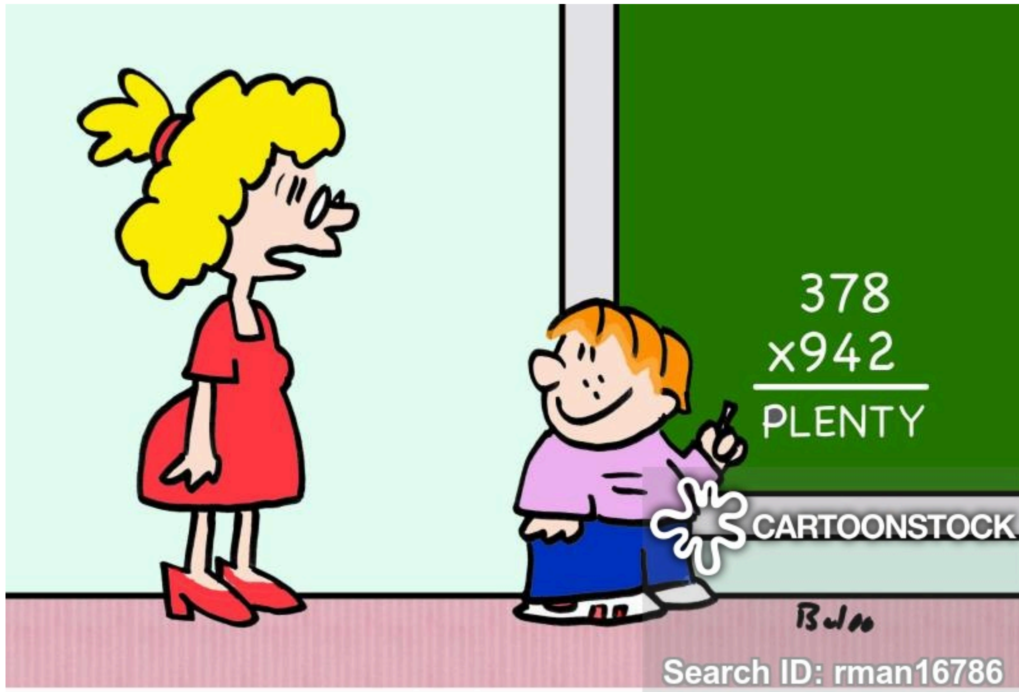
Applications of Mathematics in Computer Science (MACS)

LECTURE #1: APPROXIMATING LIBRARY FUNCTIONS



"That's true, Jeffrey, but you're missing the point."

The CS Problem:

Implementing Library Functions

Library of mathematical functions

Python standard math library

- Number-theoretic and representation functions
- Power and logarithmic functions
- Angular conversions
- Trigonometric functions
- Hyperbolic functions
- Special functions
- Constants

Uses of Functions

Power and logarithmic	efficient computation, data analysis	log — decision trees (machine learning)
Trigonometric	physics, machine learning, time series	
Hyperbolic	physics, machine learning, time series	tanh — robosoccer agent (planning)
Special functions	statistics, machine learning	erf —materials discovery and design (statistical modeling)

How do we calculate these functions?

- Our processor only knows how to preform simple operations: +,-,*,/,...
- More complex functions (sin, log,...) must be implemented in the library
  - Using only basic tools
  - Efficiently
  - With high precision

For this we use approximations!

The math technique:

Approximation theory

- Polynomials
- Analytic functions
- Taylor series
- Remez algorithm

Polynomials

Polynomial: $\sum_{i=0}^{N} a_i \cdot x^i$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_N x^N$$

- Polynomials are easy to compute:

List of coefficients

```
def poly(x, aa):
    xi, v = 1, 0
    for a in aa:
        v += a*xi
        xi *= x
    return v
```

`poly(2, [3,1,2])=13`

$$3 + 1 \cdot 2 + 2 \cdot 2^2$$

Polynomials

- Sum and product of polynomials are polynomials
- Ratios of polynomials are also easy to compute

Power series

$$N = \infty \longrightarrow \sum_{i=0}^{\infty} a_i \cdot x^i$$

Analytic Functions

Function f(x) is analytic if for every $x_0 \in D$:

$$f(x) = \sum_{i=0}^{\infty} a_i \cdot (x - x_0)^i$$

- Polynomials are analytic
- A function is analytic if can be computed as a power series.
- We can approximate: $$f(x) \approx \sum_{i=0}^{N} a_i \cdot (x - x_0)^i$$

because $|x - x_0| < 1 \implies lim_{n\to\infty}(x - x_0)^n = 0$

Examples of Analytic Functions

$$\mathtt{exp}(x) = \sum_{i=0}^{\infty} \frac{x^i}{i\,!}$$

These are **Taylor series!**

$$\mathtt{ln}(x) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{(x-1)^i}{i}$$ for 0<x<2

$$\mathtt{sin}(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)\,!} x^{2i+1}$$

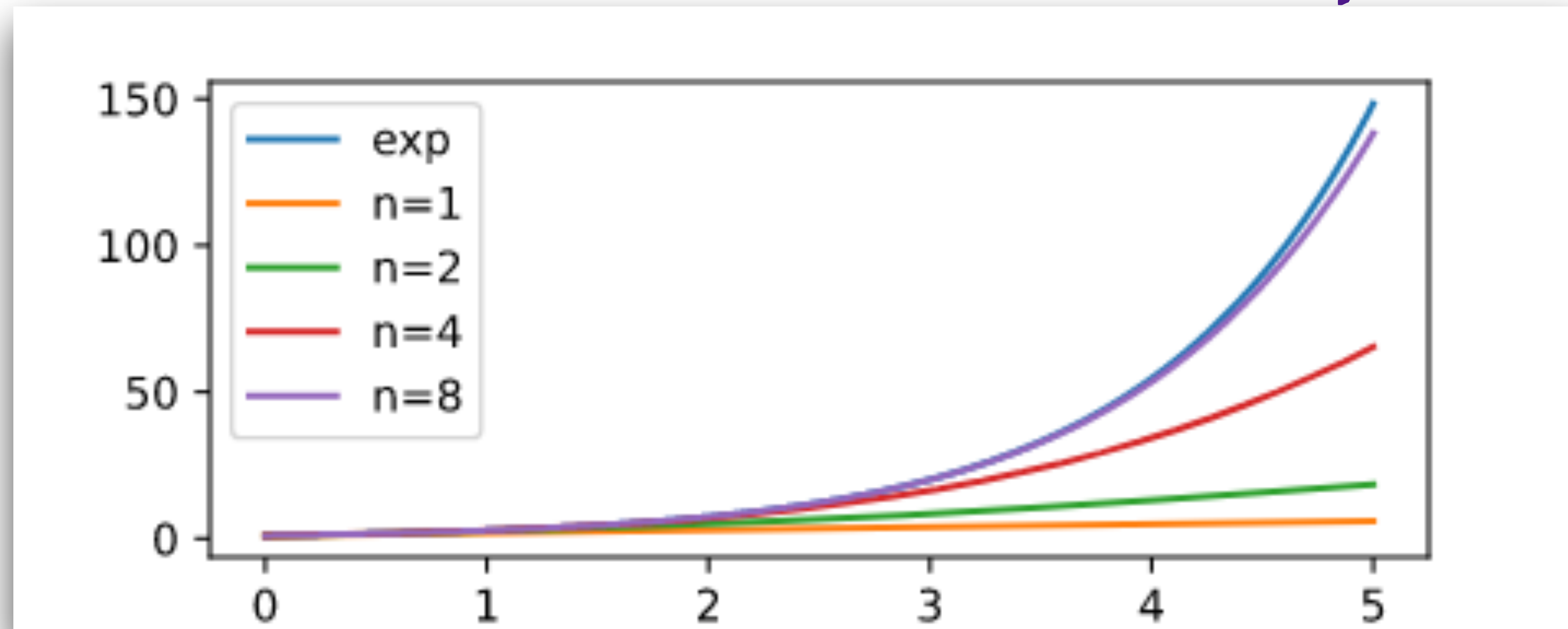
Taylor Series

If f(x) is infinitely differentiable, then at point a:

$$Taylor(f(x), a) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!} \cdot (x - a)^i$$

- Taylor series of a polynomial is the polynomial.
- For most functions, $f(x) = Taylor(f(x), a)$ for some a.
- If $f(x) = Taylor(f(x), a)$ for every a, the function is entire.

Taylor Series of Exp(x)



- Which N to take?
- What is the error?
  - $|Taylor_4(\exp(2), 0) - \exp(2)| \approx 0.002$
  - $|Taylor_4(\exp(4), 0) - \exp(4)| \approx 1.2$

Remez algorithm

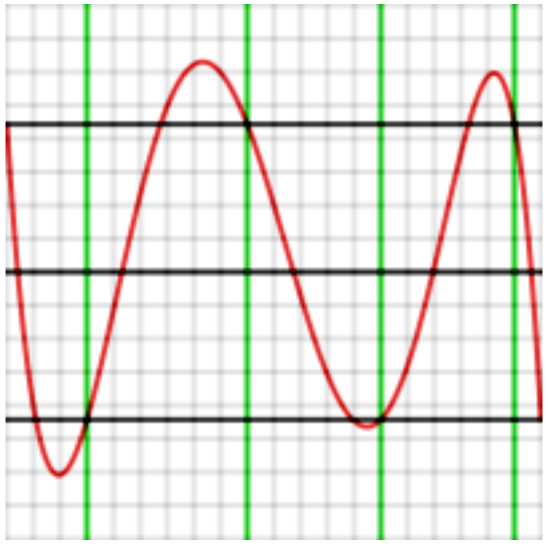
- Find optimal polynomial P(x) for $x_1, x_2, \ldots, x_{N+2}$:

$$P(x_1) - f(x_1) = +\epsilon$$

$$P(x_2) - f(x_2) = -\epsilon$$

$$\ldots$$

- Move x_i such that error $\leq \epsilon$ for any $x_1 \leq x \leq x_{N+2}$



Let's try…