IRStructuredSwapOneCurve 평가 설명서

1. **모듈설명**

   해당 모듈은 한 개의 커브를 사용하는 Callable 스왑(Range Accrual 제외)의 가격을 평가하는 모듈이다. 평가로직으로 HW 1F, 2F FDM를 사용한다.

2. **평가가능상품**

   1.1 Simple Callable Swap - Hull White 1Factor

   1.2 Range 충족시 쿠폰 지급 스왑(Accrual 제외) - Hull White 1Factor

   1.3 Spread Callable Swap - Hull White 2Factor

   1.4 Spread Range 충족시 쿠폰 지급 스왑(Accrual 제외) - Hull White 2Factor

3. **평가방법1 – HW 1Factor IFDM**

   Discount Factor

   $$P(t,T) = \hat{E}\left(e^{-\int_t^T r(u)du}\right)$$

   The instantaneous spot rate, called "short rate", is the interest rate r(t).

   HW 1F Dynamics of Short Rate

   $$\boxed{dr(t) = [\theta(t) - \kappa \cdot r(t)]dt + \sigma(t)dW = \kappa\left[\frac{\theta(t)}{\kappa} - r(t)\right]dt + \sigma(t)dW}$$

   if $r(t) > \frac{\theta(t)}{\kappa}$ Then Drift have (-) else (+)

   $$r(t) = \phi(t) + x(t)$$

   $$d\alpha(t) = [\theta(t) - \kappa\phi(t)]dt$$

   $$dx(t) = -\kappa x(t)dt + \sigma dW$$

HW 1F Partial Differential Equation

$$U_t = \kappa x U_x - \frac{1}{2}\sigma^2 U_{xx} + [\alpha(t) + x]U$$

$$\alpha(t) = -\frac{\delta P(0,t)}{\delta t} + \frac{\sigma^2}{2\kappa^2}(1 - e^{-\kappa t})^2$$

※ $[\alpha(t) + x]$ 대신 $F(t, t+dt)$를 사용

Implicit Finite Difference Method

$$\frac{u_i^{k+1} - u_i^k}{h_t} = \kappa x \cdot \frac{(u_{i+1}^k - u_{i-1}^k)}{2h_x} - \frac{1}{2}\sigma^2 \frac{(u_{i+1}^k + u_{i-1}^k - 2u_i^k)}{h_x^2} + [\alpha(t_{k+1}) + x_i]u_i^{k+1}$$

$$u_i^{k+1} - u_i^k = h_t\kappa x \cdot \frac{(u_{i+1}^k - u_{i-1}^k)}{2h_x} - \frac{h_t}{2}\sigma^2 \frac{(u_{i+1}^k + u_{i-1}^k - 2u_i^k)}{h_x^2}$$
$$+ h_t[\alpha(t_{k+1}) + x_i]u_i^{k+1}$$

$$u_i^{k+1}(1 - h_t[\alpha(t_{k+1}) + x_i])$$
$$= \left[-\frac{h_t\kappa x_i}{2h_x} - \frac{h_t\sigma^2}{2h_x^2}\right]u_{i-1}^k + \left[1 + \frac{h_t\sigma^2}{h_x^2}\right]u_i^k + \left[\frac{h_t\kappa x_i}{2h_x} - \frac{h_t\sigma^2}{2h_x^2}\right]u_{i+1}^k$$

$$u_i^{k+1}W_i = A_i u_{i-1}^k + B u_i^k + C_i u_{i+1}^k$$

$$W_i = (1 - h_t[\alpha(t_{k+1}) + x_i])$$

$$A_i = \left[-\frac{h_t\kappa x_i}{2h_x} - \frac{h_t\sigma^2}{2h_x^2}\right]$$

$$B = \left[1 + \frac{h_t\sigma^2}{h_x^2}\right]$$

$$C_i = \left[\frac{h_t\kappa x_i}{2h_x} - \frac{h_t\sigma^2}{2h_x^2}\right]$$

Boundary Condition

$$u_{-1} = 2u_0 - u_1$$

$$u_{N+1} = 2u_N - u_{N-1}$$

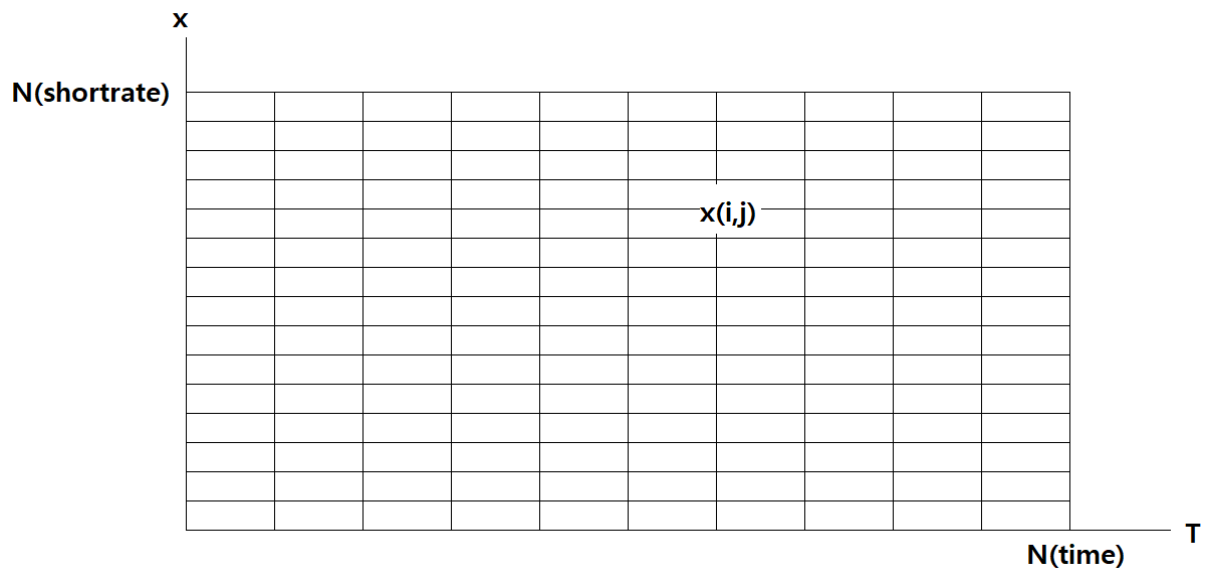$$u_0^{k+1}W_0 = A_0 u_{-1}^k + B u_0^k + C_i u_1^k = A_0(2u_0^k - u_1^k) + B u_0^k + C_0 u_1^k$$
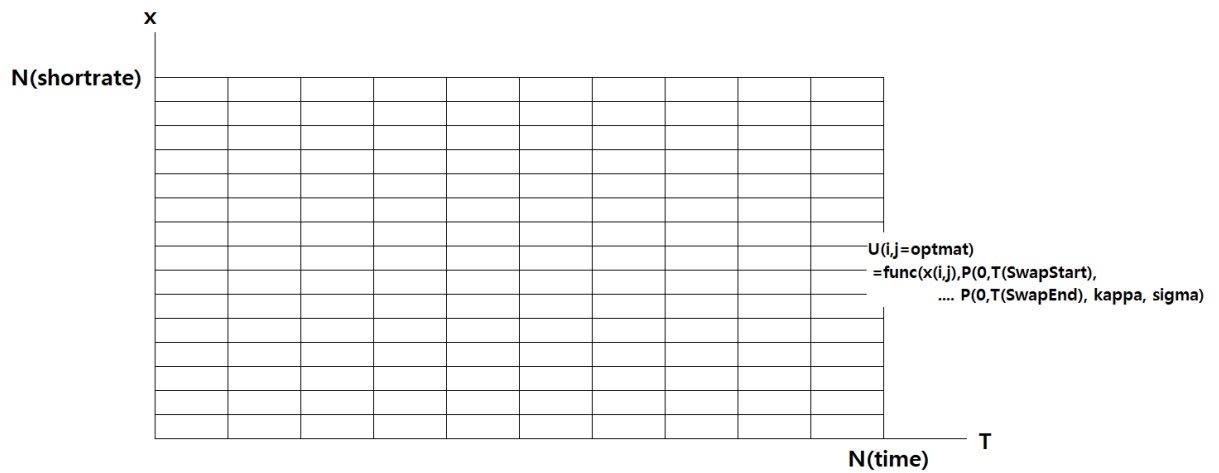
$$u_0^{k+1}W_0 = u_0^k(2A_0 + B) + u_1^k(C_0 - A_0)$$

$$u_N^{k+1}W_N = A_N u_{N-1}^k + B u_N^k + C_i u_{N+1}^k = A_N u_{N-1}^k + B u_N^k + C_N(2u_N^k - u_{N-1}^k)$$

$$u_N^{k+1} W_N = u_{N-1}^k (A_N - C_N) + u_N^k (B + 2C_N)$$

$$
\begin{bmatrix}
2A_0 + B & C_0 - A_0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 \\
A_1 & B & C_1 & 0 & \dots & \dots & 0 & 0 & 0 \\
0 & A_2 & B & C_2 & \dots & \dots & 0 & 0 & 0 \\
0 & 0 & A_3 & B & \dots & \dots & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & 0 & 0 & 0 \\
\dots & \dots & \dots & 0 & 0 & A_{N-2} & B & C_{N-2} & 0 \\
\dots & \dots & \dots & 0 & 0 & 0 & A_{N-1} & B & C_{N-1} \\
\dots & \dots & \dots & 0 & 0 & 0 & 0 & A_N - C_N & B + 2C_N
\end{bmatrix}
\begin{bmatrix}
u_0^k \\
u_1^k \\
u_2^k \\
u_3^k \\
\dots \\
\dots \\
u_{N-2}^k \\
u_{N-1}^k \\
u_N^k
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
W_0 u_0^{k+1} \\
W_1 u_1^{k+1} \\
W_2 u_2^{k+1} \\
W_3 u_3^{k+1} \\
\dots \\
\dots \\
W_{N-2} u_{N-2}^{k+1} \\
W_{N-1} u_{N-1}^{k+1} \\
W_N u_N^{k+1}
\end{bmatrix}
$$

x

N(shortrate)

U(i,j=optmat)
=func(x(i,j),P(0,T(SwapStart),
.... P(0,T(SwapEnd), kappa, sigma)

T

N(time)

4. 평가방법2 – HW 2Factor IFDM

$$U_t^x = \kappa_1 x U_x - \frac{1}{2}\sigma_1^2 U_{xx} + \frac{1}{2}F(t, t+dt)U - \frac{1}{2}\rho_{12}\sigma_1\sigma_2 U_{xy}$$

$$U_t^y = \kappa_2 y U_y - \frac{1}{2}\sigma_2^2 U_{yy} + \frac{1}{2}F(t, t+dt)U - \frac{1}{2}\rho_{12}\sigma_1\sigma_2 U_{xy}$$

Implicit Finite Difference Method to Xt

$$u_{i,j}^{k+\frac{1}{2}} + \left(u^{k+\frac{1}{2}}_{i+1,j+1} + u^{k+\frac{1}{2}}_{i-1,j-1} - u^{k+\frac{1}{2}}_{i+1,j-1} - u^{k+\frac{1}{2}}_{i-1,j+1}\right)\frac{\sigma_1\sigma_2\rho_{12}h_t}{2}\frac{1}{4h_xh_y}$$

$$= A_i u_{i-1}^k + B_i u_i^k + \Gamma_i u_{i+1}^k$$

$$A_i = \left[-\frac{h_t\kappa_1 x_i}{2h_x} - \frac{h_t\sigma_1^2}{2h_x^2}\right]$$

$$B = \left[1 + \frac{h_t\sigma_1^2}{h_x^2}\right] + \frac{1}{2}F\left(t_k, t_{k+\frac{1}{2}}\right)dt$$

$$C_i = \left[\frac{h_t\kappa_1 x_i}{2h_x} - \frac{h_t\sigma_1^2}{2h_x^2}\right]$$

Boundary Condition

$$u_{-1} = 2u_0 - u_1$$

$$u_{N+1} = 2u_N - u_{N-1}$$

$$u_0^{k+1}W_0 = A_0 u_{-1}^k + B u_0^k + C_i u_1^k = A_0(2u_0^k - u_1^k) + B u_0^k + C_0 u_1^k$$

$$u_0^{k+1} W_0 = u_0^k(2A_0 + B) + u_1^k(C_0 - A_0)$$

$$u_N^{k+1} W_N = A_N u_{N-1}^k + B u_N^k + C_i u_{N+1}^k = A_N u_{N-1}^k + B u_N^k + C_N(2u_N^k - u_{N-1}^k)$$

$$u_N^{k+1} W_N = u_{N-1}^k(A_N - C_N) + u_N^k(B + 2C_N)$$

Implicit Finite Difference Method to Yt

$$u_{i,j}^{k+1} + \left(u^{k+1}{}_{i+1,j+1} + u^{k+1}{}_{i-1,j-1} - u^{k+1}{}_{i+1,j-1} - u^{k+1}{}_{i-1,j+1}\right)\frac{\sigma_1\sigma_2\rho_{12}h_t}{2}\frac{1}{4h_x h_y}$$

$$= A_j u_{j-1}^{k+\frac{1}{2}} + B_j u_j^{k+\frac{1}{2}} + \Gamma_j u_{j+1}^{k+\frac{1}{2}}$$

$$A_i = \left[-\frac{h_t\kappa_2 y_j}{2h_y} - \frac{h_t\sigma_2^2}{2h_y^2}\right]$$

$$B = \left[1 + \frac{h_t\sigma_2^2}{h_y^2}\right] + \frac{1}{2}\hat{F}dt$$

$$C_i = \left[\frac{h_t\kappa_2 y_j}{2h_y} - \frac{h_t\sigma_2^2}{2h_y^2}\right]$$

5. **미래 ShortRate Greed별 커브 및 페이오프 계산**

   ShortRate별 커브를 구현하여 기초금리를 추정하고 Greed별 Payoff를 계산한다.

   $$P_{HW}(t,T) = E\left(e^{-\int_t^T r_u du}\middle| F_t\right) = E\left(e^{-\int_t^T(x_u + \phi_u)du}|F_t\right) \approx P^M(t,T)E\left(e^{-\int_t^T x_u du}|F_t\right)$$

   x_t 는 정규분포를 따르고, $E\left(e^{-\int_t^T x_u du}|F_t\right)$ 은 정규분포 적률생성함수(MGF) 형태이다.

   $$E(e^{tx}) = e^{\mu t + \frac{1}{2}(\sigma t)^2}$$

   여기서 $\int_t^T x(u)du$는 평균 $x_i B(t,T)$, 분산 $V(t,T)$인 정규분포를 따른다.

   $$B(t,T) = \int_t^T e^{-\kappa(\bar{T}-u)}du = \frac{1 - e^{-\kappa(T-t)}}{\kappa}$$

   $$V(t,T) = \int_t^T \sigma^2(u)B^2(u,T)du \approx \int_t^T \frac{\sigma^2[1 - 2e^{-\kappa(T-u)} + e^{-2\kappa(T-u)}]}{\kappa^2}du$$

$$= \frac{\sigma^2}{\kappa^2}\left(T - t + 2\,\frac{e^{-\kappa(T-t)} - 1}{\kappa} - \frac{e^{-2\kappa(T-t)} - 1}{2\kappa} \right)$$

따라서 위험 중립 측도에서 $P(t,T)$ 는 정규분포의 적률생성함수(Moment Generate Function)에 따라 다음과 같다.

$$※\quad E(e^{tx}) = e^{\mu t + \frac{1}{2}(\sigma t)^2}\ \text{MGF of Normal Distribution,}$$

따라서

$$P_{HW}^{x_i}(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\left(-x_i B(t,T) + \frac{1}{2} V(t,T) \right)$$

$$\therefore P^M(0,t_i)\text{는 현재시점 제로쿠폰채 가격(Discount Factor)}$$

- 소스코드 정보 $B(t,T)$, $V(t,T)$, $V^{TermVol}(t,T)$는 다음을 참고

(1) $B(t,T)$

```
double B_s_to_t(
    double kappa,
    double s,
    double t
)
{
    return (1.0 - exp(-kappa * (t - s))) / kappa;
}
```

(2) $V(t,T)$

```
double V_t_T(
    double kappa,
    double kappa2,                  Simple Volatility일 때
    double t,
    double T,
    double vol,
    double vol2
)
{
    return vol * vol2 / (kappa * kappa2) * (T - t + (exp(-kappa * (T - t)) - 1.0) /
        kappa + (exp(-kappa2 * (T - t)) - 1.0) / kappa2 - (exp(-(kappa + kappa2) * (T - t)) - 1.0) / (kappa + kappa2));
}
```

```
double V_t_T(double k1, double k2, double t, double T,
    double* tVol1, double* Vol1, long nVol1,
    double* tVol2, double* Vol2, long nVol2)
{
    long NInteg = 10;
    double RHS = 0.0;                           Term Volatility 일 때
    double du = (T - t) / (double)NInteg;
    double u, v1, v2, term;
    for (long i = 0; i < NInteg; i++)
    {
        u = t + du * (i + 0.5);
        v1 = Interpolate_Linear(tVol1, Vol1, nVol1, u);
        v2 = Interpolate_Linear(tVol2, Vol2, nVol2, u);

        term = v1 * v2 *(1.0 - exp(-k1 * (T - u))) / k1 *(1.0 - exp(-k2 * (T - u))) / k2 *du;
        RHS += term;
    }
    return RHS;
}
```

HW 2F Dynamics of Short Rate(HJM 2F Model)

$$r(t) = \alpha(t) + x(t) + y(t)$$
$$d\alpha(t) = [\theta(t) - \kappa_1\alpha(t) - \kappa_2\alpha(t)]dt$$
$$dx(t) = -\kappa_1 x(t)dt + \sigma_1 dW_1$$
$$dy(t) = -\kappa_2 y(t)dt + \sigma_2 dW_2$$

$$P_{HW2F}^{x_i y_i}(t, T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\left(-x_i B_x(t,T) + V_x(t,T) - y_i B_y(t,T) + V_y(t,T) + \text{CrossTerm}_{2F}(t,T)\right)$$

$$\mathbf{CrossTerm_{2F}(t, T)}$$
$$= 2\rho\frac{1}{2}\left(VT_{2F}(t, T, \kappa_1, \kappa_2, \sigma_1, \sigma_2) - VT_{2F}(0, T, \kappa_1, \kappa_2, \sigma_1, \sigma_2) + VT_{2F}(0, t, \kappa_1, \kappa_2, \sigma_1, \sigma_2)\right)$$

$$\mathbf{VT_{2F}(T_1, T_2, \kappa_1, \kappa_2, \sigma_1, \sigma_2)}$$

$$\approx \frac{\sigma_1 \sigma_2}{\kappa_1 \kappa_2} \left(T - t + \frac{e^{-\kappa_1(T-t)} - 1}{\kappa_1} + \frac{e^{-\kappa_2(T-t)} - 1}{\kappa_2} - \frac{e^{-(\kappa_1+\kappa_2)(T-t)} - 1}{(\kappa_1 + \kappa_2)} \right)$$

6.