Chapter2

31202008881 Bao Ze an

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2.1

What we know is the linear system must obey the superposition property.
The input-output description in Fig2.1(a) is :$y = a * u$.
Here a is a constant .It is eay to find the system(a) is a linear system.
The input-output decription in Fig2.1(b) is:$y = a * u + b$.
Here a and b are all constants.Thstify whether the system has the property of additivity Let:

$$y_1 = a * u_1 + b.$$

$$y_2 = a * u_2 + b.$$

then:

$$(y_1 + y_2) = a * (u_1 + u_2) + 2 * b$$

so it does not satisfy the property of additivity.therefore,it is a nonlinear system.
It is obviously the system in the Fig2.1(c) is a nonliear system.
When system(b) introduce $y - y_0$ as the new output,system(c) can be the linear system.

2.2

Because $g(t)$ is not zero,when $t \leq 0$,so the ideal lowpass filter is not causual and the ideal filter can't build in the real world.

2.3

It is easy to find the system is a linear system.
Testify whether the system is time-invariable:
Definine the initial time of input t_0,system input is $u(t)$,$t \geq t_0$,so it decides the output $y(t)$,$t \geq t_0$

$$y(t) = \begin{cases} u(t), & for \quad t_0 \leq t \leq \alpha \\ 0, & for \quad t \geq \alpha \end{cases}$$

Shift the initial time to $t_0 + T$.Let $t_0 + T > \alpha$,and shift the input to $u(t - T)$,$t \geq t_0 + T$.
The system output is $y'(t) = 0$.Suppose that $u(t)$ is not 0,$y'(t)$ is not equal to $y(t - T)$.
So,the system is time-invaring.
For any time t,the system output y(t) is decided by the current input u(t) exclusively.
So,it is a causual time.

2.4

Let:$y = Hu$
Because of causual property:

$$y_{(-\infty,\alpha)} = Hu_{(-\infty,+\infty)} = Hu_{(-\infty,\alpha)} = HP_\alpha u_{(-\infty,+\infty)}$$

Thus we have:

$$P_\alpha y = P_\alpha Hu = P_\alpha HP_\alpha u$$

Because$(P_\alpha Hu)(t) = 0$ for $t \geq \alpha$,but $(HP_\alpha u)(t)$can be nonzero for $t \geq \alpha$, Thus $P_\alpha Hu \neq HP_\alpha u$

2.5

* If the system is a nonlinear system,for $x(0) \neq 0$ and $x(0) = 0$, all three case are not correct.
* If the system is a linear system:

Superposition property must hold for the input and initial state.

if $x(0) \neq 0$:

case1:

$$x(0) + x(0) \neq x(0)$$

so case1 statement is not correct.

case2:

$$0.5 * x(0) + 0.5 * x(0) = x(0)$$

so case2 statement is correct.

case3:

$$x(0) - x(0) = 0 \neq x(0)$$

so case3 statement is correct.

if $x(0) = 0$:

three statement are all correct.

2.6

Suppose the system input:$u'(t) = \alpha u(t)$,here a is a constant.
The system output:

$$y'(t) = \begin{cases} \alpha u^2(t)/u(t-1) & if \quad u(t-1) \neq 0 \\ 0 & if \quad u(t-1) = 0 \end{cases}$$

so,$y'(t) = \alpha y(t)$,it satisfies the homogeneity property.
Suppose the input:$u'(t) = u_1(t) + u_2(t)$, The system output:

$$y'(t) = \begin{cases} \alpha(u_1(t) + u2(t))^2(t)/(u_1(t-1) + u_2(t-1)) & if \quad u_1(t-1) + u_2(t-1) \neq 0 \\ 0 & if \quad u_1(t-1) + u_2(t-1) = 0 \end{cases}$$

in some case,$y'(t) \neq y_1(t) + y_2(t)$,so it don't satisfy the additivity property

2.7

Any rational number $\alpha = m/n$,here m and n are both integer. Firstly,prove that if the system input-output can be decribed as following:

$$x \to y$$

then:

$$mx \to my$$

The input mx can be regarded as the sum of m input x.
It is easy to testify it satisfy the additivity.
Secondly, prove that if a system input-output can be described as following:

$$x \to y$$

then:

$$x/n \to y/n$$

Suppose:

$$x/n \to u$$

using additivity:

$$n * (x/n) = x$$

thus to say:$n * (x/n) \to y$,in the same time,from the above statement,$n * (x/n) \to nu$ so:

$$y = nu$$

$$u = y/n$$

thus:

$$x/n \to y/n$$

$$x * m/n \to y * m/n$$

$$\alpha x \to \alpha y$$

2.8

Define:

$$x = t + \tau y = t - \tau$$

so:

$$t = \frac{x+y}{2}\tau = \frac{x-y}{2}$$

for all t, τ:

$$\begin{aligned} g(t,\tau) &= g(\frac{x+y}{2}, \frac{x-y}{2}) \\ &= g(\frac{x+y}{2} + \frac{-x+y}{2}, \frac{x-y}{2} + \frac{-x+y}{2}) \\ &= g(y,0) \end{aligned}$$

so:

$$\frac{\partial g(t,\tau)}{\partial x} = \frac{\partial g(y,0)}{\partial x} = 0$$

it just prove the $g(t,\tau)$ depends only on the $t - \tau$.

2.9

i

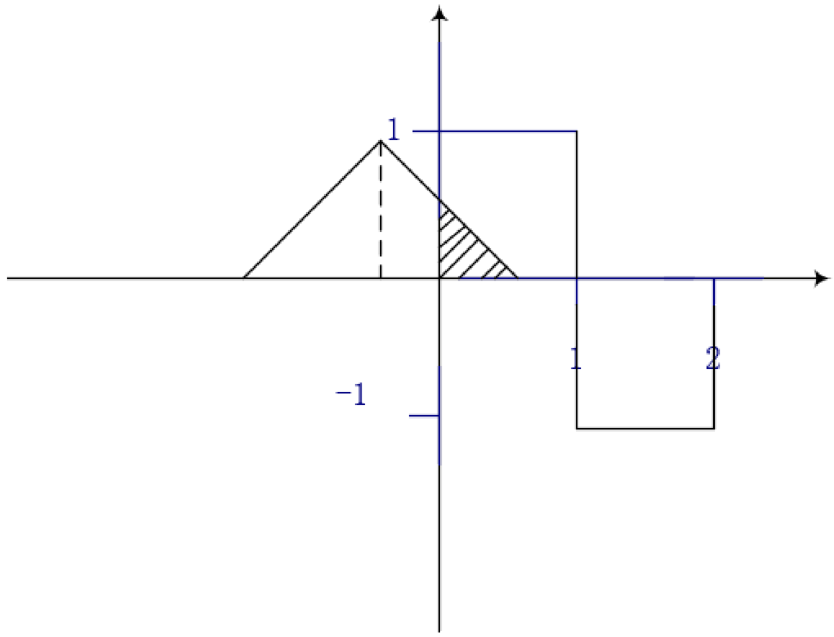
when $t < 0, y(t) = 0$

ii

when $0 \leq t \leq 1$

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

take the convolution integral:

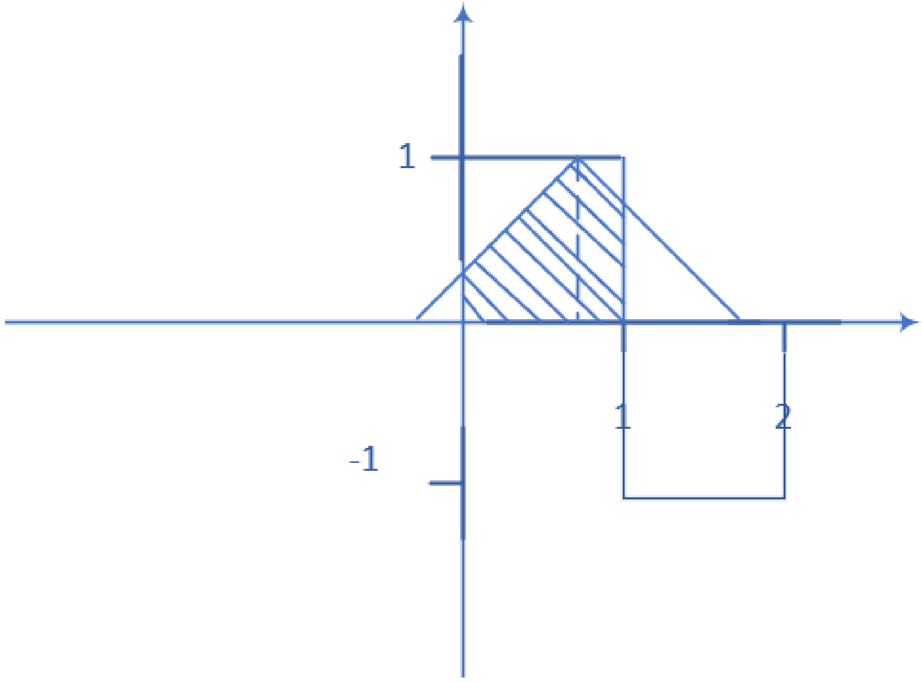


$$y(t) = \int_0^t (t-\tau)d\tau$$

$$y(t) = \frac{1}{2}t^2$$

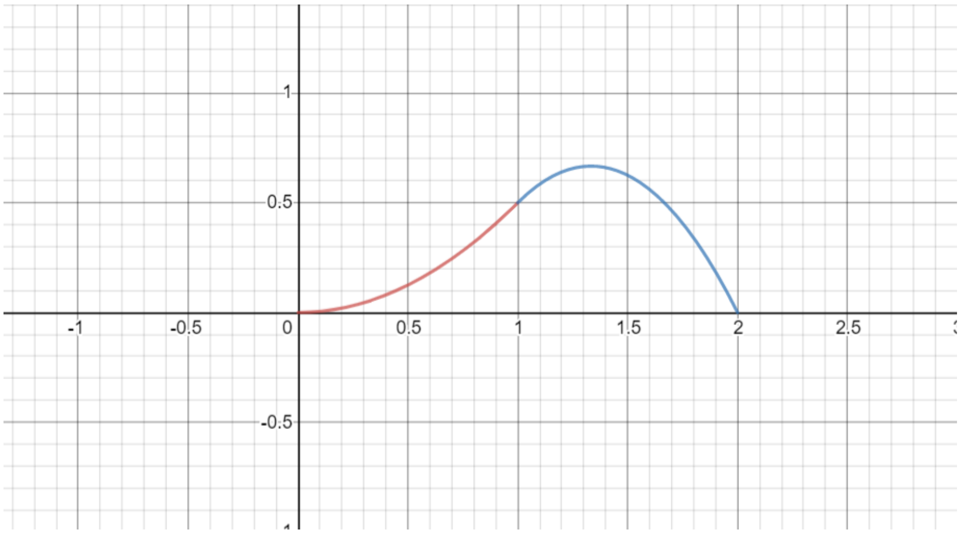
iii

when $1 < t \leq 2$ take the convolution integral:



$$y(t) = 1 - \frac{(2-t)^2}{2} - \frac{(t-1)^2}{2}$$

the image on full definition domain:



2.10

i

Take the lapalace transform to both sides of the equation:

$$s^2\hat{y(s)} + 2s\hat{y(s)} - 3\hat{y(s)} = s\hat{u(s)} - \hat{u(s)}$$

arrange the equation:

$$g(s) = \frac{y(\hat{s})}{u(\hat{s})} = \frac{s-1}{s^2+2s-3} = \frac{1}{s+3}$$

the impulse response of the system is just the inverse lapalace

$$g(t) = \mathcal{L}^{-1}[g(\hat{s})] = e^{-3t} \qquad t \geq 0$$

2.11

Let g(t) is the impulse response,u(t)=1 is the input.so the unit-step response is:

$$\overline{y}(t) = \int_0^t g(t)u(t-\tau)d\tau = \int_0^t g(\tau)d\tau$$

Therefore g(t)=$\frac{\mathrm{d}\overline{y}(t)}{\mathrm{d}t}$

2.12

Take the lapalace transform to both sides of the equations:

$$D_{11}(s)y_1(s) + D_{12}(s)y_2(s) = N_{11}(s)u_1(s) + N_{12}(s)u_2(s)$$

$$D_{21}(s)y_1(s) + D_{22}(s)y_2(s) = N_{21}(s)u_1(s) + N_{22}(s)u_2(s)$$

Rewrite them in matrix form:

$$\begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix} \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

so the transfer matrix of the system is:

$$\hat{G}(s) = \begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix}^{-1} \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}$$

2.13

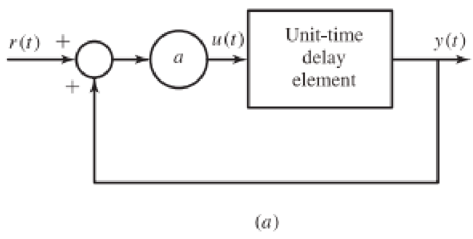


Figure 1: system with negative feedback

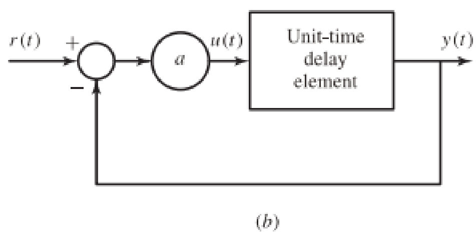


Figure 2: system with positive feedback

i:the positive feedback system

$$y(t) = u(t-1), r(t) = 1 \qquad for \quad t \geq 0$$

$a = 1:$

$$u(t) = r(t) + y(t) = 1 + u(t-1)$$

thus to say:

$$y(t+1) = 1 + y(t)$$

From the initial condition:$y(t) = 0$ for $0 \leq t < 1$,then:

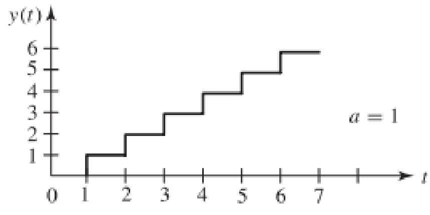
$$y(t) = 1 \qquad for \quad 1 \leq t < 2$$

$$y(t) = 2 \qquad for \quad 2 \leq t < 3$$

$$\vdots$$

$$y(t) = n \qquad for \quad n \leq t < (n+1)$$

so the image of the y(t):



(a)

$a = 0.5:$

$$u(t) = 0.5(r(t) + y(t)) = 0.5 + 0.5y(t)$$

thus to say:

$$y(t+1) = 0.5 + 0.5y(t)$$

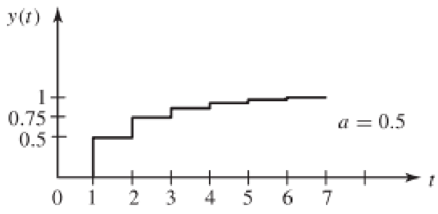
From the initial condition:$y(t) = 0$ for $0 \leq t < 1$,then:

$$y(t) = 0.5 \qquad for \quad 1 \leq t < 2$$

$$y(t) = 0.75 \qquad for \quad 2 \leq t < 3$$

$$\vdots$$

so the image of the y(t):



(b)

ii:the negative feedback system

$$y(t) = u(t-1), r(t) = 1 \qquad for \quad t \geq 0$$

$a = 1:$

$$u(t) = r(t) - y(t) = 1 - u(t-1)$$

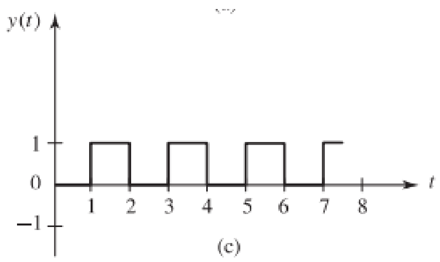
thus to say:

$$y(t+1) = 1 - y(t)$$

From the initial condition:$y(t) = 0$ for $0 \leq t < 1$,then:

$$\begin{aligned} y(t) &= 1 \qquad for \quad 1 \leq t < 2 \\ y(t) &= 0 \qquad for \quad 2 \leq t < 3 \\ y(t) &= 1 \qquad for \quad 3 \leq t < 4 \\ &\vdots \end{aligned}$$

so the image of the y(t):



(c)

$a = 0.5:$

$$u(t) = 0.5(r(t) - y(t)) = 0.5 - 0.5u(t-1)$$

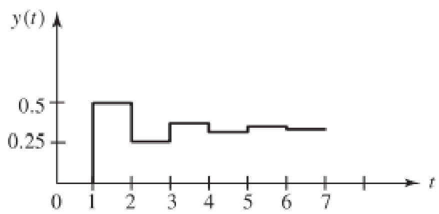
thus to say:

$$y(t+1) = 0.5 - 0.5y(t)$$

From the initial condition:$y(t) = 0$ for $0 \leq t < 1$,then:

$$\begin{aligned} y(t) &= 0.5 \qquad for \quad 1 \leq t < 2 \\ y(t) &= 0.25 \qquad for \quad 2 \leq t < 3 \\ y(t) &= 0.375 \qquad for \quad 3 \leq t < 4 \\ &\vdots \end{aligned}$$

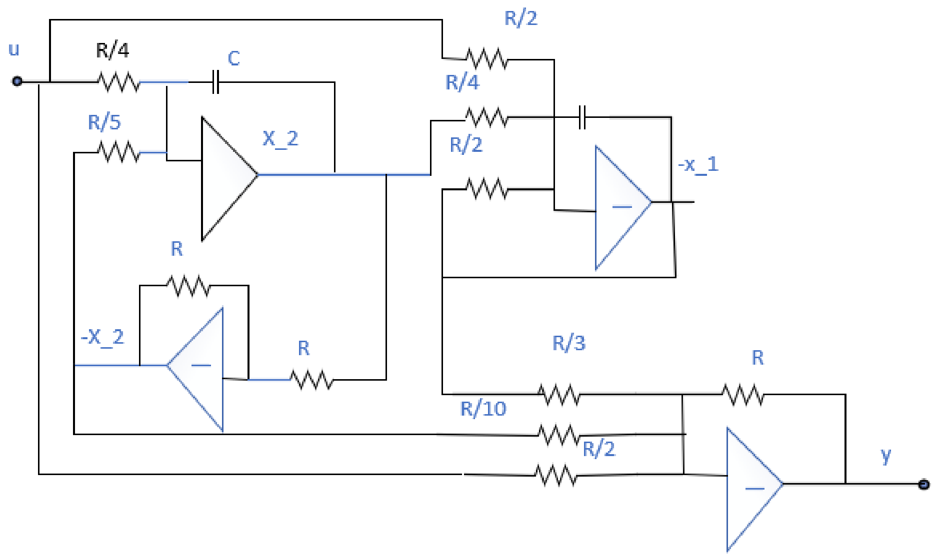
so the image of the y(t):



(d)

2.14

From the state-space equation,it has 2 dimensions,,so we need two intergerators to implement it.
We choose the output of number 1 intergerators as $+x_2$,and the output of number 2 intergerators as $-x_1$.
We suppose $RC = 1$:
The op-amp circuit diagram:



2.15

a

Application of Newton's law to the rotaional movement of the pendulum:
Moment of gravity component to pendulum:$M = (ucos\theta - mgsin\theta)l$
According to the angular momentum theorem:

$$M = I\beta = ml^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

thus to say:

$$ml^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = (ucos\theta - mgsin\theta)l$$

choose the $\dot{\theta}, \theta$ as the state variable Define $x_1 = \theta, x_2 = \dot{\theta}$

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\frac{g}{l}sinx_1 + \frac{u}{ml}cosx_1 \end{cases}$$

The system is nonlinear. If θ is very small:$sin\theta \sim \theta, cos\theta \sim 1$ so the state-space equation is:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml} \end{bmatrix} u$$

when θ is very small, we can consider the system is linear.

b

In the same way:
for m_2 and l_2:

$$m_2 l_2^2 \frac{\mathrm{d}^2\theta_2}{\mathrm{d}t^2} = (ucos\theta_2 - m_2 gsin\theta_2) l_2$$

for m_1 and l_1,define the force on the l_2 is T:

$$T = m_2 gcos\theta_2 + usin\theta_2$$

$$m_1 l_1^2 \frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} = (-m_1 gsin\theta_1 + Tsin(\theta_2 - \theta_1)) l_1$$

Define $x_1 = \theta_1, x_2 = \dot{\theta_1}, x_3 = \theta_2, x_4 = \dot{\theta_2}$

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\dfrac{g}{l_1} sinx_1 + \dfrac{m_2 gcosx_3 sin(x_3 - x_1)}{m_1 l_1} + \dfrac{usinx_3 sin(x_3 - x_1)}{m_1 l_1} \\ \dot{x_3} = x_4 \\ \dot{x_4} = -\dfrac{gsinx_3}{l_2} + \dfrac{ucosx_3}{m_2 l_2} \end{cases}$$

The system is nonlinear. If θ is very small:$sin\theta_1 \sim \theta_1, sin\theta_2 \sim \theta_2, cos\theta_1 \sim 1, cos\theta_2 \sim 1$ so the state-space equation is:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{g}{l_1} - \frac{m_1 g}{m_1 l_1} & 0 & \frac{m_2 g}{m_1 l_1} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g}{l_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2 l_2} \end{bmatrix} u$$

when θ_1, θ_2 is very small, we can consider the system is linear.

2.16

According to Newton's second law we can get the equation:

$$m\ddot{h} = f_1 - f_2 = k_1\theta - k_2 u$$

According to the angular momentum theorem,we can get the equation:

$$I\ddot{\theta} + b\dot{\theta} = (l_1 + l_2) f_2 - l_1 f_1$$

Define $x_1 = h, x_2 = \dot{h}, x_3 = \theta, x_4 = \dot{\theta}$:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \dfrac{k_1}{m} x_3 - \dfrac{k_2}{m} u \\ \dot{x_3} = x_4 \\ \dot{x_4} = -\dfrac{l_1 k_1}{I} x_3 - \dfrac{b}{I} x_4 + \dfrac{(l_1 + l_2) k_2}{I} u \end{cases}$$

If neglecting the effect of I,two equations became:

$$m\ddot{h} = f_1 - f_2 = k_1\theta - k_2 u$$

$$b\dot{\theta} = (l_1 + l_2) f_2 - l_1 f_1$$

Simultaneous equations,take Lapalace transform to both sides of equations, eliminating variable θ

$$ms^2h(s) = k_1\theta(s) - k_2u(s)$$
$$bs\theta(s) = (l_1 + l_2)k_2u(s) - l_1k_1\theta(s)$$

so,the transfer function from u to h is:

$$\hat{g}(s) = \frac{\hat{h}(s)}{\hat{u}(s)} = \frac{k_1k_2l_2 - k_2bs}{ms^2(bs + k_1l_1)}$$

2.17

A state-space equation to describe the system is:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k}{m}x_3 - g \\ \dot{x}_3 = u \end{cases}$$

2.18

Refer to example 2.9, we can get the equations following:

$$y_1 = \frac{x_1}{R_1} \quad and \quad y_2 = \frac{x_2}{R_2}$$

Changes of liquid levels are governed by:

$$A_1dx_1 = (u - y_1)dt$$

$$A_2dx_2 = (y - y_1)dt$$

Thus to say:

$$\begin{cases} A_1\dot{x_1} = u - \dfrac{x_1}{R_1} \\ A_2\dot{x_2} = \dfrac{x_1}{R_1} - \dfrac{x_2}{R_2} \end{cases}$$

Take the lapalace transform:

$$\frac{\hat{y}_1(s)}{\hat{u}(s)} = \frac{1}{A_1R_1s + 1}$$

the transfer function from y_1 to y:

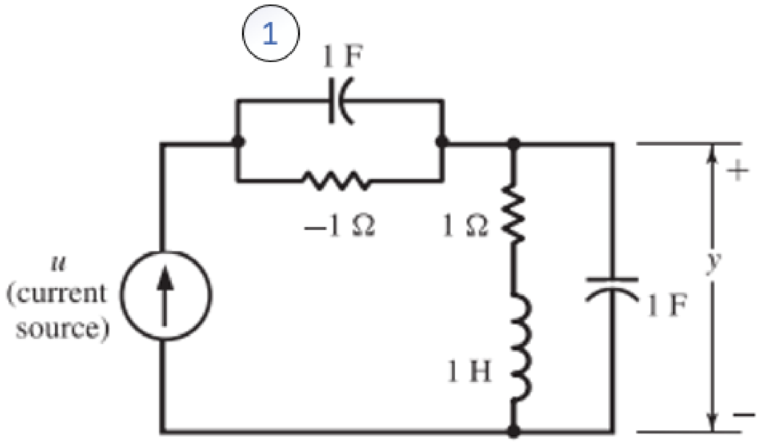
$$\frac{\hat{y}(s)}{\hat{y}_1(s)} = \frac{1}{A_2R_2s + 1}$$

the transfer function from u to y:

$$\frac{\hat{y}(s)}{\hat{u}(s)} = \frac{1}{(A_1R_1s + 1)(A_2R_2s + 1)}$$

So the transfer function from u to y equal the product of the two transfer functions.

2.19



The voltage across the 1-F capacitor number 1 is assigned x_1,then its current is $\hat{x_1}$,the voltage across the other 1-F capacitor is assigned x_2,then its current is $\hat{x_2}$,the current through the 1-H inductor is assigned as x_3, then its voltage is $\hat{x_3}$
According to the Kirchhoff's current law and Kirchhoff's voltage law,we can get the equation following:

$$\begin{cases} \dot{x_1} = x_1 + u \\ \dot{x_2} = -x_3 + u \\ \dot{x_3} = x_2 - x_3 \\ y = x_2 \end{cases}$$

Rewrite them in matrix form:

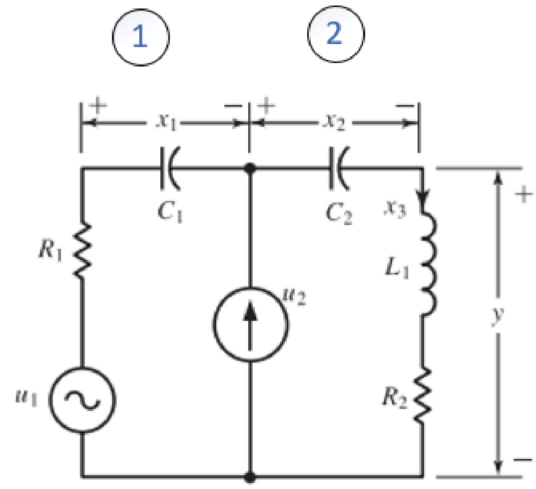
$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Its transfer matrix from the state-space equation:

$$\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = C(SI - A)^{-1}B = \frac{s+1}{s^2+s+1}$$

2.20



Select the state variable as shown in the figure,According to the Kirchhoff's current law and Kirchhoff's voltage law,we can get the equation following:

$$\begin{cases} C_1\dot{x_1} = x_3 - u_2 \\ C_2\dot{x_2} = x_3 \\ L_1\dot{x_3} = u_1 - x_1 - x_2 - R_2x_3 - C_1\dot{x_1}R_1 \\ y = u_1 - C_1\dot{x_1}R_1 - x_1 - x_2 \end{cases}$$

Rewrite them in matrix form:

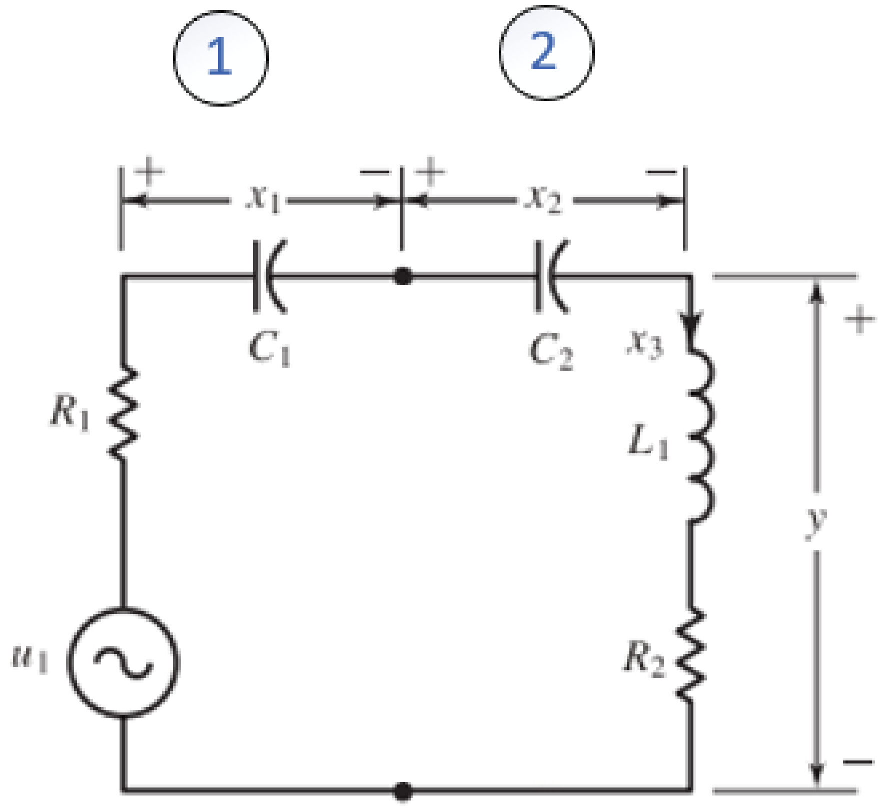
$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} \\ 0 & 0 & \frac{1}{C_2} \\ -\frac{1}{L_1} & -\frac{1}{L_1} & -\frac{R_1+R_2}{L_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{C_1} \\ 0 & 0 \\ \frac{1}{L_1} & \frac{R_1}{L_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} -1 & -1 & -R_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & R_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

It is a two-input one-output system,So its transfer function matrix is 2x1 dimension.

i: The transfer function from u_1 to y

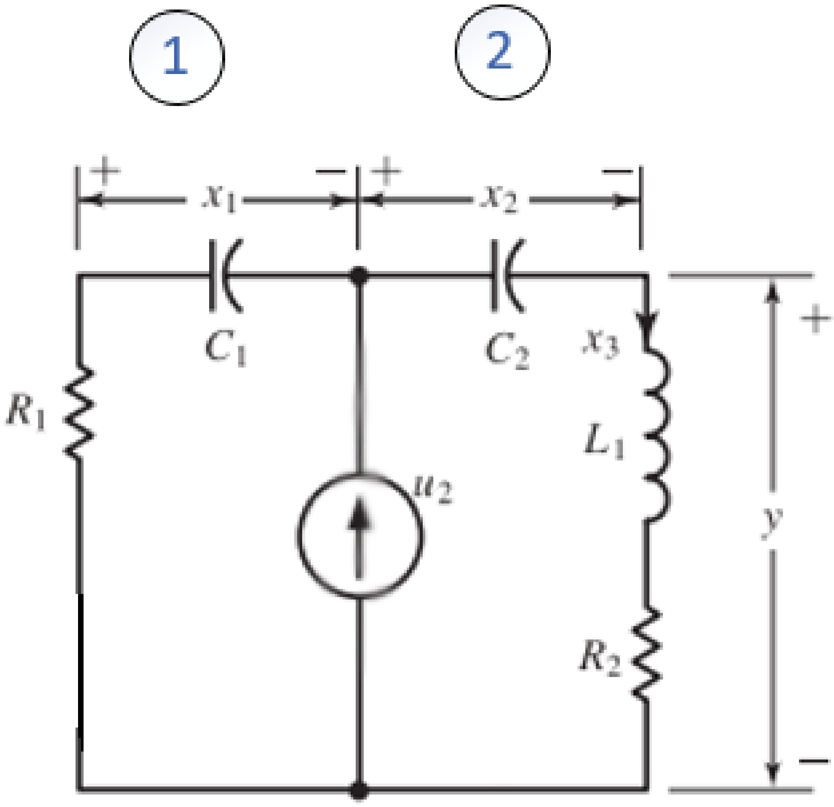
Let the u_2=0,the components are expressed in the form of complex impedance,we can get the equation:



$$\hat{g}_1(s) = \frac{L_1 s + R_2}{R_1 + \frac{1}{C_1 s} + \frac{1}{C_2 s} + L_1 s + R_2} = \frac{s^2 + \frac{R_2}{L_1} s}{s^2 + \frac{R_1 + R_2}{L_1} s + \frac{C_1 + C_2}{C_1 C_2 L_1}}$$

i: The transfer function from u_2 to y

In the same way:



According to the shunt formula: the current through L_1:

$$\hat{i}(s) = \frac{\frac{1}{C_1 s} + R_1}{R_1 + R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s} + L_1 s} \hat{u_2}(s)$$

$$\hat{y}(s) = (L_1 s + R_1)\hat{i}(s)$$

so:

$$\hat{g}_2(s) = \frac{\hat{y}(s)}{\hat{u}_2(s)} = \frac{(\frac{1}{C_1 s} + R_1)(L_1 s + R_2)}{R_1 + R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s} + L_1 s}$$

The ultimate transfer function matrix:

$$\hat{G}(s) = [\hat{g}_1(s) \qquad \hat{g}_2(s)]$$

2.21

Neglecting the mass of m_1 and m_2
If the θ is very small,according to Newton's second law and the angular momentum theorem:

$$m_2 \ddot{y} = k_2(l_2\theta - y)$$

$$I\ddot{\theta} = ul_2 - k_1(l_1\theta)l_1 - k_2(l_2\theta - y)l_2$$

Define $x_1 = \theta, x_2 = \dot{\theta}, x_3 = y, x_4 = \dot{y}$ Rewrite them in matrix form:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1 l_1^2 + k_2 l_2^2)}{I} & 0 & \frac{k_2 l_2}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2 l_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{l_2}{I} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Take the lapalace transform to both sides of equations:

$$m_2 s^2 \hat{y}(s) = k_2 l_2 \theta(s) - k_2 \hat{y}(s)$$

$$I s^2 \hat{\theta}(s) = \hat{u}(s) l_2 - k_1 l_1^2 \hat{\theta}(s) - k_2 l_2^2 \hat{\theta}(s) + k_2 l_2 \hat{y}(s)$$

arrange them:

$$\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{k_2 l_2^2}{m_2 I s^4 + (k_2 I + (k_1 l_1^2 + k_2 l_2^2) m_2) s^2 + k_1 k_2 l_1^2}$$