Chapter6

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6.1

The system is a continous linear time invariant system,its controllability matrix:

$$C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$\rho(C) = 3$,so the controllability matrix is full row rank, it is controllable. its observability matrix:

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$\rho(O) = 1 < 3$, so the system is not observable.

6.2

in the same way as in problem 6.1, but we can use the controllability index and observability index to simplify the calculation.

$$C_\mu = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

the controllability matrix is full row rank,it is controllable.

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & -2 & 4 \end{bmatrix}$$

$\rho(O) = 3$, so the system is observable.

6.3

it is not true,only if the matrix A is nonsingular, it will be true.

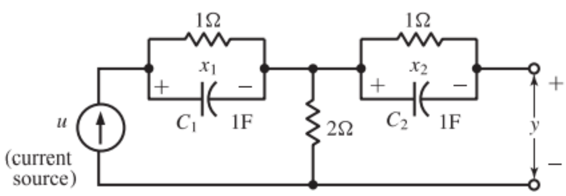
6.4

if the state equation is controllable ,so it must satisfy the PBH criterion.

$$rank\begin{bmatrix} A_{11}-\lambda I & A_{12} & B_1 \\ A_{21} & A_{22}-\lambda I & 0 \end{bmatrix} = n$$

thus just means $[A_{21}A_{22}-SI]$ has full row rank. $\iff$ A_{22}, A_{21} controllable.

6.5



Let X_i be the voltage across the capacitor with capacitance.

$$\begin{cases} \dot{x_1} = u - x_1 \\ \dot{x_2} = -x_2 \qquad \Rightarrow \\ y = 2u - x_2 \end{cases}$$

$$\dot{x}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & -1 \end{bmatrix} x + 2u$$

The state equation is in Jordan-form,There are two jordan blocks, both with oreder 1, and associated with eigenvalue -1, the entry of B corresponding to the second Jordan block is zero, so the state equation is not controllable.in the dual way, we can conclude it is not observable.

6.6

for the problem 6.1:
The controllability matrix:

$$C = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

it is controllable,so its controllability index is $\mu = 3$ because the system is not observable, so it is doesn't have a observability index.
for problem 6.2: The controllability matrix:

$$C = \begin{bmatrix} B & AB & \cdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \end{bmatrix}$$

because $Ab_2 = [0\ 0\ 0]'$, so the controllability indices are 2,1,so the controllability index is $\mu = max(\mu_1, \mu_2) = 2$ The system is observable,and the matrix C's row rank is 1,so the observability index is $v = 3$.

6.7

The controllability index is 1.

6.8

The controllability matrix is:

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}$$

$\rho(C) = 1$, take a canonical decomposition

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$PAP^{-1} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix} PB = \begin{bmatrix} 1 \\ 0 \end{bmatrix} CP^{-1} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$$\dot{\bar{x}} = PAP^{-1}\bar{x} + PBu$$
$$y = CP^{-1}\bar{x}$$

so the reduced controllable equation is:

$$\dot{\bar{x}}_c = 3\bar{x}_c + u$$
$$y = 2\bar{x}_c$$

The reduced equation is observable.

6.9

The controllable and observable equation is $y = 2u$,none of the states are controllable and observable.

6.10

The state equation is in Jordan-form. using the corollary 6.8, we can conclude that x_3 is not controllable, we rearrange the equation as:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_4} \\ \dot{x_5} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 1 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \end{bmatrix} x$$

thus we can reduce the equation as:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_4} \\ \dot{x_5} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} x$$

from the output equation,we can easily find that state x_1 and x_4 is not observable.

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_5} \\ \dot{x_1} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 1 & 0 & \lambda_1 & 0 \\ 0 & 1 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \\ x_1 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} x$$

so the controllable and observable equation is:

$$\dot{x} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

6.11

Select an arbitrary Q_2 such that $[Q_1\ Q_2]$ is nonsingular,define

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^{-1}$$

thus

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} P_1Q_1 & P_1Q_2 \\ P_2Q_1 & P_2Q_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n-n_1} \end{bmatrix}$$

we know $P_2Q_1 = 0$ and Q_1 consists of all linearly indepedent columns of

$$\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

we can conclude that $P_2B = 0$ and $P_2AQ_1 = 0$,let consider the transformation

$$\dot{x} = \begin{bmatrix} P_1 \\ p_2 \end{bmatrix} x$$

$$\bar{A} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} A \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} P_1AQ_1 & P_1AQ_2 \\ P_2AQ_1 & P_2AQ_2 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} B = \begin{bmatrix} P_1B \\ P_2B \end{bmatrix}$$

$$\bar{C} = C \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} CQ_1 & CQ_2 \end{bmatrix}$$

Because $P_2B = 0$ and $P_2AQ_1 = 0$, the equation can be reduced to the controllable equation:

$$\dot{\bar{x_1}} = P_1AQ_1\bar{x_1} + P_1Bu$$

$$y = CQ_1\bar{x_1} + Du$$

6.12

Let P be a unit-matrix,take elementary row operation to tranformate Q_1 inducto

$$PQ_1 = \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}$$

The first n_1 row of P is P_1.

6.13

consider the n-dimensional state equation:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

The rank of its observability matrix is assumed to be $n_2 < n$.Let P_2 be an n_2xn matrix whose rows are any n_2 linearly indepedent rows of the observability matrix .Let Q_2 be an nxn_2 matrix such that $P_2Q_2 = I_{n2}$ where I_{n2} is the unit matrix of order n_2,
the following n_2-dimensional state equation

$$\begin{cases} \dot{\bar{x_2}} = P_2AQ_2\bar{x_2} + P_2Bu \\ \bar{y} = CQ_2\bar{x_2} + Du \end{cases}$$

is observable and has the same transfer matrix as the original state equation.

6.14

There are three Jordan blocks,with order 2,1 and 1 associated with 2 and two Jordan blocks, with order 2,1 associated with 1. The entry of B corresponding to the last row of the the first three block are $[2,1,1],[1,1,1]$ and $[3,2,1]$, and they are linearly indepedent .The entry of B corresponding to the last row of the second two Jordan block are $[1,0,1]$ and $[1,0,0]$,they are also linearly indepedent ,so the Jordan-form state equation is controllable.
unforately,The entry of C corresponding to the first column of the first three Jordan block are $[2,1,0]',[1,1,1]'$ and $[3,2,1]'$, but they are linearly depedent. so the Jordan-form state equation is not observable.

6.15

if required Jordan-form state equation is controllable,only if the $[b_{21}\ b_{22}],[b_{41}\ b_{42}]$ and $[b_{51}\ b_{52}]$ is linearly indepedent, this is obviously impossible. so it is not impossible to find a set of b_{ij} such that the state equation is controllable.but, we can find a set of c_{ij} such that the state equation is observable,just make the

$$\begin{bmatrix} c_{11} & c_{13} & c_{15} \\ c_{21} & c_{23} & c_{25} \\ c_{31} & c_{33} & c_{35} \end{bmatrix}$$

is nonsingular.

6.16

Take an equivalence transformation ,$\bar{x} = Px$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5j & 0 & 0 \\ 0 & 0.5 & 0.5j & 0 & 0 \\ 0 & 0 & 0 & 0.5 & -0.5j \\ 0 & 0 & 0 & 0.5 & 0.5j \end{bmatrix}$$

$$\bar{A} = PAP^{-1} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 + j\beta_1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 - j\beta_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 + j\beta_2 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 - j\beta_2 \end{bmatrix}$$

$$\bar{B} = PB = \begin{bmatrix} b_1 \\ 0.5b_{11} - 0.5b_{12}j \\ 0.5b_{11} + 0.5b_{12}j \\ 0.5b_{21} - 0.5b_{22}j \\ 0.5b_{21} + 0.5b_{22}j \end{bmatrix}$$

$$\bar{C} = CP^{-1} = \begin{bmatrix} c_1 & c_{11} + jc_{12} & c_{11} - jc_{12} & c_{21} + jc_{22} & c_{21} - jc_{22} \end{bmatrix}$$

The equivalence transformation doesn't change the controllability and observability. for:

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$
$$y = \dot{C}\bar{x}$$

it is controllable $\iff$ $b_1 \neq 0, b_{i1} \neq 0$ or $b_{i2} \neq 0$ (for i=1,2)
it is observable $\iff$ $c_1 \neq 0, c_{i1} \neq 0$ or $c_{i2} \neq 0$ (for i=1,2)

6.17

Let x_1, x_2 be the states and x_3 can be expressed by x_1, x_2,so the two-dimensional state equation is :

$$\begin{cases} y = -x_1 - x_2 \\ \dot{x_2} = -3(\dot{x_1} + \dot{x_2}) \\ \dfrac{u + x_1}{2} + 2\dot{x_1} = \dot{x_2} \end{cases}$$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -\frac{11}{2} & 0 \\ \frac{3}{22} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -\frac{2}{11} \\ \frac{3}{22} \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The controllability matrix:

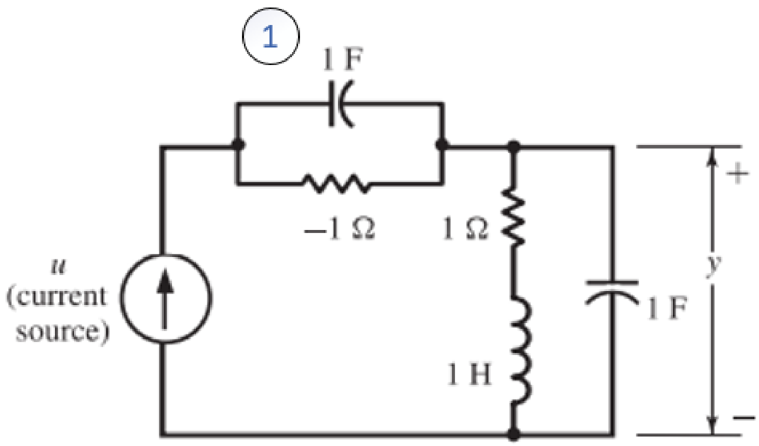
$$C = \begin{bmatrix} -\frac{2}{11} & -\frac{2}{11} \times (-\frac{2}{11}) \\ \frac{3}{22} & \frac{3}{22} \times (-\frac{2}{11}) \end{bmatrix}$$

it is easily to see that $\rho(c) = 1 < 2$,it is not controllable. The three-dimensional state equations: from $x_3 = -x_1 - x_2$, we can get $\dot{x_3} = -\dot{x_1} - \dot{x_2} = \frac{1}{22}x_1 + \frac{1}{22}u$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} -\frac{11}{2} & 0 & 0 \\ \frac{3}{22} & 0 & 0 \\ \frac{1}{22} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -\frac{2}{11} \\ \frac{3}{22} \\ \frac{1}{22} \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

6.18



The voltage across the 1-F capacitor number 1 is assigned x_1,then its current is $\hat{x_1}$,the voltage across the other 1-F capacitor is assigned x_2,then its current is $\hat{x_2}$,the current through the 1-H inductor is assigned as x_3, then its voltage is $\hat{x_3}$

According to the Kirchhoff's current law and Kirchhoff's voltage law,we can get the equation following:

$$\begin{cases} \dot{x_1} = x_1 + u \\ \dot{x_2} = -x_3 + u \\ \dot{x_3} = x_2 - x_3 \\ y = x_2 \end{cases}$$

Rewrite them in matrix form:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

its controllability matrix:

$$C = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$\rho(C) = 3$,it is have full row rank,so the state equation is controllable. its controllability matrix is:

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$\rho(O) = 2 < 3$,so the state equation is not observable. The RC loop doesn't have influence on the current source,the RC loop can be regarded as a wire, so the response of x_1 will not affect the output of the network.so the network is not observable.

6.19

Let $u(t)$ is piecewise constant,that is to say,the input changes values only at discrete-time instants. we get the discrete-time equation without the approximation:

$$x[k+1] = A_d x[k] + B_d u[k]$$
$$y[k] = C_d x[k] + D_d u[k]$$

where $A_d = e^{AT}, B_d = (\int_0^T e^{A\tau} d\tau)B, C_d = C, D_d = D$ when $T = 1$:

$$A_d = e^A = \begin{bmatrix} e^{-1}(cos1 + sin1) & e^{-1}sin1 \\ -2e^{-1}sin1 & e^{-1}(cos1 - sin1) \end{bmatrix}$$

Because A is nonsingular, so we can compute the

$$B_d = A^{-1}(A_d - I)B = \begin{bmatrix} 1.0491 \\ -0.1821 \end{bmatrix}$$

$$C_d = C = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

thus the discrete-time equation:

$$x[k+1] = \begin{bmatrix} e^{-1}(cos1 + sin1) & e^{-1}sin1 \\ -2e^{-1}sin1 & e^{-1}(cos1 - sin1) \end{bmatrix} x[k] + \begin{bmatrix} 1.0491 \\ -0.1821 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} 2 & 3 \end{bmatrix} x[k]$$

in the same way,for $T = \pi$:

$$x[k+1] = \begin{bmatrix} -0.0432 & 0 \\ 0 & -0.0432 \end{bmatrix} x[k] + \begin{bmatrix} 1.5648 \\ -1.0432 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} 2 & 3 \end{bmatrix} x[k]$$

for the original state equation(continous),its controllability matrix:

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$$

$\rho(C) = 2$,so the continous system is controllable. its observability matrix:

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -6 & -4 \end{bmatrix}$$

$\rho(O) = 2$,so the continous system is observable.
its eigenvalues are:$\lambda_1 = -1 + i, \lambda_2 = -1 - i$
the sufficient condition:$|Im[\lambda_1 - \lambda_2]| \neq \frac{2\pi m}{T}$ thus to say:$2 \neq \frac{2\pi m}{T}$,$T \neq \pi m$
for sampling period T=1:
it satisfy the condition, so it is controllable and observable.
for sampling period T=π:
it doesn't satisfy the condition,and it is a SISO problem,so it is also a necessary condition.so it is not controllable and observable.

6.20

This is a LTV system, using the theorem 6.12,$M_0(t) = B(t)$,$M_1(t) = -A(t)M_0(t) + \frac{\mathrm{d}}{\mathrm{d}t}M_0(t)$ we have:

$$\begin{bmatrix} M_0(t) & M_1(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -t \end{bmatrix}$$

for any t,$rank[M_0(t)\ M_1(t)] = 2$,so the state eqution is controllable.
in the same way,$N_0(t) = C(t)$, and $N_1(t) = N_0(t)A(t) + \frac{\mathrm{d}}{\mathrm{d}t}N_0(t)$ because the theorem 6.12 is not a necessary and sufficient condition, but we can extend the therom 6.o12

$$\begin{bmatrix} N_0(t) \\ N_1(t) \\ N_2(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & t \\ 0 & t^2 \\ \vdots & \end{bmatrix}$$

there doesn't exist a t make the

$$rank\begin{bmatrix} N_0(t) \\ N_1(t) \\ \vdots \end{bmatrix} = 2$$

so the system is not observable.using the theorem 6.o11,we can also check the observability. we can compute the solution of $x_1(t), x_2(t)$

$$\begin{cases} x_1(t) = \int_0^t x_2(0)e^{0.5t^2}dt + x_1(0) \\ x_2(t) = x_2(0)e^{0.5t^2} \end{cases}$$

let

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} or \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we can get the fundamental matrix:

$$\begin{bmatrix} 1 & \int_0^t e^{0.5t^2}dt \\ 0 & e^{0.5t^2} \end{bmatrix}$$

so,from the fundamental matrix ,we can easily get the state transition matrix:

$$\phi(t, t_0) = \begin{bmatrix} 1 & e^{-0.5t_0^2}\int_0^t e^{0.5\tau^2}d\tau \\ 0 & e^{0.5(t^2-t_0^2)} \end{bmatrix}$$

we can get:

$$C\phi(\tau, t_0) = \begin{bmatrix} 0 & e^{0.5(\tau^2-t_0^2)} \end{bmatrix}$$

so the

$$W_o(t_0, t_1) = \int_{t_0}^{t_1} \begin{bmatrix} 0 & 0 \\ 0 & e^{\tau^2-t_0^2} \end{bmatrix} d\tau$$

it is singular,so it is not observable.

6.21

This is a LTV system, using the theorem 6.12,$M_0(t) = B(t)$,$M_1(t) = -A(t)M_0(t) + \frac{\mathrm{d}}{\mathrm{d}t}M_0(t)$ we have:

$$\begin{bmatrix} M_0(t) & M_1(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e^{-t} & 0 \end{bmatrix}$$

for this sufficient condition, we can't check its controllability. its state transition matrix is:

$$\Phi(t, t_0) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(t-t_0)} \end{bmatrix}$$

we can get:

$$\Phi(t_1, \tau)B(\tau) = \begin{bmatrix} 1 \\ e^{-t_1} \end{bmatrix}$$

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} \begin{bmatrix} 1 & e^{-t_1} \\ e^{-t_1} & e^{-2t_1} \end{bmatrix} d\tau = \begin{bmatrix} t_1 - t_0 & e^{-t_1}(t_1 - t_0) \\ e^{-t_1}(t_1 - t_0) & e^{-2t_1}(t_1 - t_0) \end{bmatrix}$$

the determinant of $W_c(t_0, t_1)$ is zero,so it is singular, so the state euqation is not controllable. in the same way,$N_0(t) = C(t)$, and $N_1(t) = N_0(t)A(t) + \frac{\mathrm{d}}{\mathrm{d}t}N_0(t)$

$$\begin{bmatrix} N_0(t) \\ N_1(t) \end{bmatrix} = \begin{bmatrix} 0 & e^{-t} \\ 0 & -2e^{-t} \end{bmatrix}$$

there doesn't exist a t make the

$$rank \begin{bmatrix} N_0(t) \\ N_1(t) \end{bmatrix} = 2$$

theorem 6.o12 can not check the observability. so we use the theorem 6.o11,

$$C(\tau)\Phi(\tau, t_0) = \begin{bmatrix} 0 & e^{-(2\tau - t_0)} \end{bmatrix}$$

$$W_o(t_0, t_1) = \int_{t_0}^{t_1} \begin{bmatrix} 0 & 0 \\ 0 & e^{-4\tau - 2t_0} \end{bmatrix} d\tau$$

it is easily to find the $W_o(t_0, t_1)$ is singular,so the state equation is not observable.

6.22

Let X(t) be a fundamental matrix of $\dot{x} = A(t)x$ we know $X^{-1}(t)X(t) = I$,taking differentiation on both sides of the equation,we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}X^{-1}(t) = -X^{-1}(t)A(t)$$

in the same way,we suppose:

$$\dot{x(t)} = -A(t)'x(t)$$

Let $X_1(t)$ be a fundamental matrix of $\dot{x(t)} = -A'(t)x(t)$ we can easily find that

$$\frac{\mathrm{d}}{\mathrm{d}t}X_1(t) = -A'(t)X_1(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}X_1'(t) = -X_1'(t)A(t)$$

we can conclude

$$\begin{aligned}
&X_1'(t) = X^{-1}(t) \\
&(X_1'(t))^{-1} = X(t) \\
&\Phi(t,\tau) = X(t)X^{-1}(\tau) \\
&\Phi_1(t,\tau) = X_1(t)X_1^{-1}(\tau) \\
&\Phi_1'(t,\tau) = (X_1'(t))^{-1}X'(t) = X(\tau)X^{-1}(t) = \Phi(\tau,t)
\end{aligned}$$

if $(A(t), B(t))$ is controllable, if and only if :

$$W_c(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t,\tau)B(\tau)B'(\tau)\Phi'(t,\tau)d\tau$$

is nonsingular.we already know that $\phi(t,t_0)$ is nonsingular.

$$W_c(t_0,t_1) = \Phi(t,t_0)\int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B'(\tau)\Phi'(t_0,\tau)d\tau\Phi'(t,t_0)$$

so we just need

$$W_c(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B'(\tau)\Phi'(t_0,\tau)d\tau$$

is nonsingular. for $(-A'(t), B'(t))$ is observable at t_0, if and only if:

$$W_c(t_0,t_1) = \int_{t_0}^{t_1} \Phi_1'(\tau,t_0)B(\tau)B'(\tau)\Phi_1(\tau,t_0)d\tau$$

that is just

$$W_c(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B'(\tau)\Phi'(t_0,\tau)d\tau$$

which is equal to the condition of $(A(t), B(t))$ is controllable.

6.23

For a time-invariant system,(A, B) is controllable,its controllability matrix:

$$C = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

must be full row rank. for $(-A, B)$ is controllable,its controllability matrix is:

$$C_1 = \begin{bmatrix} B & -AB & A^2B & -A^3B & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & -I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

because the matrix:

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & -I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

is nonsingular,so C and C_1 have same rank. it is not true for time-varying system.for example,the system in problem 6.21 it is not controllable,but $(-A(t), B(t))$ is controllable.