Heat flux maps

1 The simple case

If the mercurian mantle is not convecting, and has not been for a long time, we would expect the mantle temperature distribution to approach a conductive solution. However, due to the unusual spin-orbit resonance in which Mercury is locked, there are large and persistent temperature variations on the surface. This is because some regions get significantly more insolation than others.

Here I calculate the temperature distribution in a conducting mantle with constant conductivity and temperature boundary conditions. I assume that the temperature at the CMB is constant due to efficient transport of heat in the liquid outer core. Further, I assume that the temperature at the surface is time-independent, but may vary spatially due to insolation differences. This assumes that I am taking the surface to be below the skin-depth of the regolith.

Outer radius	R_o
Inner radius	R_i
Aspect ratio	$R_i/R_o = \eta$
Inner temperature	T_i
Surface temperature	$S(\theta, \phi)$

Table 1: Symbols used

1.1 General solution

Steady state diffusion satisfies Laplace's equation:

$$\nabla^2 T = 0 \tag{1}$$

In spherical coordinates this has the general solution

$$T(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^l + B_{lm} r^{-(l+1)} \right] Y_{lm}(\theta, \phi) \tag{2}$$

Where Y_{lm} represents fully normalized real spherical harmonics, and A_{lm} and B_{lm} are coefficients.

We suplement this with the boundary conditions

$$T(R_o, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} R_o^l + B_{lm} R_o^{-(l+1)} \right] Y_{lm}(\theta, \phi) = S(\theta, \phi) \tag{3}$$

$$T(R_i,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}R_i^l + B_{lm}R_i^{-(l+1)}\right]Y_{lm}(\theta,\phi) = T_i \tag{4}$$

1.2 Solution for $l \neq 0$

The inner boundary has no θ, ϕ dependence, so the function of r multiplying the spherical harmonics must be zero for $l \neq 0$:

$$\begin{aligned} A_{lm}R_i^l + B_{lm}R_i^{-(l+1)} &= 0 \\ A_{lm}R_i^l &= -B_{lm}R_i^{-(l+1)} \\ -A_{lm}R_i^{(2l+1)} &= B_{lm} \end{aligned} \tag{5}$$

We can plug this in to the upper boundary condition to find

$$\begin{aligned} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}\left[R_o^l - R_i^{(2l+1)}R_o^{-(l+1)}\right]Y_{lm}(\theta,\phi) &= S(\theta,\phi) \\ \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}R_o^l\left[1 - R_i^{(2l+1)}R_o^{-(2l+1)}\right]Y_{lm}(\theta,\phi) &= S(\theta,\phi) \\ \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}R_o^l\left[1 - \eta^{(2l+1)}\right]Y_{lm}(\theta,\phi) &= S(\theta,\phi) \end{aligned} \tag{6}$$

If we further expand $S(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} S_{lm}Y_{lm}(\theta,\phi)$ we find

$$A_{lm}R_o^l\left[1 - \eta^{(2l+1)}\right] = S_{lm} \tag{7}$$

So that

$$A_{lm} = \frac{S_{lm}}{R_o^l\left[1 - \eta^{(2l+1)}\right]} \tag{8}$$

Finally, we may find B_{lm}:

$$\begin{aligned} B_{lm} &= -A_{lm}R_i^{(2l+1)} = \frac{-S_{lm}R_i^{(2l+1)}}{R_o^l\left[1 - \eta^{(2l+1)}\right]} \\ B_{lm} &= \frac{-S_{lm}\eta^{(2l+1)}}{R_o^{-(l+1)}\left[1 - \eta^{(2l+1)}\right]} \\ B_{lm} &= \frac{-S_{lm}}{R_o^{-(l+1)}\left[\eta^{-(2l+1)} - 1\right]} \end{aligned} \tag{9}$$

We may plug these expressions for A_{lm} and B_{lm} into the general solution to find

$$
\begin{aligned}
T(r,\theta,\phi) &= \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left[\frac{S_{lm}}{1-\eta^{(2l+1)}}\left(\frac{r}{R_o}\right)^{l} - \frac{S_{lm}}{\eta^{-(2l+1)}-1}\left(\frac{r}{R_o}\right)^{-(l+1)}\right] Y_{lm}(\theta,\phi) \\
&= \sum_{l=1}^{\infty}\sum_{m=-l}^{l} S_{lm}\left[\frac{1}{1-\eta^{(2l+1)}}\left(\frac{r}{R_o}\right)^{l} + \frac{1}{1-\eta^{-(2l+1)}}\left(\frac{r}{R_o}\right)^{-(l+1)}\right] Y_{lm}(\theta,\phi) \\
&= \sum_{l=1}^{\infty}\sum_{m=-l}^{l} S_{lm}\left[\frac{1}{1-\eta^{(2l+1)}}\left(\frac{r}{R_o}\right)^{l} - \frac{\eta^{(2l+1)}}{1-\eta^{(2l+1)}}\left(\frac{r}{R_o}\right)^{-(l+1)}\right] Y_{lm}(\theta,\phi)
\end{aligned}
\tag{10}
$$

We may verify that for $r = R_o$ this becomes $S(\theta,\phi)$ and for $r = R_i$ this is zero.

1.3 Solution for $l = 0$

Now we must do the case for $l = 0$. Using the equation for the inner boundary condition, we find

$$
\begin{aligned}
\left[A_{00} + B_{00}/R_i\right] Y_{00} &= T_i \\
B_{00} &= \left[T_i/Y_{00} - A_{00}\right] R_i
\end{aligned}
\tag{11}
$$

Plugging into the outer boundary:

$$
\begin{aligned}
\left[A_{00} + B_{00}/R_o\right] Y_{00} &= S_{00} \\
\left[A_{00} + \eta\frac{T_i}{Y_{00}} - A_{00}\eta\right] Y_{00} &= S_{00} \\
A_{00}(1-\eta) &= \frac{S_{00}}{Y_{00}} - \eta\frac{T_i}{Y_{00}} \\
A_{00}(1-\eta) &= \frac{S_{00}-\eta T_i}{Y_{00}} \\
A_{00} &= \frac{S_{00}-\eta T_i}{(1-\eta)Y_{00}}
\end{aligned}
\tag{12}
$$

Therefore

$$
\begin{aligned}
B_{00} &= \left[T_i/Y_{00} - \frac{S_{00}-\eta T_i}{(1-\eta)Y_{00}}\right] R_i \\
B_{00} &= \left[\frac{T_i(1-\eta)}{(1-\eta)} - \frac{S_{00}-\eta T_i}{(1-\eta)}\right] \frac{R_i}{Y_{00}} \\
B_{00} &= \left[T_i(1-\eta) - S_{00} + \eta T_i\right] \frac{R_i}{(1-\eta)Y_{00}} \\
B_{00} &= \frac{(T_i - S_{00})R_i}{(1-\eta)Y_{00}}
\end{aligned}
\tag{13}
$$

We can now plug these in to the general solution for $l = 0$:

$$
\begin{aligned}
T(r,\theta,\phi) &= \left[\frac{S_{00}-\eta T_i}{(1-\eta)Y_{00}} + \frac{T_i - S_{00}}{(1-\eta)Y_{00}}\frac{R_i}{r}\right] Y_{00} \\
&= \left[S_{00} - \eta T_i + (T_i - S_{00})\frac{R_i}{r}\right]\frac{1}{1-\eta} \\
&= \left[S_{00}(1-\frac{R_i}{r}) + T_i(\frac{R_i}{r} - \eta)\right]\frac{1}{1-\eta}
\end{aligned} \tag{14}
$$

Again, we may verify this by plugging in R_i and R_o to so that it satisfies boundary conditions.

1.4 Heat flux calculation

The heat flux through the CMB is calculated with Fourier's law:

$$
q = -k\nabla T \cdot \mathbf{r} = -k\partial T/\partial r \tag{15}
$$

where k is the thermal conductivity. This derivative is evaluated at R_i. First, we calculate the case for $l = 0$:

$$
\partial T/\partial r = \left[\frac{S_{00}R_i}{r^2} - \frac{T_i R_i}{r^2}\right]\frac{1}{1-\eta} \tag{16}
$$

Evaluating this at R_i we find:

$$
q_{cmb} = k\frac{(T_i - S_{00})}{R_i}\frac{1}{1-\eta} \tag{17}
$$

For the case of $l \neq 0$ we find:

$$
\partial T/\partial r = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{S_{lm}}{R_o}\left[\frac{l}{1-\eta^{(2l+1)}}\left(\frac{r}{R_o}\right)^{l-1} + \frac{(l+1)\eta^{(2l+1)}}{1-\eta^{(2l+1)}}\left(\frac{r}{R_o}\right)^{-(l+2)}\right] Y_{lm}(\theta,\phi) \tag{18}
$$

Again we evaluate this at R_i:

$$
\begin{aligned}
q_{cmb} &= -k\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{S_{lm}}{R_o}\left[\frac{l\eta^{(l-1)}}{1-\eta^{(2l+1)}} + \frac{(l+1)\eta^{(2l+1)}\eta^{-(l+2)}}{1-\eta^{(2l+1)}}\right] Y_{lm}(\theta,\phi) \\
&= -k\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{S_{lm}}{R_o}\left[\frac{l\eta^{(l-1)}}{1-\eta^{(2l+1)}} + \frac{(l+1)\eta^{(l-1)}}{1-\eta^{(2l+1)}}\right] Y_{lm}(\theta,\phi) \\
&= -k\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{S_{lm}}{R_o}\left[\frac{(2l+1)\eta^{(l-1)}}{1-\eta^{(2l+1)}}\right] Y_{lm}(\theta,\phi)
\end{aligned} \tag{19}
$$

Note that the integral of a spherical harmonic over a sphere with $l \neq 0$ is zero, so each of these components of the heat flux to not affect the overall flux, but are merely spatial variations on top of the average flux as calculated with the $l = 0$ term.