Signals and Communication Lecture 3

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1 Summary

This lecture will focus on:

1. Fourier Series

2 Mathematical Preliminaries

It is assumed that the student is familiar with integral and differential calculus and complex numbers

2.1 Complex Numbers

1. A complex number is given in rectangular coordinates by $x = x_r + jx_i$ where $j = \sqrt{-1}$.
2. In polar coordinates we have $x = re^{j\phi}$ where $r = \sqrt{x_r^2 + x_i^2}$ is the magnitude of the complex number and $\phi = \arctan \frac{x_i}{x_r}$ is the phase of the complex number.
3. The complex conjugate of x denoted by $x^* = x_r - jx_i$.
4. Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$
5. Derive $\cos\theta$ in terms of the complex exponential

2.2 Integration and differentiation

1. Compute $\frac{dcos(t)}{dt}, \frac{de^{j\omega t}}{dt}$
2. Compute $\int_a^b x^n dx, \int_a^b x\sin(cx)dx$

3 Fourier series

We have seen that communication systems can be modelled as LTI systems. In this case it is useful to model system inputs as linear combinations of basic signals. The superposition property of linear systems can then be used to obtain the output of the system.

3.1 Response of an LTI system to a complex exponential

Consider an LTI system with an impulse response $h(t)$. If the input to this system is a complex exponential $e^{j2\pi ft}$ then the output is given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-j2\pi f(t-\tau)}h(\tau)d\tau \\ &= e^{j2\pi ft}\underbrace{\int_{-\infty}^{\infty} e^{-j2\pi f\tau}h(\tau)d\tau}_{H(f)} \end{aligned}$$

We see that the output is a scaled version of the input with the same frequency. Since $H(f)$ is a complex number both the phase and amplitude of the input are modified but the frequency is unchanged. Complex exponentials are known as eigenfunctions of LTI systems. Eigenfunctions of a system are those inputs whose output is a scaled version of the input.

This property of complex exponentials makes it useful to decompose arbitrary signals as a linear combination of complex exponentials and motivates the study of Fourier series.

3.2 Fourier series and its properties

A Periodic signal $x(t)$ with period T_0 can be expanded in terms of complex exponentials if it satisfies the Dirichlet conditions

1. $x(t)$ is absolutely integrable over its period
2. The number of maxima and minima of $x(t)$ in each period is finite
3. The number of discontinuities of $x(t)$ in each period is finite

When the Dirichlet conditions are met, $x(t)$ can be written as a sum of complex exponentials. We have

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j\frac{2\pi n}{T_0}t}$$

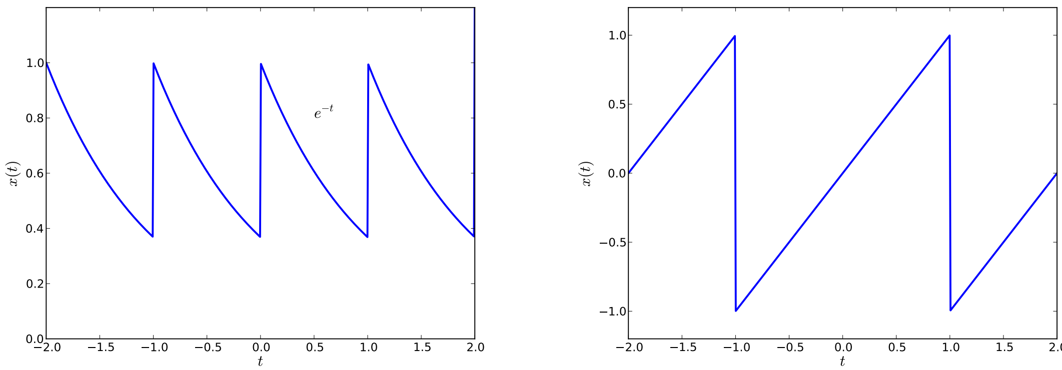
where

$$x_n = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j\frac{2\pi n}{T_0}t}dt$$

and α is arbitrary.

3.3 Computation of Fourier series coefficients

Compute the Fourier series expansion for the periodic signals in Figure 1.



(a) (b)

Figure 1:

3.4 Trigonometric representation of Fourier series

For real valued signals, we can write the complex exponential Fourier series in terms of trigonometric functions. We have

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} x_n e^{j\frac{2\pi n}{T_0}t} \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right)\right) \end{aligned}$$

where

$$\begin{aligned} a_n &= \frac{2}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt \\ b_n &= \frac{2}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\sin\left(\frac{2\pi nt}{T_0}\right)dt \end{aligned}$$

This representation is useful in exploring the approximation of signals via finite sum of sinusoids.

Consider the square wave shown in Figure 2.

We can show that $a_0 = 0$, $a_n = 0$ and

$$b_n = \frac{2}{\pi n}(1 - \cos(\pi n))$$

Thus

$$x(t) = \sum_{n=1}^{\infty} \frac{2}{\pi n}\Big(1 - \cos(\pi n)\Big)\sin(\pi nt)$$

$x(t)$ can be approximated by a finite sum $x_N(t)$ by retaining only N components.

$$x_N(t) = \sum_{n=1}^{N} \frac{2}{\pi n}\Big(1 - \cos(\pi n)\Big)\sin(\pi nt)$$

As N increases, the approximation becomes better. See Figures 3, 4 and 5.

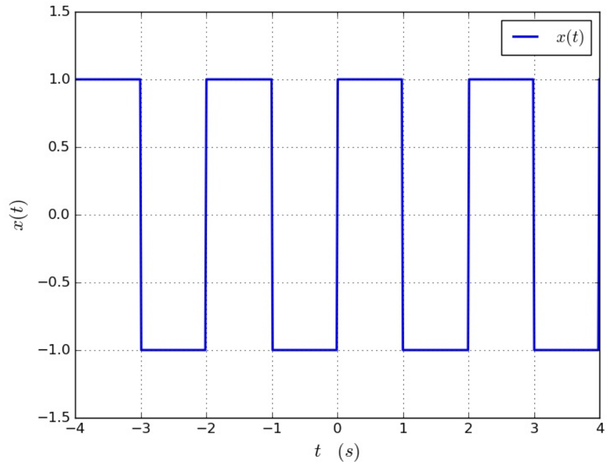


Figure 2: A square wave.

The trigonometric and complex exponential forms of the Fourier series are related and it can be shown that

$$x_n = \frac{a_n - jb_n}{2}$$

and we can move from the complex exponential to the trigonometric function representation.

Example: Let us consider the Fourier series representation of the periodic square wave.

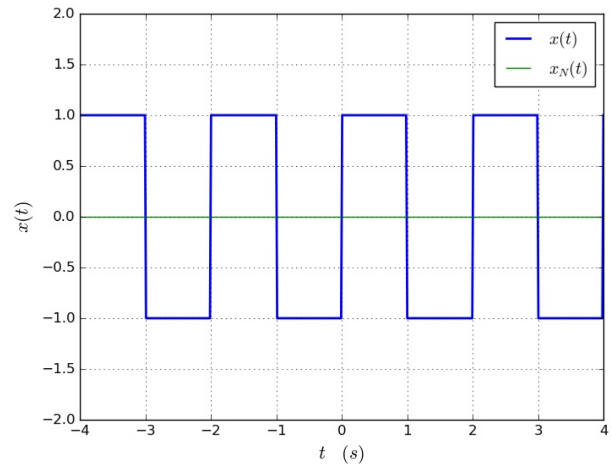


Figure 3: A square wave and $x_N(t)$ for $N = 1$.

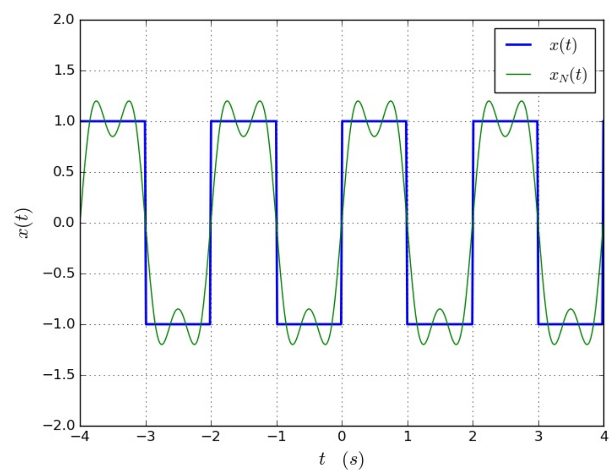


Figure 4: A square wave and $x_N(t)$ for $N = 5$.

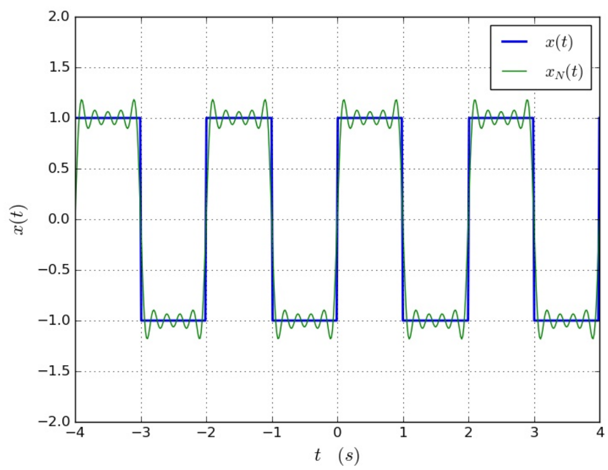


Figure 5: A square wave and $x_N(t)$ for $N = 10$.