Lecture 6
Newton-Raphson Method

- Assumptions
- Interpretation
- Examples
- Convergence Analysis

Newton-Raphson Method

(Also known as Newton's Method)

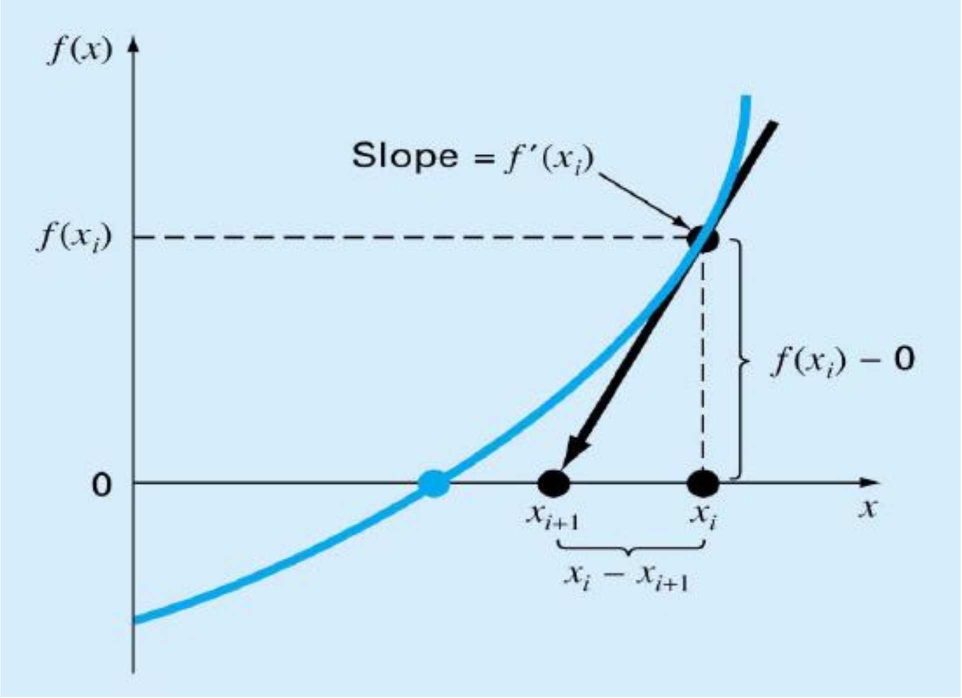
Given an initial guess of the root x_0, Newton-Raphson method uses information about the function and its derivative at that point to find a better guess of the root.

Assumptions:

- ***f(x)*** is continuous and the first derivative is known
- An initial guess x_0 such that $f'(x_0) \neq 0$ is given

Newton Raphson Method
- Graphical Depiction -

- If the initial guess at the root is $\boldsymbol{x_i}$, then a tangent to the function of $\boldsymbol{x_i}$ that is $\boldsymbol{f'(x_i)}$ is extrapolated down to the *x-axis* to provide an estimate of the root at $\boldsymbol{x_{i+1}}$.



Derivation of Newton's Method

$Given$: x_i an initial guess of the root of $f(x)=0$

$Question$: How do we obtain a better estimate x_{i+1}?

Taylor Therorem: $f(x+h) \approx f(x) + f'(x)h$

Find h such that $f(x+h)=0$.

$$\Rightarrow h \approx -\frac{f(x)}{f'(x)}$$

Newton–Raphson Formula

A new guess of the root: $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

Newton's Method

$$Given \quad f(x),\ f'(x),\ x_0$$

$$Assumption \quad f'(x_0) \neq 0$$

$$for \quad i = 0:n$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$end$$

```
C    FORTRAN PROGRAM
       F(X) = X**3 - 3*X**2 + 1
       FP(X) = 3*X**2 - 6*X
       X = 4
       DO 10 I = 1,5
        X = X - F(X)/FP(X)
        PRINT *, X
10      CONTINUE
       STOP
       END
```

Newton's Method

$$Given \quad f(x),\ f'(x),\ x_0$$
$$Assumption \quad f'(x_0) \neq 0$$

$$for \quad i = 0:n$$
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$
$$end$$

F.m

$$function\ [F] = F(X)$$
$$F = X\text{^}3 - 3 * X\text{^}2 + 1$$

FP.m

$$function\ [FP] = FP(X)$$
$$FP = 3 * X\text{^}2 - 6 * X$$

$$\%\quad MATLAB\ PROGRAM$$
$$X = 4$$
$$for\ i = 1:5$$
$$X = X - F(X) / FP(X)$$
$$end$$

Example

Find a zero of the function $f(x) = x^3 - 2x^2 + x - 3$, $x_0 = 4$

$f'(x) = 3x^2 - 4x + 1$

Iteration1: $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 4 - \dfrac{33}{33} = 3$

Iteration2: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 3 - \dfrac{9}{16} = 2.4375$

Iteration3: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 2.4375 - \dfrac{2.0369}{9.0742} = 2.2130$

Example

k (Iteration)	x_k	$f(x_k)$	$f'(x_k)$	x_{k+1}	$\lvert x_{k+1} - x_k \rvert$
0	**4**	**33**	**33**	**3**	1
1	**3**	**9**	**16**	**2.4375**	0.5625
2	**2.4375**	**2.0369**	**9.0742**	**2.2130**	0.2245
3	**2.2130**	**0.2564**	**6.8404**	**2.1756**	0.0384
4	**2.1756**	**0.0065**	**6.4969**	**2.1746**	0.0010

Convergence Analysis

Theorem:

Let $f(x), f'(x)$ and $f''(x)$ be continuous at $x \approx r$ where $f(r) = 0$. If $f'(r) \neq 0$ then there exists $\delta > 0$ such that $|x_0 - r| \leq \delta \Rightarrow \dfrac{|x_{k+1} - r|}{|x_k - r|^2} \leq C$

$$C = \frac{1}{2} \frac{\max\limits_{|x_0 - r| \leq \delta} |f''(x)|}{\min\limits_{|x_0 - r| \leq \delta} |f'(x)|}$$

Convergence Analysis

Remarks

When the guess is close enough to a simple root of the function then Newton's method is guaranteed to converge quadratically.

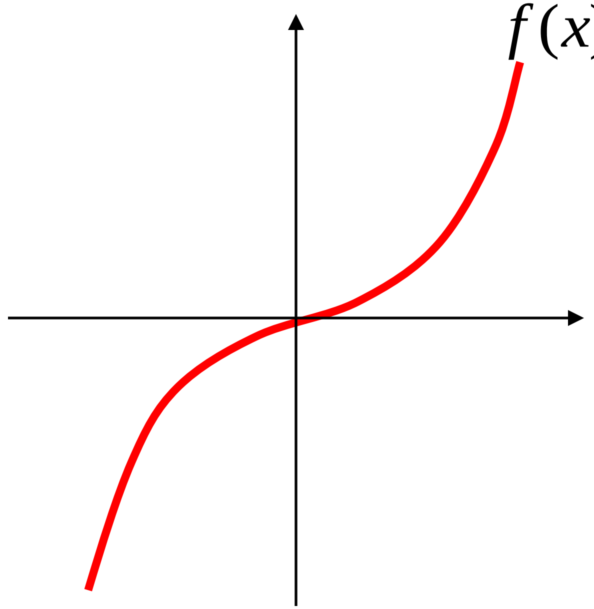
Quadratic convergence means that the number of correct digits is nearly doubled at each iteration.

Problems with Newton's Method

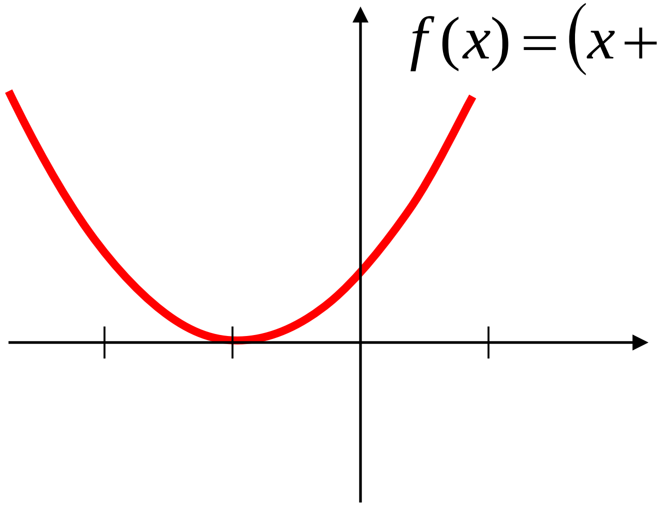
- If the initial guess of the root is far from the root the method may not converge.
- Newton's method converges linearly near multiple zeros { $f(r) = f'(r) = 0$ }. In such a case, modified algorithms can be used to regain the quadratic convergence.

Multiple Roots

$$f(x) = x^3$$



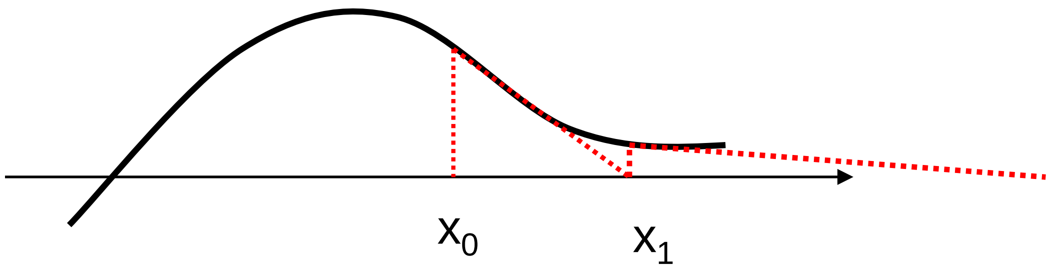
$$f(x) = (x+1)^2$$



$f(x)$ has three zeros at $x = 0$

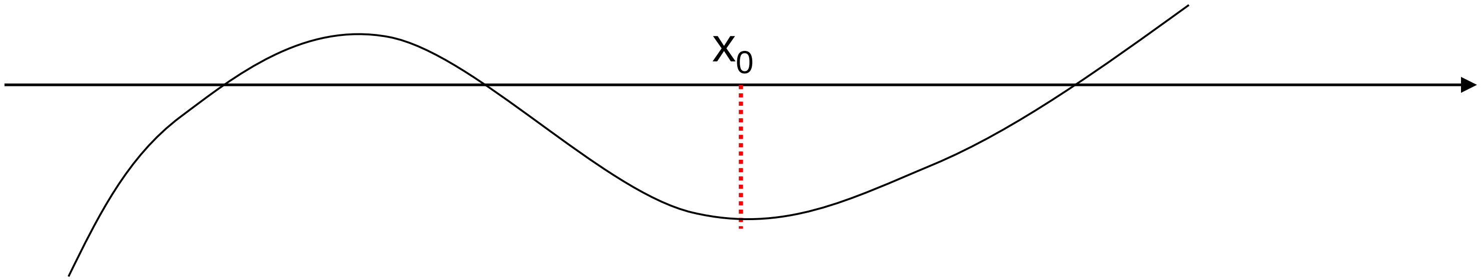
$f(x)$ has two zeros at $x = -1$

Problems with Newton's Method
- Runaway -



The estimates of the root is going away from the root.

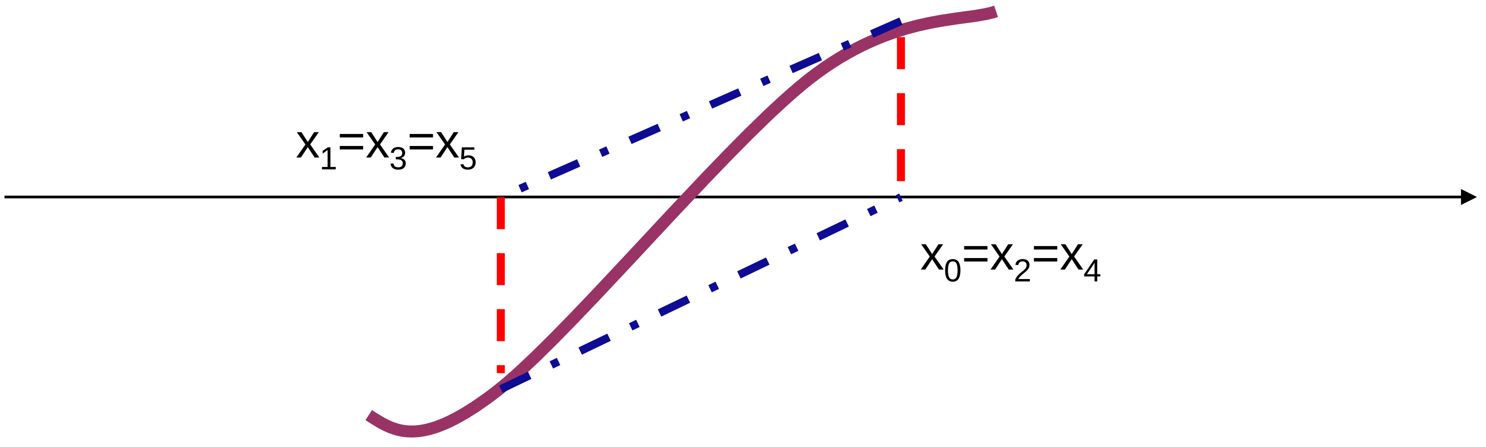
Problems with Newton's Method
- Flat Spot -



The value of ***f'(x)*** is zero, the algorithm fails.

If ***f '(x)*** is very small then $\boldsymbol{x_1}$ will be very far from $\boldsymbol{x_0}$.

Problems with Newton's Method
- Cycle -



The algorithm cycles between two values x_0 and x_1

Newton's Method for Systems of Non Linear Equations

$$\textit{Given: } X_0 \text{ an initial guess of the root of } F(x)=0$$

$$\textit{Newton's Iteration}$$

$$X_{k+1} = X_k - \left[F'(X_k)\right]^{-1} F(X_k)$$

$$F(X) = \begin{bmatrix} f_1(x_1, x_2, \ldots) \\ f_2(x_1, x_2, \ldots) \\ \vdots \end{bmatrix}, \quad F'(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \vdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \\ \vdots & & \end{bmatrix}$$

Example

- Solve the following system of equations:

$$y + x^2 - 0.5 - x = 0$$

$$x^2 - 5xy - y = 0$$

$$\text{Initial guess } x = 1, \quad y = 0$$

$$F = \begin{bmatrix} y + x^2 - 0.5 - x \\ x^2 - 5xy - y \end{bmatrix}, \quad F' = \begin{bmatrix} 2x - 1 & 1 \\ 2x - 5y & -5x - 1 \end{bmatrix}, X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution Using Newton's Method

Iteration1:

$$F = \begin{bmatrix} y + x^2 - 0.5 - x \\ x^2 - 5xy - y \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} =, \quad F' = \begin{bmatrix} 2x-1 & 1 \\ 2x-5y & -5x-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}^{-1} \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 0.25 \end{bmatrix}$$

Iteration 2:

$$F = \begin{bmatrix} 0.0625 \\ -0.25 \end{bmatrix} =, \quad F' = \begin{bmatrix} 1.5 & 1 \\ 1.25 & -7.25 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1.25 \\ 0.25 \end{bmatrix} - \begin{bmatrix} 1.5 & 1 \\ 1.25 & -7.25 \end{bmatrix}^{-1} \begin{bmatrix} 0.0625 \\ -0.25 \end{bmatrix} = \begin{bmatrix} 1.2332 \\ 0.2126 \end{bmatrix}$$

Example

Try this

- Solve the following system of equations:

$$y + x^2 - 1 - x = 0$$

$$x^2 - 2y^2 - y = 0$$

$$\text{Initial guess } x = 0, \ \ y = 0$$

$$F = \begin{bmatrix} y + x^2 - 1 - x \\ x^2 - 2y^2 - y \end{bmatrix}, \ F' = \begin{bmatrix} 2x - 1 & 1 \\ 2x & -4y - 1 \end{bmatrix}, \ X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example

Solution

$Iteration$	0	1	2	3	4	5
X_k	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -0.6 \\ 0.2 \end{bmatrix}$	$\begin{bmatrix} -0.5287 \\ 0.1969 \end{bmatrix}$	$\begin{bmatrix} -0.5257 \\ 0.1980 \end{bmatrix}$	$\begin{bmatrix} -0.5257 \\ 0.1980 \end{bmatrix}$

Lecture 7
Secant Method

- Secant Method
- Examples
- Convergence Analysis

Newton's Method (Review)

$$Assumptions: \quad f(x),\ f'(x),\ x_0 \quad are\ available,$$

$$f'(x_0) \neq 0$$

$$Newton's\ \ Method\ new\ estimate:$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Problem:

$f'(x_i)$ is not available,

or difficult to obtain analytically.

Secant Method

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

if x_i *and* x_{i-1} are two initial points:

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{\dfrac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}} = x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}}$$

Secant Method

Assumptions:

Two initial points x_i and x_{i-1}

such that $f(x_i) \neq f(x_{i-1})$

New estimate (Secant Method):

$$x_{i+1} = x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Secant Method

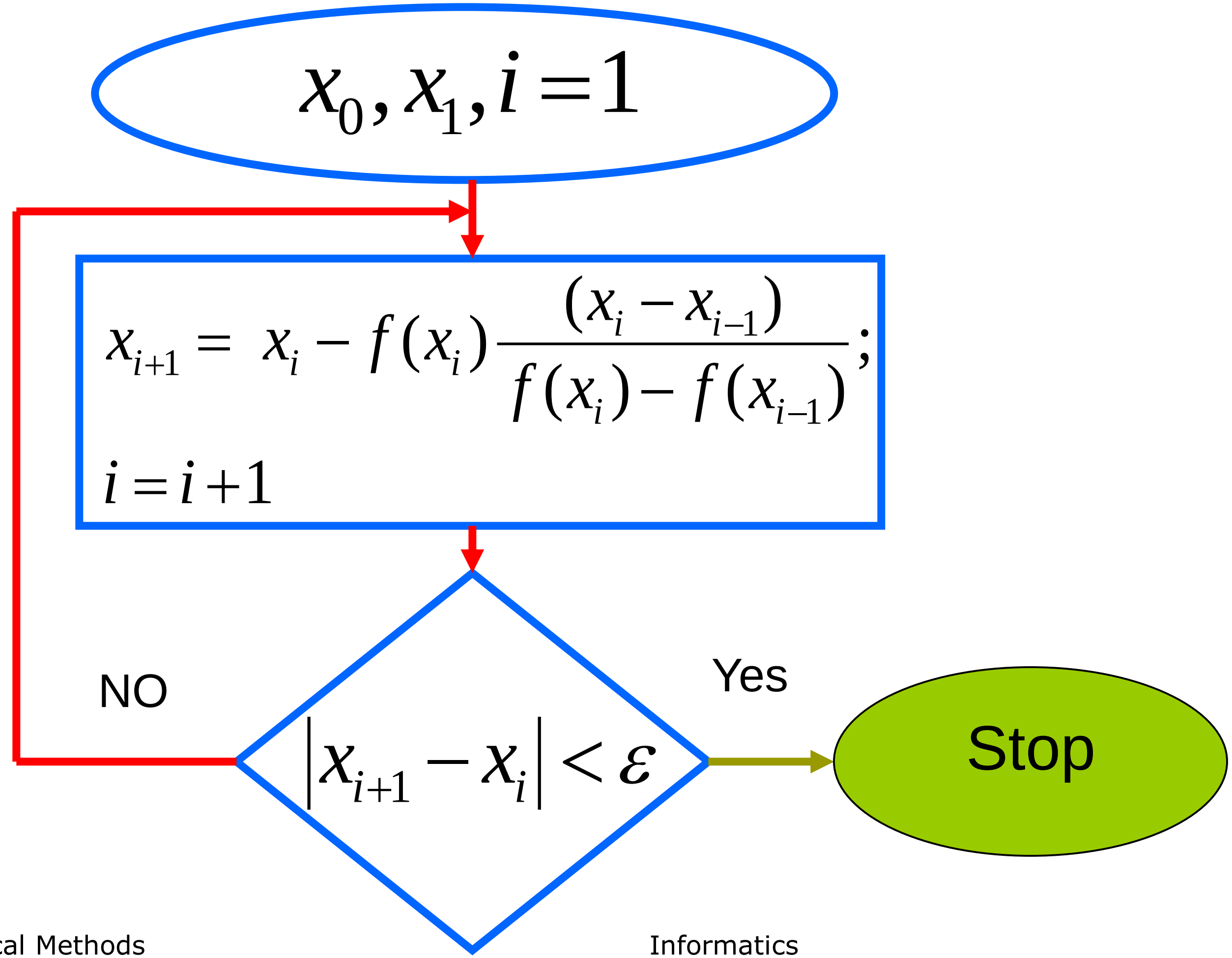
$$f(x) = x^2 - 2x + 0.5$$

$$x_0 = 0$$

$$x_1 = 1$$

$$x_{i+1} = x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Secant Method - Flowchart



Modified Secant Method

In this modified Secant method, only one initial guess is needed:

$$f'(x_i) \approx \frac{f(x_i + \delta x_i) - f(x_i)}{\delta x_i}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{\dfrac{f(x_i + \delta\ x_i) - f(x_i)}{\delta\ x_i}} = \quad x_i - \frac{\delta\, x_i\, f(x_i)}{f(x_i + \delta\ x_i) - f(x_i)}$$

Problem: How to select δ ?

If not selected properly, the method may diverge.

Example

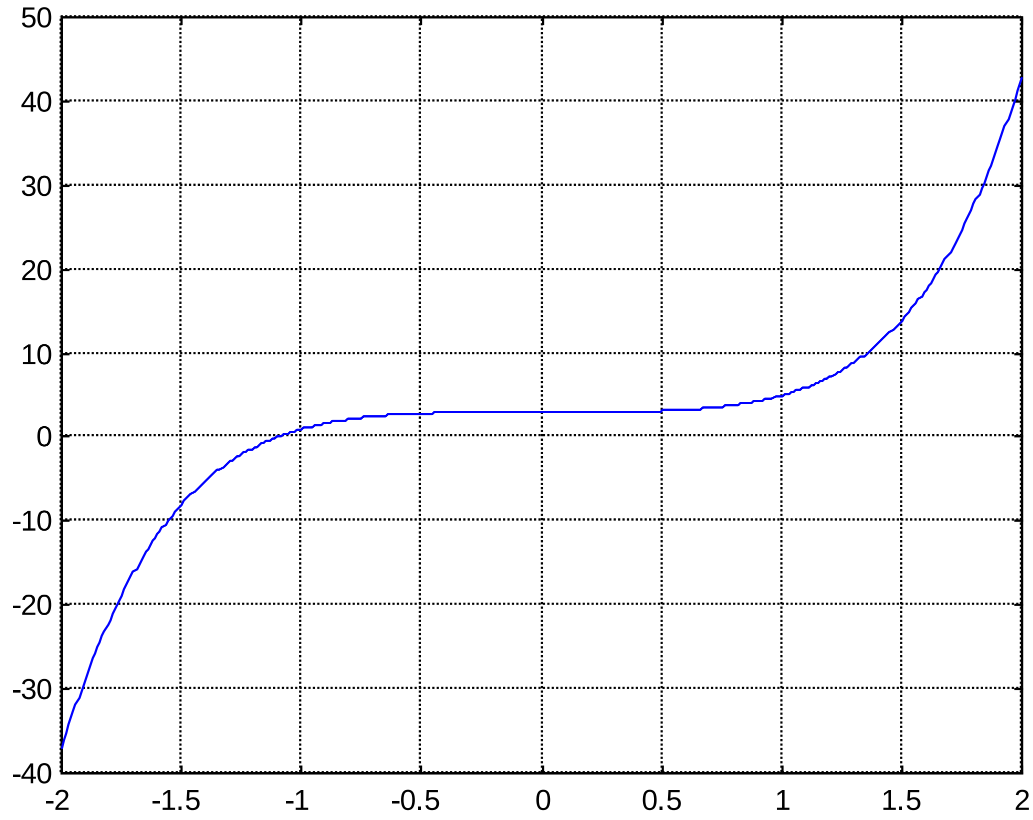
Find the roots of :

$$f(x) = x^5 + x^3 + 3$$

Initial points

$$x_0 = -1 \; and \; x_1 = -1.1$$

$$with \;\; error < 0.001$$



Example

x(i)	f(x(i))	x(i+1)	|x(i+1)-x(i)|
-1.0000	1.0000	-1.1000	0.1000
-1.1000	0.0585	-1.1062	0. 0062
-1.1062	0.0102	-1.1052	0.0009
-1.1052	0.0001	-1.1052	0.0000

Convergence Analysis

- The rate of convergence of the Secant method is super linear:

$$\frac{\left|x_{i+1} - r\right|}{\left|x_i - r\right|^{\alpha}} \le C, \qquad \alpha \approx 1.62$$

$$r : root \qquad x_i : \text{estimate of the root at the } \mathrm{i}^{\text{th}} \text{ iteration.}$$

- It is better than Bisection method but not as good as Newton's method.

Comparison of Root Finding Methods

- Advantages/disadvantages
- Examples

Summary

Method	Pros	Cons
Bisection	- Easy, Reliable, Convergent - One function evaluation per iteration - No knowledge of derivative is needed	- Slow - Needs an interval [a,b] containing the root, i.e., $f(a)f(b)<0$
Newton	- Fast (if near the root) - Two function evaluations per iteration	- May diverge - Needs derivative and an initial guess x_0 such that $f'(x_0)$ is nonzero
Secant	- Fast (slower than Newton) - One function evaluation per iteration - No knowledge of derivative is needed	- May diverge - Needs two initial points guess x_0, x_1 such that $f(x_0)- f(x_1)$ is nonzero

Example

Use Secant method to find the root of :

$$f(x) = x^6 - x - 1$$

Two initial points $x_0 = 1$ *and* $x_1 = 1.5$

$$x_{i+1} = x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Solution

k	x_k	$f(x_k)$
0	1.0000	-1.0000
1	1.5000	8.8906
2	1.0506	-0.7062
3	1.0836	-0.4645
4	1.1472	0.1321
5	1.1331	-0.0165
6	1.1347	-0.0005

Example

Use Newton's Method to find a root of:

$$f(x) = x^3 - x - 1$$

Use the initial point: $x_0 = 1$.

Stop after three iterations, or

if $\left| x_{k+1} - x_k \right| < 0.001$, or

if $\left| f(x_k) \right| < 0.0001$.

Five Iterations of the Solution

k	x_k	$f(x_k)$	$f'(x_k)$	ERROR
0	1.0000	-1.0000	2.0000	
1	1.5000	0.8750	5.7500	0.1522
2	1.3478	0.1007	4.4499	0.0226
3	1.3252	0.0021	4.2685	0.0005
4	1.3247	0.0000	4.2646	0.0000
5	1.3247	0.0000	4.2646	0.0000

Example

Use Newton's Method to find a root of:

$$f(x) = e^{-x} - x$$

Use the initial point: $x_0 = 1$.

Stop after three iterations, or

if $\left| x_{k+1} - x_k \right| < 0.001$, or

if $\left| f(x_k) \right| < 0.0001$.

Example

Use Newton's Method to find a root of:

$$f(x) = e^{-x} - x, \qquad f'(x) = -e^{-x} - 1$$

x_k	$f(x_k)$	$f'(x_k)$	$\dfrac{f(x_k)}{f'(x_k)}$
1.0000	-0.6321	-1.3679	0.4621
0.5379	0.0461	-1.5840	-0.0291
0.5670	0.0002	-1.5672	-0.0002
0.5671	0.0000	-1.5671	-0.0000

Example

Estimates of the root of: x-cos(x)=0.

0.600000000000000	Initial guess
0.744017319445980	1 correct digit
0.739090476886240	4 correct digits
0.739085133222147	10 correct digits
0.739085133215160	14 correct digits

Example

In estimating the root of: ***x-cos(x)=0***, to get more than 13 correct digits:

- 4 iterations of Newton (x_0=0.8)
- 43 iterations of Bisection method (initial interval [0.6, 0.8])
- 5 iterations of Secant method (x_0=0.6, x_1=0.8)