Chapter5

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5.1

The transfer function of the system is

$$g(s) = \frac{s}{s^2+1}$$

Let $u(t) = \sin(t)$,then the Laplace transform of $u(t)$ is $u(s) = \frac{1}{s^2+1}$,and the output will be

$$y(s) = \frac{s}{(s^2+1)^2}$$

Through inverse Laplace transform,we have $y(t) = 0.5t\sin(t)$,and it is not bounded.So the system is not BIBO stable.

5.2

$g(s) = \int_0^\infty g(t)e^{-st}dt$.And let $s = \sigma + j\omega$,then we have:

$$\begin{aligned} \left|e^{-st}\right| &= \left|e^{-\sigma t}\right|\left|e^{-j\omega t}\right| \\ &= \left|e^{-\sigma t}\right| \\ &\leq 1 \end{aligned}$$

So if the system is BIBO stable,then we have $\int_0^\infty |g(t)|\,dt < \infty$,thus we have for $Re > 0$

$|g(s)| \leq \int_0^\infty |g(t)|\,|e^{-st}|\,dt \leq \int_0^\infty |g(t)|\,dt \leq \infty$.Thus the necessarily can be proved.

5.3

For $g(t) = 1/(t+1)$:

$$\begin{aligned}\int_0^{\infty} |g(t)|\, dt &= \int_0^{\infty} |1/(t+1)|\, dt \\ &= ln(t+1)|_0^{\infty} \\ &= \infty\end{aligned}$$

So $g(t)$ is not absolutely integrable,the system is not BIBO stable.

For $g(t) = te^{-t}$:
By taking Laplace transform,$g(s) = \frac{1}{(s+1)^2}$,and all the roots have negative real parts,so the system is BIBO stable.

5.4

By taking Laplace transform,we have

$$\text{For } t \geq 2, g(t) = e^{-t+2}$$
$$\text{For } t < 2, g(t) = 0$$

Then

$$\begin{aligned}\int_0 \infty |g(t)|\, dt &= \int_2 \infty \left|e^{-t+2}\right| dt \\ &= -e^{-t+2}\Big|_2^{\infty} \\ &= 1\end{aligned}$$

So $g(t)$ is absolutely integrable,and the system is BIBO stable.

5.5

If $r(t) = \delta(t)$,then we have $g(t) = a\delta(t-1) - a^2\delta(t-2) + a^3\delta(t-3) + \cdots$.And

$$
\begin{aligned}
\int_0^\infty |g(t)|\,dt &= \int_0^\infty \left| a\delta(t-1) - a^2\delta(t-2) + a^3\delta(t-3) + \cdots \right| dt \\
&= |a| + |a|^2 + |a|^3 + \cdots \\
&= \begin{cases} \frac{|a|}{1-|a|}, |a| < 1 \\ \infty, |a| \geq 1 \end{cases}
\end{aligned}
$$

So the system is BIBO stable if $|a| < 1$.

And if $a = 1$,let $r(t) = \begin{cases} 1 & 2n < t < 2n+1, n = 0,1,2,\cdots \\ -1 & 2n+1 < t < 2n+2, n = 0,1,2,\cdots \end{cases}$.Obviously $r_m = 1 < \infty$,so the input is bounded.

Then the output will be $y(t) = i \times (-1)^{i+1}, i < t < i+1, i = 0,1,2,\cdots$,and the output is not bounded.So the system is not BIBO stable.

5.6

From theorem 5.2,we have

For $u(t) = 3$,$y(t) = 3 \times \hat{g}(0) = -6$

For $u(t) = \sin(2t)$,$y(t) = |\hat{g}(j2)| \sin(2t + \varphi(\hat{g}(j2))) = 1.26\sin(t + 1.25)$

5.7

The transfer function is

$$
\begin{aligned}
g(s) &= \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 0 \end{bmatrix} \\
&= \frac{4}{s+1}
\end{aligned}
$$

The root of the transfer matrix has negative real part,so the system is BIBO stable.

5.8

By taking z-transform,we have

$$g(z) = \frac{0.8z}{(z-0.8)^2}$$

Then the poles lie in the unit circle,so the system is BIBO stable.

5.9

$\lambda I - A = \begin{bmatrix} \lambda+1 & \lambda-10 \\ 0 & \lambda-1 \end{bmatrix}$,so the eigenvalues are -1 and 1.Since there is one eigenvalue 1 that has positive real part,the system is neither marginally stable nor asymptotically stable.

5.10

$\lambda I - A = \begin{bmatrix} \lambda+1 & 0 & -1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$,and by the determinant of it we can get that the eigenvalues of A are 0,0,-1.Thus the system is not asymptotically stable.Then for eigenvalue 0,let $\lambda = 0$,then $(\lambda I - A)x = 0$ will be

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = 0$$

The coefficient matrix has rank 1,so there are 2 independent solutions.So matrix A can be transformed as $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.And the minimal polynomial is $\lambda(\lambda+1)$,so 0 is the single root of the minimal polynomial.Thus the system is marginally stable.

5.11

$\lambda I - A = \begin{bmatrix} \lambda+1 & 0 & -1 \\ 0 & 0 & \lambda-1 \\ 0 & 0 & \lambda \end{bmatrix}$,so the characteristic polynomial is $\lambda^2(\lambda+1)$,the eigenvalues are 0,0,-1.Thus the system is not asymptotically stable.For eigenvalue 0,$(\lambda I - A)x = 0$ will be

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} x = 0$$

The coefficient matrix has rank 2,so there is only 1 independent solution.The solution will be $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}'$,and from this there exists a generated solution $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}'$.These two vectors consists of the eigenvectors of eigenvalue 0.And from the calculation we can get that A can be transform as $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.So the minimal polynomial will be $\lambda^2(\lambda+1)$,and since 0 is not single root of the minimal polynomial,the system is not marginally stable.

5.12

The eigenvalues of A are 0.9,1,1.So the system is not asymptotically stable.Then for eigenvalue 1,$I - A$ has rank 1,so there are two independent solutions for $(I - A)x = 0$.So A can be transformed into

$$\begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the minimal polynomial will be $(\lambda - 1)(\lambda - 0.9)$,so 1 is the single root of the minimal polynomial.So the system is marginally stable.

5.13

The characteristic polynomial is $(\lambda-1)^2(\lambda-0.9)$.So the system is not asymptotically stable.As for eigenvalue 1,$I - A$ will be

$$\begin{bmatrix} 0.1 & 0-1 & \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The coefficient matrix has rank 2.So there is only one independent solution for $(I - A)x = 0$.Thus A can be transformed into

$$\begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

From the Jordan form,we can get that the minimal polynomial of A is $(\lambda-1)^2(\lambda-0.9)$.So 1 is not the single root of the minimal polynomial.The system is not marginally stable.

5.14

As for the Lyapunov equation $A'M+MA=-N$,we can choose N to be identity matrix.Then the equation will be

$$\begin{bmatrix} 0 & -0.5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}+\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

And by solving the equation,we can get $M=\begin{bmatrix} 1.75 & 1 \\ 1 & 1.5 \end{bmatrix}$.Since $1.75>0$,$1.75\times1.5-1>0$,M is of positive definite.So all eigenvalues of A have negative real parts.

5.15

For the Lyapunov equation $M-A'MA=N$,we choose N to be identity matrix.Then we have

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}-\begin{bmatrix} 0 & -0.5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By solving the equation,we can get that $M=\begin{bmatrix} 2.2 & 1.6 \\ 1.6 & 4.8 \end{bmatrix}$.The matrix is of positive definite,so all eigenvalues of the A have magnitudes less than 1.

5.16

Let $\mathbf{A}=\operatorname{diag}(\lambda_1,\lambda_2,\lambda_3)$,then the Lyapunov equation will be

$$-\begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{bmatrix}=-N$$

Let $N = \bar{N}'\bar{N}$,then we have $\bar{N} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$.Then construct the following matrix:

$$\begin{bmatrix} \bar{N} \\ \bar{N}A \\ \bar{N}A^2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ \lambda_1 a_1 & \lambda_2 a_2 & \lambda_3 a_3 \\ \lambda_1^2 a_1 & \lambda_2^2 a_2 & \lambda_3^2 a_3 \end{bmatrix} \doteq \alpha.$$

And $\det(\alpha) = a_1 a_2 a_3 (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)$.Since a_i are all non-zero,and λ_i are all distinct,$\det \alpha \neq 0$.So α is of full rank and from the corollary M is of positive definite.

5.17

Let $H_1 = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$,$H_2 = \begin{bmatrix} 2 & 1 \\ 1.9 & 1 \end{bmatrix}$.

For matrix H_1,the eigenvalues are $1, 2$.Let $x = \begin{bmatrix} 4 & 1 \end{bmatrix}'$,then $x'Ax = -1 < 0$,so the matrix is not positive definite.

For matrix H_2,let $x = \begin{bmatrix} 0.5805 & -0.8142 \end{bmatrix}$.And $x'Ax = -0.0338 < 0$.The matrix is not positive definite.But all its leading principal minors are 2 and 0.1.They are both larger than 0.

So we cannot judge the positive definiteness of the matrix by eigenvalues or leading principal minors if matrix is not symmetric.

Under this condition,we can construct matrix $\hat{H} = \frac{1}{2}(H' + H)$.That is,for matrix H_1,$H_1' = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 3 \end{bmatrix}$.It is not positive definite because the leading principal minors are 0,-0.25.Similarly,$H_2' = \begin{bmatrix} 2 & 1.45 \\ 1.45 & 1 \end{bmatrix}$.It is not positive definite since the leading principal minors are 2,-0.1025.

5.18

The equation can be transformed as:

$$(A' + \mu I')M + M(A + \mu I') = -N$$

Since the equation has a unique symmetric solution M and M is positive definite for any given positive definite symmetric matrix N,$(A + \mu I)$ has eigenvalues with negative real parts.Let the eigenvalues of A be λ_i,then the eigenvalues of $A + \mu I$ can be $\lambda + \mu$.Then we have $Re(\lambda) < -\mu$.

5.19

The equation can be transformed as:

$$M - (\frac{1}{\rho}A')M(\frac{1}{\rho}A) = N$$

If $N > 0, M > 0$,then we can get that all the eigenvalues of $\frac{1}{\rho}A$ have magnitudes less than 1.Let λ_i be the eigenvalues of A,then we have $\left|\frac{1}{\rho}\lambda_i\right| < 1$,that is,$|\lambda_i| < \rho$.

5.20

$g(t, \tau) = e^{-2|t|-|\tau|}$:

By integration,we have:

$$\int_{t_0}^{t} |e^{-2|t|-|\tau|}|d\tau = e^{-2|t|}\int_{t_0}^{t} e^{-|\tau|}d\tau \doteq \alpha$$

When $t_0 \leq t < 0$,we have:

$$\begin{aligned} e^{-2|t|}\int_{t_0}^{t} e^{-|\tau|}d\tau &= e^{2t}\int_{t_0}^{t} e^{\tau}d\tau \\ &= e^{2t}(e^{t} - e^{t_0}) \\ &< \infty \end{aligned}$$

When $t_0 \leq t, t > 0$,we have:

$$\begin{aligned} \alpha &= e^{-2t}(\int_{t_0}^{0} e^{\tau}d\tau + \int_{0}^{t} e^{-\tau}d\tau) \\ &= e^{-2t}[(1 - e^{t_0}) - (e^{-t} - 1)] \\ &= e^{-2t}(-e^{-t} - e^{t_0} + 2) \\ &< \infty(t_0 < 0) \end{aligned}$$

$$\begin{aligned} \alpha &= e^{-2t}\int_{t_0}^{t} e^{-\tau}d\tau \\ &= e^{-2t}(e^{-t_0} - e^{-t}) \\ &< \infty(t_0 > 0) \end{aligned}$$

So the system is BIBO stable.

$g(t,\tau) = \sin t \left(e^{-(t-\tau)}\right) \cos\tau$:

With integration,we have:

$$\begin{aligned}
\int_{t_0}^{\infty} |g(t-\tau)| d\tau &\le \int_{t_0}^{t} e^{-(t-\tau)} d\tau \\
&= e^{-t} \int_{t_0}^{t} e^{\tau} d\tau \\
&= e^{-t}(e^{t} - e^{t_0}) \\
&= 1 - e^{-(t-t_0)} \\
&< \infty
\end{aligned}$$

So the system is BIBO stable.

5.21

For the scalar equation,we have:

$$\phi(t,t_0) = e^{\int_{t_0}^{t} e^{2t} dt} = e^{t^2 - t_0^2}$$

Thus $g(t,\tau) = e^{-\tau^2}$,and with integration,we have:

$$\begin{aligned}
\int_{t_0}^{t} |g(t,\tau)| d\tau &= \int_{t_0}^{t} e^{-\tau^2} d\tau \\
&< \infty
\end{aligned}$$

Since $e^{-\tau^2} < e^{-\tau} < \infty$ when τ approaches ∞,the system is BIBO stable.

And $|\phi(t,\tau)| = e^{t^2 - t_0^2}$ approaches ∞ when t approaches ∞,the system is not marginally stable nor asymptotically stable.

5.22

With the transform formula,we have:

$$\begin{aligned}\bar{A} &= [P(t)A(t) + \dot{P}(t)]P^{-1}(t) \\ &= (2te^{-t^2} - 2te^{-t^2})e^{t^2} \\ &= 0\end{aligned}$$

Thus the system can be transformed into

$$\dot{\bar{x}} = 0 \cdot \bar{x} + e^{-t^2} u \quad y = \bar{x}$$

Then the transfer matrix is $\phi = C$,let $C = 1$,then we have:

$$g(t, \tau) = e^{-\tau^2}$$

The impulse response remain unchanged corresponding to the last problem,so the system is BIBO stable.

And the transfer matrix is a constant number,so the system is marginally stable,but it will not approaches 0 when $t \to \infty$,so the system is not asymptotically stable.

And since $P^{-1}(t) = e^{\tau^2}$,and it doesn't approach 0 when t approaches ∞,so the transformation is not a Lyapunov transformation.

5.23

From the equation,we have:

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) \\ \dot{x}_2(t) &= -e^{-3t} x_1(t)\end{aligned}$$

Then we have:

$$\begin{aligned}x_1(t) &= e^{-t} x_1(0) \\ x_2(t) &= \frac{1}{4} e^{-4t} x_1(0) + x_2(0) - \frac{1}{4} x_1(0)\end{aligned}$$

Thus the fundamental matrix can be obtained as $\begin{bmatrix} e^{-t} & 0 \\ \frac{1}{4}e^{-4t}-\frac{1}{4} & 1 \end{bmatrix}$.And the state transfer matrix is:

$$\phi = \begin{bmatrix} e^{-t+t_0} & 0 \\ \frac{1}{4}e^{-4t+t_0}-\frac{1}{4}e^{-3t_0} & 1 \end{bmatrix}$$

For the sum of the matrix rows,$0 \leq e^{-t+t_0} \leq 1$,$\frac{3}{4} \leq 1+\frac{1}{4}e^{-4t+t_0}-\frac{1}{4}e^{-3t_0} \leq 1$.So $||\phi||_\infty \leq 1 < \infty$.Thus the matrix is marginally stable.However,since not every entry of the matrix approaches zero as t approaches ∞,the system is not asymptotically stable.