8. Lezione 22 ottobre

giovedì 22 ottobre 2020 08:42

R

CAMB. DI SDR

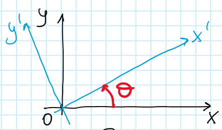
$[P]_S = R_{SS'}[P]_{S'}$

2 SDR STESSA CONFIG.

PROPRIETÀ

$R_{SS'} R_{SS'}^T = I$

$\det R_{SS'} = 1$



$R_{SS'} = R_z(\theta)$

ROTAZ. FISICA

$p(1) = R\, p(0)$

senza SDR

DUE CONF. DIVERSE

$R(r, \theta)$

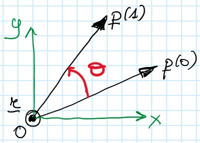
ROT. ATTORNO AD ASSE x O

$p = \vec{OP}$

PROPRIETÀ

$R R^T = I$

$\det R = 1$



$R(r,\theta) \xrightarrow{S} [R]_S = R_z(\theta)$

$r = k$

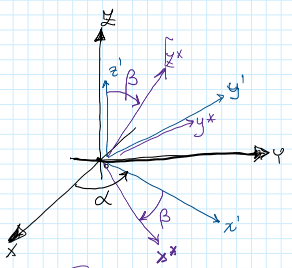
Matrici di rotazione elementari

- $R_z(\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$ $(r = i)$
- $R_y(\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$

rotx( ) roty( ) rotz( ) angolo in gradi in Matlab

Sono "comode", struttura predefinita. Spesso si usano per costruire R più complesse

ESEMPIO



S^* $\xrightarrow{\text{rot } y'}$ S' $\xrightarrow{\text{rot } z}$ S

S, S', S^* 3 TERNE

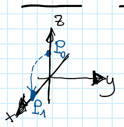
a) S': $z' \equiv z$

- $R_{SS'} = R_z(\alpha)$

b) S^*: $y^* \equiv y'$

- $R_{S'S^*} = R_y(\beta) = \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}$
 $R_y\beta$)

$R_{SS^*} = R_{SS'}\, R_{S'S^*} = R_z(\alpha)\, R_y(\beta)$ COMPOSIZ. MATRICI

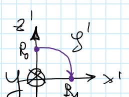


rot y di 90°

$R_y(90°)$

$[P_1]_S = R_y(90°)[P_0]_S$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$



rot y' di 90°

$R_y(90°)$

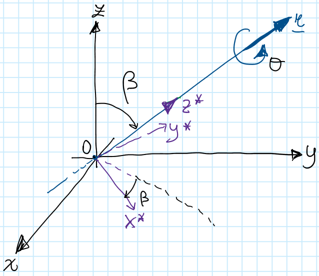
$[R_1]_S = R_y(90°)\, [R_0]_{S'}$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$[P_1]_S = R_y(90°)[P_0]_S$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

ROTAZIONE ATTORNO ASSE GENERICO



versore $\underline{r}$ ANGOLO θ

$R(\underline{r}, \theta)$

la "forma" di R dipende dal sdr

SI scrive in forma generale

$$\underline{p}(1) = R\ \underline{p}(0) \quad S,\ S^*$$

$[R]_S$ non banale, ma in S^* semplice $\underline{k}^* = \underline{r}$

quindi

$$[R(\underline{r},\theta)]_{S^*} = R_z(\theta) \rightarrow [R(\underline{r},\theta)]_S$$

COME VARIA LA MATRICE ROT. FISICA AL VARIARE DEL SDR

NON BANALE

vediamo come trovarlo

Osserviamo che

DATI:

$[\underline{p}(0)]_S$

$[\underline{r}]_S$ e θ

VEDI ANGOLI α, β IN FIG

$$[\underline{r}]_S = (s\beta c\alpha,\ s\beta s\alpha,\ c\beta)$$

si può considerare nota anche

$$R_{SS^*} = R_z(\alpha)\, R_y(\beta)$$

$$\beta = \operatorname{acos}(r_z)$$

$$\alpha = \operatorname{atan}\left(\frac{r_y}{r_x}\right) \quad \text{o atan2 meglio}$$

$$\underline{p}(1) = R\ \underline{p}(0)$$

$$\left\{\begin{aligned}
&[\underline{p}(1)]_S = \underbrace{[R]_S}_{\text{INC}} \underbrace{[\underline{p}(0)]_S}_{\text{NOTO}} && \text{in } S \\
&[\underline{p}(1)]_{S^*} = \underbrace{[R]_{S^*}}_{\text{NOTO}} \underbrace{[\underline{p}(0)]_{S^*}}_{\text{INC}} && \text{in } S^* \\
&[\underline{p}(t)]_S = \underbrace{R_{SS^*}}_{\text{NOTO}} [\underline{p}(t)]_{S^*} && \text{cambiamento sdr } S^* \rightarrow S \text{ vale per } t = 0, 1 \ldots
\end{aligned}\right.$$

$$[\underline{p}(1)]_S = R_{SS^*}[\underline{p}(1)]_{S^*}$$

$$= R_{SS^*}\, \overbrace{[R]_{S^*}\, [\underline{p}(0)]_{S^*}}$$

$$[\underline{p}(1)]_S = \underbrace{R_{SS^*}\, [R]_{S^*}\, R^T_{SS^*}}_{[R]_S} [\underline{p}(0)]_S \Rightarrow [R]_S = R_{SS^*} [R]_{S^*} R^T_{SS^*}$$

GRANDEZZE E SDR: sintesi

anche per
- t. INERZIA
- t. DEFORMAZIONE

- TENSORI $[R] = R_{SS^*}[R] \; R^T$

GRANDEZZE E SDR: sintesi

- **TENSORI** $[R]_S = R_{SS^*} [R]_{S^*} R_{SS^*}^T$ forma SANDWICH

anche per
- t. INERZIA
- t. DEFORMAZIONE
- ....

- **VETTORI** $[\underline{v}]_S = R_{SS^*} [\underline{v}]_{S^*}$

- **SCALARI** m

⊛ Ritornando all'esempio di prima

$$R_{SS^*} = R_z(\alpha) R_y(\beta)$$

$$[R]_S = R_{SS^*} [R]_{S^*} R_{SS^*}^T$$

$$= R_z(\alpha) R_y(\beta) R_z(\theta) R_y(-\beta) R_z(-\alpha)$$

ATTENZIONE! trasposta del prodotto

$$R_{SS^*}^T = (R_z(\alpha) R_y(\beta))^T = R_y^T(\beta) R_z^T(\alpha) = R_y(-\beta) R_z(-\alpha)$$

$$R_y^T(\beta) = R_y^{-1}(\beta) = R_y(-\beta)$$

OSSERVAZIONI

- usando matr. rot. elem. si otteng. R generiche
- componendo (moltiplicando) matrici di rot $\Rightarrow$ si ottiene una matrice di rotazione

Esprimere Rot con vettori

$$(\underline{r}, \theta) \longrightarrow R$$

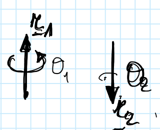
asse angolo

problema inverso

OSSERVIAMO INTANTO CHE:

data una matrice R si possono distinguere due versori opposti (con angoli opposti)

- $R = \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}$



$(\underline{r}, \theta)$? $\rightarrow$ $(\underline{j}, 45°)$, $(-\underline{j}, -45°)$ 2 soluzioni

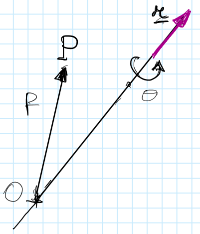
N.B. caso particolare

- $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ∞ soluzioni $\begin{cases} \underline{r} \text{ qualsiasi} \\ \theta = 0° \end{cases}$

- R SIMMETRICA

$\theta = \pm\pi \quad r_1, r_2 \to 4$ soluz.

$R \to (\underline{r}, \theta)$? come si fa la procedura inversa



$R(\underline{r}, \theta)$

$\underline{r} = R(\underline{r}, \theta)\, \underline{r}$ — versore asse

Questa relazione ricorda

$\lambda \underline{v} = A \underline{v}$ pb. autovalori autovettori

- $\underline{r}$ è autovettore associato a $\lambda = 1$

$\boxed{\text{data } R}$

↳ autov $\begin{cases} \lambda_1 = 1 \leftrightarrow \underline{r} \\ \lambda_{2,3} = \cos\theta \pm i \sin\theta \end{cases}$

θ da acos $+ - ?$ non abbinabile a $\underline{r}$ secondo convenz. $\circlearrowleft\uparrow +$

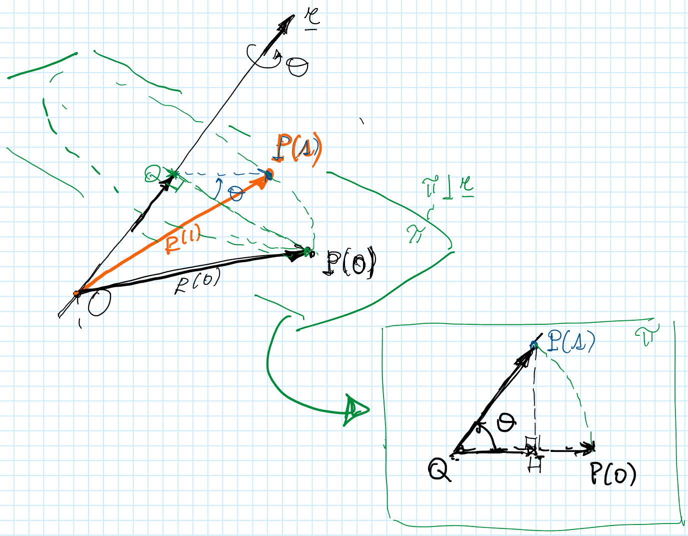
da pb. autovalori si trova $\underline{r}, \pm\theta$ NON SOLUZIONE PB. INVERSO

Il problema inverso richiede percorso di Eulero

ROT. FISICA

ASSE : O PUNTO , $\underline{r}$ VERSORE

θ : ANGOLO ORIENTATO $\circlearrowleft\uparrow^{\underline{r}}_{+}$ | INV NAT.



$$\underline{p}(1) = f(\underline{p}(0), \underline{r}, \theta) \quad (= R\, \underline{p}(0))$$

$$\underline{p}_1 = \underline{p}(1) = \overrightarrow{OP(1)} = \overrightarrow{OQ} + \overrightarrow{QP(1)} = \overrightarrow{OQ} + \overrightarrow{QH} + \overrightarrow{HP(1)}$$

$$\underline{p}_0 = \underline{p}(0) = \overrightarrow{OP(0)} = \underbrace{\overrightarrow{OQ}}_{/\!/ \underline{r}} + \underbrace{\overrightarrow{QP(0)}}_{\perp \underline{r}}$$

$$\overrightarrow{OQ} = (\underline{p}_0 \cdot \underline{r})\, \underline{r}$$

$$\overrightarrow{QP(1)} = \overrightarrow{QH} + \overrightarrow{HP(1)}$$

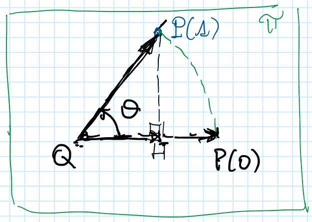
$$\left(\overrightarrow{QP(1)} \cdot \frac{\overrightarrow{QP(0)}}{|\overrightarrow{QP(0)}|} \right) \frac{\overrightarrow{QP(0)}}{|\overrightarrow{QP(0)}|} = \cos\theta \frac{|\overrightarrow{QP(1)}|\, \cancel{|\overrightarrow{QP(0)}|}}{\cancel{|\overrightarrow{QP(0)}|}} \frac{\overrightarrow{QP(0)}}{|\overrightarrow{QP(0)}|}$$

$$|\overrightarrow{QP(1)}| = |\overrightarrow{QP(0)}| \leftarrow$$

$$\left(\overrightarrow{QP}(1)\cdot\frac{\overrightarrow{QP}(0)}{|\overrightarrow{QP}(0)|}\right)\frac{\overrightarrow{QP}(0)}{|\overrightarrow{QP}(0)|}$$

$$|\overrightarrow{QP}(1)| = |\overrightarrow{QP}(0)|$$

$$\overrightarrow{QH} = \cos\theta\ \overrightarrow{QP}(0)$$



$\overrightarrow{HP}(1)$ – modulo $|\overrightarrow{HP}(1)| = |\overrightarrow{QP}(0)|\,|\sin\theta|$

– direzione $\in \pi \rightarrow \perp \underline{r}$

$\perp \overrightarrow{QP}_0$

$\underline{w} = \underline{u}\wedge\underline{v}$! $\quad \overrightarrow{HP}(1) \parallel \underline{r}\wedge\overrightarrow{QP}_0$

$\overrightarrow{QP}_0 \wedge \underline{r}$ × AVERE VERSO CORRETTO

$$\overrightarrow{HP}(1) = \sin\theta\ \underline{r}\wedge\overrightarrow{QP}_0$$

$$|\overrightarrow{HP}(1)| = |\sin\theta|\,|\overrightarrow{QP}_0|$$

$$\underline{p}_1 = \overrightarrow{OQ} + \overrightarrow{QH} + \overrightarrow{HP}_1 = (\underline{p}_0\cdot\underline{r})\underline{r} + \cos\theta\ \overrightarrow{QP}(0) + \sin\theta\ \underline{r}\wedge\overrightarrow{QP}(0)$$

$$\overrightarrow{QP}(0) = \underline{p}_0 - \overrightarrow{OQ} = \underline{p}_0 - (\underline{p}_0\cdot\underline{r})\underline{r}$$

$$\underline{p}_1 = (\underline{p}_0\cdot\underline{r})\underline{r} + \cos\theta\left(\underline{p}_0 - (\underline{p}_0\cdot\underline{r})\underline{r}\right) + \sin\theta\ \underline{r}\wedge\left(\underline{p}_0 - \cancel{(\underline{p}_0\cdot\underline{r})\underline{r}}\right)$$

$$\underline{p}_1 = \underline{r}\,(\underline{r}\cdot\underline{p}_0) + \cos\theta\left[\underline{p}_0 - \underline{r}\,(\underline{r}\cdot\underline{p}_0)\right] + \sin\theta\ \underline{r}\wedge\underline{p}_0$$

EULERO

$\Downarrow$ PASSAGGIO A
R MATRICE DI ROTAZIONE

$\underline{p}$ come $[\underline{p}]$ 3×1

$$\underline{r}\cdot\underline{p}_0 \rightarrow \underline{r}^T\underline{p}_0$$

$$\underline{p}_1 = \underset{3\times1}{\underline{r}}\ \underset{1\times3}{\underline{r}^T}\ \underset{3\times1}{\underline{p}_0} + \cos\theta\underbrace{\left(\underline{p}_0 - \underline{r}\,\underline{r}^T\underline{p}_0\right)}_{(\underset{I_{3\times3}}{1} - \underset{3\times3}{\underline{r}\,\underline{r}^T})\,\underline{p}_0} + \sin\theta\ \underbrace{\underline{r}\wedge\underline{p}_0}_{S\,\underline{p}_0}$$

SI INTRODUCE MATRICE PROD. VETTORIALE

$$S = \begin{bmatrix} 0 & -r_z & r_y \\ +r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$ si prova da

$$\det\begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ r_x & r_y & r_z \\ p_{0x} & p_{0y} & p_{0z} \end{bmatrix} = \overset{S}{\begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}}\begin{bmatrix} p_{0x} \\ p_{0y} \\ p_{0z} \end{bmatrix} = \begin{bmatrix} r_y p_{0z} - r_z p_{0y} \\ -r_x p_{0z} + r_z p_{0x} \\ r_x p_{0y} - r_y p_{0x} \end{bmatrix}$$

NOTO $\underline{r} \rightarrow S$

S matrice antisimmetrica

$S = -S^T$ (elem. nulli su diag.)

$$\underline{p}_1 = \underline{r}\,\underline{r}^T\underline{p}_0 + \cos\theta\,(I - \underline{r}\,\underline{r}^T)\,\underline{p}_0 + \sin\theta\ S\,\underline{p}_0$$

$$\underline{p}_1 = \underbrace{\left[\underline{r}\,\underline{r}^T + (I - \underline{r}\,\underline{r}^T)\cos\theta + \sin\theta\ S\right]}_{R}\underline{p}_0$$

$\underline{r}$ VERSORE ! $\quad [R(\underline{r},\theta)]_S$ ↗ come tensore ↘ $[\underline{r}]_S$ FORM. EULERO-RODRIGUEZ

$$\boxed{R = \underline{r}\,\underline{r}^T + (I - \underline{r}\,\underline{r}^T)\cos\theta + S\sin\theta}$$

PROCEDURA INVERSA:

$$\mathrm{tr}\, R = 1 + (3-1)\cos\theta$$

SOMMA EL. DIAG. $\quad \mathrm{tr}(\underline{r}\,\underline{r}^T)\quad \mathrm{tr}\, I \quad (\mathrm{tr}\, S = 0)$

$$\underline{r}\,\underline{r}^T = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} [r_x \; r_y \; r_z] = \begin{bmatrix} r_x^2 & r_x r_y & \cdot \\ & r_y^2 & \\ & & r_z^2 \end{bmatrix}$$

- $\mathrm{tr}\, R = 1 + 2\cos\theta = \lambda_1 + \lambda_2 + \lambda_3$

$$\theta = \mathrm{acos}\left(\frac{\mathrm{tr}\, R - 1}{2}\right) \begin{array}{l} \nearrow \theta_1 \\ \searrow \theta_2 \quad (= -\theta_1) \end{array}$$

- $\underline{r}$? cerchiamo $\underline{r}$ da S - ANTISIMM.

$$R = \underbrace{\underline{r}\,\underline{r}^T + (I - \underline{r}\,\underline{r}^T)\cos\theta}_{\text{parte SIMMETRICA}} + \underbrace{S\sin\theta}_{\text{PARTE ASIMM}}$$

$\underline{r}\,\underline{r}^T$ e I simmetriche

una qualsiasi matrice A si può scomporre in parti SIMM e ASIMM

$$A = A_{SIMM} + A_{ASIMM}$$

$$A_{SIMM} = \frac{A + A^T}{2} \qquad A_{ASIMM} = \frac{A - A^T}{2}$$

quindi, nota R se ne determina la parte asimmetrica $\quad R_{ASIMM} = \dfrac{R - R^T}{2}$

Le componenti di $\underline{r}$ si trovano nella S,

$$R_{ASIMM} = S\sin\theta \rightarrow S = \frac{1}{\sin\theta} R_{ASIMM}$$

Da $\mathrm{tr}\, R$ abbiamo ricavato due angoli

$$S = \frac{1}{\sin\theta_1} R_{ASIMM} \longrightarrow \underline{r}_1$$

$$S = \frac{1}{\sin\theta_2} R_{ASIMM} \longrightarrow \underline{r}_2$$

due soluzioni come mi aspettavo

Questo percorso inverso evidentemente richiede che $\quad \sin\theta_{1,2} \neq 0 \quad (R_{ASIMM} \neq 0)$

si torna ai casi particolari già evidenziati

$$\sin\theta_{1,2} = 0 \begin{cases} \cos\theta_{1,2} = 1 \rightarrow 0°, \ \infty \text{ soluz. } \underline{r} \text{ INDETERM} \\ \cos\theta_{1,2} = -1 \rightarrow \theta_{1,2} = \pm\pi \quad \underline{r}_{1,2} = \pm\underline{r}^* \end{cases}$$

questo si risolve da $R = R_{SIMM}$

$$R = \underline{r}\,\underline{r}^T + (I - \underline{r}\,\underline{r}^T)\cos\theta = 2\underline{r}\,\underline{r}^T - I$$

NOTA

$$= 2\begin{bmatrix} r_x^2 & r_x r_y & r_x r_z \\ \cdot & r_y^2 & r_y r_z \\ \cdot & \cdot & r_z^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ \cdot & 1 & 0 \\ \cdot & \cdot & 0 \end{bmatrix}$$

INCOGNITA

$$R_{1,1} = 2r_x^2 - 1 \longrightarrow \text{estraggo } r_x = \pm\sqrt{\frac{R_{11}+1}{2}}$$

scegliendo segno

scegliendo
segno

$r_x = \sqrt{\frac{R_{1,1}+1}{2}}$, r_y e $r_{\cancel{y}}$ da $R_{1,2}$ e $R_{1,3}$