EEE 4106 Signals and Communication I.

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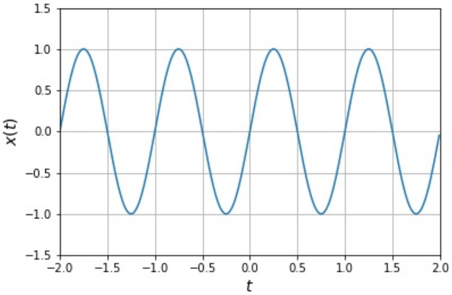
Today's Lecture

1. Fourier Transform

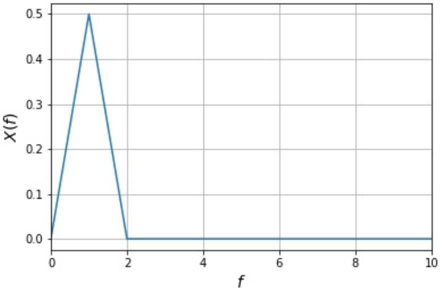
Introduction

- The Fourier transform provides the link between the time domain and frequency domain
- Consider the following sinusoid. What is its frequency?



Introduction

- The Fourier transform provides the link between the time domain and frequency domain
- The same signal can be viewed in the frequency domain



Fourier Transform

- For a nonperiodic signal $x(t)$, its Fourier transform is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

where the variable f denotes frequency.

- The time domain signal can be recovered from $X(f)$ as

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

- $x(t)$ and $X(f)$ constitute a Fourier Transform pair

Fourier Transform

- We can use the notation

$$X(f) = \mathcal{F}[x(t)]$$

and

$$x(t) = \mathcal{F}^{-1}[X(f)]$$

Fourier Transform

- The Fourier transform allows us to express nonperiodic signals as sums of complex exponentials
- Recall

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

- We have

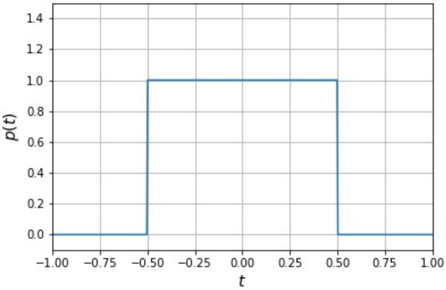
$$\lim_{\Delta f \to 0} \sum_{n=-\infty}^{\infty} X(n\Delta f) e^{j2\pi n\Delta ft} \Delta f = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

- Thus we can view the signal $x(t)$ as consisting of complex exponentials at frequency $n\Delta f$ with each component weighted by $X(n\Delta f)\Delta f$

Example 1

- Consider the pulse $p(t)$ given by

$$p(t) = \begin{cases} 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

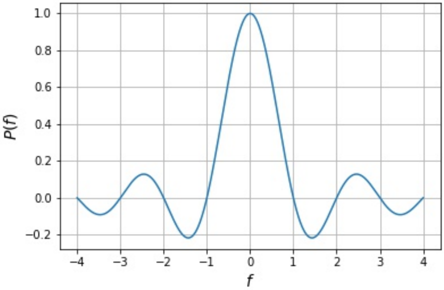


Example 1

- We see that the pulse consists of regions with no variation and two transitions at $t = -1/2$ and $t = 1/2$
- The regions with no variation correspond to low frequency content $f = 0$ while the sharp transitions correspond to high frequency content
- We have

$$
\begin{aligned}
P(f) &= \int_{-\infty}^{\infty} p(t) e^{-j2\pi f t} dt = \int_{-1/2}^{1/2} e^{-j2\pi f t} dt \\
&= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} = \frac{\sin(\pi f)}{\pi f} \\
&= \operatorname{sinc}(f)
\end{aligned}
$$

Example 1



- Note that $\mathrm{sinc}(0) = 1$
- We see that most of the energy of the signal is located around $f = 0$ as expected

Example 2

- Consider the Dirac delta function $\delta(t)$.
- Recall the sifting property.

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0)$$

- Let's compute its Fourier transform

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}dt = e^{-j2\pi ft}\Big|_{t=0} = 1$$

- The Dirac delta function has equal energy over the entire frequency spectrum

Properties of the Fourier Transform

- Linearity

$$\mathcal{F}[a_1 x_1(t) + a_2 x_2(t)] = a_1 X_1(f) + a_2 X_2(f)$$

- Time Shifting

$$\mathcal{F}[x(t - t_0)] = X(f) e^{-j2\pi f t_0}$$

$$\begin{aligned}
\mathcal{F}[x(t - t_0)] &= \int_{-\infty}^{\infty} x(t - t_0) e^{-j2\pi f t} dt \\
&= \int_{-\infty}^{\infty} x(t') e^{-j2\pi f (t' + t_0)} dt' \quad t - t_0 = t' \\
&= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} x(t') e^{-j2\pi f t'} dt' \\
&= e^{-j2\pi f t_0} X(f)
\end{aligned}$$

Properties of the Fourier Transform

- Frequency Shifting

$$\mathcal{F}[x(t)e^{j2\pi f_c t}] = X(f - f_c)$$

$$\begin{aligned}
\mathcal{F}[x(t)e^{j2\pi f_c t}] &= \int_{-\infty}^{\infty} x(t)e^{j2\pi f_c t} e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f - f_c)t} dt \\
&= X(f - f_c)
\end{aligned}$$

Properties of the Fourier Transform

- Duality. If

$$\mathcal{F}[x(t)] = X(f)$$

Then

$$\mathcal{F}[X(t)] = x(-f)$$

- Example: Since $\mathcal{F}[\delta(t)] = 1$, then $\mathcal{F}[1] = \delta(-f) = \delta(f)$
- Thus all the energy of a DC signal is located at $f = 0$

Example 3

- Determine the Fourier transform of $e^{j2\pi f_c t}$
- Recall

$$\mathcal{F}[x(t)e^{j2\pi f_c t}] = X(f - f_c)$$

  In this case $x(t) = 1$ and $\mathcal{F}[x(t)] = \delta(f)$. Thus

$$\mathcal{F}[e^{j2\pi f_c t}] = \delta(f - f_c)$$

- Similarly by exploiting linearity we can show that

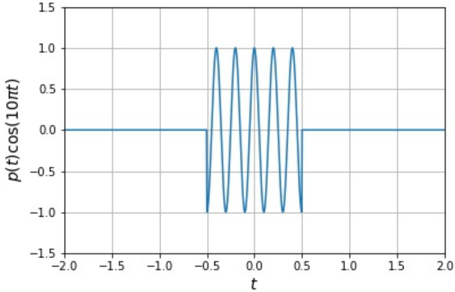
$$\mathcal{F}[\cos(2\pi f_c t)] = \frac{1}{2}\Big(\delta(f - f_c) + \delta(f + f_c)\Big)$$

Example 4

- Recall the pulse $p(t)$ given by

$$p(t) = \begin{cases} 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

- Determine the Fourier transform of $p(t)\cos(10\pi t)$

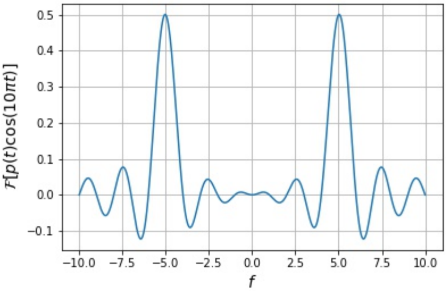


Example 4

- We have $P(f) = \mathrm{sinc}(f)$. Thus

$$\mathcal{F}[p(t)\cos(2\pi f_c t)] = \frac{1}{2}\Big(P(f - f_c) + P(f + f_c)\Big)$$

with $f_c = 5$



Properties of the Fourier Transform

- Convolution in time.

$$\mathcal{F}\Big[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\Big] = H(f)X(f)$$

- Recall that the output $y(t)$ of an LTI system in response to $x(t)$ is given by the convolution integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- We have $Y(f) = H(f)X(f)$. $H(f)$ is also known as the *transfer function* of the system as it relates the Fourier transform of the output to that of the input.
- $H(f) = \mathcal{F}[h(t)]$

Example 5

- Consider an LTI system with impulse response $h(t) = e^{-t}u(t)$
- Show that

$$H(f) = \frac{1}{1 + j2\pi f}$$

- In general, $H(f)$ is complex and we can write

$$H(f) = |H(f)|e^{j\theta(f)}$$

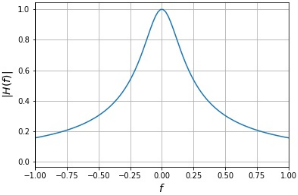
where $|H(f)|$ is known as the magnitude spectrum and $\theta(f)$ is known as the phase spectrum.

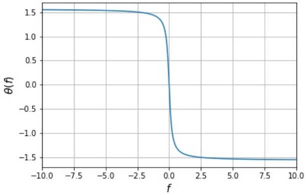
Example 5

- Show that

$$|H(f)| = \frac{1}{\sqrt{1 + 4\pi^2 f^2}}$$

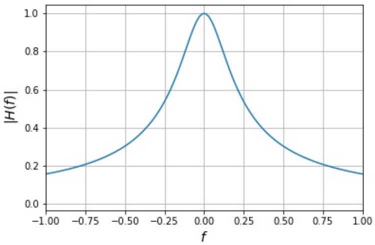
$$\theta(f) = -\tan^{-1}(2\pi f)$$





Filter

- The linear time invariant system in example 5 is a low pass filter
- Filters are frequency selective systems
- Frequency components of the input signal that are present in the output lie within the passband of the filter.
- Frequency components that are present in the input but absent from the output lie in the stopband of the filter.



Filters

- The bandwidth of a filter is a number used to measure the extent of significant frequency content (in the positive frequency range) that is allowed to pass through the filter.
- An ideal lowpass filter has a frequency response given by

$$H_{LP}(f) = \begin{cases} 1 & |f| \leq W \\ 0 & \text{otherwise} \end{cases}$$

The bandwidth of this filter is W.

Filters

- An ideal bandpass filter has a frequency response given by

$$H_{BP}(f) = \begin{cases} 1 & W_1 \leq |f| \leq W_2 \\ 0 & \text{otherwise} \end{cases}$$

  The bandwidth of this filter is $W_2 - W_1$.

Filters

- An ideal highpass filter has a frequency response given by

$$H_{HP}(f) = \begin{cases} 1 & |f| \geq W \\ 0 & \text{otherwise} \end{cases}$$

Filters

- For a non-ideal low pass filter with maximum magnitude response at $f = 0$, the 3dB bandwidth is the frequency at which the magnitude response is $|H(0)|/\sqrt{2}$.
- At this frequency the power transfer is half that at the origin.
- For the low pass filter with $h(t) = e^{-t}u(t)$ show that

$$f_{3dB} = \frac{1}{2\pi}$$