Chapter4

31202008881 Bao Ze an

Monday 28th December, 2020

4.1

The system is a LTI system,so the solution can be obtained by:$\mathcal{L}^{-1}(SI-A)^{-1}$

$$(SI-A)^{-1}=\begin{bmatrix} S & -1 \\ 1 & S \end{bmatrix}^{-1}=\begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$$

$$\mathcal{L}^{-1}(SI-A)^{-1}=\begin{bmatrix} cost & sint \\ -sint & cost \end{bmatrix}$$

so the solution is:

$$x(t)=\begin{bmatrix} cost & sint \\ -sint & cost \end{bmatrix}x(0)$$

4.2

The first method: $G(s)=C(SI-A)^{-1}B+D$

$$G(s)=\begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} \frac{s+2}{s^2+s2+2} & \frac{1}{s^2+2s+2} \\ \frac{-2}{s^2+2s+2} & \frac{s}{s^2+2s+2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\frac{5s}{s^2+2s+2}$$

so the unit-step response is:$y(s)=G(s)u(s)=\frac{5s}{s^2+2s+2}\cdot\frac{1}{s}=\frac{5}{(s+1)^2+1}$

$$y(t)=\mathcal{L}^{-1}(y(s))=5e^{-t}sint$$

The second method:calculate the solution of $x(t)$,then we can simply get the response by $y(t)=Cx(t)+Du(t)$

$$x(t)=e^{At}x(0)+\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

because of zero initial state,so:

$$y(t)=C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

from the first method:

$$(SI-A)^{-1}\begin{bmatrix} \frac{s+2}{s^2+s2+2} & \frac{1}{s^2+2s+2} \\ \frac{-2}{s^2+2s+2} & \frac{s}{s^2+2s+2} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-t}(cost + sint) & e^{-t}sint \\ -2e^{-t}sint & e^{-t}(cost - sint) \end{bmatrix}$$

take a substitution,we can get:

$$y(t) = \int_0^t 5e^{-(t-\tau)}[cos(t-\tau) - sin(t-\tau)]d\tau = 5e^{-t}sint$$

5.3

Let $u(t)$ is piecewise constant,that is to say,the input changes values only at discrete-time instants. we get the discrete-time equation without the approximation:

$$x[k+1] = A_d x[k] + B_d u[k]$$
$$y[k] = C_d x[k] + D_d u[k]$$

where $A_d = e^{AT}, B_d = (\int_0^T e^{A\tau}d\tau)B, C_d = C, D_d = D$ when $T = 1$:

$$A_d = e^{A} = \begin{bmatrix} e^{-1}(cos1 + sin1) & e^{-1}sin1 \\ -2e^{-1}sin1 & e^{-1}(cos1 - sin1) \end{bmatrix}$$

Because A is nonsingular, so we can compute the

$$B_d = A^{-1}(A_d - I)B = \begin{bmatrix} 1.0491 \\ -0.1821 \end{bmatrix}$$

$$C_d = C = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

thus the discrete-time equation:

$$x[k+1] = \begin{bmatrix} e^{-1}(cos1 + sin1) & e^{-1}sin1 \\ -2e^{-1}sin1 & e^{-1}(cos1 - sin1) \end{bmatrix} x[k] + \begin{bmatrix} 1.0491 \\ -0.1821 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} 2 & 3 \end{bmatrix} x[k]$$

in the same way,for $T = \pi$:

$$x[k+1] = \begin{bmatrix} -0.0432 & 0 \\ 0 & -0.0432 \end{bmatrix} x[k] + \begin{bmatrix} 1.5648 \\ -1.0432 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} 2 & 3 \end{bmatrix} x[k]$$

4.4

For companion form,we can compute the tansformation matrix:

$$Q = \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & -4 \\ 1 & -2 & 0 \end{bmatrix}$$

$$\dot{\overline{x}} = Q^{-1}AQ\overline{x} + Q^{-1}bu$$

$$y = CQ\overline{x}$$

substitue with Q, A, B, C,we can get:

$$\dot{\overline{x}} = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -6 \\ 0 & 1 & -4 \end{bmatrix} \overline{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & -4 & 8 \end{bmatrix} \overline{x}$$

For modal form: The characteristic polynomial of A,

$$det(SI - A) = (s+2)(s^2 + 2s + 2)$$

The eigenvalues of A are:-2,-1+j,-1-j, their corresponding eigenvectors are q_1, q_2, q_3

$$q_1 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad q_2 = \begin{bmatrix} 0 \\ 0.5774j \\ -0.5774 - 0.5774j \end{bmatrix} \quad q_3 = \begin{bmatrix} 0 \\ -0.5774j \\ -0.5774 + 0.5774j \end{bmatrix}$$

so,the transformation matrix is:

$$Q = \begin{bmatrix} q_1 & Re(q_2) & Im(q_2) \end{bmatrix} = \begin{bmatrix} 0.7071 & 0 & 0 \\ 0 & 0 & 0.5774 \\ -0.7071 & -0.5774 & -0.5774 \end{bmatrix}$$

we can get the modal form by:

$$\dot{\overline{x}} = Q^{-1}AQ\overline{x} + Q^{-1}bu$$

$$y = CQ\overline{x}$$

thus:

$$\dot{\overline{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \overline{x} + \begin{bmatrix} -3.4638 \\ 0 \\ 1.4138 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & -0.5774 & 0.7071 \end{bmatrix} \overline{x}$$

4.5

For unit step input ,we can get $|y|_{max} = 0.55, |x_1|_{max} = 0.5, |x_2|_{max} = 1.05, |x_3|_{max} = 0.52$. Define $\bar{x}_1 = x_1, \bar{x}_2 = 0.5x_2, \bar{x}_3 = x_3$, so

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \dot{\bar{\boldsymbol{x}}} &= PAP^{-1}\bar{\boldsymbol{x}} + Pbu \\ &= \begin{bmatrix} -2 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & -4 & -2 \end{bmatrix} \bar{\boldsymbol{x}} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u \\ y &= cP^{-1}\bar{\boldsymbol{x}} = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} \bar{\boldsymbol{x}} \end{aligned}$$

The largest permissible a is $\frac{10}{0.55} = 18.2$

4.6

$$\begin{aligned}
\dot{\bar{\boldsymbol{x}}} &= Q^{-1} \left[\begin{array}{cc} \lambda & 0 \\ 0 & \bar{\lambda} \end{array} \right] Q\bar{\boldsymbol{x}} + Q^{-1} \left[\begin{array}{c} b_1 \\ b_2 \end{array} \right] u \\
&= \tfrac{1}{(\lambda-\bar{\lambda})b_1\bar{b}_1} \left[\begin{array}{cc} \bar{b}_1 & -b_1 \\ \lambda\bar{b}_1 & -\bar{\lambda}b_1 \end{array} \right] \left[\begin{array}{cc} \lambda & 0 \\ 0 & \bar{\lambda} \end{array} \right] \left[\begin{array}{cc} -\bar{\lambda}b_1 & b_1 \\ -\lambda\bar{b}_1 & \bar{b}_1 \end{array} \right] \bar{\boldsymbol{x}} + \tfrac{1}{(\lambda-\bar{\lambda})b_1\bar{b}_1} \left[\begin{array}{cc} \bar{b}_1 & -b_1 \\ \lambda\bar{b}_1 & -\bar{\lambda}b_1 \end{array} \right] \left[\begin{array}{c} b_1 \\ b_2 \end{array} \right] u \\
&= \tfrac{1}{(\lambda-\bar{\lambda})b_1\bar{b}_1} \left[\begin{array}{cc} \bar{b}_1\lambda & -b_1\bar{\lambda} \\ \lambda^2\bar{b}_1 & -\bar{\lambda}^2 b_1 \end{array} \right] \left[\begin{array}{cc} -\bar{\lambda}b_1 & b_1 \\ -\lambda\bar{b}_1 & \bar{b}_1 \end{array} \right] \bar{\boldsymbol{x}} + \tfrac{1}{(\lambda-\bar{\lambda})b_1\bar{b}_1} \left[\begin{array}{c} 0 \\ (\lambda-\bar{\lambda})b_1\bar{b}_1 \end{array} \right] u \\
&= \left[\begin{array}{cc} 0 & 1 \\ -\lambda\bar{\lambda} & \lambda+\bar{\lambda} \end{array} \right] \bar{\boldsymbol{x}} + \left[\begin{array}{c} 0 \\ 1 \end{array} \right] u = \bar{A}\bar{\boldsymbol{x}} + \bar{B}u \\
y &= \left[\begin{array}{cc} c_1 & \bar{c}_1 \end{array} \right] Q\bar{\boldsymbol{x}} = \left[\begin{array}{cc} c_1 & \bar{c}_1 \end{array} \right] \left[\begin{array}{cc} -\bar{\lambda}b_1 & b_1 \\ -\lambda\bar{b}_1 & \bar{b}_1 \end{array} \right] \bar{\boldsymbol{x}} = \left[\begin{array}{cc} -\bar{\lambda}b_1c_1 - \lambda\bar{b}_1\bar{c}_1 & c_1b_1 + \bar{c}_1\bar{b}_1 \end{array} \right] \bar{\boldsymbol{x}} = \bar{C}_1\bar{\boldsymbol{x}}
\end{aligned}$$

4.7

Change the order of the state variables from $\begin{bmatrix} \dot{x_1} & \dot{x_2} & \dot{x_3} & \dot{x_4} & \dot{x_5} & \dot{x_6} \end{bmatrix}'$ to $\begin{bmatrix} \dot{x_1} & \dot{x_4} & \dot{x_2} & \dot{x_5} & \dot{x_3} & \dot{x_6} \end{bmatrix}'$ we can get

$$\dot{\bar{\boldsymbol{x}}} = \begin{bmatrix} \dot{x_1} \\ \dot{x_4} \\ \dot{x_2} \\ \dot{x_5} \\ \dot{x_3} \\ \dot{x_6} \end{bmatrix} = \begin{bmatrix} \lambda & & 1 & & & \\ & \bar{\lambda} & & 1 & & \\ & & \lambda & & 1 & \\ & & & \bar{\lambda} & & 1 \\ & & & & \lambda & \\ & & & & & \bar{\lambda} \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \\ x_2 \\ x_5 \\ x_3 \\ x_6 \end{bmatrix} + \begin{bmatrix} b_1 \\ \bar{b}_1 \\ b_2 \\ \bar{b}_2 \\ b_3 \\ \bar{b}_3 \end{bmatrix} = \begin{bmatrix} \bar{A} & I_2 & 0 \\ 0 & \bar{A} & I_2 \\ 0 & 0 & \bar{A} \end{bmatrix} \bar{\boldsymbol{x}} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{bmatrix} u$$

$$y = \left[\begin{array}{cccccc} c_1 & \bar{c}_1 & c_2 & \bar{c}_2 & c_3 & \bar{c}_3 \end{array} \right] \bar{\boldsymbol{x}} = \left[\begin{array}{ccc} \bar{C}_1 & \bar{C}_2 & \bar{C}_3 \end{array} \right] \bar{\boldsymbol{x}}$$

4.8

The eigenvalues of the first equation are $2, 2, 1$, and eigenvalues of the second equation are $2, 2, -1$. So they are not equivalent.

$$\hat{G}_1(s) = C_1(sI - A_1)^{-1}B_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} s-2 & -1 & -2 \\ 0 & s-2 & -2 \\ 0 & 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}}{(s-2)^2(s-1)} \begin{bmatrix} (s-2)(s-1) & (s-1) & 2(s-1) \\ 0 & (s-2)(s-1) & 2(s-2) \\ 0 & 0 & (s-2)^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(s-2)^2}$$

$$\hat{G}_2(s) = C_2(sI - A_2)^{-1}B_2 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} s-2 & -1 & -1 \\ 0 & s-2 & -1 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}}{(s-2)^2(s+1)} \begin{bmatrix} (s-2)(s+1) & (s+1) & (s-1) \\ 0 & (s-2)(s+1) & (s-2) \\ 0 & 0 & (s-2)^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(s-2)^2}$$

$\hat{G}_1(s) = \hat{G}_2(s)$, so they are zero-state equivalent.

4.9

Let us define

$$Z := \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_r \end{bmatrix} := C(sI - A)^{-1}$$

where Z_i is $q \times q$ and Z is $q \times rq$. Then the transfer matrix of (4.34) equals

$$C(sI - A)^{-1}B = N_1 Z_1 + N_2 Z_2 + \cdots + N_r Z_r$$

$$Z = C(sI - A)^{-1} \quad \Rightarrow \quad ZA = sZ - C$$

Then

$$sZ - I_q = -\alpha_1 Z_1 - \alpha_2 Z_2 - \cdots - -\alpha_r Z_r$$

$$Z_1 = sZ_2 \quad \Rightarrow \quad Z_2 = \frac{1}{s} Z_1$$

$$Z_2 = sZ_3 \quad \Rightarrow \quad Z_3 = \frac{1}{s^2} Z_1$$

$$\vdots$$

$$Z_{r-1} = sZ_r \quad \Rightarrow \quad Z_r = \frac{1}{s^{r-1}} Z_1$$

Then

$$sZ_1 - I_q = (-\alpha_1 - \frac{1}{s}\alpha_2 - \cdots - \frac{1}{s^{r-1}}\alpha_r)Z_1$$
$$\Downarrow$$
$$Z_1 = \frac{s^{r-1}}{d(s)}I_q \quad Z_2 = \frac{s^{r-2}}{d(s)}I_q \quad Z_r = \frac{1}{d(s)}I_q$$
$$\Downarrow$$
$$\begin{aligned} C(sI-A)^{-1}B &= N_1Z_1 + N_2Z_2 + \cdots + N_rZ_r \\ &= \frac{1}{d(s)}\begin{bmatrix} N_1 s^{r-1} & N_2 s^{r-2} & \cdots & N_r \end{bmatrix} \\ &= \hat{G}_{sp}(s) \end{aligned}$$

So this is a realization of $\hat{G}_{sp}(s)$.

4.10

In this case,$r = 4, q = 1$, then

$$\begin{aligned} I_q &= 1 \\ N_1 &= \begin{bmatrix} \beta_{11} & \beta_{12} \end{bmatrix} \\ N_2 &= \begin{bmatrix} \beta_{21} & \beta_{22} \end{bmatrix} \\ N_3 &= \begin{bmatrix} \beta_{31} & \beta_{32} \end{bmatrix} \\ N_4 &= \begin{bmatrix} \beta_{41} & \beta_{42} \end{bmatrix} \end{aligned}$$

We can get the conclusion.

4.11

$$\begin{aligned} \hat{G}(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{2s-3}{(s+1)(s+2)} \\ \frac{s-2}{s+1} & \frac{s}{s+2} \end{bmatrix} &= \begin{bmatrix} \frac{2}{s+1} & \frac{2s-3}{(s+1)(s+2)} \\ \frac{-3}{s+1} & \frac{-2}{s+2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{s^2+3s+2}\left[s\begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix} + \begin{bmatrix} 4 & -3 \\ -6 & -2 \end{bmatrix} \right] + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

so the realization is

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -3 & 0 & -2 & 0 \\ 0 & -3 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\boldsymbol{u}$$

$$\boldsymbol{y} = \begin{bmatrix} 2 & 2 & 4 & -3 \\ -3 & -2 & -6 & -2 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\boldsymbol{u}$$

4.12

$$\hat{G}_1(s) = \frac{1}{s+1}\begin{bmatrix} 2 \\ s-2 \end{bmatrix} = \frac{1}{s+1}\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\dot{\boldsymbol{x}}_1 = -\boldsymbol{x}_1 + u_1$$

$$\boldsymbol{y}_{c_1} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}\boldsymbol{x}_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1$$

$$\hat{G}_2(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} 2s-3 \\ -2s-2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(s+1)(s+2)}\left[s\begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix}\right] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\dot{\boldsymbol{x}}_2 = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}\boldsymbol{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2$$

$$\boldsymbol{y}_{c_2} = \begin{bmatrix} 2 & -3 \\ -2 & -2 \end{bmatrix}\boldsymbol{x}_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2$$

Then we can get

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & -2 \\ 0 & 1 & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{u}$$

$$\boldsymbol{y} = \begin{bmatrix} 2 & 2 & -3 \\ -3 & -2 & -2 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\boldsymbol{u}$$

The dimension of this realization is 3, and dimension in Problem 4.11 is 4.

4.13

$$\hat{G}_1(s) = \frac{1}{(s+1)(s+2)}[2s+4 \quad 2s-3] = \frac{1}{s^2+3s+2}\left[s\begin{bmatrix} 2 & 2 \end{bmatrix} + \begin{bmatrix} 4 & -3 \end{bmatrix}\right]$$

$$\dot{\boldsymbol{x}}_1 = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}\boldsymbol{x}_1 + \begin{bmatrix} 2 & 2 \\ 4 & -3 \end{bmatrix}\boldsymbol{u}_1$$

$$y_{c_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}_1 + \begin{bmatrix} 0 & 0 \end{bmatrix}\boldsymbol{u}_1$$

$$\hat{G}_2(s) = \frac{1}{(s+1)(s+2)}[-3(s+2) - 2(s+1)] + \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)}\left[s\begin{bmatrix} -3 & -2 \end{bmatrix} + \begin{bmatrix} -6 & -2 \end{bmatrix}\right] + \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\dot{\boldsymbol{x}}_2 = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}\boldsymbol{x}_2 + \begin{bmatrix} -3 & -2 \\ -6 & -2 \end{bmatrix}\boldsymbol{u}_2$$

$$y_{c_2} = \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}_2 + \begin{bmatrix} 1 & 1 \end{bmatrix}\boldsymbol{u}_2$$

Then we can get

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 2 & 2 \\ 4 & -3 \\ -3 & -2 \\ -6 & -2 \end{bmatrix}\boldsymbol{u}$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\boldsymbol{u}$$

The dimension of this realization is 4, which equal to dimension in Problem 4.11 and one more then dimension in Problem 4.12.

4.14

$$\hat{G}(s) = \begin{bmatrix} \frac{-(12s+6)}{3s+34} & \frac{22s+23}{3s+34} \end{bmatrix} = \begin{bmatrix} -4 & \frac{22}{3} \end{bmatrix} + \frac{1}{s+\frac{34}{3}}\begin{bmatrix} \frac{130}{3} & -\frac{679}{9} \end{bmatrix}$$

$$\dot{x} = -\frac{34}{3}x + \begin{bmatrix} \frac{130}{3} & -\frac{679}{9} \end{bmatrix} U$$

$$y = x + \begin{bmatrix} -4 & \frac{22}{3} \end{bmatrix} U$$

4.15

$$\hat{g}(s) = c(sI-A)^{-1}b = \frac{cR_0bs^{n-1} + cR_1bs^{n-2} + ... + cR_{n-2}bs + R_{n-1}}{\Delta s}$$

The numerator of $\hat{g}(s)$ has degree m if and only if $cR_{n-m-1}b \neq 0$ and $cR_i b = 0 \quad for \quad i = 1, 2, ..., n-m-2$.

$$\begin{aligned}
cR_0 b &= cb = 0 \\
cR_1 b &= cAb + \alpha_1 cb = 0 \quad \Rightarrow cAb = 0 \\
cR_2 b &= cA^2 b + \alpha_2 cR_1 b = 0 \quad \Rightarrow cA^2 b = 0 \\
&\vdots \\
cR_{n-m-2} b &= cA^{n-m-2} b + cR_{n-m-3} b = 0 \quad \Rightarrow cA^{n-m-2} b = 0 \\
cR_{n-m-1} b &= cA^{n-m-1} b + cR_{n-m-2} b \neq 0 \quad \Rightarrow cA^{n-m-1} b \neq 0
\end{aligned}$$

Then $\hat{g}(s)$ has m zeros if and only if $cR_{n-m-1}b \neq 0$ and $cR_i b = 0 \quad for \quad i = 1, 2, ..., n-m-2$.

4.16

(1)

$$\dot{x}_1 = x_2 \Rightarrow x_1(t) = \int_0^t x_2(t)dt + x_1(0)$$
$$\dot{x}_2 = tx_2 \Rightarrow x_2(t) = x_2(0)e^{0.5t^2}$$

We have

$$X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow X(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow X(t) = \begin{bmatrix} \int_0^t e^{0.5\tau^2 d\tau} \\ e^{0.5t^2} \end{bmatrix}$$

Then the fundamental matrix and transition matrix are

$$X(t) = \begin{bmatrix} 1 & \int_0^t e^{0.5\tau^2 d\tau} \\ 0 & e^{0.5t^2} \end{bmatrix}$$
$$\Phi(t, t_0) = \begin{bmatrix} 1 & \int_0^t e^{0.5\tau^2 d\tau} \\ 0 & e^{0.5t^2} \end{bmatrix} \begin{bmatrix} 1 & \int_0^{t_0} e^{0.5\tau^2 d\tau} \\ 0 & e^{0.5t_0^2} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 1 & \int_0^t e^{0.5\tau^2 d\tau} \\ 0 & e^{0.5t^2} \end{bmatrix} \begin{bmatrix} 1 & -e^{-0.5t_0^2} \int_0^{t_0} e^{0.5\tau^2 d\tau} \\ 0 & e^{-0.5t_0^2} \end{bmatrix} = \begin{bmatrix} 1 & e^{-0.5t_0^2 d\tau} \int_{t_0}^t e^{0.5\tau^2 d\tau} \\ 0 & e^{0.5\left(t^2 - t_0^2\right)} \end{bmatrix}$$

(2)

$$\dot{x}_1 = -x_1 + e^{2t}x_2(t) = -x_1 + e^t x_2(0) \Rightarrow x_1(t) = 0.5x_2(0)\left(e^t - e^{-t}\right) + x_1(0)e^{-t}$$
$$\dot{x}_2 = -x_2 \Rightarrow x_2(t) = x_2(0)e^{-t}$$

Then

$$X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow X(t) = \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}$$
$$X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow X(t) = \begin{bmatrix} 0.5\left(e^t - e^{-t}\right) \\ e^{-t} \end{bmatrix}$$

Then the fundamental matrix and transition matrix are

$$X(t) = \begin{bmatrix} e^{-t} & 0.5\left(e^t - e^{-t}\right) \\ 0 & e^{-t} \end{bmatrix}$$
$$\Phi\left(t, t_0\right) = \begin{bmatrix} e^{-t} & 0.5\left(e^t - e^{-t}\right) \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} e^{-t_0} & 0.5\left(e^{t_0} - e^{-t_0}\right) \\ 0 & e^{-t_0} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} e^{t_0-t} & 0.5e^{-t}\left(e^{t_0} - e^{-3t_0}\right) + 0.5\left(e^t - e^{-t}\right)e^{t_0} \\ 0 & e^{t_0-t} \end{bmatrix}$$

4.17

$$\frac{\partial}{\partial t}X^{-1}(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(X(t)X^{-1}(t)) = \dot{X}(t)X^{-1}(t) + X(t)\frac{\mathrm{d}}{\mathrm{d}t}X^{-1}(t) = 0$$
$$\Downarrow$$
$$\frac{\mathrm{d}}{\mathrm{d}t}X^{-1}(t) = -X^{-1}(t)\dot{X}(t)X^{-1}(t) = -X^{-1}(t)A(t)X(t)X^{-1}(t) = -X^{-1}(t)A(t)$$

Then we have

$$\frac{\partial}{\partial t}\Phi(t_0, t) = -X(t_0)X^{-1}(t)A(t) = -\Phi(t_0, t)A(t)$$

4.18

$$\frac{\partial}{\partial t}\Phi(t, t_0) = \begin{bmatrix} \frac{\partial}{\partial t}\phi_{11} & \frac{\partial}{\partial t}\phi_{12} \\ \frac{\partial}{\partial t}\phi_{21} & \frac{\partial}{\partial t}\phi_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial t}(\det\Phi) &= \frac{\partial}{\partial t}\left(\phi_{11}\phi_{22} - \phi_{21}\phi_{12}\right) \\
&= \frac{\partial}{\partial t}\phi_{11}\phi_{22} - \frac{\partial}{\partial t}\phi_{21}\phi_{12} + \phi_{11}\frac{\partial}{\partial t}\phi_{22} - \phi_{21}\frac{\partial}{\partial t}\phi_{12} \\
&= \left(a_{11}\phi_{11} + a_{12}\phi_{21}\right)\phi_{22} - \left(a_{11}\phi_{12} + a_{12}\phi_{22}\right)\phi_{21} + \phi_{11}\left(a_{21}\phi_{12} + a_{22}\phi_{22}\right) - \phi_{12}\left(a_{21}\phi_{11} + a_{22}\phi_{21}\right) \\
&= \left(a_{11} + a_{22}\right)\left(\phi_{11}\phi_{22} - \phi_{21}\phi_{12}\right) \\
&= \left(a_{11} + a_{22}\right)\det\Phi
\end{aligned}$$

Then $\det\Phi\left(t,t_0\right) = ce^{\int_{t_0}^{t}\left(a_{11}(\tau)+a_{22}(\tau)\right)d\tau}$ and $\Phi\left(t_0,t_0\right) = I$, so we can get $c = 1, \det\Phi\left(t,t_0\right) = e^{\int_{t_0}^{t}\left(a_{11}(\tau)+a_{22}(\tau)\right)d\tau}$

4.19

$$\Phi(t_0,t_0) = \begin{bmatrix} \phi_{11}(t_0,t_0) & \phi_{12}(t_0,t_0) \\ \phi_{21}(t_0,t_0) & \phi_{22}(t_0,t_0) \end{bmatrix} = I$$

Then $\phi_{21}(t_0,t_0) = 0, \phi_{22}(t_0,t_0) = I$.

$$\frac{\partial}{\partial t}\Phi(t,t_0) = \begin{bmatrix} \frac{\partial}{\partial t}\phi_{11}(t,t_0) & \frac{\partial}{\partial t}\phi_{12}(t,t_0) \\ \frac{\partial}{\partial t}\phi_{21}(t,t_0) & \frac{\partial}{\partial t}\phi_{22}(t,t_0) \end{bmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix}\begin{bmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) \\ \phi_{21}(t,t_0) & \phi_{22}(t,t_0) \end{bmatrix}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial t}\phi_{11}(t,t_0) &= A_{11}(t)\phi_{11}(t,t_0) + A_{12}(t)\phi_{21}(t,t_0) \\
\frac{\partial}{\partial t}\phi_{21}(t,t_0) &= A_{22}(t)\phi_{21}(t,t_0) \\
\frac{\partial}{\partial t}\phi_{22}(t,t_0) &= A_{22}(t)\phi_{22}(t,t_0)
\end{aligned}$$

The equation $\frac{\partial}{\partial t}\phi_{22}(t,t_0) = A_{22}(t)\phi_{22}(t,t_0)$ with $\phi_{22}(t_0,t_0) = I$ has the unique solution $\phi_{22}(t,t_0)$. The equation $\frac{\partial}{\partial t}\phi_{21}(t,t_0) = A_{22}(t)\phi_{21}(t,t_0)$ with $\phi_{21}(t_0,t_0) = 0$ has the unique solution $\phi_{21}(t,t_0) \equiv 0$. So $\frac{\partial}{\partial t}\phi_{11}(t,t_0) = A_{11}(t)\phi_{11}(t,t_0) + A_{12}(t)\phi_{21}(t,t_0) = A_{11}(t)\phi_{11}(t,t_0)$

4.20

$$\begin{aligned}
\dot{x}_1 &= -\sin t x_1 \Rightarrow x_1(t) = x_1(0)e^{\cos t} \\
\dot{x}_2 &= -\cos t x_2 \Rightarrow x_2(t) = x_2(0)e^{-\sin t}
\end{aligned}$$

then we have

$$X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow X(t) = \begin{bmatrix} e^{\cos t} \\ 0 \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow X(t) = \begin{bmatrix} 0 \\ e^{-\sin t} \end{bmatrix}$$

Then the fundamental matrix is

$$X(t) = \begin{bmatrix} e^{\cos t} & 0 \\ 0 & e^{-\sin t} \end{bmatrix}$$

So the state transition matrix is

$$\Phi\left(t, t_0\right) = \begin{bmatrix} e^{\cos t} & 0 \\ 0 & e^{-\sin t} \end{bmatrix} \begin{bmatrix} e^{-\cos t_0} & 0 \\ 0 & e^{\sin t_0} \end{bmatrix} = \begin{bmatrix} e^{\cos t - \cos t_0} & 0 \\ 0 & e^{-\sin t + \sin t_0} \end{bmatrix}$$

4.21

Because $\dot{X}(t) = Ae^{At}Ce^{Bt} + e^{At}Ce^{Bt}B = AX + XB$ and $X(0) = e^{A\dot{0}}Ce^{B\dot{0}} = C$,so $X(t) = e^{At}Ce^{Bt}$ is the solution.

4.22

We can get the solution of $\dot{A}(t) = A_1A(t) + A(t)(-A_1)$ is $A(t) = e^{A_1t}A(0)e^{-A_1t}$ from the conclusion of Problem 4.20.

$$\begin{aligned} det(\lambda I - A(t)) &= det(e^{A_1t}\lambda Ie^{-A_1t} - A(t)) \\ &= det(e^{A_1t}(\lambda I - A(0))e^{-A_1t}) \\ &= dete^{A_1t}dete^{-A_1t}det(\lambda I - A(0)) \\ &= det(\lambda I - A(0)) \end{aligned}$$

So the eigenvalues of $A(t)$ are incdependent of t.

4.23

Define $T = 2\pi$, then

$$X(t) = \begin{bmatrix} e^{\cos t} & 0 \\ 0 & e^{-\sin t} \end{bmatrix}, X(t+T) = \begin{bmatrix} e^{\cos(t+2\pi)} & 0 \\ 0 & e^{-\sin(t+2\pi)} = X(t) \end{bmatrix}$$

$$X(t+T) = X(t)e^{\bar{A}t} \Rightarrow \bar{A} = 0$$

Then

$$P(t) = e^{\bar{A}t}X^{-1}(t) = \begin{bmatrix} e^{-\cos t} & 0 \\ 0 & e^{\sin t} \end{bmatrix}$$

Let $\bar{\boldsymbol{x}} = P(t)\boldsymbol{x}$, then the state equation is $\dot{\bar{\boldsymbol{x}}} = \bar{A}\boldsymbol{x} = 0$.

4.24

$$X(t) = e^{At} \quad \Rightarrow X^{-1}(t) = e^{-At}$$

Let $P(t) = e^{\bar{A}t}X^{-1}(t) = X^{-1}(t) = e^{-At}$ and $\bar{\boldsymbol{x}} = P(t)\boldsymbol{x}$, then we can get

$$\bar{A}(t) = 0 \qquad \bar{B}(t) = e^{-At}B \qquad \bar{C}(t) = Ce^{At}$$

4.25

time-varying realization:

$$g(t-\tau) = (t-\tau)^2 e^{\lambda(t-\tau)} = \begin{bmatrix} t^2e^{\lambda t} & -2te^{\lambda t} & e^{\lambda t} \end{bmatrix} \begin{bmatrix} e^{-\lambda\tau} \\ \tau e^{-\lambda\tau} \\ \tau^2 e^{-\lambda\tau} \end{bmatrix}$$

Thus a time-varying realization is

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} e^{-\lambda t} \\ te^{-\lambda t} \\ t^2e^{-\lambda t} \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} t^2e^{\lambda t} & -2te^{\lambda t} & e^{\lambda t} \end{bmatrix} \boldsymbol{x}$$

time-invariant realization: The Laplace transform of the impluse response is

$$\hat{g}(s) = \frac{2}{(s-\lambda)^2} = \frac{2}{s^3 - 3\lambda s^2 + 3\lambda^2 s + \lambda^3}$$

Then we can get

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 3\lambda & -3\lambda^2 & \lambda^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} \boldsymbol{x}$$

4.26

$$g(t,\tau) = sinte^{-t}e^{\tau}cos\tau$$

Thus a time-varying realization is

$$\dot{x} = 0x + e^t costu$$

$$y = sinte^{-t}u$$

Because $g(t,\tau)$ can not be expended as $g(t-\tau)$, it is not possible to find a time-invariant state equation realization.