^{-j2\pi fn} = L(z)\Big|_{z=e^{j2\pi f}}$$

- We are interested in the functional relations that link the 1st and 2nd order moments of a random process at the input of the LTI system with those at the output.

Linear Time-Invariant systems (3/7)

Linear Time-Invariant system

$$w[n] \rightarrow \boxed{l[n]} \rightarrow x[n]$$

$$x[n] = w[n] \otimes l[n] = \sum_{k=-\infty}^{+\infty} l[k]w[n-k] = \sum_{k=-\infty}^{+\infty} w[k]l[n-k]$$

- The mean value functions of the input and output random processes are related by the following expression:

$$\eta_X[n] = E\{x[n]\} = E\{w[n] \otimes l[n]\} = E\{w[n]\} \otimes l[n] = \eta_W[n] \otimes l[n]$$

- If the input is a w.s.s. random process, then

$$\eta_X[n] = \eta_W \otimes l[n] = \sum_{k=-\infty}^{+\infty} \eta_W l[k] = \eta_W \sum_{k=-\infty}^{+\infty} l[k] = \eta_W L\left(e^{j2\pi f}\right)\Big|_{f=0} = \eta_W L(1)$$

Linear Time-Invariant systems (4/7)

Linear Time-Invariant system

$$w[n] \longrightarrow \boxed{\;l[n]\;} \longrightarrow x[n]$$

$$x[n] = w[n] \otimes l[n] = \sum_{k=-\infty}^{+\infty} l[k] w[n-k] = \sum_{k=-\infty}^{+\infty} w[k] l[n-k]$$

- If *w*[*n*] is a w.s.s. random process, then the output *x*[*n*] is w.s.s. as well. Moreover, *w*[*n*] and *x*[*n*] are *jointly wide-sense stationary*.

- The ACF of the output of an LTI system can be expressed:

$$
\begin{aligned}
R_X[m] &= E\{x[n]x[n+m]\} = E\left\{\sum_{k=-\infty}^{+\infty} l[k]w[n-k]x[n+m]\right\} \\
&= \sum_{k=-\infty}^{+\infty} l[k]E\{w[n-k]x[n+m]\} = \sum_{k=-\infty}^{+\infty} l[k]R_{WX}[m+k] \\
&= l[m] \otimes R_{WX}[-m]
\end{aligned}
$$

where $R_{WX}[m]$ is the cross-correlation function between the input and the output.

Linear Time-Invariant systems (5/7)

- The cross-correlation function $R_{WX}[m]$ can be explicitly evaluated as

$$R_{WX}[m] = E\{w[n]x[n+m]\} = E\left\{w[n]\sum_{k=-\infty}^{+\infty} l[k]w[n+m-k]\right\}$$
$$= \sum_{k=-\infty}^{+\infty} l[k]E\{w[n]w[n+m-k]\} = \sum_{k=-\infty}^{+\infty} l[k]R_W[m-k]$$
$$= l[m] \otimes R_W[m]$$

- Finally, the ACF of the output of a LTI system is:

$$R_X[m] = l[m] \otimes R_{WX}[-m] = l[m] \otimes l[-m] \otimes R_W[-m]$$
$$= l[m] \otimes l[-m] \otimes R_W[m]$$

$$R_X[m] = l[m] \otimes l[-m] \otimes R_W[m]$$

Linear Time-Invariant systems (6/7)

- By exploiting the properties of the convolution operator and of the Fourier Transform, the relation between the PSDs of the input and output is:

$$
\begin{aligned}
S_X\left(e^{j2\pi f}\right) &= FT\left\{R_X[m]\right\} = FT\left\{l[m]\otimes l[-m]\otimes R_W[m]\right\} \\
&= L\left(e^{j2\pi f}\right)L\left(e^{-j2\pi f}\right)S_W\left(e^{j2\pi f}\right) \\
&= L\left(e^{j2\pi f}\right)L^*\left(e^{j2\pi f}\right)S_W\left(e^{j2\pi f}\right) \\
&= \left|L\left(e^{j2\pi f}\right)\right|^2 S_W\left(e^{j2\pi f}\right)
\end{aligned}
$$

- Similarly, using the properties of the Z-transform, we can derive the input-output relation for the complex PSD:

$$
\begin{aligned}
S_X(z) &= ZT\left\{R_X[m]\right\} = ZT\left\{l[m]\otimes l[-m]\otimes R_W[m]\right\} \\
&= L(z)L(1/z)S_W(z)
\end{aligned}
$$

Linear Time-Invariant systems (7/7)

Linear Time-Invariant system

$$w[n] \longrightarrow \boxed{l[n]} \longrightarrow x[n]$$

$$x[n] = w[n] \otimes l[n] = \sum_{k=-\infty}^{+\infty} l[k] w[n-k] = \sum_{k=-\infty}^{+\infty} w[k] l[n-k]$$

- If the input process *w*[*n*] is a white process, i.e.

$$R_W[m] = \sigma_W^2 \delta[m], \qquad S_W\left(e^{j2\pi f}\right) = \sigma_W^2$$

the previous relations simplify to:

$$R_X[m] = R_W[m] \otimes l[m] \otimes l[-m] = \sigma_W^2 l[m] \otimes l[-m]$$

$$S_X\left(e^{j2\pi f}\right) = S_W\left(e^{j2\pi f}\right) \left|L\left(e^{j2\pi f}\right)\right|^2 = \sigma_W^2 \left|L\left(e^{j2\pi f}\right)\right|^2$$

$$S_X(z) = L(z) L(1/z) S_W(z) = \sigma_W^2 L(z) L(1/z)$$

The Autoregressive process of order 1: AR(1)

Linear Time-Invariant system

$$w[n] \longrightarrow \boxed{\;l[n]\;} \longrightarrow x[n]$$

$$x[n] = w[n] \otimes l[n] = \sum_{k=-\infty}^{+\infty} l[k]w[n-k] = \sum_{k=-\infty}^{+\infty} w[k]l[n-k]$$

- The input process $w[n]$ is a white process, i.e.

$$R_W[m] = \sigma_W^2 \delta[m] \qquad\qquad S_W\left(e^{j2\pi f}\right) = \sigma_W^2$$

- An **AR(1) process** is characterized by the following linear difference equation:

$$x[n] = \rho x[n-1] + w[n], \quad |\rho| < 1$$

- An **AR(1) process**, also called **Markov process**, can be thought as the output of an IIR(1) LTI system with a suitable impulsive response driven by a white process.

The Autoregressive process of order 1: AR(1)

- The **impulse response** $l[n]$ of the IIR(1) filter that that generates an AR(1) process can be evaluated by solving the relevant linear difference equation with a Kronecker delta as input sequence:

$$x[n] = \rho x[n-1] + w[n]$$

$$\Downarrow$$

$$l[n] = \rho l[n-1] + \delta[n], \quad \text{with } l[-1] = 0 \text{ and } |\rho| < 1$$

$$\Downarrow$$

$$\begin{cases} l[0] = \rho l[-1] + \delta[0] = 1 \\ l[1] = \rho l[0] + \delta[1] = \rho \\ l[2] = \rho l[1] + \delta[2] = \rho^2 \\ \vdots \\ l[n] = \rho l[n-1] + \delta[n] = \rho^n \end{cases}$$

- Then, finally: $l[n] = \rho^n u[n], \quad |\rho| < 1$

The Autoregressive process of order 1: AR(1)

- In order to evaluate the **frequency response**, we rely on the Z-transform of both sides of the linear difference equation characterizing the AR(1) process:

$$x[n] = \rho x[n-1] + w[n], \quad |\rho| < 1$$

$$\Downarrow$$

$$X(z) = \rho z^{-1} X(z) + W(z), \quad |\rho| < 1$$

$$\Downarrow$$

$$\left(1 - \rho z^{-1}\right) X(z) = W(z), \quad |\rho| < 1$$

$$\Downarrow$$

$$L(z) = \frac{X(z)}{W(z)} = \frac{1}{1 - \rho z^{-1}} = \frac{z}{z - \rho}, \quad |\rho| < 1$$

L(*z*) has 1 pole at **z**=ρ and one zero at *z*=0.

- Then: $L\left(e^{j2\pi f}\right) = L(z)\big|_{z=e^{j2\pi f}} = \dfrac{1}{1 - \rho e^{-j2\pi f}}, \quad |\rho| < 1$

The Autoregressive process of order 1: AR(1)

- Alternatively, the **frequency response** can be derived as the Fourier Transform (FT) of the impulse response *l*[*n*]:

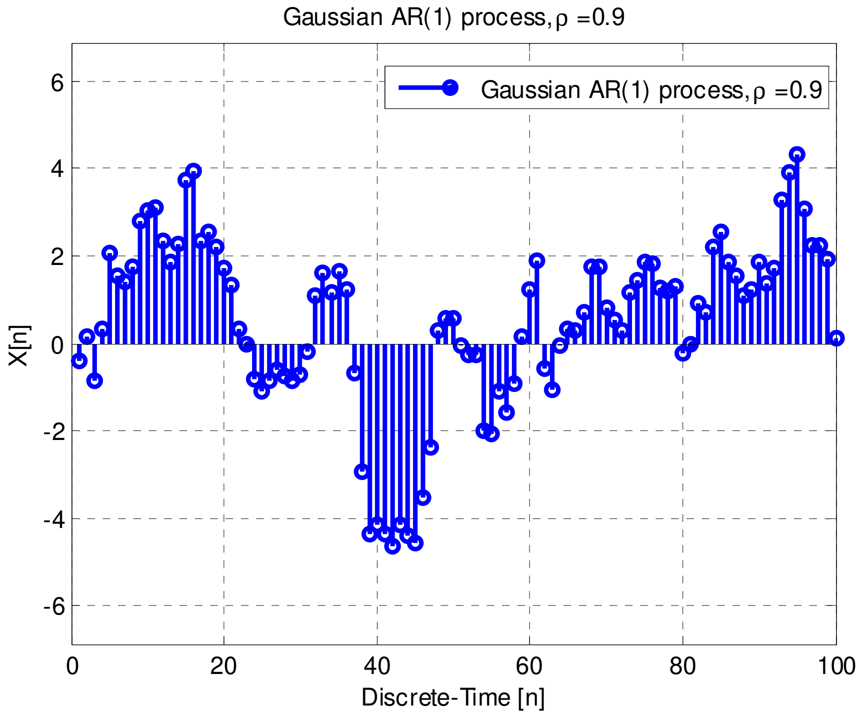
$$L\left(e^{j2\pi f}\right) = FT\left\{l[n]\right\} = \sum_{n=-\infty}^{+\infty} l[n]e^{-j2\pi fn} = \sum_{n=-\infty}^{+\infty} \rho^n u[n]e^{-j2\pi fn}$$

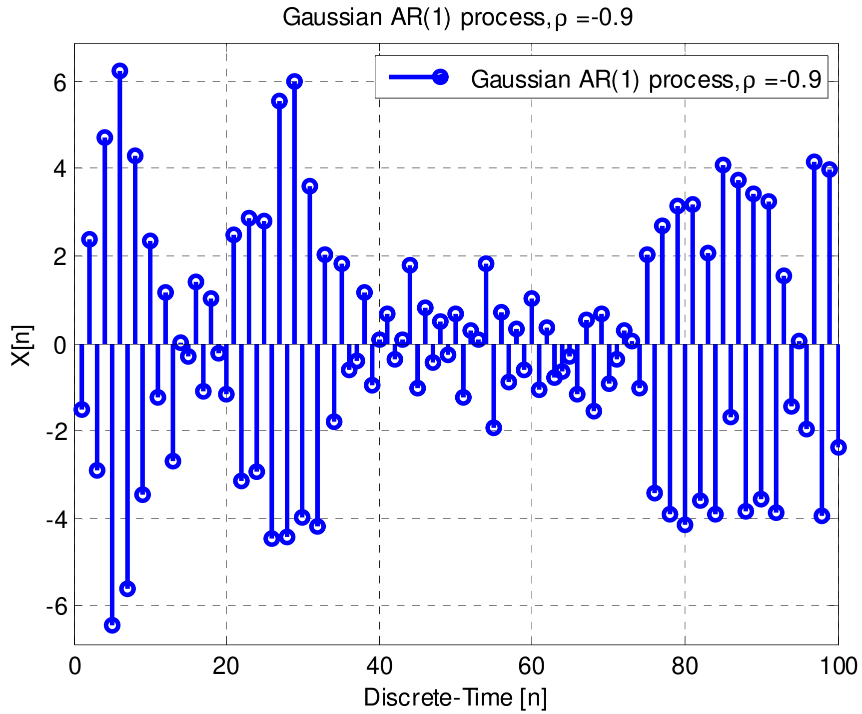
$$= \sum_{n=0}^{+\infty} \rho^n \, e^{-j2\pi fn} = \sum_{n=0}^{+\infty} \left(\rho e^{-j2\pi f} \right)^n$$

$$= \frac{1}{1-\rho e^{-j2\pi f}} \quad iff \quad \left|\rho e^{j2\pi f}\right| = |\rho| < 1$$

The Autoregressive process of order 1: AR(1)

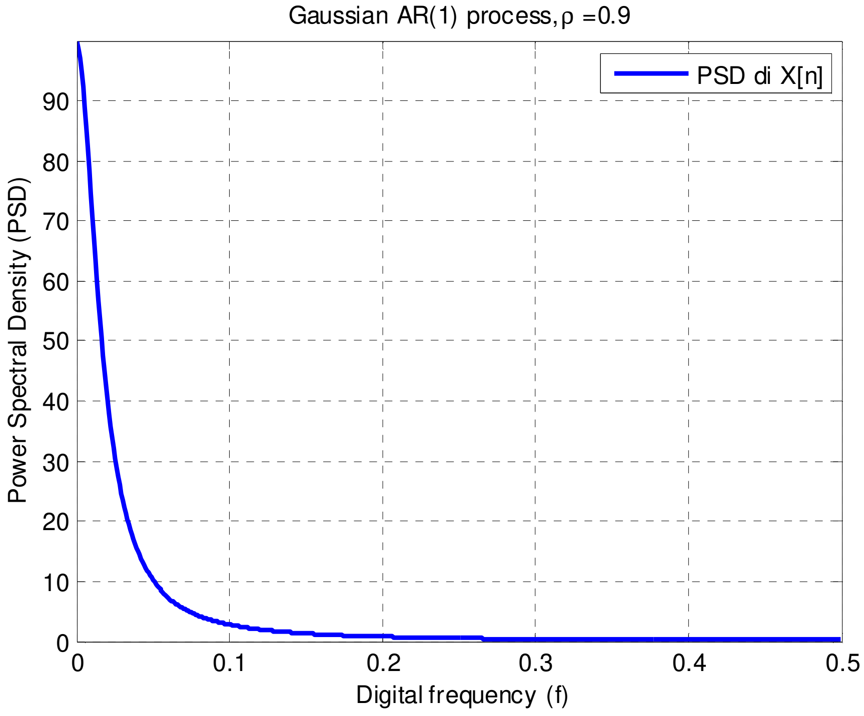
- Parameter ρ represents the **one-lag correlation coefficent** of the process.

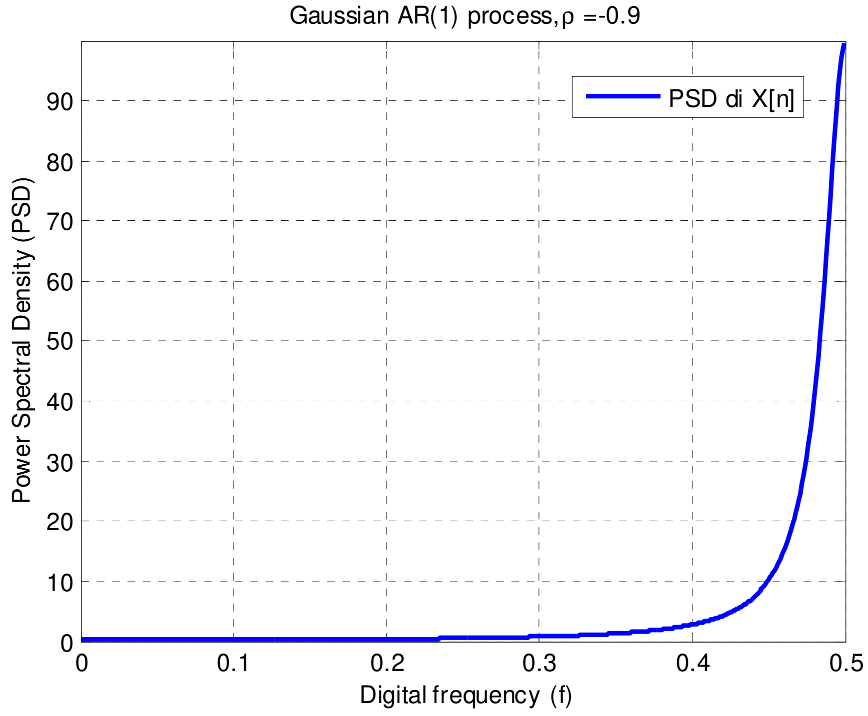




The Autoregressive process of order 1: AR(1)

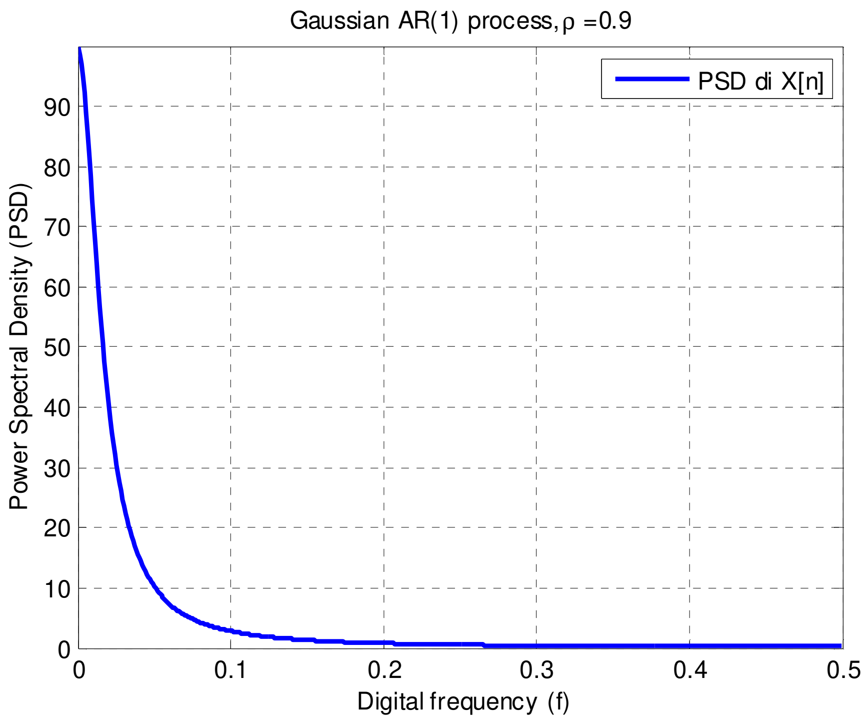
$$S_X\left(e^{j2\pi f}\right)=\sigma_W^2\left|L\left(e^{j2\pi f}\right)\right|^2=\sigma_W^2\left|\frac{1}{1-\rho e^{-j2\pi f}}\right|^2=\frac{\sigma_W^2}{1+\rho^2-2\rho\cos\left(2\pi f\right)}$$



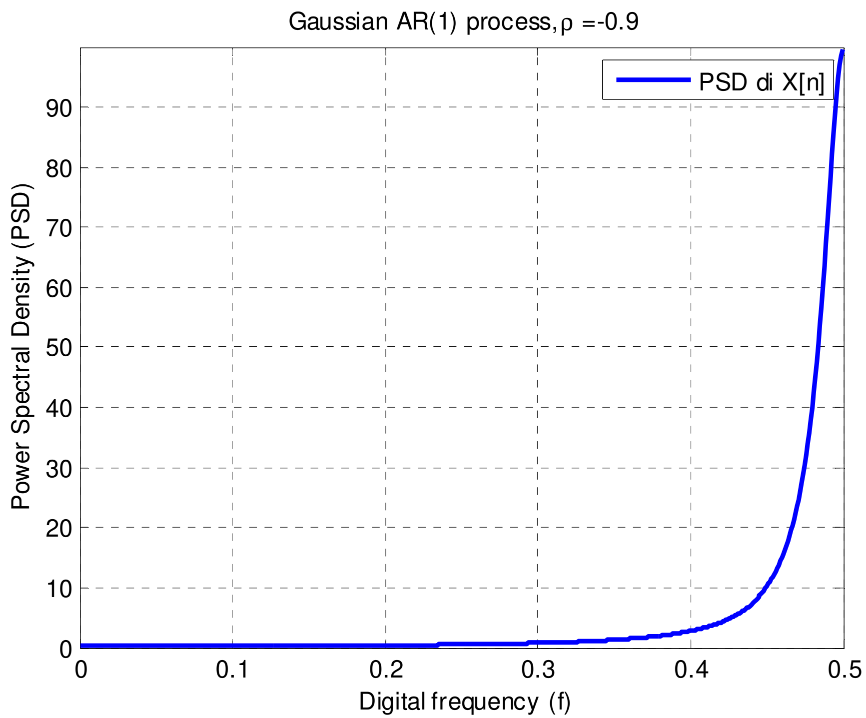


The Autoregressive process of order 1: AR(1)

$\rho > 0 \rightarrow$ low-pass process



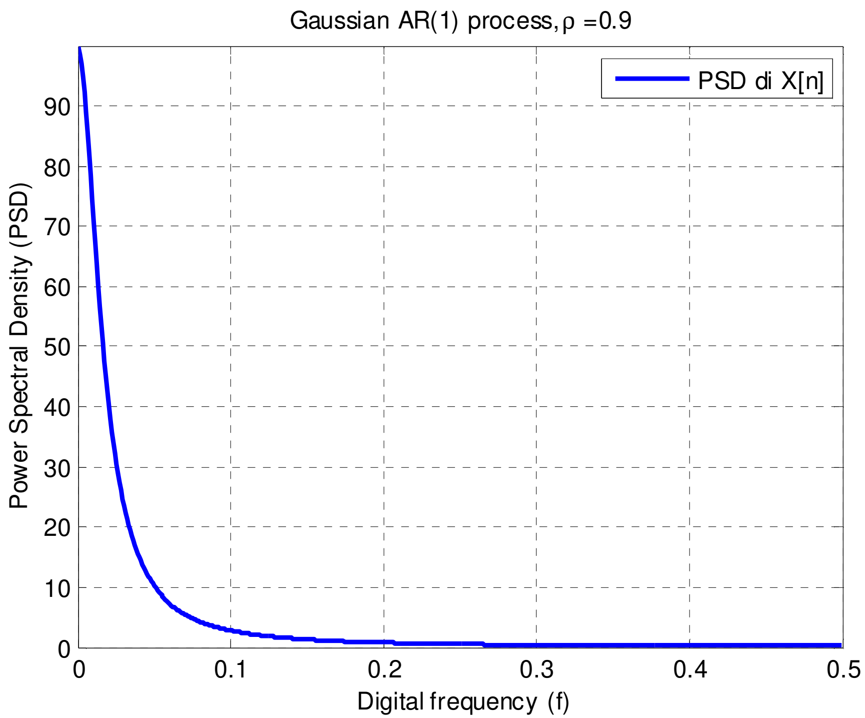
$\rho < 0 \rightarrow$ high-pass process

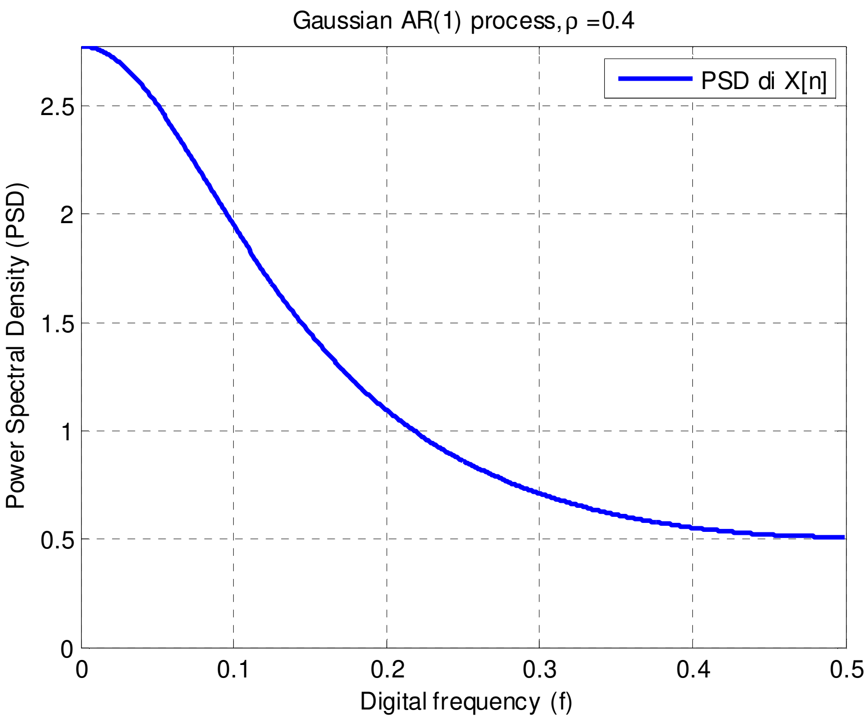


- With an AR(1) model we can obtain/represent only low-pass or high-pass processes, but not pass-band processes.

The Autoregressive process of order 1: AR(1)

- The bandwidth B of the process depends on how close the pole is to the unit circle: B gets lower when ρ gets closer to the unit circle.





The Autoregressive process of order 1: AR(1)

- **Autocorrelation function (ACF)** of an AR(1) process:

$$
\begin{aligned}
R_X[m] &= E\{x[n]x[n+m]\} \\
&= E\{x[n](\rho x[n+m-1]+w[n+m])\} \\
&= E\{\rho x[n]x[n+m-1]+x[n]w[n+m]\} \\
&= \rho R_X[m-1], \quad |\rho|<1 \text{ and } m>0
\end{aligned}
$$

in fact $E\{x[n]w[n+m]\}=0 \quad \text{if} \ \ m>0$

- This is 0, since the output at time *n* cannot depend on the input at time *n+m>n*, being the filter causal.

$$R_X[m]-\rho R_X[m-1]=0, \quad |\rho|<1 \text{ and } m>0$$

$$R_X[m]=R_X[0]\rho^m, \quad |\rho|<1 \text{ and } m>0$$

The Autoregressive process of order 1: AR(1)

- To derive the "initial condition" $R_X[0]$:

$$\sigma_X^2 = R_X[0] = E\left\{x^2[n]\right\} = E\left\{\left(\rho x[n-1] + w[n]\right)^2\right\}$$
$$= \rho^2 E\left\{x^2[n-1]\right\} + 2\rho E\left\{x[n-1]w[n]\right\} + E\left\{w^2[n]\right\}$$
$$= \rho^2 \sigma_X^2 + \sigma_W^2$$

$$\sigma_X^2 = \frac{\sigma_W^2}{1-\rho^2}, \quad |\rho| < 1$$

- this is 0, since the output at time n-1 cannot depend on the input at time n, being the filter causal.

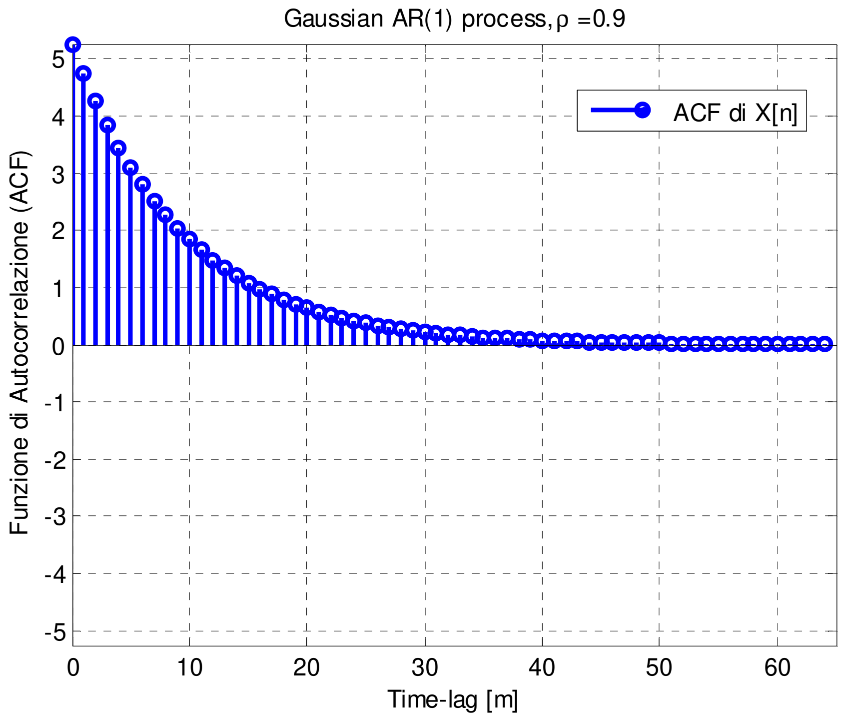
- Taking into account that for real processes $R_X[-m] = R_X[m]$, we finally have:

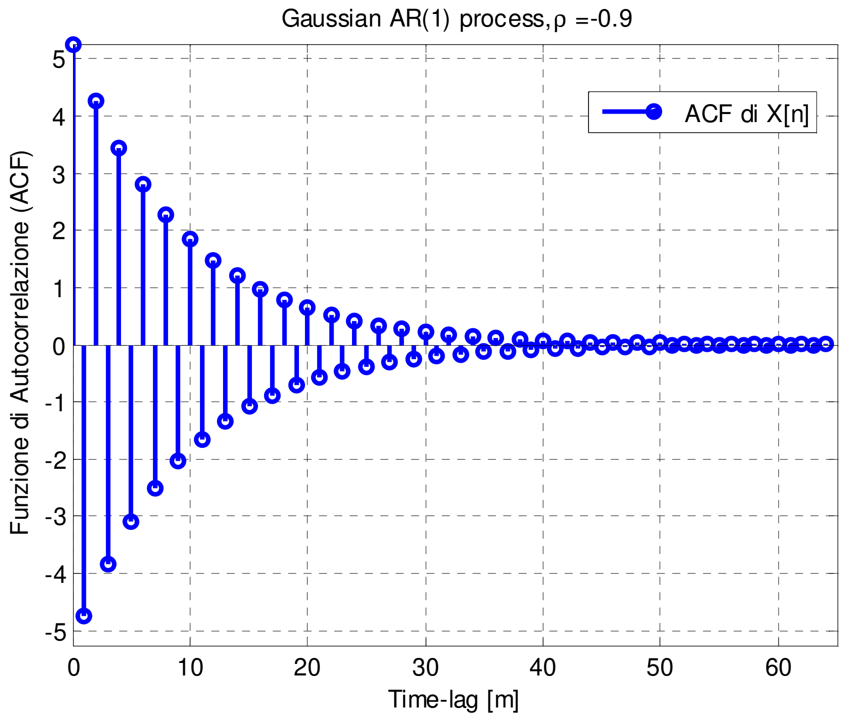
$$R_X[m] = \frac{\sigma_W^2}{1-\rho^2}\rho^{|m|}, \quad |\rho| < 1$$

The Autoregressive process of order 1: AR(1)

$$R_X[m] = \frac{\sigma_W^2}{1-\rho^2}\rho^{|m|} \quad \Rightarrow \quad \rho_X = \frac{\operatorname{cov}\{X[n], X[n+1]\}}{\sqrt{\operatorname{var}\{X[n]\}\operatorname{var}\{X[n+1]\}}} = \frac{R_X[1]}{R_X[0]} = \rho$$

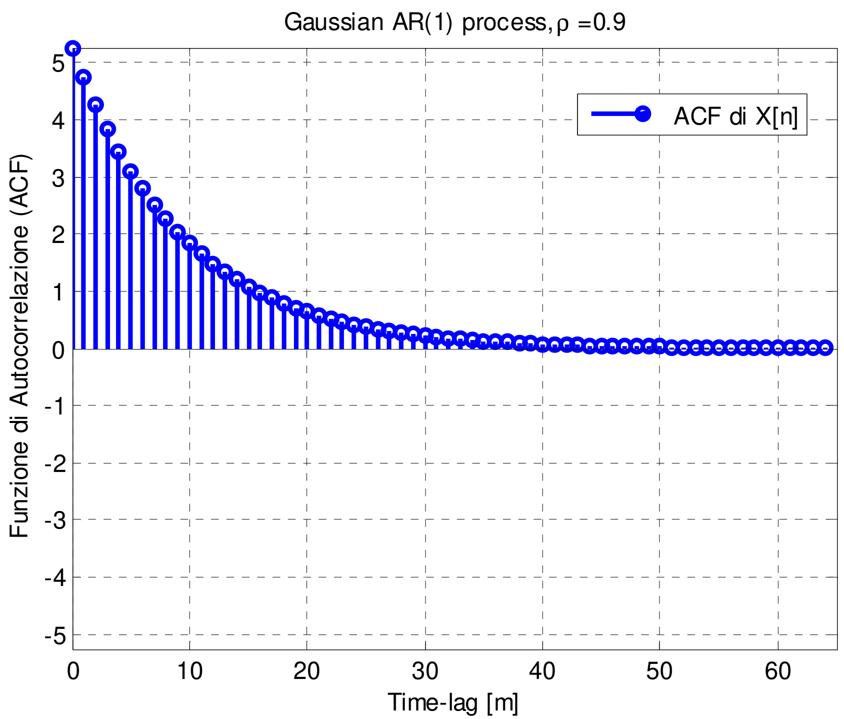
- Hence, the parameter ρ is **one-lag correlation coefficient**.

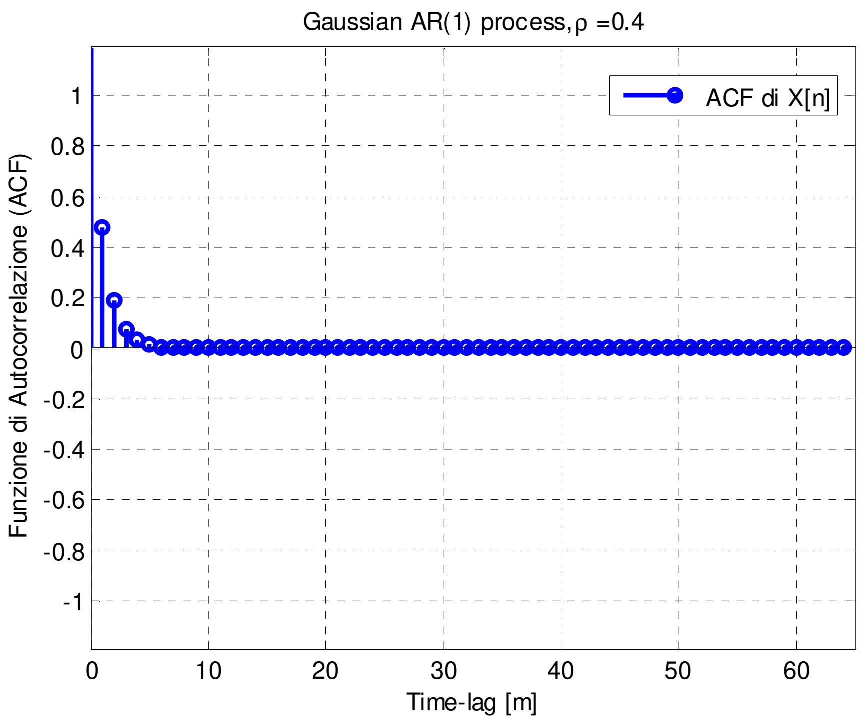




The Autoregressive process of order 1: AR(1)

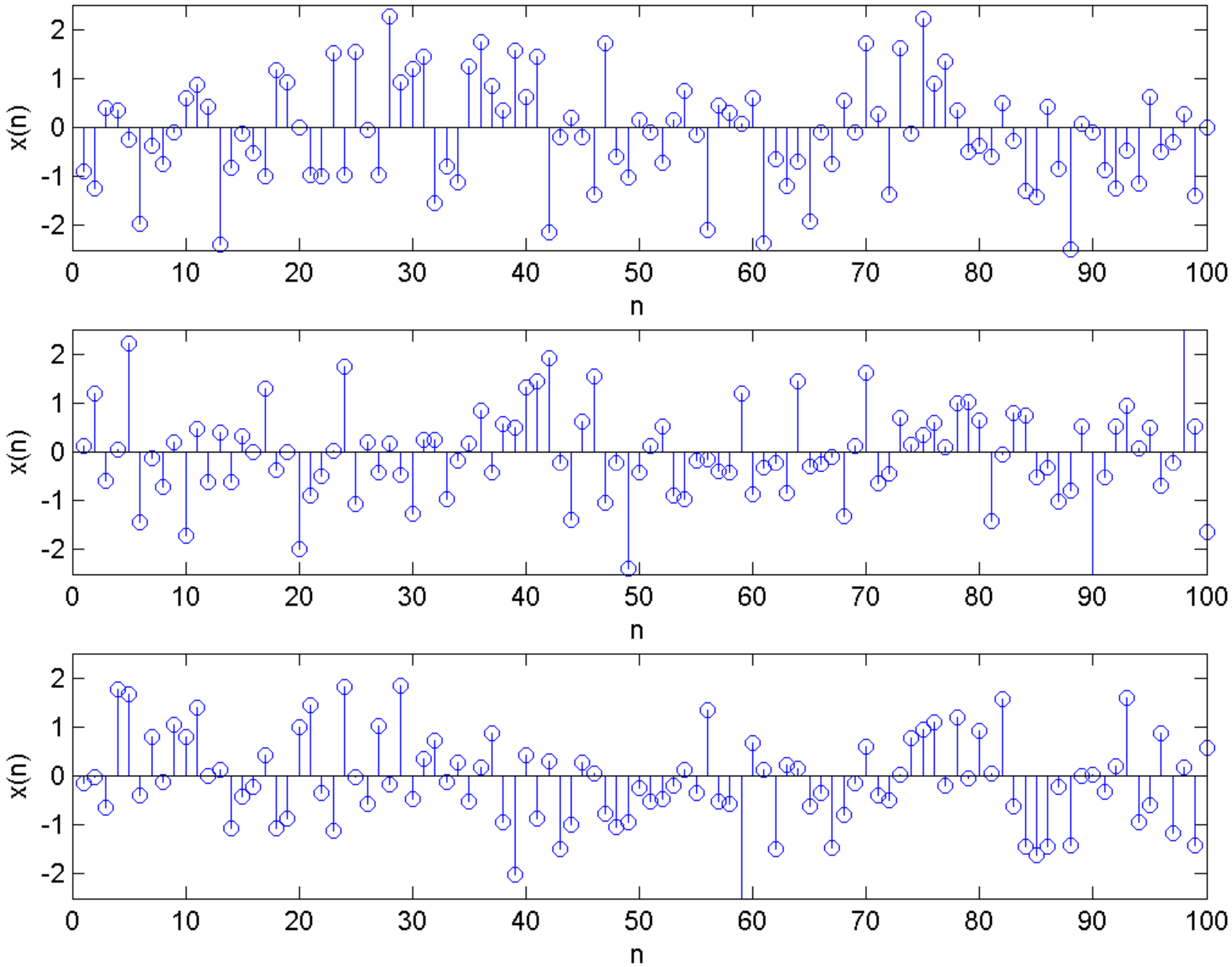
- The duration of the ACF (**correlation time** or **coherence time**) is larger when $|\rho|$ is closer to the unit circle.





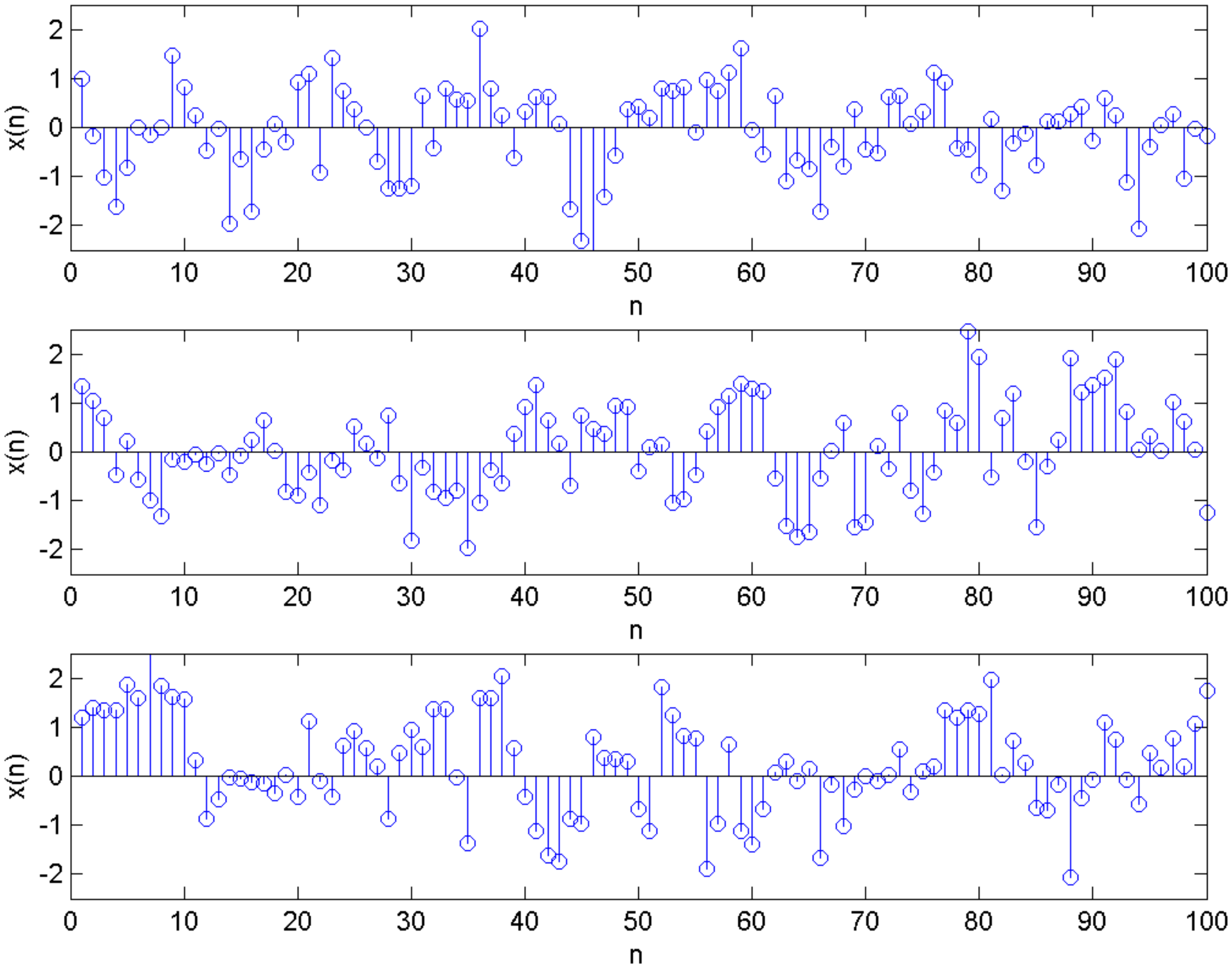
The Autoregressive process of order 1: AR(1)

$\rho = 0.1$



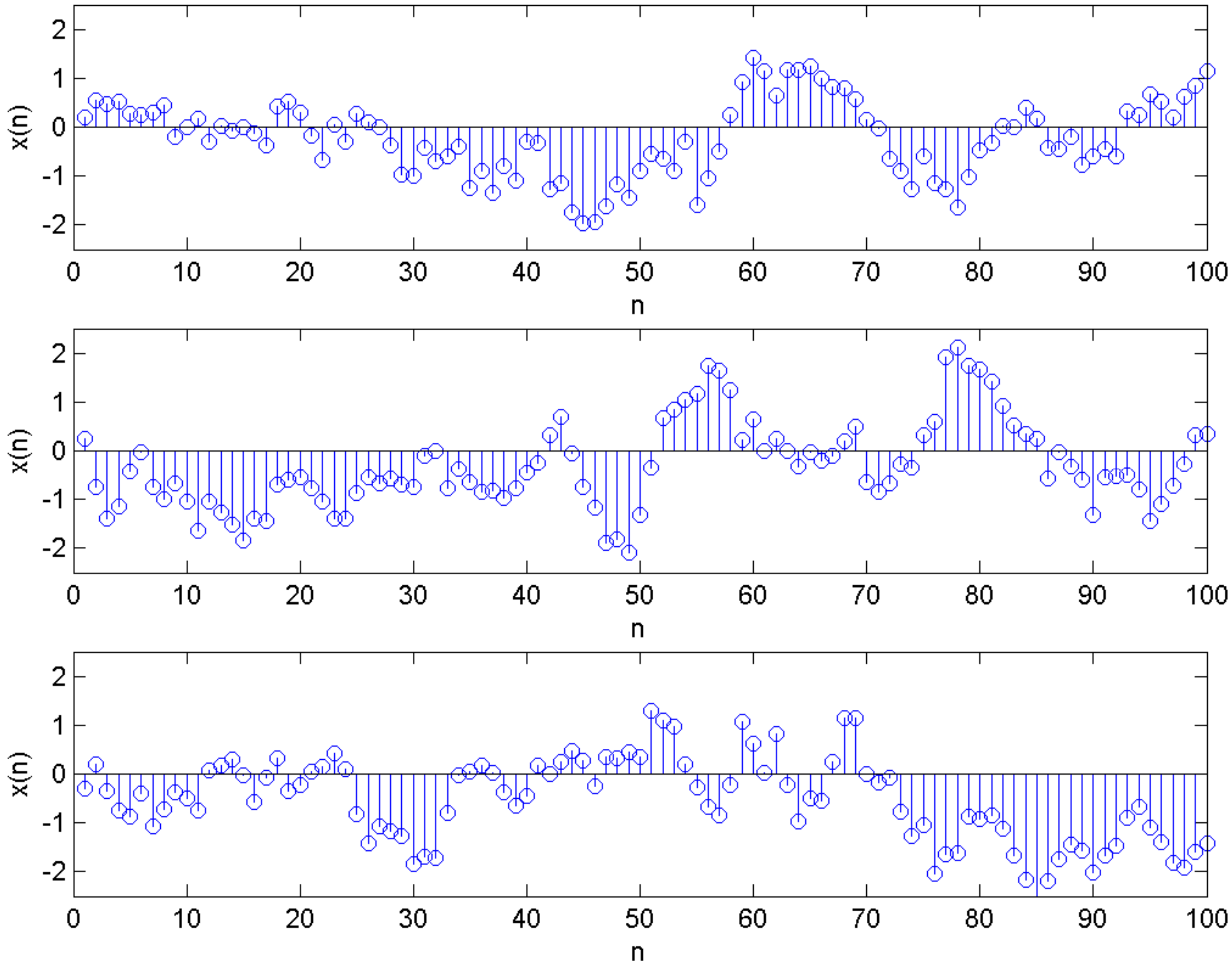
The Autoregressive process of order 1: AR(1)

$\rho = 0.5$



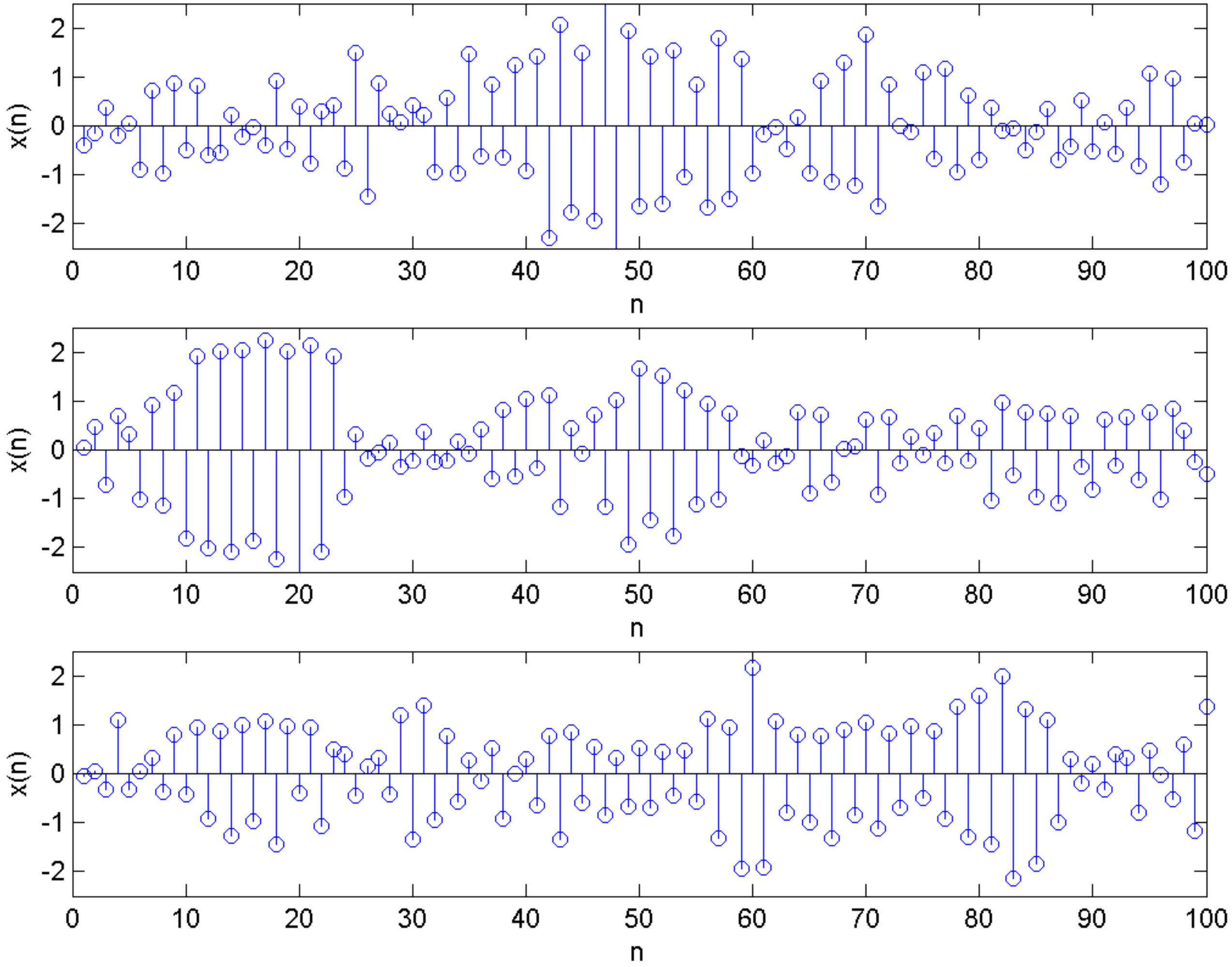
The Autoregressive process of order 1: AR(1)

$\rho = 0.9$



The Autoregressive process of order 1: AR(1)

$\rho = -0.9$



The Moving Average process of order 1: MA(1)

Linear Time-Invariant system

$$w[n] \longrightarrow \boxed{l[n]} \longrightarrow x[n]$$

$$x[n] = w[n] \otimes l[n] = \sum_{k=-\infty}^{+\infty} l[k]w[n-k] = \sum_{k=-\infty}^{+\infty} w[k]l[n-k]$$

- The input process $w[n]$ is a white process, i.e.

$$R_W[m] = \sigma_W^2 \delta[m] \qquad S_W\left(e^{j2\pi f}\right) = \sigma_W^2$$

- An **MA(1) process** is characterized by the following linear difference equation:

$$x[n] = w[n] + b_1 w[n-1], \quad |b_1| < 1$$

- An **MA(1) process** can be thought as the output of an FIR(1) LTI system with a suitable impulsive response driven by a white process.

The Moving Average process of order 1: MA(1)

- Let us first derive the **impulse response** $l[n]$ of the FIR(1) LTI filter and its **frequency response**:

$$x[n] = w[n] + b_1 w[n-1], \quad |b_1| < 1$$

$$l[n] = \delta[n] + b_1 \delta[n-1] \quad \Rightarrow \quad l[n] = \begin{cases} 1, & n = 0 \\ b_1, & n = 1 \\ 0, & otherwise \end{cases}$$

$$L\left(e^{j2\pi f}\right) = FT\left\{l[n]\right\} = \sum_{n=-\infty}^{+\infty} \left(\delta[n] + b_1 \delta[n-1]\right) e^{-j2\pi fn} = 1 + b_1 e^{-j2\pi f}$$

$$X(z) = W(z) + b_1 z^{-1} W(z) \quad \Rightarrow \quad L(z) = \frac{X(z)}{W(z)} = 1 + b_1 z^{-1}$$

$$L(z) = 0 \quad \Rightarrow \quad z = -b_1 \quad \Rightarrow \quad 1 \text{ zero } z_1 = -b_1$$

The Moving Average process of order 1: MA(1)

- Let us now derive the **Autocorrelation Function (ACF)**:

$$R_X[m] = E\left\{x[n]x[n+m]\right\}$$

$$= E\left\{\left(w[n]+b_1 w[n-1]\right)\left(w[n+m]+b_1 w[n+m-1]\right)\right\}$$

$$= R_W[m]+b_1 R_W[m-1]+b_1 R_W[m+1]+b_1^2 R_W[m]$$

$$= \left(1+b_1^2\right)\sigma_W^2\delta[m]+b_1\sigma_W^2\delta[m-1]+b_1\sigma_W^2\delta[m+1]$$

The Moving Average process of order 1: MA(1)

$$R_X[m] = \left(1+b_1^2\right)\sigma_W^2\delta[m] + b_1\sigma_W^2\delta[m-1] + b_1\sigma_W^2\delta[m+1]$$

$$R_X[m] = \begin{cases} \left(1+b_1^2\right)\sigma_W^2, & m=0 \\ b_1\sigma_W^2, & m=\pm 1 \\ 0, & otherwise \end{cases}$$

- The power of the process and the one-lag correlation coefficient are given by:

$$R_X[0] = \sigma_X^2 = \left(1+b_1^2\right)\sigma_W^2, \qquad \rho_X[1] = \frac{R_X[1]}{R_X[0]} = \frac{b_1}{1+b_1^2}$$

The Moving Average process of order 1: MA(1)

- The **Power Spectral Density (PSD)** can be derived as the FT of the ACF:

$$S_X\left(e^{j2\pi f}\right) = FT\left\{R_X[m]\right\} = \sum_{m=-\infty}^{+\infty} R_X[m] e^{-j2\pi fm}$$

$$= \sum_{m=-\infty}^{+\infty} \left[\left(1+b_1^2\right)\sigma_W^2 \delta[m] + b_1 \sigma_W^2 \delta[m-1] + b_1 \sigma_W^2 \delta[m+1]\right] e^{-j2\pi fm}$$

$$= \left(1+b_1^2\right)\sigma_W^2 + b_1 \sigma_W^2 e^{-j2\pi f} + b_1 \sigma_W^2 e^{j2\pi f}$$

$$= \sigma_W^2 \left(1 + b_1^2 + 2b_1 \cos\left(2\pi f\right)\right)$$

$$b_1 < 0 \quad \Rightarrow \quad \text{high-pass process}, \qquad b_1 > 0 \quad \Rightarrow \quad \text{low-pass process}$$

The Moving Average process of order 1: MA(1)

- The PSD can also be obtained from the frequency response of the FIR(1) filter:

$$S_X\left(e^{j2\pi f}\right) = \sigma_W^2 \left|L\left(e^{j2\pi f}\right)\right|^2 = \sigma_W^2 \left|1 + b_1 e^{-j2\pi f}\right|^2$$

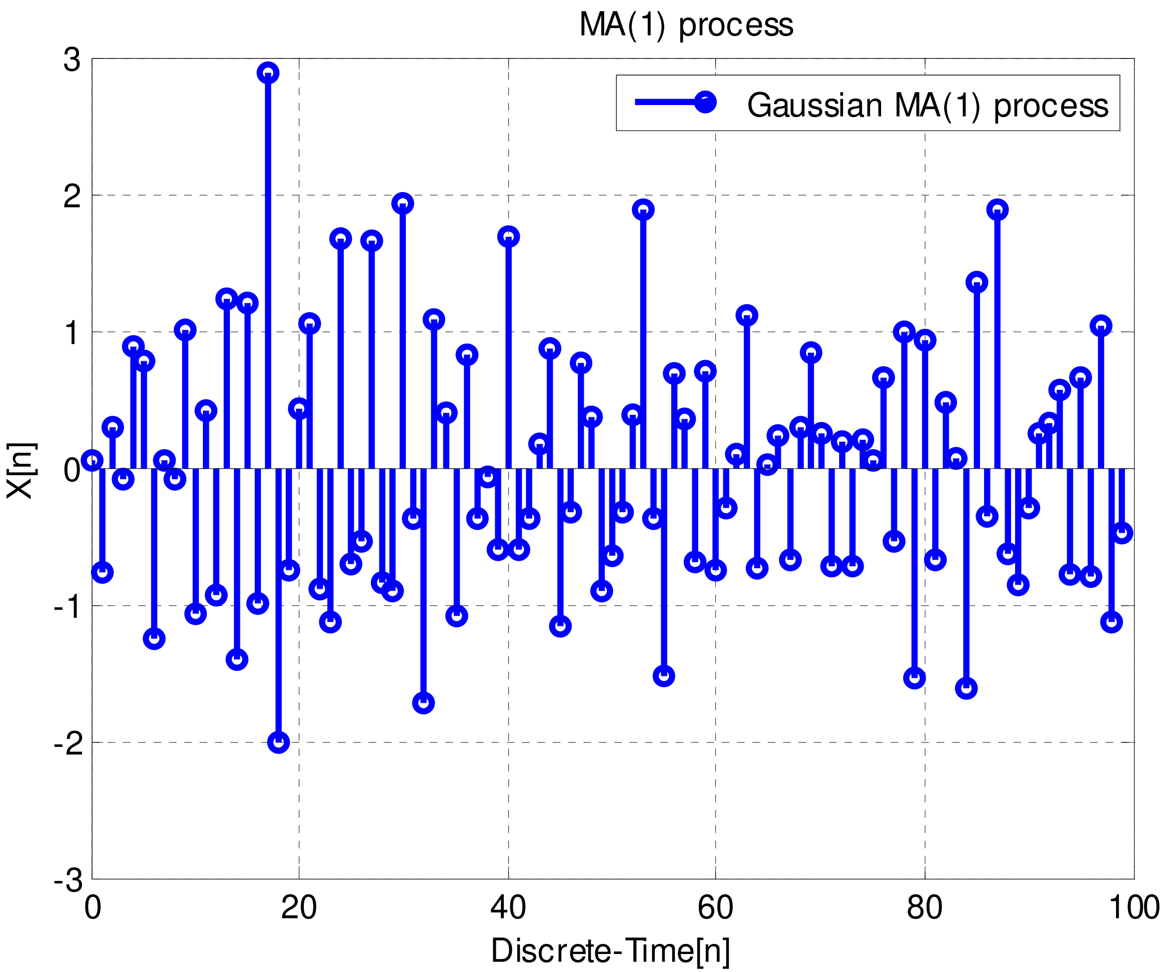
$$= \sigma_W^2 \left(1 + b_1 e^{-j2\pi f}\right)\left(1 + b_1 e^{j2\pi f}\right)$$

$$= \sigma_W^2 \left(1 + b_1^2 + b_1 e^{-j2\pi f} + b_1 e^{j2\pi f}\right)$$

$$= \sigma_W^2 \left(1 + b_1^2 + 2b_1 \cos\left(2\pi f\right)\right)$$

The Moving Average process of order 1: MA(1)

$$b_1 = -0.98, \quad z_1 = 0.98, \quad \sigma_X^2 = 1 \quad \Rightarrow \quad X[n] \text{ high-pass process}$$



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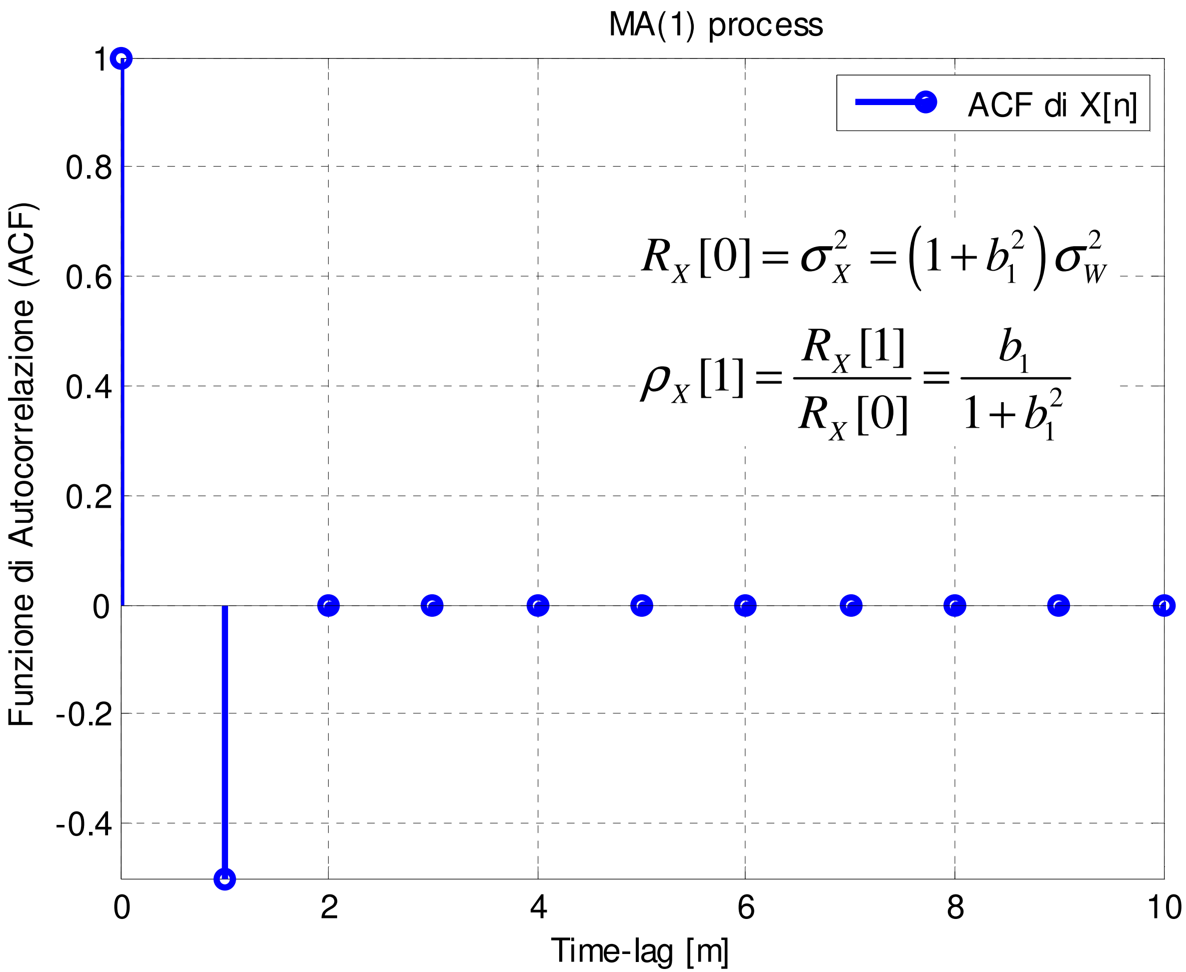
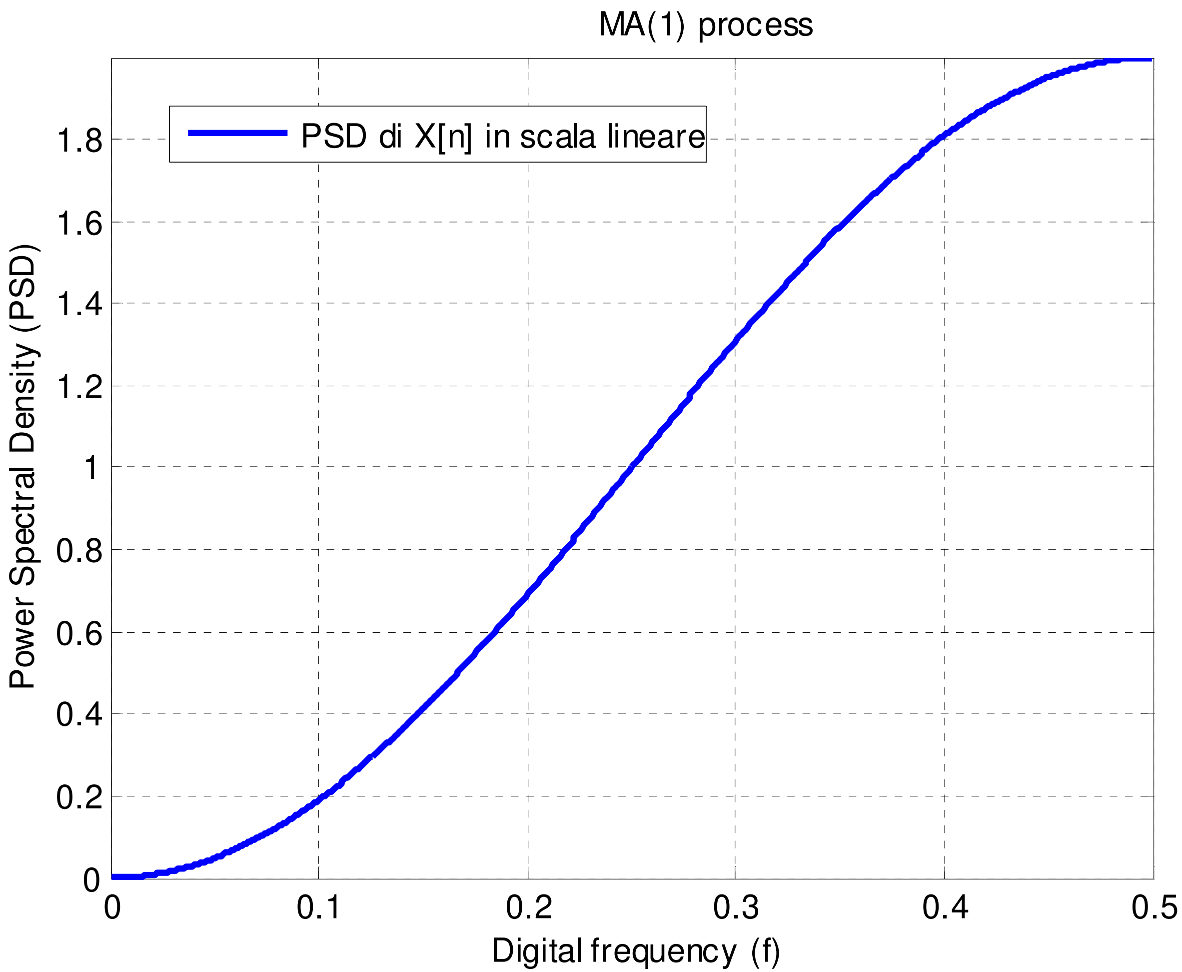


$$R_X[0] = \sigma_X^2 = \left(1 + b_1^2\right)\sigma_W^2$$

$$\rho_X[1] = \frac{R_X[1]}{R_X[0]} = \frac{b_1}{1 + b_1^2}$$

The Moving Average process of order 1: MA(1)

$$b_1 = -0.98, \quad z_1 = 0.98, \quad \sigma_X^2 = 1 \quad \Rightarrow \quad X[n] \text{ high-pass process}$$



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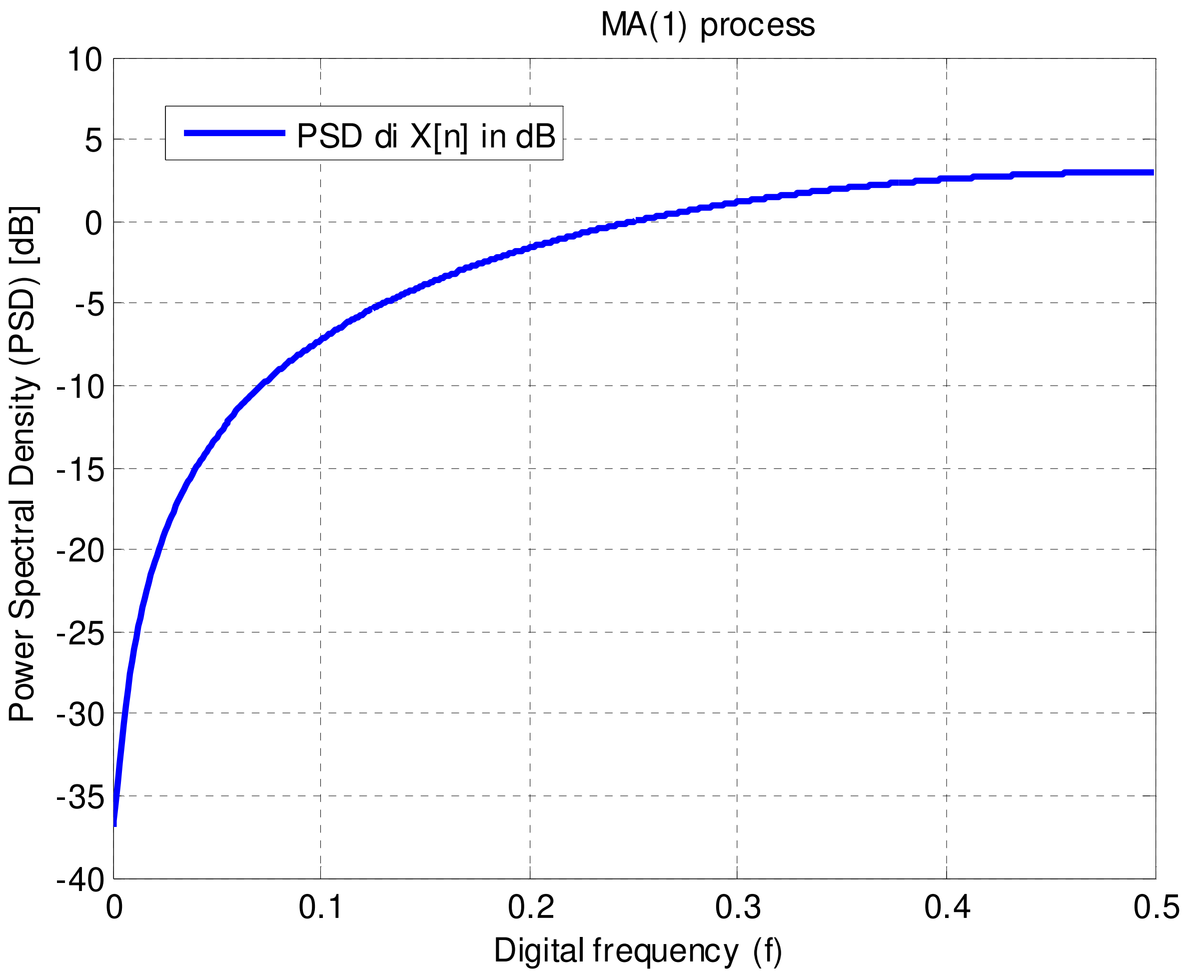
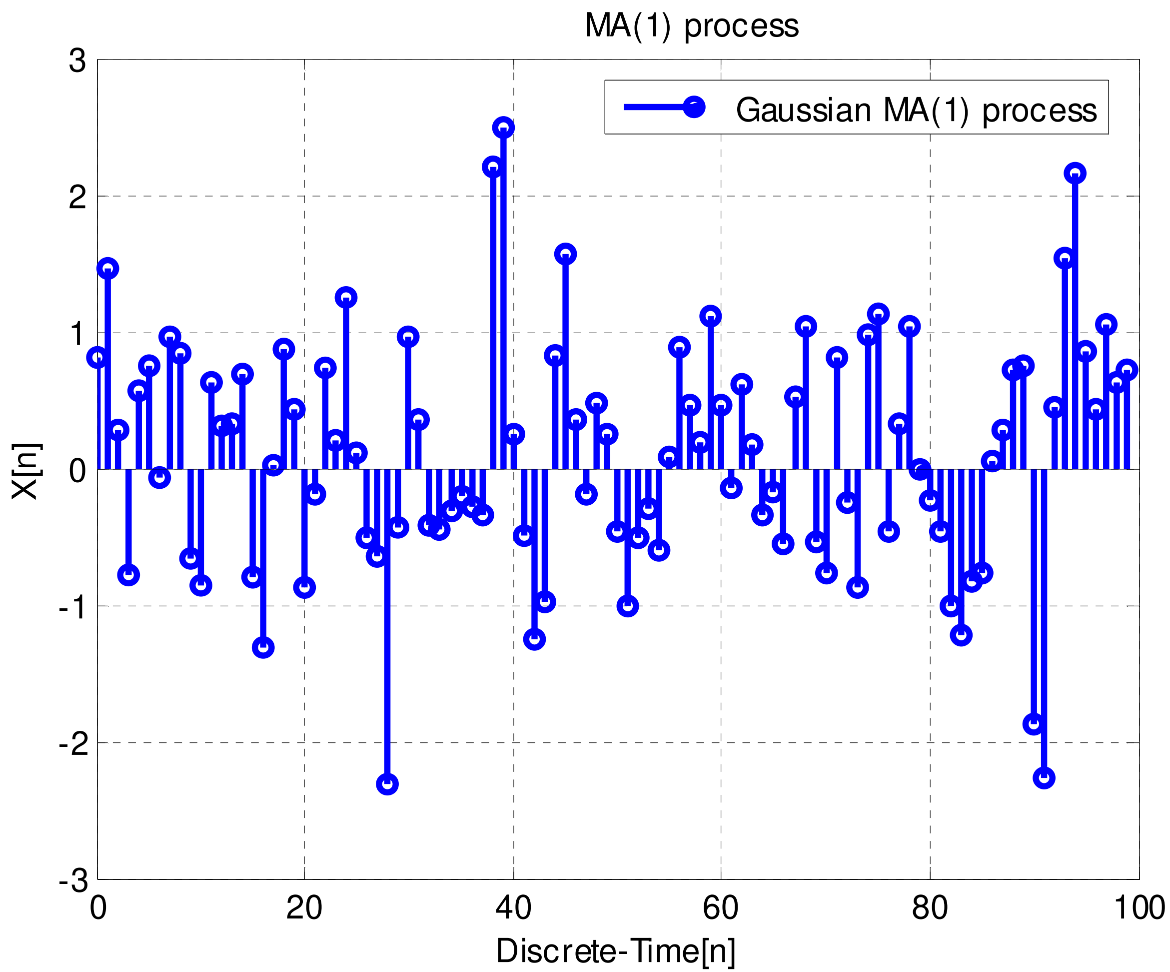


The Moving Average process of order 1: MA(1)

$$b_1 = 0.98, \quad z_1 = -0.98, \quad \sigma_X^2 = 1 \quad \Rightarrow \quad X[n] \text{ low-pass process}$$



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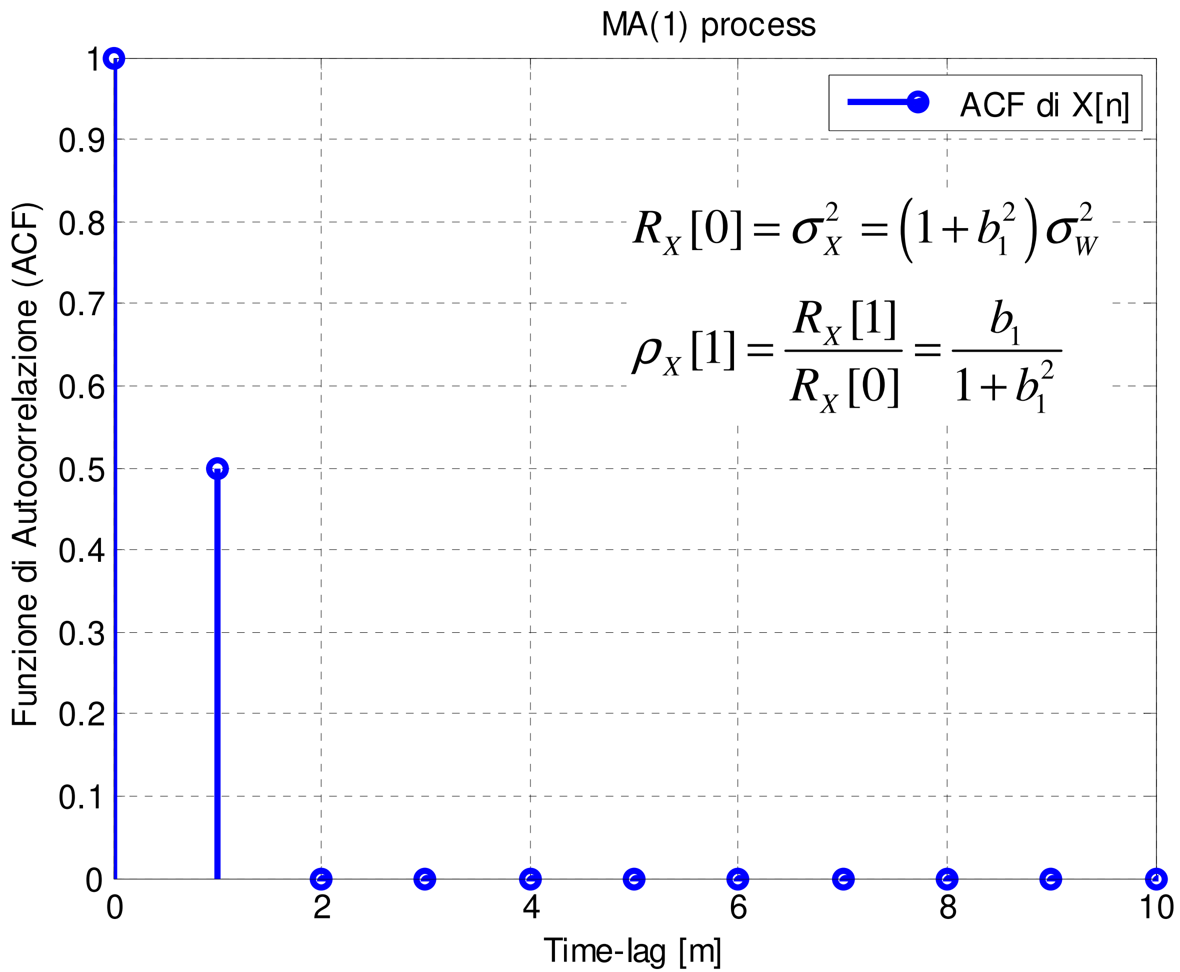

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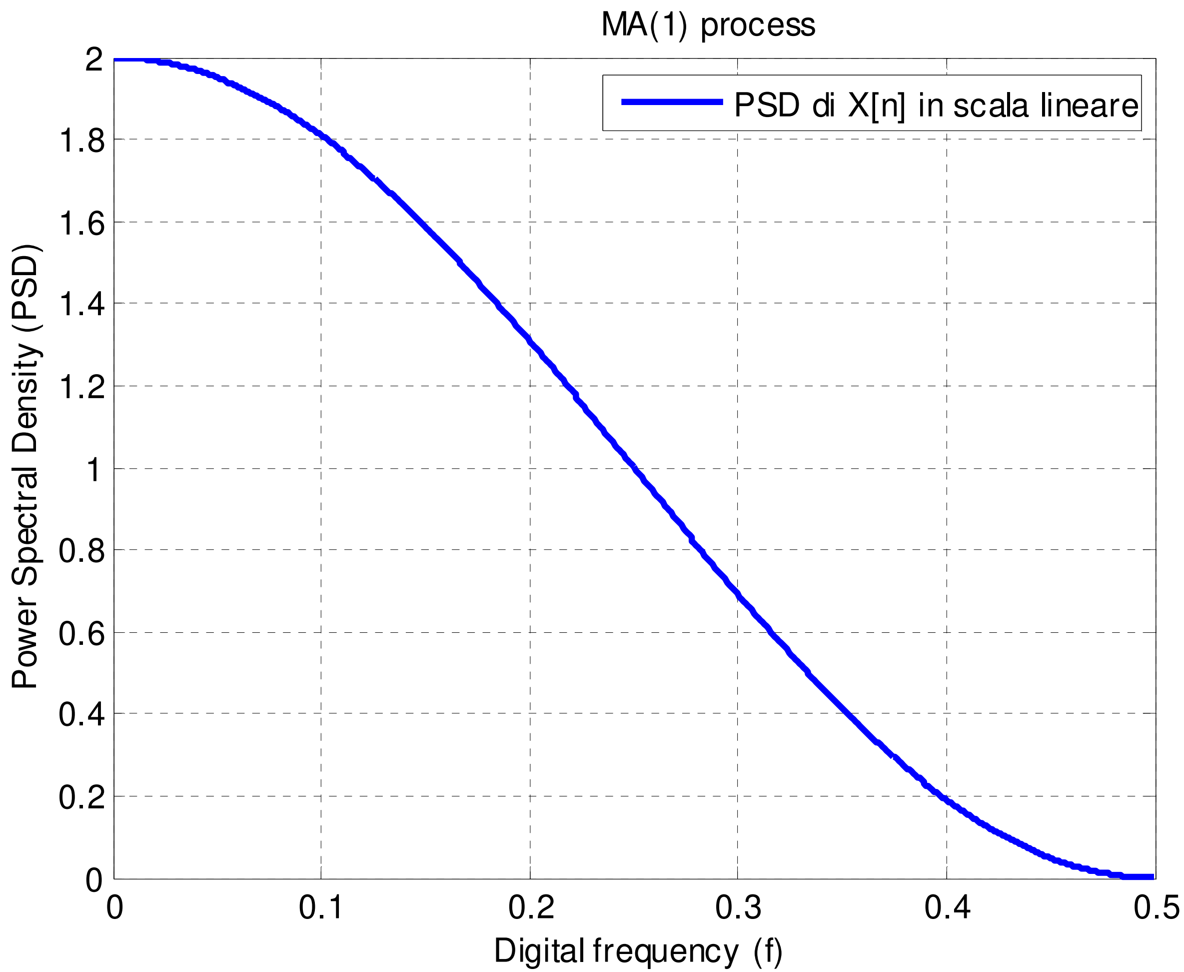

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