Time-varying extreme pattern with dynamic models

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Why do we need to study extremes?

Introduction

- Extreme Value Theory (EVT):
  required when extreme pattern is relevant
- Extrapolation beyond data points

- Useful tool in many areas of application
- Examples include
  Environment (climate change), Finance (stock returns), ...
- Beautiful theory but...
  usually applied for independent observations

- Many examples in fact consist of time series
- Models should incorporate temporal dependence

EVT for exceedances

- **Starting point**: Pickands (1975) theorem

$$\lim_{u \to x_F} P(X - u \leq x | X > u) = G(x)$$

  x_F is upper limit of F, the distribution of X
  G is the d.f. of the GPD
  valid for a large number of scenarios for F
- **Standard approach**: set/choose finite value for u
  Perform EV analysis for exceedances $X - u$ with fixed u
- Problem 1: *Throws away* information from bulk of distribution
- Problem 2: u is assumed to be known and $< x_F$

Our approach for exceedances

Nascimento, Gamerman and Lopes (S&C, 2012)

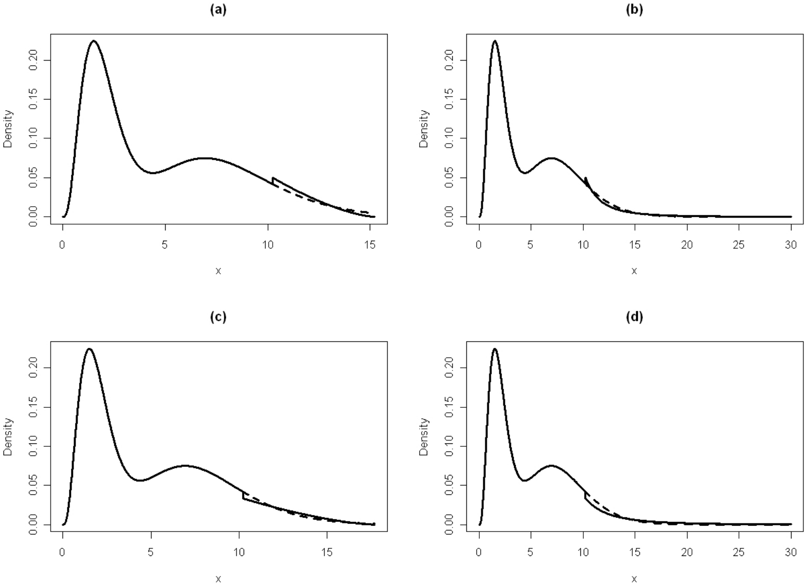
- We treat threshold u as a model parameter!
- Use **complete** data to estimate the threshold
- GPD distribution for exceedances beyond the threshold
- *Nonparametric* approach to center of distribution

- GPD density

$$g(x|\xi,\sigma,u) = \begin{cases} \sigma^{-1}\left(1+\xi x/\sigma\right)^{-(1+\xi)/\xi}, & \text{if} \quad \xi \neq 0 \\ \sigma^{-1}\exp\{-x/\sigma\}, & \text{if} \quad \xi = 0 \end{cases}, \qquad (1)$$

for $x > 0$ when $\xi \geq 0$ and for $0 \leq x < -\sigma/\xi$ when $\xi < 0$.

3 different regimes:
Weibull ($\xi < 0$), Gumbel ($\xi = 0$), Frechet ($\xi > 0$)



(a)
(b)
(c)
(d)
Density
x
0.00
0.05
0.10
0.15
0.20
0
5
10
15
20
25
30

Model

The distribution function is given by

$$F(x|\theta,\Psi) = \begin{cases} H(x \mid \theta), & \text{if} \quad x < u \\ H(u \mid \theta) + [1 - H(u \mid \theta)]\, G(x|\Psi), & \text{if} \quad x > u, \end{cases}$$

H: mixture of Gamma's with parameters (μ_j, ν_j) and weights p_j
Why mixture of Gamma's? It is dense!

Parameters: $\theta = (\mu, \nu, p)$ and $\Psi = (u, \sigma, \xi)$

Observations are conditionally independent given (θ, Ψ)

Provides likelihood for all parameters

Maxima and higher quantiles

Extreme analysis: extrapolation
evaluation in the tail of distribution (beyond data points)
Extreme quantiles: q s.t. $P(X > q) = 1 - p$, for large p

$$q = \frac{[(1-p^*)^{-\xi} - 1]\sigma}{\xi}, \text{ where } p^* = \frac{p - H(u \mid \theta)}{1 - H(u \mid \theta)}.$$

s-period return level $=$ quantile $1 - 1/s$

H is the d.f. of the central part
θ - parameters of H
Note: when $\xi < 0$, the distribution is bounded by $u - \sigma/\xi$

Dynamics on tail

- Rainfall and temperature can be related to seasonality and level rise over the years
- Recession periods may have influence on economic indices
- Propose a model to estimate time-varying extreme events
- Dynamic structure should imply better estimation and forecasting
- Dynamic extremes: evolution of tail parameters
- $(u, \sigma, \xi) \rightarrow (u_t, \sigma_t, \xi_t)$

Maxima and higher quantiles

Extreme analysis: extrapolation
evaluation in the tail of distribution (beyond data points)
Extreme quantiles: q_t s.t. $P(X_t > q_t) = 1 - p_t$, for large p_t

$$q_t = \frac{[(1 - p_t^*)^{-\xi_t} - 1]\sigma_t}{\xi_t}, \text{ where } p_t^* = \frac{p - H(u_t \mid \theta)}{1 - H(u_t \mid \theta)}.$$

s-period return level = quantile $1 - 1/s$

H is the d.f. of the central part
θ - parameters of H
Note: when $\xi_t < 0$, the distribution is bounded by $u_t - \sigma_t/\xi_t$

Dynamic model

- Suppose a temporal dependence of the data
- σ_t are parameters in $(0, \infty)$ and ξ_t in $(-1, \infty)$
- We used $l\sigma_t = \log(\sigma_t)$ and $l\xi_t = \log(\xi_t + 1)$

Example: First order dynamic model

$$\begin{aligned}
l\xi_t &= \theta_{\xi,t} + v_{\xi,t} & v_{\xi,t} &\sim N(0, 1/V_\xi),\\
\theta_{\xi,t} &= \theta_{\xi,t-1} + w_{\xi,t} & w_{\xi,t} &\sim N(0, 1/W_\xi),\\
l\sigma_t &= \theta_{\sigma,t} + v_{\sigma,t} & v_{\sigma,t} &\sim N(0, 1/V_\sigma),\\
\theta_{\sigma,t} &= \theta_{\sigma,t-1} + w_{\sigma,t} & w_{\sigma,t} &\sim N(0, 1/W_\sigma),
\end{aligned}$$

Threshold is static in the example

Model

The distribution function is given by

$$F_t(x_t|\theta,\Psi_t) = \begin{cases} H(x_t \mid \theta), & \text{if} \quad x_t < u \\ H(u \mid \theta) + [1 - H(u \mid \theta)]\, G(x_t|\Psi_t), & \text{if} \quad x_t > u, \end{cases}$$

H: mixture of Gamma's with parameters (μ_j, ν_j) and weights p_j
Parameters: $\theta = (\mu, \nu, p)$ and $\Psi_t = (u_t, \sigma_t, \xi_t)$, $t = 1, \ldots, n$

Provides likelihood for all the above parameters

Observations are conditionally independent given
$\theta, \{\Psi_t\ ,\ t = 1, ..., n\}$
Temporal dependence provided through latent states

Inference

- Bayesian approach to inference $\rightarrow$ requires prior for all parameters

- Prior distribution for dynamic structure of the example

$$
\begin{aligned}
\pi(I\xi, \theta_\xi, V_\xi, W_\xi) &= \prod_{t=1}^{n} \left(\pi(I\xi_t|\theta_{\xi,t}, V_\xi)\pi(\theta_{\xi,t}|\theta_{\xi,t-1}, W_\xi)\right) \pi(\theta_{\xi,0})\pi(V_\xi)\pi(W_\xi) \\
\pi(I\sigma, \theta_\sigma, V_\sigma, W_\sigma) &= \prod_{t=1}^{n} \left(\pi(I\sigma_t|\theta_{\sigma,t}, V_\sigma)\pi(\theta_{\sigma,t}|\theta_{\sigma,t-1}, W_\sigma)\right) \pi(\theta_{0,\sigma})\pi(V_\sigma)\pi(W_\sigma)
\end{aligned}
$$

Hyperparameters: $(\theta_\xi, V_\xi, W_\xi, \theta_\sigma, V_\sigma, W_\sigma)$

- combined with likelihood $\rightarrow$ posterior distribution
- Approximate estimation using MCMC
- Usual mix of full conditionals and M-H steps

Simulations

- Simulations of proposed model, with n=1,000 , 2,500 and 10,000
- Gamma distribution to nontail and GPD for tail
- Results shown the efficiency on detection of true parameters
- Precise estimation of threshold
- Dynamic structure is better estimated when the sample size increase

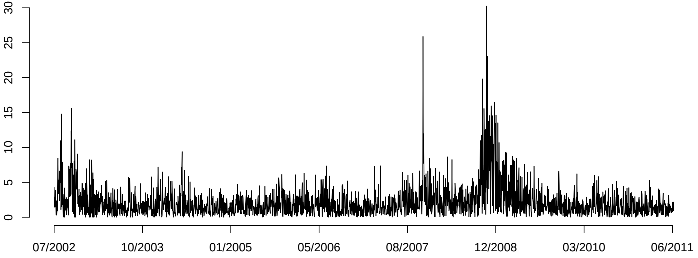
Applications

Indexes of stock market

Absolute returns $r_t = \mid x_t / x_{t-1} - 1 \mid \times 100$

Petrobras and SP500

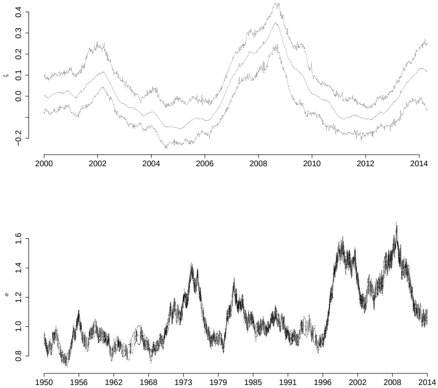
Figura: Petrobras Stocks



Model comparison for Petrobrás

	DIC values					
Time series	MG_k	$MGPD_k$	SV	$Prop$	$Prop_\sigma$	$Prop_\xi$
Petrobrás	$4529_{(3)}$	$4524_{(2)}$	4885	$4467_{(2)}$	$\mathbf{4444}_{(2)}$	$4525_{(2)}$
SP500	$9412_{(3)}$	$9438_{(2)}$	9338	$9297_{(2)}$	$9320_{(2)}$	$\mathbf{9295}_{(2)}$

MG_k mixture of Gamma's with the smallest DIC
$MGPD_k$ mixture of Gamma's + GPD with the smallest DIC
SV is the standard stochastic volatility model
$Prop$ is the proposed model
$Prop_\sigma$ ($Prop_\xi$): proposed model with σ (ξ) constant.



95% credibility intervals to tail parameters

Top: ξ for Petrobrás; bottom: σ for SP500

Figura: Petrobras data with 95% and 99% quantiles.

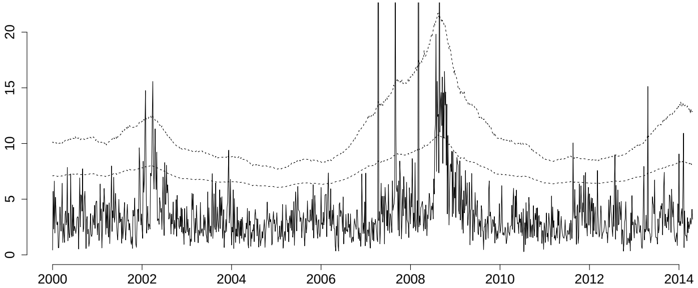
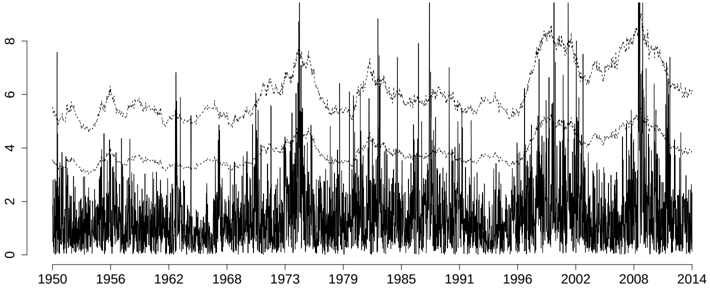
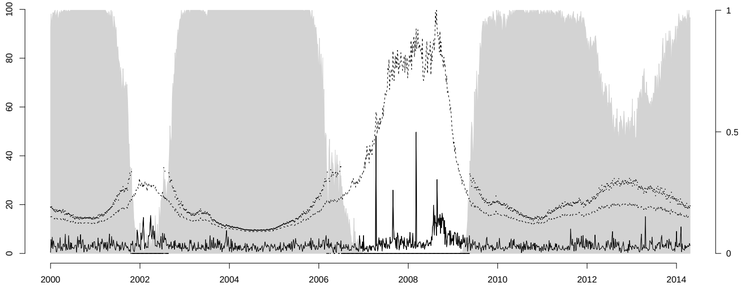
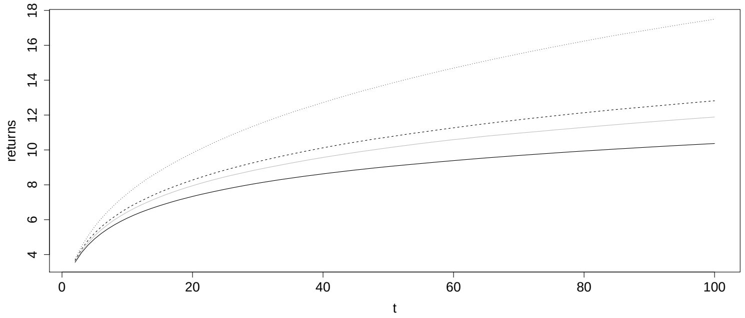


Figura: SP500 data with 95% and 99% quantiles.

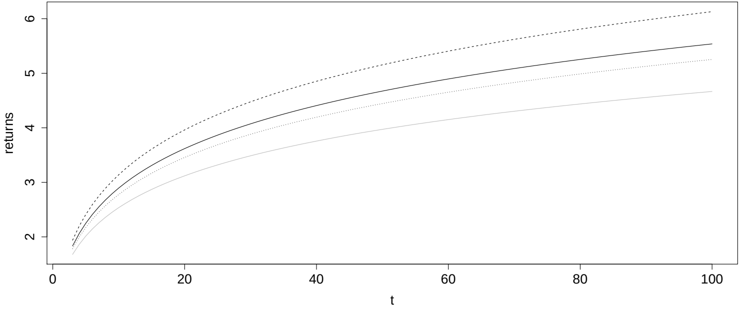




Absolute returns of Petrobrás (full)
99.9999% quantiles (dashed)
maximum when $median(\xi|\mathbf{x}) < \mathbf{0}$ (dotted)
Shaded area: posterior probability of finite maximum
$P(\xi_t < 0|\mathbf{x})$, for all t.



Posterior mean of the return level plot for Petrobras, in different points in time: October 2000 (Full), February 2003 (Grey) and July 2008 (Dotted) and February 2014 (Dashed).



Posterior mean of the return level plot for SP500, in different points in time: January 1950 (Full), April 1988 (Grey), March 2008 (Dashed) and February 2014 (Dotted).

Extension 1: Dynamic regression

Regression on tail:
Nascimento et al. (2011) and Cabras et al. (2011)

Use a dynamic model where $F_t' = (1, x_{1,t}, \ldots, x_{t,p})$, $G = I_n$ and $\theta_t' = (\beta_{t,0}, \ldots, \beta_{t,p})$.

For example, to scale parameter $l\sigma_t = \log(\sigma_t)$,

$$\begin{aligned} l\sigma_t &= \beta_{t,0} + \beta_{t,1} x_{t,1} + \ldots + \beta_{t,p} x_{t,p} + v_t \\ \beta_{t,i} &= \beta_{t-1,i} + w_{i,t}, \quad i = 0, \ldots, p. \end{aligned}$$

Extend these ideas to the threshold: $u \to u_t$

Extension 2: Regime identification

Our approach treated ξ as continuous
Shape parameter ξ defines 3 different extremal regimes:
Weibull ($\xi < 0$), Gumbel ($\xi = 0$), Frechet ($\xi > 0$)

Nascimento et al. (2016)
These regimes could be treated separately
Allows identification of the regime

Regime may switch over time $\rightarrow$ Markov switching model
Current work with Fernando Nascimento

Extension 3: Multivariate extremes

Our approach considers only univariate extremes
Many situations involve joint distributions
Example: wind speed and wave height

Possible static model:
univariate models MGPD +
dependence through flexible copulas

Time-varying scenario:
Parameters may be allowed to change over time
Current work with Manuele Leonelli (Glasgow)

Conclusions

- Model for extreme values with dynamic structure
- Model captures qualitative changes in tail behaviour
- Advantages over other standard models
- Temporal dependence in central part?
- Point process approach?

Obrigado!

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