Chapter7

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Thursday 18th February, 2021

7.1

$$\hat{g(s)} = \frac{s-1}{(s^2-1)(s+2)} = \frac{s-1}{s^3+2s^2-s-2}$$

so,from the equation (7.9),we can get the three-dimensional controllable realization:

$$\dot{x} = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} x$$

it is obviously the $\hat{g(s)}$ is not coprime fraction,so the controllable realization is not observable.

7.2

from the quation (7.14), we can easily get the three-dimensional observable realization:

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

similarly, it is not controllable.

7.3

from the inverse canonical decomposition,we can add an uncontrollable state to problem 7.1:

$$\dot{x} = \begin{bmatrix} -2 & 1 & 2 & a_1 \\ 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & a_3 \\ 0 & 0 & 0 & a_4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & -1 & c_4 \end{bmatrix} x$$

where a_i and c_i are arbitrary value,this is an uncontrollable and unobservable realization. the transfer function can be reduced to a coprime fraction,which is:

$$\hat{g(s)} = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

A minimal realization can be realized through controllable realization, this realization is two-dimensional

$$\dot{x} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

7.4

from the equation (7.27),we can get:

$$\begin{aligned} D(s) &= -2 - s + 2s^2 + s^3 \\ N(s) &= -1 + s \\ \overline{D}(s) &= \overline{D}_0 + \overline{D}_1 s + \overline{D}_2 s^2 \\ \overline{N}(s) &= \overline{N}_0 + \overline{N}_1 s + \overline{N}_2 s^2 \end{aligned}$$

the Sylvester resultant:

$$\mathbf{SM} := \begin{bmatrix} -2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -2 & -1 & 0 & 0 \\ 2 & 0 & -1 & 1 & -2 & -1 \\ 1 & 0 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\overline{N}_0 \\ \overline{D}_0 \\ -\overline{N}_1 \\ \overline{D}_1 \\ -\overline{N}_2 \\ \overline{D}_2 \end{bmatrix} = 0$$

the rank of **S** is 5, so there are 2 linearly indepedent N-columns,the degree of the transfer function is 2 .we can also calculate a monic null vector

$$z = \begin{bmatrix} -1 & 2 & 0 & 3 & 0 & 1 \end{bmatrix}'$$

7.5

just as the problem (7.4) show,in the same way:

$$D(s) = -1 + 4s^2$$
$$N(s) = -1 + 2s$$
$$\overline{D}(s) = \overline{D}_0 + \overline{D}_1 s$$
$$\overline{N}(s) = \overline{N}_0 + \overline{N}_1 s$$

the Sylvester resultant:

$$\mathbf{SM} := \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 4 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} -\overline{N}_0 \\ \overline{D}_0 \\ -\overline{N}_1 \\ \overline{D}_1 \end{bmatrix} = 0$$

the rank of $\mathbf{S}$ is 3,we can calculate a monic null vector

$$z = \begin{bmatrix} -0.5 & 0.5 & 0 & 1 \end{bmatrix}'$$

so the coprime fraction is:

$$\hat{g(s)} = \frac{1}{2s+1}$$

7.6

The Sylvester resultant by arranging the coefficients of $N(s)$ and $D(s)$ in descending powers:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

the second D-columns is linearly depedent of its LHS columns, so it is not true that all D-columns are linearly indepedent of their LHS columns. the degree of $\hat{g(s)}$ is 1,but the linearly indepedent N-columns is 2.

7.7

the realization is a controllable realization,the controllable realization is observable if and only if the transfer function is coprime fraction.$D(s)$ and $N(s)$ are coprime if and only if the Sylvester resultant is nonsingular.

$$\hat{g(s)} = \frac{N(s)}{D(s)} = \frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2}$$

the Sylvester resultant:

$$\mathbf{S} := \begin{bmatrix} \alpha_2 & \beta_2 & 0 & 0 \\ \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\ 1 & 0 & \alpha_1 & \beta_1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The determinant of $\mathbf{S}$ is $-\alpha_2\beta_1^2 + \alpha_1\beta_1\beta_2 - \beta_2^2$ for the controllable realization, its observability matrix:

$$O = \begin{bmatrix} \beta_1 & \beta_2 \\ -\alpha_1\beta_1 + \beta_2 & -\alpha_2\beta_1 \end{bmatrix}$$

The realization is observable if and only if the observability matrix is full column rank,the determinant of O is $-\alpha_2\beta_1^2 + \alpha_1\beta_1\beta_2 - \beta_2^2$ Thus the two condition is same.

7.8

Let us consider a transfer function:

$$\hat{g(s)} = \frac{N(s)}{D(s)} = \frac{\beta_1 s^2 + \beta_2 s + \beta_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}$$

its Sylvester resultant:

$$\mathbf{S} := \begin{bmatrix} \alpha_3 & \beta_3 & 0 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_3 & \beta_3 & 0 & 0 \\ \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \alpha_3 & \beta_3 \\ 1 & 0 & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\ 0 & 0 & 1 & 0 & \alpha_1 & \beta_1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The observability matrix:

$$O = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\ -\alpha_1\beta_1 + \beta_2 & -\alpha_2\beta_1 + \beta_3 & -\alpha_3\beta_1 \\ (\alpha_1^2 - \alpha_2)\beta_1 - \alpha_1\beta_2 + \beta_3 & (\alpha_1\alpha_2 - \alpha_3)\beta_1 - \alpha_2\beta_2 & \alpha_1\alpha_3\beta_1 - \alpha_3\beta_2 \end{bmatrix}$$

7.9

its controllable realization is:

$$\dot{x} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

its observability matrix is:

$$O = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

its determinant is nonzero, so the realization is observable. The Sylvester resultant of D(s) and N(S) is:

$$\mathbf{S} := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

its determinant is also nonzero,so the Sylvester resultant is nonsingular.

7.10

$$\hat{g(s)} = \frac{1}{(s+1)^2} = s^{-2} - 2s^{-3} + 3s^3$$

the irreducible companion form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

7.11

from the problem (7.10)

$$T(2,2) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\tilde{T}(2,2) = OAC = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$$

using the singular value decomposition to express $T(2,2)$ as

$$T(2,2) = OC = K \wedge L'$$

where K and L' are orthogonal matrix. Let $O = K\wedge^{\frac{1}{2}}$ and $C = \wedge^{\frac{1}{2}}L'$

$$O = \begin{bmatrix} 0.5946 & -0.5946 \\ -1.4355 & -0.2463 \end{bmatrix} C = \begin{bmatrix} -0.5946 & -1.4355 \\ 0.5946 & -0.2463 \end{bmatrix}$$

$$A = O^{-1}\tilde{T}(2,2)C^{-1} = \begin{bmatrix} 1.000 & 3.4142 \\ -0.5858 & -1.000 \end{bmatrix}$$

$$b = \begin{bmatrix} -0.5946 \\ 0.5946 \end{bmatrix}$$

$$c = \begin{bmatrix} 0.5946 & -0.5946 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 1.000 & 3.4142 \\ -0.5858 & -1.000 \end{bmatrix} x + \begin{bmatrix} -0.5946 \\ 0.5946 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0.5946 & -0.5946 \end{bmatrix} x$$

7.12

$$\hat{g(s)} = \frac{2s+2}{s^2-s-2} = \frac{2}{s-2}$$

we can see that the transfer function can be reduced to a coprime fucntion withd degree 1. so they are not minimal realizations,and they are not algebraically equivalent,because they have differnert eigenvalues.

7.13

The character polynomials of $\hat{G_1(s)}$ is $s(s+1)(s+3)$ and the degree is 3.
The character polynomials of $\hat{G_2(s)}$ is $(s+1)^3(s+2)^2$ and the degree is 5.
The character polynomials of $\hat{G_2(s)}$ is $s(s+1)^(s+2)(s+3)^2(s+4)(s+5)$ and the degree is 8.

7.14

$$\hat{G(s)} = \begin{bmatrix} s & 1 \\ -s & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\overline{D}(s) = \begin{bmatrix} s & 1 \\ -s & s \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} s$$

$$\overline{N}(s) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The generalized resultant:

$$\mathbf{SM} := \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\overline{N}_0 \\ \overline{D}_0 \\ -\overline{N}_1 \\ \overline{D}_1 \end{bmatrix} = 0$$

The nullity of S is

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so we can get:

$$N_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} D_0 = 0 N_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} D_1 = 1$$

so we have

$$\hat{G(s)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} s^{-1}$$

The degree of $G(s)$ is 1,deg det $\overline{D}(s) = 2$,so the left fraction is not coprime.

7.15

we form from

$$\hat{G(s)} = \begin{bmatrix} s & 1 \\ -s & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

where $D(s)$ and $N(s)$ are arranged in descending powers of s,it is not true that all D-columns are linearly indepedent of their LHS columns, bescuse the second block of D_1-columns is linearly depedent of its LHS columns. The number of linearly indepedent N-columns is 2,so the degree is not equal the number of linearly indepedent N-columns,so the theorem is not hold either.

7.16

$$\hat{G(s)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} s^{-1}$$

The generalized resultant

$$T := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

N2-row in the first N block-row is linearly depedent of its preceding rows, Let

$$t1 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} t1 = 0$$

N1-row in the second block-row is linearly depedent of its preceding rows,let(deleting the N2-row in the first block)

$$t2 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} t2 = 0$$

$$\begin{bmatrix} -\overline{N}_0 & \overline{D}_0 & -\overline{N}_1 & \overline{D}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

we can get

$$\overline{N}(s) = \overline{N}_0 + \overline{N}_1 s$$
$$\overline{D}(s) = \overline{D}_0 + \overline{D}_1 s$$

$$\hat{G}(s) = \begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

7.17

$$\hat{G}(s)=\begin{bmatrix}\frac{s^2+1}{s^3} & \frac{2s+1}{s^2}\\ \frac{s+2}{s^2} & \frac{2}{s}\end{bmatrix}=:\bar{D}^{-1}(S)\bar{N}(S)$$

$$\bar{D}(s)=\begin{bmatrix}s^3 & 0\\ 0 & s^2\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}s+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}s^2+\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}s^3$$

where.

$$\bar{N}(s)=\begin{bmatrix}s^2+1 & s(2s+1)\\ s+2 & 2s\end{bmatrix}=\begin{bmatrix}1 & 0\\ 2 & 0\end{bmatrix}+\begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}s+\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}s^2$$

the generalized resultant-is

$$S=\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 1 & & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

rank $s=9$, $\quad\mu_1=2,\mu_2=1$ the monic null vectors of the submatrices that consist of the primary dependent- $\bar{N}_2$ -columns and $\bar{N}_1$ -columns are , respectively $Z2=\begin{bmatrix}-2.5 & -2.5 & 0 & -0.5 & 0 & 0 & 0.5 & 1\end{bmatrix}$
$Z1=\begin{bmatrix}-0.5 & -2.5 & 0 & -0.5 & -1 & -1 & 0.5 & 0 & 0 & 1\end{bmatrix}$

$$\therefore\begin{bmatrix}-N_0\\ D_0\\ -N_1\\ D_1\\ N_2\\ D_2\end{bmatrix}=\begin{bmatrix}-0.5 & -2.5 & 0 & -0.5 & -1 & -1 & 0.5 & 0 & 0 & 0 & 1 & 0\\ -2.5 & -2.5 & 0 & -0.5 & 0 & 0 & 0.5 & 1 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$D(s) = \begin{bmatrix} 0 & 0 \\ -0.5 & -0.5 \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^2 = \begin{bmatrix} s^2+0.5s & 0.5s \\ -0.5 & s-0.5 \end{bmatrix}$$

$$N(s) = \begin{bmatrix} 0.5 & 2.5 \\ 2.5 & 2.5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} s = \begin{bmatrix} s+0.5 & 2.5 \\ s+2.5 & 2.5 \end{bmatrix}$$

<u>thus</u> - right coprime fraction of $\hat{G}(s)\cdot$ is $\hat{G}(s) = \begin{bmatrix} s+0.5 & 2.5 \\ s+2.5 & 2.5 \end{bmatrix} \begin{bmatrix} s^2+0.5s & 0.5s \\ -0.5 & s-0.5 \end{bmatrix}$

we define $H(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}$ $\quad L(s) = \begin{bmatrix} s & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

then -we have

$$D(s) = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} H(s) + \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -0.5 & -0.5 \end{bmatrix} L(s)$$

$$N(s) = \begin{bmatrix} 1 & 0.5 & 2.5 \\ 1 & 2.5 & 2.5 \end{bmatrix} L(s)$$

$$D_{hc}^{-1} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix}$$

$$D_{hc}^{-1} D_{lc} = \begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -0.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0 & -0.5 & -0.5 \end{bmatrix}$$

thus - minimal realization of $\hat{G}(s)$ is

$$\dot{x} = \begin{bmatrix} -0.5 & -0.25 & -0.25 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} x + \begin{bmatrix} 1 & -0.5 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0.5 & 2.5 \\ 1 & 2.5 & 2.5 \end{bmatrix} x$$