

Analytical solution to two layer conduction with heating in spherical coordinates

Brent G. Delgbridge

December 8, 2014

1 Problem Formulation:

1.1 Thermal Conduction Equation in Spherical Conduction:

In order to determine the energy balance at the base of the stagnant lid in parameterized convection models we must determine the thermal gradient at the base of the lid. It is assumed that heat within the stagnant lid heat is transported conductively, and thus in order to determine the thermal profile within the lid we must solve the spherical conduction equation given by

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 k_l \frac{\partial T}{\partial r} \right) + q_l = 0 \quad . \quad (1)$$

Here r is the radial distance from the center of the planet, k_l and q_l are the thermal conductivity and heat production in the stagnant lid respectively. Due to the preferential differentiation of the radiogenic nucleotides into the partial melts which form the crust, we expect a considerably higher level of heating in the crust which will drastically alter the temperature profile within the stagnant lid. Additionally, we expect that the thermal conductivities will differ between the crust and mantle. So to first order, we model the stagnant lid as two layers, with different thermal conductivities and rates of internal heating, k_m , q_m and k_c , q_c , in the mantle and crust respectively. We will solve for the the thermal profiles T_m and T_c in each layer separately and then match the solutions at the boundary between the two layers. In this geometry both the thermal conductivity and heat production rates are constants representing their volume averaged quantities. We can now obtain the solution within each layer by simply integrating EQ(1) twice with respect to r .

Multiplying by r^2 and integrating we obtain:

$$\int \frac{\partial}{\partial r} \left(r^2 k_l \frac{\partial T}{\partial r} \right) = - \int r^2 q_l \Rightarrow \frac{\partial T}{\partial r} = -\frac{q_l r}{3k_l} + \frac{c_1}{r^2} \quad ,$$

where we have absorbed k_l into the integration constant c_1 to be determined by the boundary conditions later.

Integrating again we obtain the following solution:

$$T(r) = \begin{cases} T_c(r) = -\frac{q_c}{6k_l}r^2 + \frac{c_1}{r} + c_2 & \text{if } r_s \geq r \geq r_c \\ T_m(r) = -\frac{q_m}{6k_l}r^2 + \frac{m_1}{r} + m_2 & \text{if } r_c > r \geq r_l \end{cases} \quad (2)$$

2 Boundary Conditions:

2.1 Fixed Temperature at the planet surface (r_s) :

The temperature at the surface of the planet (T_s) is assumed to be fixed to $T_c(r_s) = T_s$, from which we obtain

$$T_s = -\frac{q_c}{6k_l}r_s^2 + \frac{c_1}{r_s} + c_2 \quad . \quad (3)$$

2.2 Fixed Temperature at base of the stagnant lid (r_l) :

The temperature at the base of the stagnant lid (T_l) is fixed to $T_m(r_l) = T_l$, from which we obtain

$$T_l = -\frac{q_m}{6k_m}r_l^2 + \frac{m_1}{r_l} + m_2 \quad . \quad (4)$$

2.3 Temperature matching conditions at the interior boundary (r_c) :

Both solutions must agree on the temperature at the boundary between the crust and mantle. From setting the two equations equal at the boundary $T_c(r_s) = T_m(r_s)$, we obtain

$$-\frac{q_c}{6k_l}r_c^2 + \frac{c_1}{r_c} + c_2 = -\frac{q_m}{6k_m}r_c^2 + \frac{m_1}{r_c} + m_2 \quad . \quad (5)$$

2.4 Flux matching conditions at the interior boundary (r_c) :

Both solutions must agree on the flux at the boundary between the crust and mantle. By setting the flux of the two equations equal at the boundary ($-k_c \frac{\partial T_c}{\partial r} = -k_m \frac{\partial T_m}{\partial r}$) we obtain

$$\text{fluxes} = \begin{cases} F_c(r_c) & = -k_c \left(-\frac{q_c r_c}{3k_c} - \frac{c_1}{r_c^2} \right) \\ F_m(r_c) & = -k_m \left(-\frac{q_m r_c}{3k_m} - \frac{m_1}{r_c^2} \right) \end{cases}$$

Simplifying the above expression and equating we obtain

$$\frac{q_c r_c}{3} + \frac{k_c c_1}{r_c^2} = \frac{q_m r_c}{3} + \frac{k_m m_1}{r_c^2} \quad (6)$$

3 Solution for integration constants (c_1, c_2, m_1, m_2) :

3.1 Solving for m_1 from the flux matching condition EQ(6):

Multiplying EQ(6) by r_c^2 and dividing by k_m and solving for m_1 :

$$\begin{aligned}
 m_1 &= \frac{q_c r_c^3}{3k_m} + \frac{k_c c_1}{k_m} - \frac{q_m r_c^3}{3k_m} \\
 &= \frac{r_c^3}{3k_m} (q_c - q_m) + \frac{k_c}{k_m} c_1 \\
 &= \alpha + k c_1
 \end{aligned} \tag{7}$$

where we have defined,

$$\begin{cases} \alpha &= \frac{r_c^3}{3k_m} (q_c - q_m) \\ k &= \frac{k_c}{k_m} \end{cases} .$$

3.2 Solving for m_2 in the temperature matching condition

$$\begin{aligned}
 m_2 &= -\frac{q_c r_c^2}{6k_c} + \frac{c_1}{r_c} + c_2 + \frac{q_m r_c^2}{6k_m} - \frac{m_1}{r_c} \\
 &= \frac{r_c^2}{6} \left(\frac{q_m}{k_m} - \frac{q_c}{k_c} \right) + \frac{c_1}{r_c} + c_2 - \frac{m_1}{r_c} \\
 &= \frac{r_c^2}{6} \left(\frac{q_m}{k_m} - \frac{q_c}{k_c} \right) + \frac{c_1}{r_c} + c_2 - \left(\frac{\alpha + k c_1}{r_c} \right) \\
 &= \frac{r_c^2}{6} \left(\frac{q_m}{k_m} - \frac{q_c}{k_c} \right) - \frac{\alpha}{r_c} + \left(\frac{1 - k}{r_c} \right) c_1 + c_2 \\
 &= \beta + \left(\frac{1 - k}{r_c} \right) c_1 + c_2
 \end{aligned} \tag{8}$$

where we have defined,

$$\begin{aligned}
 \beta &= \frac{r_c^2}{6} \left(\frac{q_m}{k_m} - \frac{q_c}{k_c} \right) - \frac{\alpha}{r_c} \\
 &= \frac{r_c^2}{6} \left(\frac{q_m}{k_m} - \frac{q_c}{k_c} \right) - \frac{r_c^2}{3k_m} (q_c - q_m) \\
 &= \frac{r_c^2}{6} \left(\frac{q_m}{k_m} - \frac{q_c}{k_c} - 2 \frac{(q_c - q_m)}{k_m} \right) \\
 &= \frac{r_c^2}{6} \left(\frac{2q_c - q_m}{k_m} - \frac{q_c}{k_c} \right)
 \end{aligned}$$

3.3 Solving for c_2 from the temperature boundary condition at the base of the stagnant lid :

$$\begin{aligned}
T_l &= -\frac{q_m r_l^2}{6k_m} + \frac{m_1}{r_l} + m_2 \\
&= -\frac{q_m r_l^2}{6k_m} + \frac{(\alpha + k c_1)}{r_l} + \beta + \left(\frac{1-k}{r_c}\right) c_1 + c_2 \\
&= -\frac{q_m r_l^2}{6k_m} + \frac{\alpha}{r_l} + \beta + \left(\frac{k}{r_l} + \frac{1-k}{r_c}\right) c_1 + c_2 \\
&= \gamma + \left(\frac{k}{r_l} + \frac{1-k}{r_c}\right) c_1 + c_2 \\
\Rightarrow c_2 &= T_l - \gamma - \left(\frac{k}{r_l} + \frac{1-k}{r_c}\right) c_1
\end{aligned} \tag{9}$$

where we have defined,

$$\gamma = -\frac{q_m r_l^2}{6k_m} + \frac{\alpha}{r_l} + \beta \tag{10}$$

3.4 Solving for c_1 from the temperature boundary condition at the surface :

$$\begin{aligned}
T_s &= -\frac{q_c r_s^2}{6k_c} + \frac{c_1}{r_s} + c_2 \\
&= -\frac{q_c r_s^2}{6k_c} + \frac{c_1}{r_s} + T_l - \gamma - \left(\frac{k}{r_l} + \frac{1-k}{r_c}\right) c_1 \\
&= -\frac{q_c r_s^2}{6k_c} + T_l - \gamma - \left(\frac{k}{r_l} + \frac{1-k}{r_c} - \frac{1}{r_s}\right) c_1 \\
\Rightarrow c_1 &= \frac{-\frac{q_c r_s^2}{6k_c} - \gamma + (T_l - T_s)}{\left(\frac{k}{r_l} + \frac{1-k}{r_c} - \frac{1}{r_s}\right)}
\end{aligned} \tag{11}$$