

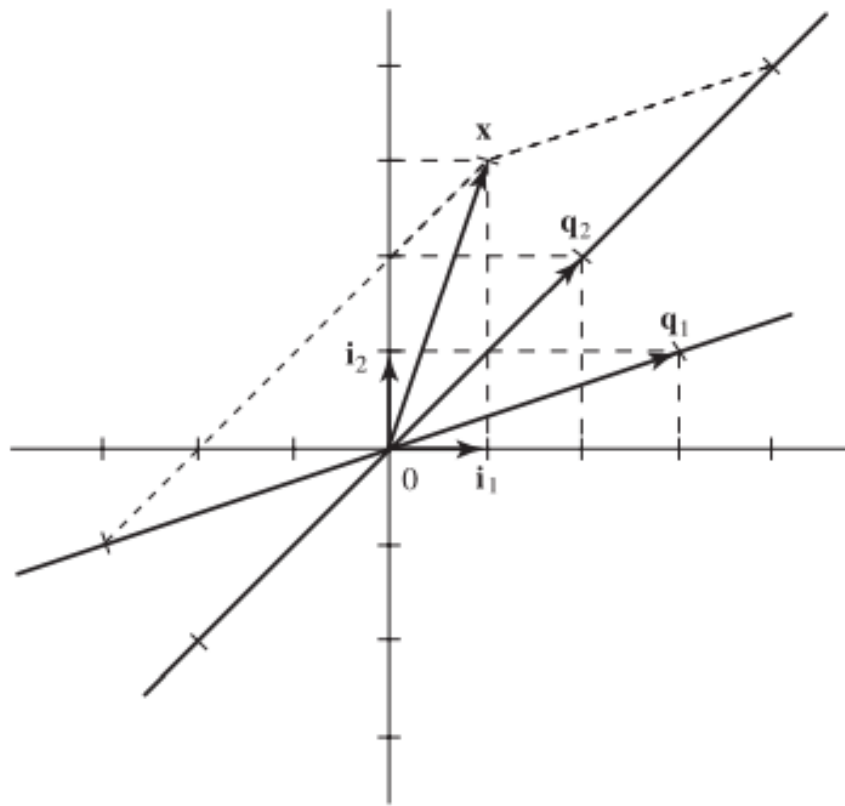
Chapter2

31202008881

Bao Ze an

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3.1



From the above figure, The three vectors $\mathbf{q}_1 = \begin{bmatrix} 3 & 1 \end{bmatrix}'$, $\mathbf{i}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}'$ and $\mathbf{q}_2 = \begin{bmatrix} 2 & 2 \end{bmatrix}'$
 The representation of x with respect to $\{\mathbf{q}_1, \mathbf{i}_2\}$ is $\begin{bmatrix} \frac{1}{3} & \frac{8}{3} \end{bmatrix}'$
 The representation of \mathbf{q}_1 with respect to $\{\mathbf{i}_2, \mathbf{q}_2\}$ is $\begin{bmatrix} -2 & \frac{3}{2} \end{bmatrix}'$

These can be verified like this:

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{i}_2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \end{bmatrix}$$

3.2

i:The norm of x_1

$$1\text{-norm: } \|x_1\|_1 = \sum_{i=1}^3 |x_i| = |2| + |-3| + |1| = 6$$

$$2\text{-norm: } \|x_1\|_2 = \left(\sum_{i=1}^3 |x_i|^2 \right)^{\frac{1}{2}} = \sqrt{2^2 + |-3|^2 + 1^2} = \sqrt{14}$$

$$\text{infinite-norm: } \|x_1\|_\infty = \max_i |x_i| = 3$$

ii:The norm of x_2

$$1\text{-norm: } \|x_2\|_1 = \sum_{i=1}^3 |x_i| = |1| + |1| + |1| = 3$$

$$2\text{-norm: } \|x_2\|_2 = \left(\sum_{i=1}^3 |x_i|^2 \right)^{\frac{1}{2}} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\text{infinite-norm: } \|x_2\|_\infty = \max_i |x_i| = 1$$

3.3

This is just the orthonormalization procedure.

$$\begin{cases} u_1 = \alpha_1 & q_1 = u_1 / \|u_1\| \\ u_2 = \alpha_2 - (q_1' \alpha_2) q_1 & q_2 = u_2 / \|u_2\| \end{cases}$$

This is the ordinary method, what we find is the two vector are orthogonal. so, we just need to make the length of vector is 1 .

$$q_1 = \frac{u_1}{\|u_1\|} = \begin{bmatrix} \frac{2}{\sqrt{14}} & -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{bmatrix}'$$

$$q_2 = \frac{u_2}{\|u_2\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}'$$

3.4

a

if $n > m$, $\mathbf{A}\mathbf{A}'$ is a ordinary vector, which has the rank m

b

if $m=n$, so \mathbf{A} is a nonsingular square matrix, we already have $\mathbf{A}'\mathbf{A} = \mathbf{I}_m$, so $\mathbf{A}' = \mathbf{A}^{-1}$. $\mathbf{A}\mathbf{A}' = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$

3.5

According to the principle:

$$\text{Nullity}(\mathbf{A}) = \text{number of columns of } \mathbf{A} - \text{rank}(\mathbf{A})$$

i:

$$\text{Rank}(\mathbf{A}_1) = 2$$

$$\text{Nullity}(\mathbf{A}_1) = 3 - 2 = 1$$

ii:

$$\text{Rank}(\mathbf{A}_2) = 3$$

$$\text{Nullity}(\mathbf{A}_2) = 3 - 3 = 0$$

iii:

$$\text{Rank}(\mathbf{A}_3) = 3$$

$$\text{Nullity}(\mathbf{A}_3) = 4 - 3 = 1$$

3.6

For \mathbf{A}_1 :

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The set of $\{\mathbf{a}_1, \mathbf{a}_2\}$ can be the basis of the range spaces.

The independent vectors of null space can get by solving the equation:

$$\mathbf{A}_1 \boldsymbol{\eta}_i = 0$$
$$\boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The set of $\boldsymbol{\eta}_1$ is the basis of the null space

in the same way, we can get the basis of the range space and null space of \mathbf{A}_2

$$\mathbf{a}_1 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mathbf{a}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

The set of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is the basis of the range space.

because the \mathbf{A}_2 is full rank, so the basis of the null space is $\{\mathbf{0}\}$

3.7

$$\rho(\mathbf{A}) = \rho([\mathbf{A} \ \mathbf{y}]) = 2$$

so a solution \mathbf{x} exist with respect to this equation.

Because coefficient matrix is full column rank, so the solution is unique.

if $\mathbf{y} = [1 \ 1 \ 1]'$, $\rho(\mathbf{A}) = 2 \neq \rho([\mathbf{A} \ \mathbf{y}]) = 3$ so, when $\mathbf{y} = [1 \ 1 \ 1]'$, the solution is not exist.

3.8

$\mathbf{x}_p = [0 \ -2 \ 1 \ 1]'$ is a solution, a basis of the null space of \mathbf{A} :

$$\mathbf{A}\boldsymbol{\eta}_i = \mathbf{0}$$

$$\boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Thus the general solution can be expressed as:

$$\mathbf{x} = \mathbf{x}_p + \alpha_1 \boldsymbol{\eta}_1$$

3.9

From the example 3.3, we can know the general solution is:

$$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ -4 + \alpha_1 + 2\alpha_2 \\ -\alpha_1 \\ -\alpha_2 \end{bmatrix}$$

$$\|\mathbf{x}\|_2 = \sqrt{\alpha_1^2 + (\alpha_1 + 2\alpha_2 - 4)^2 + \alpha_1^2 + \alpha_2^2}$$

adjusting polynomials into sum of squares:

$$\sqrt{3(\alpha_1 + \frac{2}{3}(\alpha_2 - 2))^2 + \frac{11}{3}(\alpha_2 - \frac{16}{11})^2 + \frac{32}{11} - (\frac{16}{11})^2}$$

When all the square terms are zero, the Euclidean norm of the solution is the smallest so, we can get

$$\begin{cases} \alpha_1 = \frac{4}{11} \\ \alpha_2 = \frac{16}{11} \end{cases}$$

so the solution, which can get the smallest Euclidean is :

$$\mathbf{x} = \begin{bmatrix} \frac{4}{11} \\ -\frac{8}{11} \\ -\frac{4}{11} \\ -\frac{16}{11} \end{bmatrix}$$

3.9

In the same way as in the problem 3.9, but we can find the extremum by derivation

$$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ -2 - \alpha_1 \\ 1 + \alpha_1 \\ 1 \end{bmatrix}$$

$$\|\mathbf{x}\|_2^2 = 6\alpha_1^2 + 10\alpha_1 + 6$$

$$\|\mathbf{x}\|_2^2 = 12\alpha_1 + 10 = 0$$

$$\alpha_1 = -\frac{5}{6}$$

so the solution, which have the smallest Euclidean is:

$$\mathbf{x} = \begin{bmatrix} -\frac{5}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \\ 1 \end{bmatrix}$$

3.11

There will exist $u[0], u[1], \dots, u[n-1]$ to meet the equation for any $\mathbf{x}[n]$ and $\mathbf{x}[0]$, which means for any $\mathbf{x}[n]$ and $\mathbf{x}[0]$, the equation is always have the solution, so $\mathbf{b}, \mathbf{A}\mathbf{b}, \dots, \mathbf{A}^{n-1}\mathbf{b}$ must be linearly independent

3.12

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{Ab} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{A^2b} = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} \quad \mathbf{A^3b} = \begin{bmatrix} 6 \\ 12 \\ 8 \\ 1 \end{bmatrix}$$

Thus the representation of A with respect to the basis $\mathbf{b}, \mathbf{Ab}, \mathbf{A^2b}, \mathbf{A^3b}$ is

$$\overline{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

The other basis:

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \quad \mathbf{Ab} = \begin{bmatrix} 4 \\ 7 \\ 6 \\ 1 \end{bmatrix} \quad \mathbf{A^2b} = \begin{bmatrix} 15 \\ 20 \\ 12 \\ 1 \end{bmatrix} \quad \mathbf{A^3b} = \begin{bmatrix} 50 \\ 52 \\ 24 \\ 1 \end{bmatrix}$$

the representation of A with respect to the basis $\overline{\mathbf{b}}, \overline{\mathbf{Ab}}, \overline{\mathbf{A^2b}}, \overline{\mathbf{A^3b}}$ is the same as above.

3.13

The Jordan-form representation of the matrices respectively is: $\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4$

$$\hat{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \hat{A}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+j & 0 \\ 0 & 0 & -1-j \end{bmatrix} \quad \hat{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \hat{A}_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because A_4 can not be diagonalized, so we calculate the Q , which meets $Q^{-1}A_4Q = \hat{A}_4$ from the definition:

$$\begin{cases} \mathbf{Av}_1 = \lambda \mathbf{v}_1 \\ \mathbf{Av}_2 = \lambda \mathbf{v}_2 + \mathbf{v}_1 \\ \mathbf{Av}_3 = \lambda \mathbf{v}_3 + \mathbf{v}_2 \end{cases}$$

we can calculate :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -7 \\ 9 \end{bmatrix}$$

$$Q = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$$

3.14

The characteristic polynomial of $\delta(\lambda) = |\lambda E - A|$

$$|\lambda E - A| = \begin{vmatrix} \lambda + \alpha_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = \lambda^4 + \alpha_1 \lambda^3 + \alpha_2 \lambda^2 + \alpha_3 \lambda + \alpha_4$$

for $\alpha = [\lambda_i^3 \quad \lambda_i^2 \quad \lambda_i \quad 1]'$ we can verify that $(\lambda_i E - A)\alpha = 0$
 so $\alpha = [\lambda_i^3 \quad \lambda_i^2 \quad \lambda_i \quad 1]'$ is an eigenvector of A associated with λ_i .

3.15

$$\begin{vmatrix} \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{vmatrix} = \begin{vmatrix} 0 & \lambda_2^3 - \lambda_1 \lambda_2^2 & \lambda_3^3 - \lambda_1 \lambda_3^2 & \lambda_4^3 - \lambda_1 \lambda_4^2 \\ 0 & \lambda_2^2 - \lambda_1 \lambda_2 & \lambda_3^2 - \lambda_1 \lambda_3 & \lambda_4^2 - \lambda_1 \lambda_4 \\ 0 & \lambda_2 - \lambda_1 & \lambda_3 - \lambda_1 & \lambda_4 - \lambda_1 \end{vmatrix} = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1) \begin{vmatrix} \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \prod_{1 \leq i < j \leq 4} (\lambda_j - \lambda_i)$$

3.16

$$|A| = \begin{vmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

so, when $\alpha_4 \neq 0$, the companion-form matrix is nonsingular,
 when the matrix is nonsingular, so matrix A can inverse.

$$AA^{-1} = A^{-1}A = E$$