Logistic regression for a binary classification with a non-linear classification boundary

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In [114]:
```
import numpy as np
import matplotlib as mpl
import matplotlib.pyplot as plt
from matplotlib import cm
```

1. Training Data

- loading datas from `data-nonlinear.txt`
- each row $\{x^{(i)}, y^{(i)}, l^{(i)}\}$ od the data consists of 2D point (x, y) with its label l
- since it is binary classification problem, and with real point (x, y)

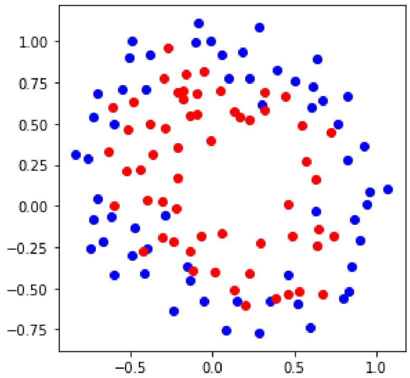
$$(x, y) \in \mathbb{R}, l \in 0, 1$$

In [116]:
```
raw_data = np.genfromtxt("data-nonlinear.txt", delimiter=',')

pointX, pointY, label = raw_data[:, 0], raw_data[:, 1], raw_data[:, 2]

pointX0, pointY0 = pointX[label == 0], pointY[label == 0]
pointX1, pointY1 = pointX[label == 1], pointY[label == 1]

plt.scatter(pointX0, pointY0, c='b')
plt.scatter(pointX1, pointY1, c='r')
plt.tight_layout()
plt.gca().set_aspect('equal', adjustable='box')
```



1-1. Matrix representations

1) Defining the train data

$$x_i = [1, x^{(i)}, y^{(i)}]^T$$

where this can be formed in Matrix representaion

$$D = \begin{bmatrix} x_1^T \\ \dots \\ x_N^T \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ & \dots & \\ 1 & x_N & y_N \end{bmatrix}$$

for the label $l_i \in \{0, 1\}$

$$L = \begin{bmatrix} l_1 \\ l_2 \\ \dots \\ l_N \end{bmatrix}$$

2) Defining the high dimensional function of x and y

$$\begin{matrix} f_0(x, y) = 1, & f_1(x, y) = x^2, & f_2(x, y) = y^2 \\ f_3(x, y) = xy, & f_4(x, y) = x, & f_5(x, y) = y \end{matrix}$$

3) Defining the Linear combination g

$$g(x, y, \theta) = \sum_{k=0}^{M} \theta_k f_k(x, y)$$

4) All-in-one

since we are using 6 parameter ($M = 6$), real training matrix becomes

$$F = \begin{bmatrix} 1 & x_1^2 & y_1^2 & x_1 y_1 & x_1 & y_1 \\ & & & \dots & & \\ 1 & x_N^2 & y_N^2 & x_N y_N & x_N & y_N \end{bmatrix}, \Theta = [\theta_0, \theta_1, \theta_2, \theta_3 \theta_4, \theta_5]^T$$

where, the result of linear combination

$$G = F\Theta$$

In [118]:
```
N, M = len(label), 6
Xs = np.matrix(pointX.reshape(N, 1))
Ys = np.matrix(pointY.reshape(N, 1))
F = np.concatenate(
    (np.ones((N, 1)),
    np.multiply(Xs, Xs),
    np.multiply(Ys, Ys),
    np.multiply(Xs, Ys),
    Xs,
    Ys
    ), axis=1)
L = np.matrix(label).reshape(N, 1)
```

2. Logistic regression

- the sigmoid function for classification

$$\sigma(z) = \frac{1}{1 + \exp\{-z\}}$$

- derivative of the sigmoid

$$\sigma'(z) = \sigma(z)(1 - \sigma(z))$$

- the linear combination

$$g(x, y, \theta) = \sum_{k=0}^{5} \theta_k f_k(x, y)$$

In [120]:
```
def sigmoid(z):
    return 1.0 / (1.0 + np.exp(-1.0 * z))

def lin_comb(F, T):
    return np.matmul(F, T)
```

3. Objective Function

$$J(\theta) = \frac{1}{N} \sum_{i=1}^{N} (-l^{(i)} \log(\sigma(g(x^{(i)}, y^{(i)}; \theta))) - (1 - l^{(i)}) \log(1 - \sigma(g(x^{(i)}, y^{(i)}; \theta))))$$

In [122]:
```
def loss(F, T, L):
    normalizer_denominator = (1.0 / N)
    left_term = -1.0 * np.multiply(L, np.log(sigmoid(lin_comb(F, T))))
    right_term = np.multiply((1.0 - L), np.log(1.0 - sigmoid(lin_comb(F, T))))
    loss_in_row = left_term - right_term
    return normalizer_denominator * np.sum(loss_in_row)
```

4. Gradient Descent

$$\theta_k^{(t+1)} := \theta_0^{(t)} - \alpha \frac{1}{N} \sum_{i=1}^{N} (\sigma(g(x^{(i)}, y^{(i)}; \theta)) - l^{(i)}) \frac{\partial g(x^{(i)}, y^{(i)}; \theta^{(t)})}{\partial \theta_k}$$

where the partial derivative $\frac{\partial g}{\partial \theta_k}$ is just f_k

$$\frac{\partial g}{\partial \theta_k} = f_k(x, y)$$

the gradient of g is

$$\nabla g = [f_0, f_1, f_2, f_3, f_4, f_5]$$

In [124]:
```
def getGradient(F, T, L):
    normalizer = 1.0 / N
    difference = sigmoid(lin_comb(F, T)) - L
    grad = []
    for idx in range(M):
        grad.append(normalizer * np.sum(np.multiply(difference, F[:, idx])))

    return np.matrix(grad).reshape(T.shape)

def gradientDescent(F, T, L, lr):
    gradients = getGradient(F, T, L)
    return T - lr * gradients
```

5. Training

In [140]:
```
t = np.matrix([[0.0], [0.0], [0.0], [0.0], [0.0], [0.0]])
learning_rate = 1.0

T_record = t
loss_record = [10.0]
accuracy_record = []
iterations = 0

decay_count = 0
decay_max_count = 5

while True:
    loss_record.append(loss(F, t, L))
    t = gradientDescent(F, t, L, learning_rate)
    T_record = np.concatenate((T_record, t), axis=1)
    iterations = iterations + 1

    correct = np.sum(np.where(np.abs(sigmoid(lin_comb(F, t)) - L) < 0.5, 1.0, 0.0))
    accuracy_record.append(correct / N)

    if loss_record[-1] <= loss(F, t, L):
        if decay_count != decay_max_count:
            print("lr decay! ("+str(iterations)+ ") : " + str(learning_rate) + " -> " + str(learning_rate * 0.1))
            learning_rate = learning_rate * 0.1
            decay_count = decay_count + 1
        else:
            break

_ = loss_record.pop(0)
T_record = np.delete(T_record, 0, axis=1)
T_predict = t
```

```
lr decay! (19743) : 1.0 -> 0.1
lr decay! (19745) : 0.1 -> 0.010000000000000002
lr decay! (19746) : 0.010000000000000002 -> 0.0010000000000000002
lr decay! (19748) : 0.0010000000000000002 -> 0.00010000000000000003
lr decay! (19749) : 0.00010000000000000003 -> 1.0000000000000004e-05
```

6. Compute the training accuracy

$$\frac{\text{number of correct predictions}}{\text{total number of predictions}}$$

In [139]:
```
correct = np.sum(np.where(np.abs(sigmoid(lin_comb(F, T_predict)) - L) < 0.5, 1.0, 0.0))

print("accuracy of " + str(correct / N))
```

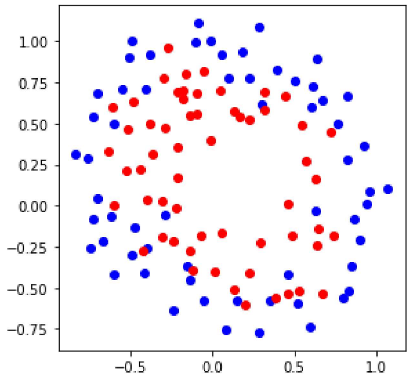
```
accuracy of 0.8559322033898306
```

Answers for the submission

1. Plot the training data

In [130]:
```
plt.scatter(pointX0, pointY0, c='b')
plt.scatter(pointX1, pointY1, c='r')
plt.tight_layout()
plt.gca().set_aspect('equal', adjustable='box')

plt.show()
```



2. Write down the high dimensional function $g(x, y; \theta)$

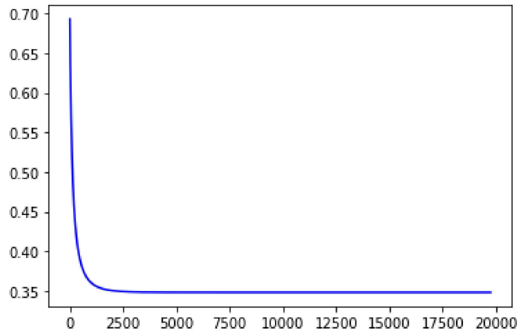
$$f_0(x,y) = 1, \quad f_1(x,y) = x^2, \quad f_2(x,y) = y^2$$
$$f_3(x,y) = xy, \quad f_4(x,y) = x, \quad f_5(x,y) = y$$

$$g(x,y,\theta) = \sum_{k=0}^{5} \theta_k f_k(x,y)$$

3. Plot the training error

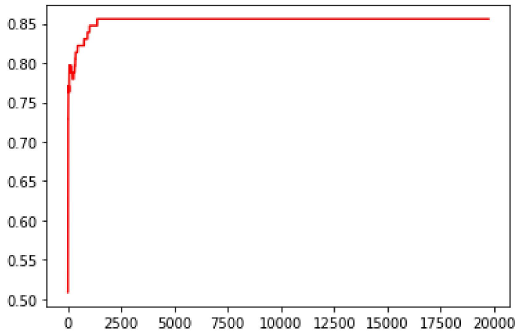
In [132]:
```
iter_x = list(range(iterations))

plt.plot(iter_x, loss_record, color='blue')
plt.show()
```



4. Plot the training accuracy

In [134]:
```
plt.plot(iter_x, accuracy_record, color = 'red')
plt.show()
```



5. Write down the final training accuracy

In [136]:
```
correct = np.sum(np.where(np.abs(sigmoid(lin_comb(F, T_predict)) - L) < 0.5, 1.0, 0.0))

print("final accuracy of " + str(correct / N))
```

```
final accuracy of 0.8559322033898306
```

6. Plot the optimal classifier superimposed on the training data

In [137]:
```
_x = np.arange(-1.0, 1.3, 0.01)
_y = np.arange(-1.0, 1.3, 0.01)

_x, _y = np.meshgrid(_x, _y)

_x2 = np.multiply(_x, _x)
_y2 = np.multiply(_y, _y)
_xy = np.multiply(_x, _y)


_z = T_predict[0].item(0) + np.multiply(_x2, T_predict[1].item(0)) + np.multiply(_y2, T_predict[2].item(0))
_z = _z + np.multiply(_xy, T_predict[3].item(0)) + np.multiply(_x, T_predict[4].item(0))
_z = _z + np.multiply(_y, T_predict[5].item(0))
_z = sigmoid(_z)

_z = np.where(_z > 0.5, 1.0, 0.0)

plt.contour(_x, _y, _in, colors='green')
plt.contour(_x, _y, _out, colors='green')

plt.scatter(pointX0, pointY0, c='b')
plt.scatter(pointX1, pointY1, c='r')
plt.tight_layout()
plt.gca().set_aspect('equal', adjustable='box')


plt.show()
```