Digital Signal Processing

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1 Summary

This lecture will focus on:

1. Sampling continuous time signals
2. Frequency Domain Representation of Sampling
3. The sampling theorem
4. Signal Reconstruction From Samples

2 Sampling

Over the last few decades, there has been a move towards digital transmission of analog signals. The first step in creating a digital signal from an analog one involves sampling the continuous time signal $x(t)$ at regular intervals to form a discrete time signal. Discrete time signals are defined at discrete times only indexed by the integers. Thus a discrete time signal is a sequence of numbers where the nth number is denoted by $x[n]$. When the discrete time signal $x[n]$ arises from the sampling of a continuous time signal $x(t)$ at regular intervals we have

$$x[n] = x(nT_s)$$

T_s is known as the sampling period. The sampling frequency is $\frac{1}{T_s}$. Figure 1 shows a continuous time signal (a) and the discrete time signal obtained from sampling the continuous signal at regular intervals (b).

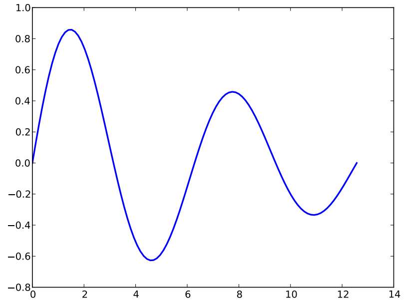
Two different time domain signal can result in the same discrete time signal after sampling. This means that an appropriate sampling rate must be chosen to allow reconstruction of signals.

To form a digital signal from the discrete time signal, the sample values are quantized into discrete values.

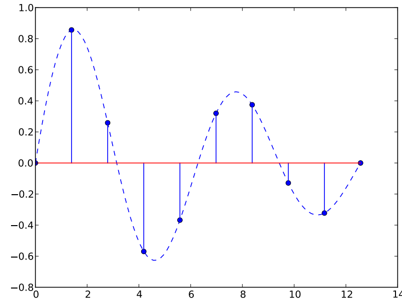
3 Frequency Domain Representation of Sampling

Mathematically, we can represent the sampled signal $x_s(t)$ as the product of the continuous time signal and an impulse train of Dirac delta functions given by

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



(a)



(b)

Figure 1: Continuous time signal (a) and the discrete time signal obtained from sampling the continuous signal at regular intervals (b).

That is

$$
\begin{aligned}
x_s(t) &= x(t)s(t) \\
&= x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)
\end{aligned}
$$

From the sifting property of the Dirac delta function, we have

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

In order to derive the Fourier transform of $x_s(t)$ we note that it is the product of two functions and therefore

$$X_s(f) = X(f) * S(f)$$

where $*$ denotes convolution.

Recall that if a periodic function is formed from a sequence of pulses we have

$$\mathcal{F}[\sum_{n=-\infty}^{\infty} x(t-nT_0)] = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_0})\delta(f-\frac{n}{T_0})$$

where T_0 is the period and $x(t)$ is the pulse whose Fourier transform is $X(f)$

Since $s(t)$ is a periodic sequence of Dirac delta functions and $\mathcal{F}[\delta(t)] = 1$ we have

$$S(f) = f_s\sum_{n=-\infty}^{\infty}\delta(f-nf_s)$$

where $f_s = \frac{1}{T_s}$. We have

$$
\begin{aligned}
X_s(f) &= X(f) * S(f) \\
&= f_s\sum_{n=-\infty}^{\infty} X(f-nf_s)
\end{aligned}
$$

If $x(t)$ is a bandlimited continuous time signal with bandwidth W Hertz with the frequency spectrum $X(f)$ shown in Figure 2 and the sampling frequency $f_s = 2W$ we see that

$$X(f) = \frac{1}{2W} X_s(f) \quad |f| < W$$

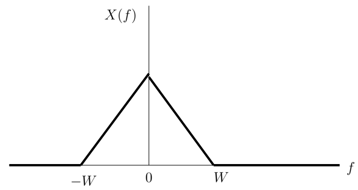


Figure 2: The frequency spectrum $X(f)$.

In general when $f_s > 2W$, the replicas of $X(f)$ in $X_s(f)$ do not overlap and $x(t)$ can be recovered from $x_s(t)$ with an ideal low pass filter.

If $f_s < 2W$, the copies of $X(f)$ overlap and we can no longer recover $x(t)$ from $x_s(t)$ via low pass filtering. The output of low pass filtering will suffer *aliasing distortion* where high frequency components take on the identity of low frequency signals.

Example: Consider the sampling of $\cos(2\pi f_0 t)$ when $f_s > 2f_0$ and when $f_s < 2f_0$. When $f_s > 2f_0$ the output of an ideal LPF with cutoff frequency $\frac{f_s}{2}$ in response to the sampled signal is $\cos(2\pi f_0 t)$. When $f_s < 2f_0$ aliasing occurs and the output of the LPF is $\cos(2\pi(f_s - f_0)t)$. The high frequency signal $\cos(2\pi f_0 t)$ has taken the alias of the lower frequency signal $\cos(2\pi(f_s - f_0)t)$.

From the above we can state the *Sampling Theorem*: A bandlimited signal of finite energy which only has frequency components below WHz, is completely specified by samples taken at a rate $f_s \geq 2W$Hz. The frequency $2W$ is sometimes called the Nyquist rate.

4 Signal Reconstruction From Samples

When the conditions of the sampling theorem are met, it is possible to recover the signal exactly from its samples and the Fourier transforms of the continuous time signal $x(t)$ and the sampled signal $x_s(t)$ are related by

$$X(f) = \frac{1}{f_s} X_s(f) \quad |f| < \frac{f_s}{2}$$

where f_s is the sampling frequency. In order to recover the signal, we pass the sampled signal through an ideal lowpass filter with gain $\frac{1}{f_s}$ over the passband $|f| < \frac{f_s}{2}$.

Recall

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Then the output of the LPF is given by $x_s(t) * h(t)$ where $h(t)$ is the impulse response of the ideal lowpass filter. We can show that

$$h(t) = \frac{\sin(\frac{\pi}{T_s} t)}{\frac{\pi}{T_s} t}$$

And the response of the LPF is given by

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin(\frac{\pi}{T_s}(t - nT_s))}{\frac{\pi}{T_s}(t - nT_s)}$$

This expression is known as the interpolation formula and allows the reconstruction of the original signal from its samples.

We can also arrive at the interpolation formula by noting that $X_s(f)$ can also be written as

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi n T_s f}$$

and therefore

$$\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \\
&= \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} \frac{1}{f_s} X_s(f) e^{j2\pi f t} df \\
&= \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi n T_s f} e^{j2\pi f t} df \\
&= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{1}{f_s} \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} e^{j2\pi f (t - nT_s)} df \\
&= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin(\pi f_s (t - nT_s))}{\pi f_s (t - nT_s)}
\end{aligned}$$

From the above development we see that a bandlimited signal can be recovered exactly from its samples. In practice signals are not bandlimited and to allow reconstruction after sampling the signals are first passed through a lowpass anti-aliasing filter to limit the bandwidth to WHz. This signal is then sampled at a frequency slightly higher than the Nyquist rate of $2W$Hz. This has the benefit of allowing the reconstruction filter to have a non-zero transition band making it realizable.

Example: The range of human hearing is upto approximately 20kHz which corresponds to a Nyquist rate of 40kHz. Audio CDs are sampled at 44.1kHz.

5 Discrete Time Processing of Continuous Signals

Often it is desirable to process continuous signals using discrete time filters. To accomplish this, the first step involves sampling. In addition, to design discrete time filters that will process continuous time signals, we need to establish the relationship between the frequency domain representation of discrete time signals and the Fourier transform of corresponding continuous time signals.

Starting with a continuous time signal $x(t)$ with Fourier transform $X(f)$ assumed bandlimited with bandwidth W and sampled at $f_s \geq 2W$ to form a discrete time signal $x[n]$ we have

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s).$$

Recall that the Discrete Time Fourier Transform is

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Taking the Fourier transform of $x_s(t)$ we have

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j2\pi nT_s f}.$$

We see that

$$X_s(f) = X(\omega)\Big|_{\omega=2\pi T_s f}$$

Or equivalently

$$X(\omega) = X_s(f)\Big|_{f=\frac{\omega}{2\pi T_s}} = f_s X(f)\Big|_{f=\frac{\omega}{2\pi T_s}}$$

6 Discrete Fourier Transform

To work with the Discrete Time Fourier Transform. We sample it at regular intervals leading to the Discrete Fourier Transform. We take N samples of the DTFT to approximate the DTFT. The value of samples N is chosen to ensure accurate representation of $X(\omega)$ from its samples. The DFT is defined as

$$\begin{aligned} X[k] &= X(\omega)\Big|_{\omega=\frac{2\pi k}{N}} \quad k = 0, \ldots, N-1 \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-\frac{j2\pi k}{N}n} \end{aligned}$$

We work with finite duration signals of length N and we get

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-\frac{j2\pi k}{N}n} \quad k = 0, \ldots, N-1$$

If $x[n]$ was obtained by sampling a continuous time signal $x(t)$ with frequency spectrum $X(f)$, we can use the DFT to estimate $X(f)$. This is a fundamental application in digital signal processing. We have

$$X(f_k) \approx T_s X[k]$$

with $f_k = \frac{k}{N} f_s$ for $k = 0, \ldots, \frac{N}{2} - 1$.