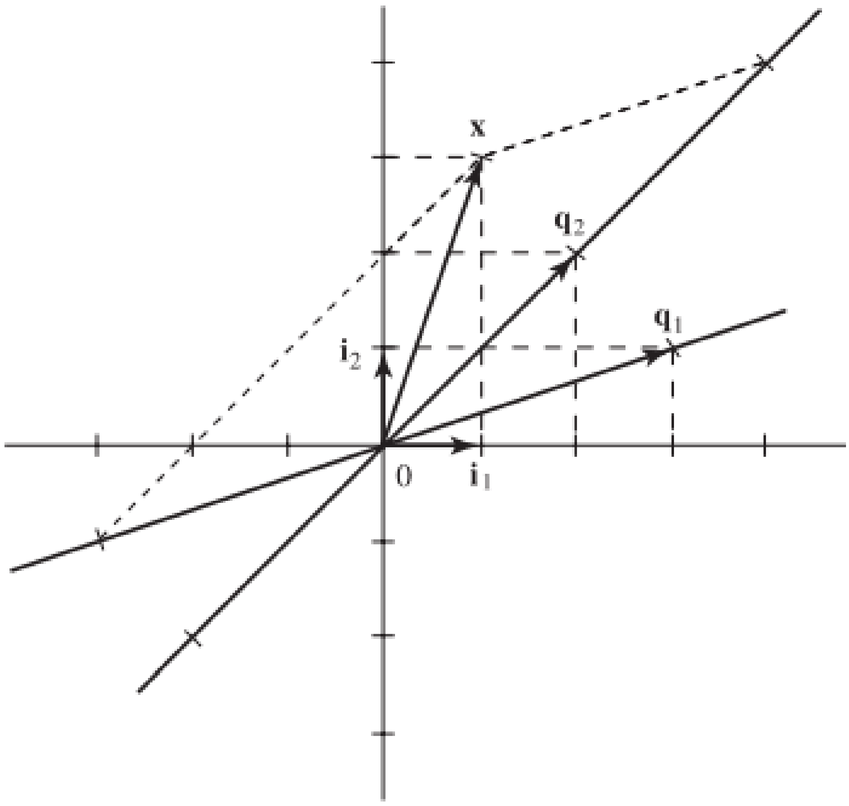
Chapter2

31202008881 Bao Ze an

Friday 27th November, 2020

3.1



From the above figure, The three vectors $\boldsymbol{q}_1 = \begin{bmatrix} 3 & 1 \end{bmatrix}'$, $\boldsymbol{i}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}'$ and $\boldsymbol{q}_2 = \begin{bmatrix} 2 & 2 \end{bmatrix}'$
The representation of x with respect to $\{\boldsymbol{q}_1, \boldsymbol{i}_2\}$ is $\begin{bmatrix} \frac{1}{3} & \frac{8}{3} \end{bmatrix}'$
The representation of $\boldsymbol{q}_1$ with respect to $\{\boldsymbol{i}_2, \boldsymbol{q}_2\}$ is $\begin{bmatrix} -2 & \frac{3}{2} \end{bmatrix}'$

These can be verified like this:

$$x=\left[\begin{array}{l}1 \\ 3\end{array}\right]=\left[\begin{array}{ll}\boldsymbol{q}_{1} & \boldsymbol{i}_{2}\end{array}\right]\left[\begin{array}{c}\frac{1}{3} \\ \frac{8}{3}\end{array}\right]=\left[\begin{array}{ll}3 & 0 \\ 1 & 1\end{array}\right]\left[\begin{array}{c}\frac{1}{3} \\ \frac{8}{3}\end{array}\right]$$

3.2

i:The norm of x_1

1-norm: $\|x_1\|_1=\sum_{i=1}^{3}|x_i|=|2|+|-3|+|1|=6$
2 -norm: $\|\boldsymbol{x}_1\|_2=\left(\sum_{i=1}^{3}|x_i|^2\right)^{\frac{1}{2}}=\sqrt{2^2+|-3|^2+1^2}=\sqrt{14}$
infinite-norm: $\|\boldsymbol{x}_1\|_\infty=\max_i|x_i|=3$

ii:The norm of x_2

1 -norm: $\|\boldsymbol{x}_2\|_1=\sum_{i=1}^{3}|x_i|=|1|+|1|+|1|=3$
2-norm: $\|\boldsymbol{x}_2\|_2=\left(\sum_{i=1}^{3}|x_i|^2\right)^{\frac{1}{2}}=\sqrt{1^2+1^2+1^2}=\sqrt{3}$
infinite-norm: $\|\boldsymbol{x}_2\|_\infty=\max_i|x_i|=1$

3.3

This is just the orthonormalization procedure.

$$\left\{\begin{array}{ll}u_1=\alpha_1 & q_1=u_1/\|u_1\| \\ u_2=\alpha_2-\left(q_1'\alpha_2\right)q_1 & q_2=u_2/\|u_2\|\end{array}\right.$$

This is the ordinary method,what we find is the two vector are orthogonal. so,we just need to make the length of vector is 1 .

$$q_1=\frac{u_1}{\|u_1\|}=\left[\begin{array}{lll}\frac{2}{\sqrt{14}} & -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}}\end{array}\right]'$$
$$q_1=\frac{u_2}{\|u_2\|}=\left[\begin{array}{lll}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{array}\right]'$$

3.4

a

if n>m,$\boldsymbol{AA}'$ is a ordinary vector,which has the rank m

b

if m=n,so $\boldsymbol{A}$ is a nonsingular square matrix,we already have $\boldsymbol{A}^{'}\boldsymbol{A} = \boldsymbol{I}_m$,so $\boldsymbol{A}' = \boldsymbol{A}^{-1}$. $\boldsymbol{A}\boldsymbol{A}^{'} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$

3.5

Accoridng to the principle:

$$Nillity(\boldsymbol{A}) = number of columns of \boldsymbol{A} - rank(A)$$

i:

Rank $(A_1) = 2$
Nullity $(A_1) = 3 - 2 = 1$

ii:

Rank $(A_2) = 3$
Nullity $(A_2) = 3 - 3 = 0$

iii:

Rank $(A_3) = 3$
Nullity $(A_3) = 4 - 3$

3.6

For A_1:

$$\boldsymbol{a_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{a_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The set of $\{\boldsymbol{a_1}, \boldsymbol{a_2}\}$ can be the basis of the range spaces.
The independent vectors of null space can get by solving the equation:

$$\boldsymbol{A}_1\boldsymbol{\eta}_i = 0$$

$$\boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The set of $\boldsymbol{\eta}_1$ is the basis of the null space
in the same way,we can get the basis of the range space and null space of $\boldsymbol{A}_2$

$$\boldsymbol{a_1} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \boldsymbol{a_2} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \boldsymbol{a_3} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

The set of $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$ is the basis of the range space.
because the $\boldsymbol{A}_2$ is full rank,so the basis of the null space is $\{\mathbf{0}\}$

3.7

$$\rho(\boldsymbol{A}) = \rho([\boldsymbol{A} \quad y]) = 2$$

so a solution $\boldsymbol{x}$ exist with respect to this equation.
Because coefficient matrix is full column rank,so the solution is unique.
if y=$[1 \quad 1 \quad 1]'$,$\rho(\boldsymbol{A}) = 2 \neq \rho([\boldsymbol{A} \quad y]) = 3$ so,when $\boldsymbol{y}$=$[1 \quad 1 \quad 1]'$,the solution is not exist.

3.8

$\boldsymbol{x}_p = [0 \quad -2 \quad 1 \quad 1]'$ is a solution, a basis of the null space of A:

$$\boldsymbol{A}\boldsymbol{\eta}_i = 0$$

$$\boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Thus the gengeral solution can be expressed as:

$$\boldsymbol{x} = \boldsymbol{x}_p + \alpha_1 \boldsymbol{\eta}_1$$

3.9

From the example 3.3,we can know the gengeral solution is:

$$\boldsymbol{x} = \begin{bmatrix} \alpha_1 \\ -4 + \alpha_1 + 2\alpha_2 \\ -\alpha_1 \\ -\alpha_2 \end{bmatrix}$$

$$\|\boldsymbol{x}\|_2 = \sqrt{\alpha_1^2 + (\alpha_1 + 2 * \alpha_2 - 4)^2 + \alpha_1^2 + \alpha_2^2}$$

adjusting polynomials into sum of squares:

$$\sqrt{3(\alpha_1 + \frac{2}{3}(\alpha_2 - 2))^2 + \frac{11}{3}(\alpha_2 - \frac{16}{11})^2 + \frac{32}{11} - (\frac{16}{11})^2}$$

When all the square terms are zero, the Euclidean norm of the solution is the smallest so,we can get

$$\begin{cases} \alpha_1 = \dfrac{4}{11} \\ \alpha_2 = \dfrac{16}{11} \end{cases}$$

so the solution,which can get the smallest Euclidean is :

$$\boldsymbol{x} = \begin{bmatrix} \frac{4}{11} \\ -\frac{8}{11} \\ -\frac{4}{11} \\ -\frac{16}{11} \end{bmatrix}$$

3.9

In the same way as in the problem 3.9,but we can find the extremum by derivation

$$\boldsymbol{x} = \begin{bmatrix} \alpha_1 \\ -2 - \alpha_1 \\ 1 + \alpha_1 \\ 1 \end{bmatrix}$$

$$\|\boldsymbol{x}\|_2^2 = 6\alpha_1^2 + 10\alpha + 6$$

$$\dot{\|\boldsymbol{x}\|_2^2} = 12\alpha_1 + 10 = 0$$

$$\alpha_1 = -\frac{5}{6}$$

so the solution,which have the smallest Euclidean is:

$$\boldsymbol{x} = \begin{bmatrix} -\frac{5}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \\ 1 \end{bmatrix}$$

3.11

There will exist $u[0], u[1], ..., u[n-1]$ to meet the equation for any x[n] and x[0],which means for any x[n] and x[0],the equation is always have the solution,so $\boldsymbol{b}, \boldsymbol{Ab}, ..., \boldsymbol{A^{n-1}b}$ must be linearly indepedent

3.12

$$\boldsymbol{b}=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\ \boldsymbol{Ab}=\begin{bmatrix}0\\1\\2\\1\end{bmatrix}\ \boldsymbol{A^2b}=\begin{bmatrix}1\\4\\4\\1\end{bmatrix}\ \boldsymbol{A^3b}=\begin{bmatrix}6\\12\\8\\1\end{bmatrix}$$

Thus the representation of A with respect to the basis $\boldsymbol{b}$, $\boldsymbol{Ab}$, $\boldsymbol{A^2b}$, $\boldsymbol{A^3b}$ is

$$\overline{A}=\begin{bmatrix}0&0&0&0\\1&0&0&20\\0&1&0&-18\\0&0&1&7\end{bmatrix}$$

The other basis:

$$\boldsymbol{b}=\begin{bmatrix}1\\2\\3\\1\end{bmatrix}\ \boldsymbol{Ab}=\begin{bmatrix}4\\7\\6\\1\end{bmatrix}\ \boldsymbol{A^2b}=\begin{bmatrix}15\\20\\12\\1\end{bmatrix}\ \boldsymbol{A^3b}=\begin{bmatrix}50\\52\\24\\1\end{bmatrix}$$

the representation of A with respect to the basis $\overline{\boldsymbol{b}}$, $\boldsymbol{A\overline{b}}$, $\boldsymbol{A^2overlineb}$, $\boldsymbol{A^3\overline{b}}$ is the same as above.

3.13

The Jordan-form representation of the matrices respectively is:$\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4$

$$\hat{A}_1=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\ \hat{A}_2=\begin{bmatrix}-1&0&0\\0&-1+j&0\\0&0&-1-j\end{bmatrix}\ \hat{A}_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}\ \hat{A}_3=\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$$

Because A_4 can not be diagonalized,so we calculate the Q, which meets $Q^{-1}A_4Q=\hat{A}_4$ from the definition:

$$\begin{cases}\boldsymbol{Av_1}=\lambda\boldsymbol{v_1}\\ \boldsymbol{Av_2}=\lambda\boldsymbol{v_2}+\boldsymbol{v_1}\\ \boldsymbol{Av_3}=\lambda\boldsymbol{v_3}+\boldsymbol{v_2}\end{cases}$$

we can calculate :

$$\boldsymbol{v_1}=\begin{bmatrix}1\\0\\0\end{bmatrix}\ \boldsymbol{v_2}=\begin{bmatrix}1\\-4\\5\end{bmatrix}\ \boldsymbol{v_3}=\begin{bmatrix}1\\-7\\9\end{bmatrix}$$

$$\boldsymbol{Q}=[\boldsymbol{v_1},\boldsymbol{v_2},\boldsymbol{v_3}]$$

3.14

The characteristic polynomial of $\delta(\lambda) = |\lambda E - A|$

$$|\lambda E - A| = \begin{vmatrix} \lambda+\alpha_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4$$

for $\alpha = [\lambda_i^3 \quad \lambda_i^2 \quad \lambda_i \quad 1]'$ we can vertify that $(\lambda_i E - A)\alpha = 0$
so $\alpha = [\lambda_i^3 \quad \lambda_i^2 \quad \lambda_i \quad 1]'$ is an eigenvector of A associated with λ_i.

3.15

$$\begin{vmatrix} \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{vmatrix} = \begin{vmatrix} 0 & \lambda_2^3-\lambda_1\lambda_2^2 & \lambda_3^3-\lambda_1\lambda_3^2 & \lambda_4^3-\lambda_1\lambda_4^2 \\ 0 & \lambda_2^2-\lambda_1\lambda_2 & \lambda_3^2-\lambda_1\lambda_3 & \lambda_4^2-\lambda_1\lambda_4 \\ 0 & \lambda_2-\lambda_1 & \lambda_3-\lambda_1 & \lambda_4-\lambda_1 \end{vmatrix} = (\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)\begin{vmatrix} \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \prod_{1\le i<j\le 4} (\lambda_j - \lambda_i)$$

3.16

$$|\boldsymbol{A}| = \begin{vmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

so,when $\alpha_4 \neq 0$,the companion-form matrix is nonsingular,
when the matrix is nonsingular,so matrix A can inverse.

$$\boldsymbol{AA^{-1} = A^{-1}A = E}$$

3.17

$$(\boldsymbol{A} - \lambda\boldsymbol{I})^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\boldsymbol{A} - \lambda\boldsymbol{I})^2 = \begin{bmatrix} 0 & 0 & \lambda^2T^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For $v = [0 \quad 0 \quad 1]'$,we can vertify $(\boldsymbol{A} - \lambda \boldsymbol{I})^3\boldsymbol{v} = \boldsymbol{0}$,$(\boldsymbol{A} - \lambda \boldsymbol{I})^2\boldsymbol{v} \neq \boldsymbol{0}$
so from the definition,we can say $[0 \quad 0 \quad 1]'$ is a generalized eigenvector of grade 3.

$$\boldsymbol{v_1} = \begin{bmatrix} \lambda^2 T^2 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{v_2} = \begin{bmatrix} \lambda T^2 \\ \lambda T \\ 0 \end{bmatrix} \boldsymbol{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we can vertify that

$$\begin{cases} \boldsymbol{A v_1} = \lambda \boldsymbol{v_1} \\ \boldsymbol{A v_2} = \lambda \boldsymbol{v_2} + \boldsymbol{v_1} \\ \boldsymbol{A v_3} = \lambda \boldsymbol{v_3} + \boldsymbol{v_2} \end{cases}$$

so the three columns of Q constitute a chain of generalized eigenvectors of length 3.
It is also easy to vertify that

$$Q^{-1}AQ = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

3.18

The characteristic polynomials of A, B, C, D(from left to right) is $(\lambda - \lambda_1)^3(\lambda - \lambda_2)$,$(\lambda - \lambda_1)^4$,$(\lambda - \lambda_1)^4$,$(\lambda - \lambda_1)^4$ resepectively.

The minimal polynomials of A, B, C, D(from left to right)is $(\lambda - \lambda_1)^3(\lambda - \lambda_2)$,$(\lambda - \lambda_1)^3$,$(\lambda - \lambda_1)^2$,$(\lambda - \lambda_1)$ resepectively.

3.19

Because:$\boldsymbol{Ax} = \lambda x$,we have $\boldsymbol{A^k x} = \lambda^k \boldsymbol{x}$

without loss of generality,we specify that:$f(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_0$

so,for:

$$f(\boldsymbol{A})\boldsymbol{x} = a_n\boldsymbol{A^n x} + a_{n-1}\boldsymbol{A^{n-1}x} + \cdots + a_0\boldsymbol{x} = (a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0)x = f(\lambda)x$$

so, $f(\lambda)$ is an eigenvector of $f(\boldsymbol{A})$ with the same eigenvector x.

3.20

There exists a nonsingular matrix Q made $Q^{-1}AQ = \hat{J}$, where $\hat{J}$ is a Jordan form.If A has any nonzero eigenvalues,then $A^k \neq 0$ for all interger k.

so,when $A^k = 0$, we can conclude that its all eigenvalues must be zero.Thus to say,A has eigenvalues 0 with multiplicity n. Let $\hat{J} = diag\{\hat{J}_1, \hat{J}_2, \cdots\}$,then $\hat{J}^k = diag\{\hat{J}_1^k, \hat{J}_2^k, \cdots\}$. each $\hat{J}_i$ is a Jordan block associated with eigenvalue 0.then,we can obtain from(3.40) that $\boldsymbol{A}_i^m = 0$ where the m is the largest order of all Jordan blocks,so for $k \geq m$,we have $\boldsymbol{A}_i^k = \mathbf{0}$. so $\boldsymbol{A}^k = \mathbf{0}$

3.21

First,calculating the eigenvalues of the A:

$$|\lambda E - A| = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 \lambda = 0$$

in this question,I want to use a complex method . so, the eigenvalues are $\lambda_1 = 0, \lambda_2 = \lambda_3 = 1$
for $\lambda_2 = \lambda_3 = 1$,rank(E-A)=2,so,for eigenvalue 1,it dosen't have two indepedent eigenvectors,so there exists a nonsingular matrix Q made the $Q^{-1}AQ = \hat{J}$,where $\hat{J}$ is a Jordan form .
from $Ax_1 = x_1$,we can calculate a $x_1 = [1 \quad 0 \quad 0]'$
form $Ax_2 = x_2 + x_1$,we can calculate the $x_2 = [1 \quad 1 \quad 1]'$
from $Ax = 0$,we can calculate a $x = [1 \quad -1 \quad 0]'$
so for $Q = [x, x_1, x_2]$,we have

$$Q^{-1}AQ = J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

we can split matrxi J into B and C,where:

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$J = B + C$$

$$A^{10} = QJ^{10}Q^{-1} = Q(B + C)^{10}Q^{-1} = Q(B^{10} + C_{10}^1 B^9 C)Q^{-1} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

in the same way,we can get:

$$A^{103} = \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

from $f(\hat{J}) = Q^{-1}f(A)Q$ and from (3.48) we can readily know:

$$e^{\hat{J}t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

so

$$e^{At} = Qe^{\hat{J}t}Q^{-1} = \begin{bmatrix} e^t & e^t-1 & te^t-e^t+1 \\ 0 & 1 & e^t-1 \\ 0 & 0 & e^t \end{bmatrix}$$

3.22

$$A_1 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} A_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

First method:$f(A) = h(A)$

for A_1,define $h(\lambda) = \beta + \beta_1\lambda + \beta_2\lambda^2$,$f(\lambda) = e^{\lambda t}$ from the problem3.13,the eigenvalues of the A_1 are 1,2,3

$$\begin{cases} f(1) = h(1) \\ f(2) = h(2) \\ f(3) = h(3) \end{cases}$$

we can calculate that:

$$\begin{cases} \beta_0 = 3e^t - 3e^{2t} + e3t \\ \beta_1 = -2.5e^t + 4e^{2t} - 1.5e^{3t} \\ \beta_2 = 0.5e^t - e^{2t} + 0.5e^{3t} \end{cases}$$

$$e^{A_1t} = h(A) = \begin{bmatrix} e^t & 4\left(e^{2t}-e^t\right) & 5\left(e^{3t}-e^t\right) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

in the same way,$for A_4$:

$$\begin{aligned} e^{A,t} &= \beta_0 I + \beta_1 A_4 + \beta_1 A_4^2 \\ &= I + t\begin{bmatrix} 0 & 4 & 3t+2t^2 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} + t^2/2\begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \downarrow \\ \text{thus } &= \begin{bmatrix} 1 & 4t+5t^2/2 & 3t+2t^2 \\ 0 & 20t+1 & 16t \\ 0 & -25t & -20t+1 \end{bmatrix} \end{aligned}$$

The second way:$f(\hat{J}) = Qf(A)Q^{-1}$ for A_1:we can calculate the eigenvectors of the eigenvalues and use them to form the Q,$QAQ^{-1}=J$,where

$$Q=\begin{bmatrix}1&4&5\\0&1&0\\0&0&1\end{bmatrix}J=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$$

so the

$$e^{At}=Qe^{Jt}Q^{-1}=\begin{bmatrix}1&4&5\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}e^t&0&0\\0&e^{2t}&0\\0&0&e^{3t}\end{bmatrix}\begin{bmatrix}1&4&5\\0&1&0\\0&0&1\end{bmatrix}^{-1}=\begin{bmatrix}e^t&4\left(e^{3t}-e^t\right)&5\left(e^{3t}-e^t\right)\\0&e^{2t}&0\\0&0&e^{3t}\end{bmatrix}$$

in the same way,we can get

$$e^{A_4t}=\begin{bmatrix}1&4t+5t^2/2&3t+2t^2\\0&20t+1&16t\\0&-25t&-20t+1\end{bmatrix}$$

3.23

if A is nxn,we have:

$$f(A)=\alpha_0+\alpha_1A+\cdots+\alpha_{n-1}A^{n-1}$$

$$g(A)=\beta_0+\beta_1A+\cdots+\beta_{n-1}A^{n-1}$$

Because A is communte with itself,so we can conclude that$f(A)g(A)=g(A)f(A)$ and in particular $Ae^t=e^tA$

3.24

for

$$C=\begin{bmatrix}\lambda_1&0&0\\0&\lambda&0\\0&0&\lambda_3\end{bmatrix}$$

then

$$B=\ln C=\begin{bmatrix}\ln\lambda_1&0&0\\0&\ln\lambda_2&0\\0&0&\ln\lambda_3\end{bmatrix}$$

if $\lambda_i=0$,then the $\ln(\lambda_i)$ is not defined,then B does not exist.
owing to the C is a Jordan from,we can using (3.48) so,we have:

$$B=\ln C=\begin{bmatrix}\ln\lambda&\ln'\lambda&0\\0&\ln\lambda&0\\0&0&\ln\lambda\end{bmatrix}=\begin{bmatrix}\ln\lambda&1/\lambda&0\\0&\ln\lambda&0\\0&0&\ln\lambda\end{bmatrix}$$

for an nonsingular C,there exists a nonsingular Q, made $Q^{-1}CQ = J$,where J is a Jordan from, and its $\lambda_i \neq= 0$ so,for any nonsingular C, there exists a matrix B.

3.25

$$A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, (SI - A_3) = \begin{bmatrix} s-1 & 0 & 1 \\ 0 & s-1 & 0 \\ 0 & 0 & s-2 \end{bmatrix}$$

$$Adj(SI - A_3) \begin{bmatrix} (s-1)(s-2) & 0 & -(s-1) \\ 0 & (s-1)(s-2) & 0 \\ 0 & 0 & (s-1)^2 \end{bmatrix}$$

we can conclude that m(s)=(s-1),the characteristic polynomial is$\Delta(s) = (s-1)^2(s-2)$ so $\Delta(s)/m(s) = (s-1)(s-2)$,its consequence is correct.

3.26

$$\Delta(s)I = (SI - A)[R_0 s^{n-1} + R_1 s^{n-2} + \cdots + R_{n-2}s + R_{n-1}]$$

Expanding the polynomial on the right，we can just get:

$$s^n I + \alpha_1 s^{n-1} I + \cdots + \alpha_n I = R_0 s^n + (R_1 - R_0 A)s^{n-1} + \cdots - R_{n-1}A$$

Equating the coefficient matrix of s^k:

$$\begin{cases} \boldsymbol{R_0} = \boldsymbol{I} \\ \boldsymbol{R_1} = \boldsymbol{AR_0} + \alpha_1 \boldsymbol{I} \\ \boldsymbol{R_2} = \boldsymbol{AR_1} + \alpha_2 \boldsymbol{I} \\ \vdots \\ \boldsymbol{R_{n-1}} = \boldsymbol{AR_{n-2}} + \alpha_{n-1} \boldsymbol{I} \\ \boldsymbol{AR_{n-1}} + \alpha_n \boldsymbol{I} = \boldsymbol{0} \end{cases}$$

Let $\lambda_i \qquad i = 1, 2, \cdots, n$ be the eigenvalues of A
Define $\Lambda_1 = \sum_{i=1}^n \lambda_i$,$\Lambda_k = \sum_{i=1}^n \lambda_i^k$,$k = 1, 2, \cdots$
There exists a matrix Q,made $A = Q\hat{J}Q^{-1}$,where $\hat{J}$ is a Jordan form.
Because $tr(BC) = tr(CB)$,we have:

$$tr(A) = tr(Q\hat{J}Q^{-1}) = tr(Q^{-1}Q\hat{J}) = tr(\hat{J}) = \sum_i^n \lambda_i = \Lambda_1$$

similarily:$tr(A^k) = \Lambda_k$
Owing to the following Newton's identity:

$$\Lambda_k + \alpha_1\Lambda_{k-1} + \cdots + \alpha_{k-1}\Lambda_1 + k\alpha_k = 0$$

thus:

$$\begin{gathered}\alpha_k = -\frac{1}{k}[\Lambda_k + \alpha_1\Lambda_{k-1} + \cdots + \alpha_{k-1}\Lambda_1] \\ = -\frac{1}{k}[tr(A^k) + \alpha_1 tr(A^{k-1}) + \cdots + \alpha_{k-1}tr(A)] \\ = -\frac{1}{k}tr[A^k + \alpha_1 A^{k-1} + \cdots + \alpha_{k-1}A] \\ = -\frac{1}{k}tr(AR_{k-1})\end{gathered}$$

This vertify the formula.

3.27

From the last two equations,we can get:

$$\begin{cases} R_{n-1} = A^{n-1} + \alpha_1 A^{n-2} + \cdots + \alpha_{n-2}A + \alpha_{n-1}I \\ 0 = AR_{n-1} + \alpha_n I \end{cases}$$

Multiply the first equation by matrix A,we can get:

$$AR_{n-1} = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-2}A^2 + \alpha_{n-1}A$$

Because,$AR_{n-1} = -\alpha_n I$,we can readily get:

$$A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-2}A^2 + \alpha_{n-1}A + \alpha_n I = 0$$

The calyley-Hamilton therom is vertified.

3.28

Owing to:

$$\begin{cases} \boldsymbol{R_0} = \boldsymbol{I} \\ \boldsymbol{R_1} = \boldsymbol{A} + \alpha_1\boldsymbol{I} \\ \boldsymbol{R_2} = \boldsymbol{A}^2 + \alpha_1\boldsymbol{A} + \alpha_2 I \\ \vdots \\ \boldsymbol{R_{n-1}} = \boldsymbol{A^{n-1}} + \alpha_1\boldsymbol{A^{n-2}} + \cdots + \alpha_{n-2}A + \alpha_{n-1}I \end{cases}$$

Let's substitute these formulas into the equation:

$$(SI - A)^{-1} = \frac{1}{\Delta(s)}[R_0 s^{n-1} + R_1 s^{n-2} + \cdots + R_{n-2}s + R_{n-1}]$$

we can get the equation as show in the problem.

3.29

Because the eigenvalues of A are all distinct,so matrix A can be diagonalized. so,there exists a nonsingular matrix Q,which can made $Q^{-1}AQ = diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$