# Chapter2

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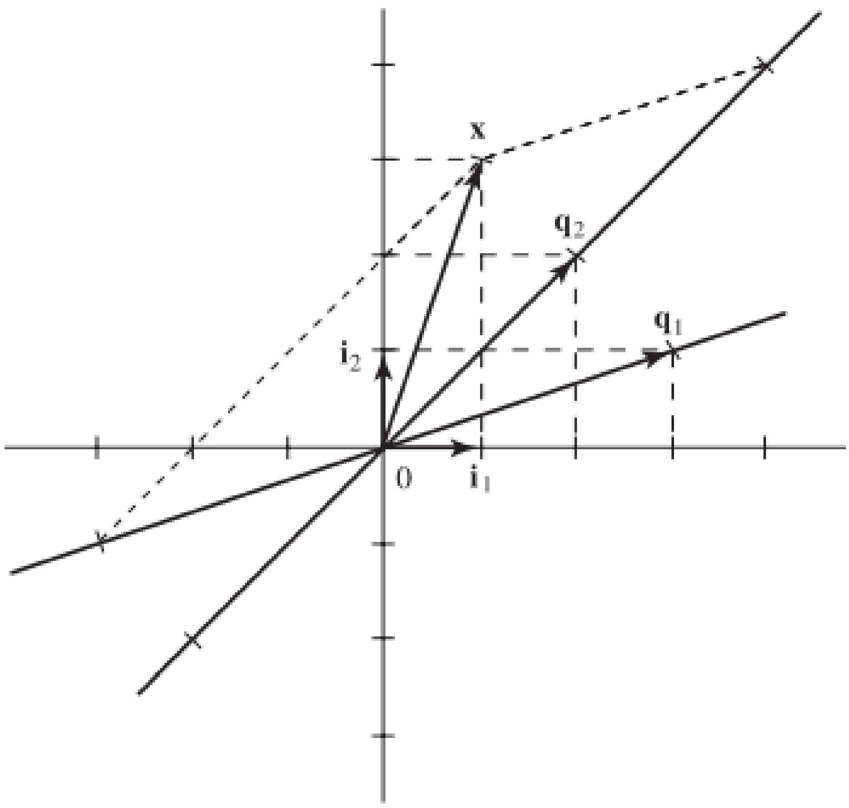
## 3.1




From the above figure, The three vectors $\boldsymbol{q}_1 = \begin{bmatrix} 3 & 1 \end{bmatrix}'$, $\boldsymbol{i}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}'$ and $\boldsymbol{q}_2 = \begin{bmatrix} 2 & 2 \end{bmatrix}'$
The representation of $x$ with respect to $\{\boldsymbol{q}_1, \boldsymbol{i}_2\}$ is $\begin{bmatrix} \frac{1}{3} & \frac{8}{3} \end{bmatrix}'$
The representation of $\boldsymbol{q}_1$ with respect to $\{\boldsymbol{i}_2, \boldsymbol{q}_2\}$ is $\begin{bmatrix} -2 & \frac{3}{2} \end{bmatrix}'$
These can be verified like this:

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{i}_2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \end{bmatrix}$$

## 3.2

### i:The norm of $x_1$

1-norm: $\|x_1\|_1 = \sum_{i=1}^{3} |x_i| = |2| + |-3| + |1| = 6$
2 -norm: $\|\boldsymbol{x}_1\|_2 = \left(\sum_{i=1}^{3} |x_i|^2\right)^{\frac{1}{2}} = \sqrt{2^2 + |-3|^2 + 1^2} = \sqrt{14}$
infinite-norm: $\|\boldsymbol{x}_1\|_\infty = \max_i |x_i| = 3$

### ii:The norm of $x_2$

1 -norm: $\|\boldsymbol{x}_2\|_1 = \sum_{i=1}^{3} |x_i| = |1| + |1| + |1| = 3$
2-norm: $\|\boldsymbol{x}_2\|_2 = \left(\sum_{i=1}^{3} |x_i|^2\right)^{\frac{1}{2}} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$
infinite-norm: $\|\boldsymbol{x}_2\|_\infty = \max_i |x_i| = 1$

## 3.3

This is just the orthonormalization procedure.

$$\begin{cases} u_1 = \alpha_1 & q_1 = u_1 / \|u_1\| \\ u_2 = \alpha_2 - (q_1' \alpha_2) q_1 & q_2 = u_2 / \|u_2\| \end{cases}$$

This is the ordinary method,what we find is the two vector are orthogonal. so,we just need to make the length of vector is 1 .

$$q_1 = \frac{u_1}{\|u_1\|} = \begin{bmatrix} \frac{2}{\sqrt{14}} - \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{bmatrix}'$$
$$q_1 = \frac{u_2}{\|u_2\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}'$$

## 3.4

### a

if n>m,$\boldsymbol{A}\boldsymbol{A}^{'}$ is a ordinary vector,which has the rank m

### b

if m=n,so $\boldsymbol{A}$ is a nonsingular square matrix,we already have $\boldsymbol{A}^{'}\boldsymbol{A} = \boldsymbol{I}_m$,so $\boldsymbol{A}' = \boldsymbol{A}^{-1}$. $\boldsymbol{A}\boldsymbol{A}^{'} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$

## 3.5

Accoridng to the principle:

$$Nillity(\boldsymbol{A}) = numberofcolumnsof \boldsymbol{A} - rank(A)$$

### i:

Rank $(A_1) = 2$
Nullity $(A_1) = 3 - 2 = 1$

### ii:

Rank $(A_2) = 3$
Nullity $(A_2) = 3 - 3 = 0$

### iii:

Rank $(A_3) = 3$
Nullity $(A_3) = 4 - 3$

## 3.6

For $A_1$:

$$\boldsymbol{a_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{a_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The set of $\{\boldsymbol{a_1}, \boldsymbol{a_2}\}$ can be the basis of the range spaces.
The independent vectors of null space can get by solving the equation:

$$\boldsymbol{A}_1 \boldsymbol{\eta}_i = 0$$

$$\boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The set of $\boldsymbol{\eta}_1$ is the basis of the null space
in the same way,we can get the basis of the range space and null space of $\boldsymbol{A}_2$

$$\boldsymbol{a_1} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \boldsymbol{a_2} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \boldsymbol{a_3} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

The set of $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$ is the basis of the range space.
because the $\boldsymbol{A}_2$ is full rank,so the basis of the null space is $\{\mathbf{0}\}$

## 3.7

$$\rho(\boldsymbol{A}) = \rho([\boldsymbol{A} \quad y]) = 2$$

so a solution $\boldsymbol{x}$ exist with respect to this equation.
Because coefficient matrix is full column rank,so the solution is unique.
if y=$[1 \quad 1 \quad 1]'$,$\rho(\boldsymbol{A}) = 2 \neq \rho([\boldsymbol{A} \quad y]) = 3$ so,when $\boldsymbol{y}$=$[1 \quad 1 \quad 1]'$,the solution is not exist.

## 3.8

$\boldsymbol{x}_p = [0 \quad -2 \quad 1 \quad 1]'$ is a solution, a basis of the null space of $A$:

$$\boldsymbol{A}\boldsymbol{\eta}_i = 0$$

$$\boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Thus the gengeral solution can be expressed as:

$$\boldsymbol{x} = \boldsymbol{x}_p + \alpha_1 \boldsymbol{\eta}_1$$

# 3.9

From the example 3.3,we can know the gengeral solution is:

$$\boldsymbol{x} = \begin{bmatrix} \alpha_1 \\ -4 + \alpha_1 + 2\alpha_2 \\ -\alpha_1 \\ -\alpha_2 \end{bmatrix}$$

$$\|\boldsymbol{x}\|_2 = \sqrt{\alpha_1^2 + (\alpha_1 + 2 * \alpha_2 - 4)^2 + \alpha_1^2 + \alpha_2^2}$$

adjusting polynomials into sum of squares:

$$\sqrt{3(\alpha_1 + \frac{2}{3}(\alpha_2 - 2))^2 + \frac{11}{3}(\alpha_2 - \frac{16}{11})^2 + \frac{32}{11} - (\frac{16}{11})^2}$$

When all the square terms are zero, the Euclidean norm of the solution is the smallest so,we can get

$$\begin{cases} \alpha_1 = \dfrac{4}{11} \\ \alpha_2 = \dfrac{16}{11} \end{cases}$$

so the solution,which can get the smallest Euclidean is :

$$\boldsymbol{x} = \begin{bmatrix} \frac{4}{11} \\ -\frac{8}{11} \\ -\frac{4}{11} \\ -\frac{16}{11} \end{bmatrix}$$

# 3.9

In the same way as in the problem 3.9,but we can find the extremum by derivation

$$\boldsymbol{x} = \begin{bmatrix} \alpha_1 \\ -2 - \alpha_1 \\ 1 + \alpha_1 \\ 1 \end{bmatrix}$$

$$\|\boldsymbol{x}\|_2^2 = 6\alpha_1^2 + 10\alpha + 6$$

$$\|\dot{\boldsymbol{x}}\|_2^2 = 12\alpha_1 + 10 = 0$$

$$\alpha_1 = -\frac{5}{6}$$

so the solution,which have the smallest Euclidean is:

$$\boldsymbol{x} = \begin{bmatrix} -\frac{5}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \\ 1 \end{bmatrix}$$

## 3.11

There will exist $u[0], u[1], ..., u[n-1]$ to meet the equation for any x[n] and x[0],which means for any x[n] and x[0],the equation is always have the solution,so $\boldsymbol{b}, \boldsymbol{Ab}, ..., \boldsymbol{A^{n-1}b}$ must be linearly indepedent

## 3.12

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \boldsymbol{Ab} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \boldsymbol{A^2b} = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} \boldsymbol{A^3b} = \begin{bmatrix} 6 \\ 12 \\ 8 \\ 1 \end{bmatrix}$$

Thus the representation of $A$ with respect to the basis $\boldsymbol{b}, \boldsymbol{Ab}, \boldsymbol{A^2b}, \boldsymbol{A^3b}$ is

$$\overline{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

The other basis:

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \boldsymbol{Ab} = \begin{bmatrix} 4 \\ 7 \\ 6 \\ 1 \end{bmatrix} \boldsymbol{A^2b} = \begin{bmatrix} 15 \\ 20 \\ 12 \\ 1 \end{bmatrix} \boldsymbol{A^3b} = \begin{bmatrix} 50 \\ 52 \\ 24 \\ 1 \end{bmatrix}$$

the representation of $A$ with respect to the basis $\boldsymbol{\overline{b}}, \boldsymbol{A\overline{b}}, \boldsymbol{A^2 overline b}, \boldsymbol{A^3\overline{b}}$ is the same as above.

## 3.13

The Jordan-form representation of the matrices respectively is:$\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4$

$$\hat{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \hat{A}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+j & 0 \\ 0 & 0 & -1-j \end{bmatrix} \hat{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \hat{A}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because $A_4$ can not be diagonalized,so we caculate the $Q$,
which meets $Q^{-1}A_4Q = \hat{A}_4$