Chapter2

31202008881 Bao Ze an

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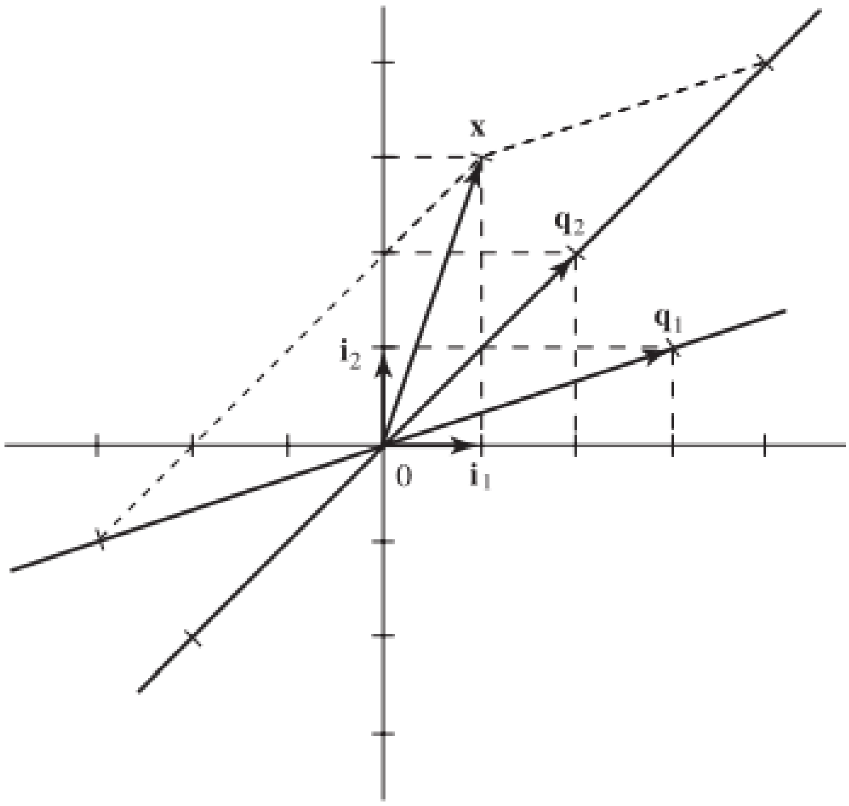
3.1



From the above figure, The three vectors $\boldsymbol{q}_1 = \begin{bmatrix} 3 & 1 \end{bmatrix}'$, $\boldsymbol{i}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}'$ and $\boldsymbol{q}_2 = \begin{bmatrix} 2 & 2 \end{bmatrix}'$
The representation of x with respect to $\{\boldsymbol{q}_1, \boldsymbol{i}_2\}$ is $\begin{bmatrix} \frac{1}{3} & \frac{8}{3} \end{bmatrix}'$
The representation of $\boldsymbol{q}_1$ with respect to $\{\boldsymbol{i}_2, \boldsymbol{q}_2\}$ is $\begin{bmatrix} -2 & \frac{3}{2} \end{bmatrix}'$

These can be verified like this:

$$x=\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}\boldsymbol{q}_1 & \boldsymbol{i}_2\end{bmatrix}\begin{bmatrix}\frac{1}{3}\\\frac{8}{3}\end{bmatrix}=\begin{bmatrix}3 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{3}\\\frac{8}{3}\end{bmatrix}$$

3.2

i:The norm of x_1

1-norm: $\|x_1\|_1=\sum_{i=1}^{3}|x_i|=|2|+|-3|+|1|=6$
2 -norm: $\|\boldsymbol{x}_1\|_2=\left(\sum_{i=1}^{3}|x_i|^2\right)^{\frac{1}{2}}=\sqrt{2^2+|-3|^2+1^2}=\sqrt{14}$
infinite-norm: $\|\boldsymbol{x}_1\|_\infty=\max_i|x_i|=3$

ii:The norm of x_2

1 -norm: $\|\boldsymbol{x}_2\|_1=\sum_{i=1}^{3}|x_i|=|1|+|1|+|1|=3$
2-norm: $\|\boldsymbol{x}_2\|_2=\left(\sum_{i=1}^{3}|x_i|^2\right)^{\frac{1}{2}}=\sqrt{1^2+1^2+1^2}=\sqrt{3}$
infinite-norm: $\|\boldsymbol{x}_2\|_\infty=\max_i|x_i|=1$

3.3

This is just the orthonormalization procedure.

$$\begin{cases} u_1=\alpha_1 & q_1=u_1/\|u_1\| \\ u_2=\alpha_2-(q_1'\alpha_2)\,q_1 & q_2=u_2/\|u_2\| \end{cases}$$

This is the ordinary method,what we find is the two vector are orthogonal. so,we just need to make the length of vector is 1 .

$$q_1=\frac{u_1}{\|u_1\|}=\left[\frac{2}{\sqrt{14}}-\frac{3}{\sqrt{14}}\quad\frac{1}{\sqrt{14}}\right]'$$
$$q_1=\frac{u_2}{\|u_2\|}=\left[\frac{1}{\sqrt{3}}\quad\frac{1}{\sqrt{3}}\quad\frac{1}{\sqrt{3}}\right]'$$

3.4

a

if n>m,$\boldsymbol{A}\boldsymbol{A}'$ is a ordinary vector,which has the rank m

b

if m=n,so $\boldsymbol{A}$ is a nonsingular square matrix,we already have $\boldsymbol{A}^{'}\boldsymbol{A} = \boldsymbol{I}_m$,so $\boldsymbol{A}' = \boldsymbol{A}^{-1}$. $\boldsymbol{A}\boldsymbol{A}^{'} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$

3.5

Accoridng to the principle:

$$Nillity(\boldsymbol{A}) = number of columns of \boldsymbol{A} - rank(A)$$

i:

Rank $(A_1) = 2$
Nullity $(A_1) = 3 - 2 = 1$

ii:

Rank $(A_2) = 3$
Nullity $(A_2) = 3 - 3 = 0$

iii:

Rank $(A_3) = 3$
Nullity $(A_3) = 4 - 3$

3.6

For A_1:

$$\boldsymbol{a_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{a_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The set of $\{\boldsymbol{a_1}, \boldsymbol{a_2}\}$ can be the basis of the range spaces.
The independent vectors of null space can get by solving the equation:

$$\boldsymbol{A}_1\boldsymbol{\eta}_i = 0$$

$$\boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The set of $\boldsymbol{\eta}_1$ is the basis of the null space
in the same way,we can get the basis of the range space and null space of $\boldsymbol{A}_2$

$$\boldsymbol{a_1} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \boldsymbol{a_2} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \boldsymbol{a_3} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

The set of $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$ is the basis of the range space.
because the $\boldsymbol{A}_2$ is full rank,so the basis of the null space is $\{\boldsymbol{0}\}$

3.7

$$\rho(\boldsymbol{A}) = \rho([\boldsymbol{A} \quad y]) = 2$$

so a solution $\boldsymbol{x}$ exist with respect to this equation.
Because coefficient matrix is full column rank,so the solution is unique.
if y=$[1 \quad 1 \quad 1]^{'}$,$\rho(\boldsymbol{A}) = 2 \neq \rho([\boldsymbol{A} \quad y]) = 3$ so,when $\boldsymbol{y}$=$[1 \quad 1 \quad 1]^{'}$,the solution is not exist.

3.8

$\boldsymbol{x}_p = [0 \quad -2 \quad 1 \quad 1]^{'}$ is a solution, a basis of the null space of A:

$$\boldsymbol{A}\boldsymbol{\eta}_i = 0$$

$$\boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Thus the gengeral solution can be expressed as:

$$\boldsymbol{x} = \boldsymbol{x}_p + \alpha_1 \boldsymbol{\eta}_1$$

3.9

From the example 3.3,we can know the gengeral solution is:

$$\boldsymbol{x} = \begin{bmatrix} \alpha_1 \\ -4 + \alpha_1 + 2\alpha_2 \\ -\alpha_1 \\ -\alpha_2 \end{bmatrix}$$

$$\|\boldsymbol{x}\|_2 = \sqrt{\alpha_1^2 + (\alpha_1 + 2 * \alpha_2 - 4)^2 + \alpha_1^2 + \alpha_2^2}$$

adjusting polynomials into sum of squares:

$$\sqrt{3(\alpha_1+\frac{2}{3}(\alpha_2-2))^2+\frac{11}{3}(\alpha_2-\frac{16}{11})^2+\frac{32}{11}-(\frac{16}{11})^2}$$

When all the square terms are zero, the Euclidean norm of the solution is the smallest so,we can get

$$\begin{cases}\alpha_1=\dfrac{4}{11}\\ \alpha_2=\dfrac{16}{11}\end{cases}$$

so the solution,which can get the smallest Euclidean is :

$$\boldsymbol{x}=\begin{bmatrix}\frac{4}{11}\\ -\frac{8}{11}\\ -\frac{4}{11}\\ -\frac{16}{11}\end{bmatrix}$$

3.9

In the same way as in the problem 3.9,but we can find the extremum by derivation

$$\boldsymbol{x}=\begin{bmatrix}\alpha_1\\ -2-\alpha_1\\ 1+\alpha_1\\ 1\end{bmatrix}$$

$$\|\boldsymbol{x}\|_2^2=6\alpha_1^2+10\alpha+6$$

$$\dot{\|\boldsymbol{x}\|_2^2}=12\alpha_1+10=0$$

$$\alpha_1=-\frac{5}{6}$$

so the solution,which have the smallest Euclidean is:

$$\boldsymbol{x}=\begin{bmatrix}-\frac{5}{6}\\ -\frac{1}{3}\\ \frac{1}{6}\\ 1\end{bmatrix}$$

3.11

There will exist $u[0], u[1], ..., u[n-1]$ to meet the equation for any x[n] and x[0],which means for any x[n] and x[0],the equation is always have the solution,so $\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, ..., \boldsymbol{A}^{\boldsymbol{n-1}}\boldsymbol{b}$ must be linearly indepedent

3.12

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \boldsymbol{Ab} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \boldsymbol{A^2b} = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} \boldsymbol{A^3b} = \begin{bmatrix} 6 \\ 12 \\ 8 \\ 1 \end{bmatrix}$$

Thus the representation of A with respect to the basis $\boldsymbol{b}$, $\boldsymbol{Ab}$, $\boldsymbol{A^2b}$, $\boldsymbol{A^3b}$ is

$$\overline{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

The other basis:

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \boldsymbol{Ab} = \begin{bmatrix} 4 \\ 7 \\ 6 \\ 1 \end{bmatrix} \boldsymbol{A^2b} = \begin{bmatrix} 15 \\ 20 \\ 12 \\ 1 \end{bmatrix} \boldsymbol{A^3b} = \begin{bmatrix} 50 \\ 52 \\ 24 \\ 1 \end{bmatrix}$$

the representation of A with respect to the basis $\overline{\boldsymbol{b}}$, $\boldsymbol{A\overline{b}}$, $\boldsymbol{A^2 overlineb}$, $\boldsymbol{A^3\overline{b}}$ is the same as above.

3.13

The Jordan-form representation of the matrices respectively is:$\hat{A}_1$, $\hat{A}_2$, $\hat{A}_3$, $\hat{A}_4$

$$\hat{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \hat{A}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+j & 0 \\ 0 & 0 & -1-j \end{bmatrix} \hat{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \hat{A}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because A_4 can not be diagonalized,so we caculate the Q, which meets $Q^{-1}A_4Q = \hat{A}_4$
from the definition:

$$\begin{cases} \boldsymbol{Av_1} = \lambda \boldsymbol{v_1} \\ \boldsymbol{Av_2} = \lambda \boldsymbol{v_2} + \boldsymbol{v_1} \\ \boldsymbol{Av_3} = \lambda \boldsymbol{v_3} + \boldsymbol{v_2} \end{cases}$$

we can calculate :

$$\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{v_2} = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix} \boldsymbol{v_3} = \begin{bmatrix} 1 \\ -7 \\ 9 \end{bmatrix}$$

$$\boldsymbol{Q} = [\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}]$$

3.14

The characteristic polynomial of $\delta(\lambda) = |\lambda E - A|$

$$|\lambda E - A| = \begin{vmatrix} \lambda + \alpha_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4$$

for $\alpha = [\lambda_i^3 \quad \lambda_i^2 \quad \lambda_i \quad 1]'$ we can vertify that $(\lambda_i E - A)\alpha = 0$
so $\alpha = [\lambda_i^3 \quad \lambda_i^2 \quad \lambda_i \quad 1]'$ is an eigenvector of A associated with λ_i.

3.15

$$\begin{vmatrix} \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{vmatrix} = \begin{vmatrix} 0 & \lambda_2^3 - \lambda_1\lambda_2^2 & \lambda_3^3 - \lambda_1\lambda_3^2 & \lambda_4^3 - \lambda_1\lambda_4^2 \\ 0 & \lambda_2^2 - \lambda_1\lambda_2 & \lambda_3^2 - \lambda_1\lambda_3 & \lambda_4^2 - \lambda_1\lambda_4 \\ 0 & \lambda_2 - \lambda_1 & \lambda_3 - \lambda_1 & \lambda_4 - \lambda_1 \end{vmatrix} = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1) \begin{vmatrix} \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \prod_{1 \le i < j \le 4} (\lambda_j - \lambda_i)$$

3.16

$$|\boldsymbol{A}| = \begin{vmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

so,when $\alpha_4 \neq 0$,the companion-form matrix is nonsingular,
when the matrix is nonsingular,so matrix A can inverse.

$$\boldsymbol{AA^{-1} = A^{-1}A = E}$$

3.17

$$(\boldsymbol{A} - \lambda\boldsymbol{I})^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\boldsymbol{A} - \lambda\boldsymbol{I})^2 = \begin{bmatrix} 0 & 0 & \lambda^2 T^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For $v = [0 \quad 0 \quad 1]'$,we can vertify $(\boldsymbol{A} - \lambda\boldsymbol{I})^3\boldsymbol{v} = \mathbf{0}$,$(\boldsymbol{A} - \lambda\boldsymbol{I})^2\boldsymbol{v} \neq \mathbf{0}$
so from the definition,we can say $[0 \quad 0 \quad 1]'$ is a generalized eigenvector of grade 3.

$$\boldsymbol{v_1} = \begin{bmatrix} \lambda^2 T^2 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{v_2} = \begin{bmatrix} \lambda T^2 \\ \lambda T \\ 0 \end{bmatrix} \boldsymbol{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we can vertify that

$$\begin{cases} \boldsymbol{A v_1} = \lambda \boldsymbol{v_1} \\ \boldsymbol{A v_2} = \lambda \boldsymbol{v_2} + \boldsymbol{v_1} \\ \boldsymbol{A v_3} = \lambda \boldsymbol{v_3} + \boldsymbol{v_2} \end{cases}$$

so the three columns of Q constitute a chain of generalized eigenvectors of length 3.
It is also easy to vertify that

$$Q^{-1}AQ = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$