

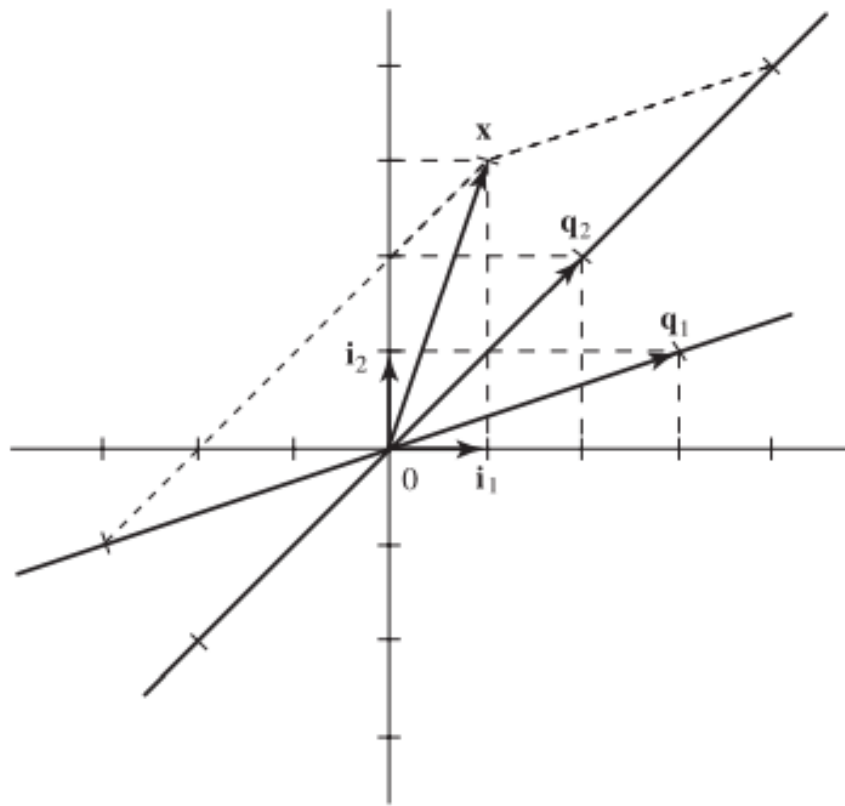
Chapter3

31202008881

Bao Ze an

Tuesday 16th February, 2021

3.1



From the above figure, The three vectors $q_1 = \begin{bmatrix} 3 & 1 \end{bmatrix}'$, $i_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}'$ and $q_2 = \begin{bmatrix} 2 & 2 \end{bmatrix}'$

The representation of x with respect to $\{q_1, i_2\}$ is $\begin{bmatrix} \frac{1}{3} & \frac{8}{3} \end{bmatrix}'$

The representation of q_1 with respect to $\{i_2, q_2\}$ is $\begin{bmatrix} -2 & \frac{3}{2} \end{bmatrix}'$

These can be verified like this:

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{i}_2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \end{bmatrix}$$

3.2

i:The norm of x_1

$$1\text{-norm: } \|x_1\|_1 = \sum_{i=1}^3 |x_i| = |2| + |-3| + |1| = 6$$

$$2\text{-norm: } \|x_1\|_2 = \left(\sum_{i=1}^3 |x_i|^2 \right)^{\frac{1}{2}} = \sqrt{2^2 + |-3|^2 + 1^2} = \sqrt{14}$$

$$\text{infinite-norm: } \|x_1\|_\infty = \max_i |x_i| = 3$$

ii:The norm of x_2

$$1\text{-norm: } \|x_2\|_1 = \sum_{i=1}^3 |x_i| = |1| + |1| + |1| = 3$$

$$2\text{-norm: } \|x_2\|_2 = \left(\sum_{i=1}^3 |x_i|^2 \right)^{\frac{1}{2}} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\text{infinite-norm: } \|x_2\|_\infty = \max_i |x_i| = 1$$

3.3

This is just the orthonormalization procedure.

$$\begin{cases} u_1 = \alpha_1 & q_1 = u_1 / \|u_1\| \\ u_2 = \alpha_2 - (q_1' \alpha_2) q_1 & q_2 = u_2 / \|u_2\| \end{cases}$$

This is the ordinary method, what we find is the two vector are orthogonal. so, we just need to make the length of vector is 1 .

$$q_1 = \frac{u_1}{\|u_1\|} = \begin{bmatrix} \frac{2}{\sqrt{14}} & -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{bmatrix}'$$

$$q_2 = \frac{u_2}{\|u_2\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}'$$

3.4

a

if $n > m$, $\mathbf{A}\mathbf{A}'$ is a ordinary vector, which has the rank m

b

if $m=n$, so \mathbf{A} is a nonsingular square matrix, we already have $\mathbf{A}'\mathbf{A} = \mathbf{I}_m$, so $\mathbf{A}' = \mathbf{A}^{-1}$. $\mathbf{A}\mathbf{A}' = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$

3.5

According to the principle:

$$\text{Nullity}(\mathbf{A}) = \text{number of columns of } \mathbf{A} - \text{rank}(\mathbf{A})$$

i:

$$\text{Rank}(\mathbf{A}_1) = 2$$

$$\text{Nullity}(\mathbf{A}_1) = 3 - 2 = 1$$

ii:

$$\text{Rank}(\mathbf{A}_2) = 3$$

$$\text{Nullity}(\mathbf{A}_2) = 3 - 3 = 0$$

iii:

$$\text{Rank}(\mathbf{A}_3) = 3$$

$$\text{Nullity}(\mathbf{A}_3) = 4 - 3 = 1$$

3.6

For \mathbf{A}_1 :

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The set of $\{\mathbf{a}_1, \mathbf{a}_2\}$ can be the basis of the range spaces.

The independent vectors of null space can get by solving the equation:

$$\mathbf{A}_1 \boldsymbol{\eta}_i = 0$$
$$\boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The set of $\boldsymbol{\eta}_1$ is the basis of the null space

in the same way, we can get the basis of the range space and null space of \mathbf{A}_2

$$\mathbf{a}_1 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mathbf{a}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

The set of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is the basis of the range space.

because the \mathbf{A}_2 is full rank, so the basis of the null space is $\{\mathbf{0}\}$

3.7

$$\rho(\mathbf{A}) = \rho([\mathbf{A} \quad \mathbf{y}]) = 2$$

so a solution \mathbf{x} exist with respect to this equation.

Because coefficient matrix is full column rank, so the solution is unique.

if $\mathbf{y} = [1 \quad 1 \quad 1]'$, $\rho(\mathbf{A}) = 2 \neq \rho([\mathbf{A} \quad \mathbf{y}]) = 3$ so, when $\mathbf{y} = [1 \quad 1 \quad 1]'$, the solution is not exist.

3.8

$\mathbf{x}_p = [0 \quad -2 \quad 1 \quad 1]'$ is a solution, a basis of the null space of \mathbf{A} :

$$\mathbf{A}\boldsymbol{\eta}_i = \mathbf{0}$$

$$\boldsymbol{\eta}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Thus the general solution can be expressed as:

$$\mathbf{x} = \mathbf{x}_p + \alpha_1 \boldsymbol{\eta}_1$$

3.9

From the example 3.3, we can know the general solution is:

$$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ -4 + \alpha_1 + 2\alpha_2 \\ -\alpha_1 \\ -\alpha_2 \end{bmatrix}$$

$$\|\mathbf{x}\|_2 = \sqrt{\alpha_1^2 + (\alpha_1 + 2\alpha_2 - 4)^2 + \alpha_1^2 + \alpha_2^2}$$

adjusting polynomials into sum of squares:

$$\sqrt{3(\alpha_1 + \frac{2}{3}(\alpha_2 - 2))^2 + \frac{11}{3}(\alpha_2 - \frac{16}{11})^2 + \frac{32}{11} - (\frac{16}{11})^2}$$

When all the square terms are zero, the Euclidean norm of the solution is the smallest so, we can get

$$\begin{cases} \alpha_1 = \frac{4}{11} \\ \alpha_2 = \frac{16}{11} \end{cases}$$

so the solution, which can get the smallest Euclidean is :

$$\mathbf{x} = \begin{bmatrix} \frac{4}{11} \\ -\frac{8}{11} \\ -\frac{4}{11} \\ -\frac{16}{11} \end{bmatrix}$$

3.9

In the same way as in the problem 3.9, but we can find the extremum by derivation

$$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ -2 - \alpha_1 \\ 1 + \alpha_1 \\ 1 \end{bmatrix}$$

$$\|\mathbf{x}\|_2^2 = 6\alpha_1^2 + 10\alpha_1 + 6$$

$$\|\mathbf{x}\|_2^2 = 12\alpha_1 + 10 = 0$$

$$\alpha_1 = -\frac{5}{6}$$

so the solution, which have the smallest Euclidean is:

$$\mathbf{x} = \begin{bmatrix} -\frac{5}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \\ 1 \end{bmatrix}$$

3.11

There will exist $u[0], u[1], \dots, u[n-1]$ to meet the equation for any $\mathbf{x}[n]$ and $\mathbf{x}[0]$, which means for any $\mathbf{x}[n]$ and $\mathbf{x}[0]$, the equation is always have the solution, so $\mathbf{b}, \mathbf{A}\mathbf{b}, \dots, \mathbf{A}^{n-1}\mathbf{b}$ must be linearly independent

3.12

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{Ab} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{A^2b} = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} \quad \mathbf{A^3b} = \begin{bmatrix} 6 \\ 12 \\ 8 \\ 1 \end{bmatrix}$$

Thus the representation of A with respect to the basis $\mathbf{b}, \mathbf{Ab}, \mathbf{A^2b}, \mathbf{A^3b}$ is

$$\overline{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

The other basis:

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \quad \mathbf{Ab} = \begin{bmatrix} 4 \\ 7 \\ 6 \\ 1 \end{bmatrix} \quad \mathbf{A^2b} = \begin{bmatrix} 15 \\ 20 \\ 12 \\ 1 \end{bmatrix} \quad \mathbf{A^3b} = \begin{bmatrix} 50 \\ 52 \\ 24 \\ 1 \end{bmatrix}$$

the representation of A with respect to the basis $\overline{\mathbf{b}}, \overline{\mathbf{Ab}}, \overline{\mathbf{A^2b}}, \overline{\mathbf{A^3b}}$ is the same as above.

3.13

The Jordan-form representation of the matrices respectively is: $\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4$

$$\hat{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \hat{A}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+j & 0 \\ 0 & 0 & -1-j \end{bmatrix} \quad \hat{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \hat{A}_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because A_4 can not be diagonalized, so we calculate the Q , which meets $Q^{-1}A_4Q = \hat{A}_4$ from the definition:

$$\begin{cases} \mathbf{Av}_1 = \lambda \mathbf{v}_1 \\ \mathbf{Av}_2 = \lambda \mathbf{v}_2 + \mathbf{v}_1 \\ \mathbf{Av}_3 = \lambda \mathbf{v}_3 + \mathbf{v}_2 \end{cases}$$

we can calculate :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -7 \\ 9 \end{bmatrix}$$

$$Q = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$$

3.14

The characteristic polynomial of $\delta(\lambda) = |\lambda E - A|$

$$|\lambda E - A| = \begin{vmatrix} \lambda + \alpha_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = \lambda^4 + \alpha_1 \lambda^3 + \alpha_2 \lambda^2 + \alpha_3 \lambda + \alpha_4$$

for $\alpha = [\lambda_i^3 \quad \lambda_i^2 \quad \lambda_i \quad 1]'$ we can verify that $(\lambda_i E - A)\alpha = 0$

so $\alpha = [\lambda_i^3 \quad \lambda_i^2 \quad \lambda_i \quad 1]'$ is an eigenvector of A associated with λ_i .

3.15

$$\begin{vmatrix} \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{vmatrix} = \begin{vmatrix} 0 & \lambda_2^3 - \lambda_1 \lambda_2^2 & \lambda_3^3 - \lambda_1 \lambda_3^2 & \lambda_4^3 - \lambda_1 \lambda_4^2 \\ 0 & \lambda_2^2 - \lambda_1 \lambda_2 & \lambda_3^2 - \lambda_1 \lambda_3 & \lambda_4^2 - \lambda_1 \lambda_4 \\ 0 & \lambda_2 - \lambda_1 & \lambda_3 - \lambda_1 & \lambda_4 - \lambda_1 \end{vmatrix} = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1) \begin{vmatrix} \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \prod_{1 \leq i < j \leq 4} (\lambda_j - \lambda_i)$$

3.16

$$|\mathbf{A}| = \begin{vmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

so, when $\alpha_4 \neq 0$, the companion-form matrix is nonsingular,

when the matrix is nonsingular, so matrix A can inverse.

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{E}$$

3.17

$$(\mathbf{A} - \lambda \mathbf{I})^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\mathbf{A} - \lambda \mathbf{I})^2 = \begin{bmatrix} 0 & 0 & \lambda^2 T^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For $v = [0 \ 0 \ 1]'$, we can verify $(\mathbf{A} - \lambda \mathbf{I})^3 \mathbf{v} = \mathbf{0}, (\mathbf{A} - \lambda \mathbf{I})^2 \mathbf{v} \neq \mathbf{0}$
so from the definition, we can say $[0 \ 0 \ 1]'$ is a generalized eigenvector of grade 3.

$$\mathbf{v}_1 = \begin{bmatrix} \lambda^2 T^2 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} \lambda T^2 \\ \lambda T \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we can verify that

$$\begin{cases} \mathbf{A}\mathbf{v}_1 = \lambda\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_3 = \lambda\mathbf{v}_3 + \mathbf{v}_2 \end{cases}$$

so the three columns of Q constitute a chain of generalized eigenvectors of length 3.

It is also easy to verify that

$$Q^{-1}AQ = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

3.18

The characteristic polynomials of A, B, C, D (from left to right) is $(\lambda - \lambda_1)^3(\lambda - \lambda_2), (\lambda - \lambda_1)^4, (\lambda - \lambda_1)^4, (\lambda - \lambda_1)^4$ respectively.

The minimal polynomials of A, B, C, D (from left to right) is $(\lambda - \lambda_1)^3(\lambda - \lambda_2), (\lambda - \lambda_1)^3, (\lambda - \lambda_1)^2, (\lambda - \lambda_1)$ respectively.

3.19

Because: $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, we have $\mathbf{A}^k\mathbf{x} = \lambda^k\mathbf{x}$

without loss of generality, we specify that: $f(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0$

so, for:

$$f(\mathbf{A})\mathbf{x} = a_n\mathbf{A}^n\mathbf{x} + a_{n-1}\mathbf{A}^{n-1}\mathbf{x} + \dots + a_0\mathbf{x} = (a_n\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0)\mathbf{x} = f(\lambda)\mathbf{x}$$

so, $f(\lambda)$ is an eigenvector of $f(\mathbf{A})$ with the same eigenvector \mathbf{x} .

3.20

There exists a nonsingular matrix Q made $Q^{-1}AQ = \hat{J}$, where \hat{J} is a Jordan form. If A has any nonzero eigenvalues, then $A^k \neq 0$ for all interger k .

so,when $A^k = 0$, we can conclude that its all eigenvalues must be zero. Thus to say, A has eigenvalues 0 with multiplicity n . Let $\hat{J} = \text{diag}\{\hat{J}_1, \hat{J}_2, \dots\}$, then $\hat{J}^k = \text{diag}\{\hat{J}_1^k, \hat{J}_2^k, \dots\}$. each \hat{J}_i is a Jordan block associated with eigenvalue 0. then, we can obtain from (3.40) that $\mathbf{A}_i^m = 0$ where the m is the largest order of all Jordan blocks, so for $k \geq m$, we have $\mathbf{A}_i^k = \mathbf{0}$. so $\mathbf{A}^k = \mathbf{0}$

3.21

First, calculating the eigenvalues of the A :

$$|\lambda E - A| = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 \lambda = 0$$

in this question, I want to use a complex method. so, the eigenvalues are $\lambda_1 = 0, \lambda_2 = \lambda_3 = 1$ for $\lambda_2 = \lambda_3 = 1$, $\text{rank}(E - A) = 2$, so, for eigenvalue 1, it doesn't have two independent eigenvectors, so there exists a nonsingular matrix Q made the $Q^{-1}AQ = \hat{J}$, where \hat{J} is a Jordan form.

from $Ax_1 = x_1$, we can calculate a $x_1 = [1 \ 0 \ 0]'$

from $Ax_2 = x_2 + x_1$, we can calculate the $x_2 = [1 \ 1 \ 1]'$

from $Ax = 0$, we can calculate a $x = [1 \ -1 \ 0]'$

so for $Q = [x, x_1, x_2]$, we have

$$Q^{-1}AQ = J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

we can split matrix J into B and C , where:

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$J = B + C$$

$$A^{10} = QJ^{10}Q^{-1} = Q(B + C)^{10}Q^{-1} = Q(B^{10} + C_{10}^1 B^9 C)Q^{-1} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

in the same way, we can get:

$$A^{103} = \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

from $f(\hat{J}) = Q^{-1}f(A)Q$ and from (3.48) we can readily know:

$$e^{\hat{J}t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

so

$$e^{At} = Qe^{\hat{J}t}Q^{-1} = \begin{bmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}$$

3.22

$$A_1 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

First method: $f(A) = h(A)$

for A_1 , define $h(\lambda) = \beta + \beta_1\lambda + \beta_2\lambda^2$, $f(\lambda) = e^{\lambda t}$ from the problem 3.13, the eigenvalues of the A_1 are 1, 2, 3

$$\begin{cases} f(1) = h(1) \\ f(2) = h(2) \\ f(3) = h(3) \end{cases}$$

we can calculate that:

$$\begin{cases} \beta_0 = 3e^t - 3e^{2t} + e^{3t} \\ \beta_1 = -2.5e^t + 4e^{2t} - 1.5e^{3t} \\ \beta_2 = 0.5e^t - e^{2t} + 0.5e^{3t} \end{cases}$$

$$e^{A_1 t} = h(A) = \begin{bmatrix} e^t & 4(e^{2t} - e^t) & 5(e^{3t} - e^t) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

in the same way, for A_4 :

$$\begin{aligned} e^{A_4 t} &= \beta_0 I + \beta_1 A_4 + \beta_2 A_4^2 \\ &= I + t \begin{bmatrix} 0 & 4 & 3t + 2t^2 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} + t^2/2 \begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \downarrow \\ \text{thus } &= \begin{bmatrix} 1 & 4t + 5t^2/2 & 3t + 2t^2 \\ 0 & 20t + 1 & 16t \\ 0 & -25t & -20t + 1 \end{bmatrix} \end{aligned}$$

The second way: $f(\hat{J}) = Qf(A)Q^{-1}$ for A_1 : we can calculate the eigenvectors of the eigenvalues and

use them to form the $Q, QAQ^{-1} = J$, where

$$Q = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

so the

$$e^{At} = Qe^{Jt}Q^{-1} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} e^t & 4(e^{3t} - e^t) & 5(e^{3t} - e^t) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

in the same way, we can get

$$e^{A_4t} = \begin{bmatrix} 1 & 4t + 5t^2/2 & 3t + 2t^2 \\ 0 & 20t + 1 & 16t \\ 0 & -25t & -20t + 1 \end{bmatrix}$$

3.23

if A is $n \times n$, we have:

$$f(A) = \alpha_0 + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}$$

$$g(A) = \beta_0 + \beta_1 A + \cdots + \beta_{n-1} A^{n-1}$$

Because A is commutative with itself, so we can conclude that $f(A)g(A) = g(A)f(A)$ and in particular $Ae^t = e^t A$

3.24

for

$$C = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

then

$$B = \ln C = \begin{bmatrix} \ln \lambda_1 & 0 & 0 \\ 0 & \ln \lambda_2 & 0 \\ 0 & 0 & \ln \lambda_3 \end{bmatrix}$$

if $\lambda_i = 0$, then the $\ln(\lambda_i)$ is not defined, then B does not exist.

owing to the C is a Jordan form, we can use (3.48) so, we have:

$$B = \ln C = \begin{bmatrix} \ln \lambda & \ln' \lambda & 0 \\ 0 & \ln \lambda & 0 \\ 0 & 0 & \ln \lambda \end{bmatrix} = \begin{bmatrix} \ln \lambda & 1/\lambda & 0 \\ 0 & \ln \lambda & 0 \\ 0 & 0 & \ln \lambda \end{bmatrix}$$

for an nonsingular C,there exists a nonsingular Q, made $Q^{-1}CQ = J$,where J is a Jordan from, and its $\lambda_i \neq 0$ so,for any nonsingular C, there exists a matrix B .

3.25

$$A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, (SI - A_3) = \begin{bmatrix} s-1 & 0 & 1 \\ 0 & s-1 & 0 \\ 0 & 0 & s-2 \end{bmatrix}$$

$$Adj(SI - A_3) = \begin{bmatrix} (s-1)(s-2) & 0 & -(s-1) \\ 0 & (s-1)(s-2) & 0 \\ 0 & 0 & (s-1)^2 \end{bmatrix}$$

we can conclude that $m(s)=(s-1)$,the characteristic polynomial is $\Delta(s) = (s-1)^2(s-2)$ so $\Delta(s)/m(s) = (s-1)(s-2)$,its consequence is correct.

3.26

$$\Delta(s)I = (SI - A)[R_0s^{n-1} + R_1s^{n-2} + \cdots + R_{n-2}s + R_{n-1}]$$

Expanding the polynomial on the right, we can just get:

$$s^n I + \alpha_1 s^{n-1} I + \cdots + \alpha_n I = R_0 s^n + (R_1 - R_0 A) s^{n-1} + \cdots - R_{n-1} A$$

Equating the coefficient matrix of s^k :

$$\left\{ \begin{array}{l} R_0 = I \\ R_1 = AR_0 + \alpha_1 I \\ R_2 = AR_1 + \alpha_2 I \\ \vdots \\ R_{n-1} = AR_{n-2} + \alpha_{n-1} I \\ AR_{n-1} + \alpha_n I = 0 \end{array} \right.$$

Let $\lambda_i \quad i = 1, 2, \cdots, n$ be the eigenvalues of A

Define $\Lambda_1 = \sum_{i=1}^n \lambda_i, \Lambda_k = \sum_{i=1}^n \lambda_i^k, k = 1, 2, \cdots$

There exists a matrix Q ,made $A = Q\hat{J}Q^{-1}$,where \hat{J} is a Jordan form.

Because $tr(BC) = tr(CB)$,we have:

$$tr(A) = tr(Q\hat{J}Q^{-1}) = tr(Q^{-1}Q\hat{J}) = tr(\hat{J}) = \sum_i^n \lambda_i = \Lambda_1$$

similarly: $\text{tr}(A^k) = \Lambda_k$

Owing to the following Newton's identity:

$$\Lambda_k + \alpha_1 \Lambda_{k-1} + \cdots + \alpha_{k-1} \Lambda_1 + k \alpha_k = 0$$

thus:

$$\begin{aligned} \alpha_k &= -\frac{1}{k} [\Lambda_k + \alpha_1 \Lambda_{k-1} + \cdots + \alpha_{k-1} \Lambda_1] \\ &= -\frac{1}{k} [\text{tr}(A^k) + \alpha_1 \text{tr}(A^{k-1}) + \cdots + \alpha_{k-1} \text{tr}(A)] \\ &= -\frac{1}{k} \text{tr}[A^k + \alpha_1 A^{k-1} + \cdots + \alpha_{k-1} A] \\ &= -\frac{1}{k} \text{tr}(AR_{k-1}) \end{aligned}$$

This verify the formula.

3.27

From the last two equations, we can get:

$$\begin{cases} R_{n-1} = A^{n-1} + \alpha_1 A^{n-2} + \cdots + \alpha_{n-2} A + \alpha_{n-1} I \\ 0 = AR_{n-1} + \alpha_n I \end{cases}$$

Multiply the first equation by matrix A , we can get:

$$AR_{n-1} = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-2} A^2 + \alpha_{n-1} A$$

Because, $AR_{n-1} = -\alpha_n I$, we can readily get:

$$A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-2} A^2 + \alpha_{n-1} A + \alpha_n I = 0$$

The Cayley-Hamilton theorem is verified.

3.28

Owing to:

$$\begin{cases} R_0 = I \\ R_1 = A + \alpha_1 I \\ R_2 = A^2 + \alpha_1 A + \alpha_2 I \\ \vdots \\ R_{n-1} = A^{n-1} + \alpha_1 A^{n-2} + \cdots + \alpha_{n-2} A + \alpha_{n-1} I \end{cases}$$

Let's substitute these formulas into the equation:

$$(SI - A)^{-1} = \frac{1}{\Delta(s)} [R_0 s^{n-1} + R_1 s^{n-2} + \cdots + R_{n-2} s + R_{n-1}]$$

we can get the equation as show in the problem.

3.29

Because the eigenvalues of A are all distinct, so matrix A can be diagonalized. so, there exists a nonsingular matrix Q , which can make $Q^{-1}AQ = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ thus:

$$Q^{-1}A = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}Q^{-1}$$

$$PA = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}P^{-1}$$

so, we have $p_i A = \lambda_i p_i$

3.30

Because all eigenvalues of A are all distinct, so there exists a matrix Q , made $Q^{-1}AQ = \hat{J}$ and owing to $Q^{-1}f(A)Q = f(\hat{J})$, then $Q^{-1}(SI - A)Q = (SI - \hat{J})$, so, we have:

$$\begin{aligned} (SI - A)^{-1} &= Q(SI - \hat{J})^{-1}Q^{-1} = [q_1 \quad q_2 \cdots q_n] \text{diag}\left\{\frac{1}{s - \lambda_1}, \frac{1}{s - \lambda_2}, \cdots, \frac{1}{s - \lambda_n}\right\} [p_1 \quad p_2 \cdots p_n]' \\ &= \sum \frac{1}{s - \lambda_i} q_i p_i \end{aligned}$$

3.31

from the Lyapunov equation $AM + MB = C$ in this problem, M is 2×1 dimension, we assume $M = [m_1, m_2]'$ take into the Lyapunov equation:

$$\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} 3 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

solve the equation, we can get: $M = [0 \quad 3]'$ The eigenvalues of A are $\lambda_1 = -1 + i, \lambda_2 = -1 - i$, The eigenvalues of B is $\mu_1 = 3$ so the eigenvalues of the Lyapunov equation are :

$$\begin{cases} \eta_1 = \lambda_1 + \mu_1 = 2 + i \\ \eta_2 = \lambda_2 + \mu_1 = 2 - i \end{cases}$$

Because the eigenvalues of the Lyapunov equation are not zero, so the Lyapunov equation is non-singular, and the solution is unique.

3.32

i

if $C = C_1 = [3 \ 3]'$

$$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

from the matrix equation, we can find the M is not exist.

ii

in the same way:

$$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

we can solve from the equation: $M = [3 - k \ k]'$, where k is an arbitrary constant. The eigenvalues of A are -1, -1, the eigenvalues of B is 1, so the eigenvalues of Lyapunov equation are 0, 0. Because all the eigenvalues of Lyapunov equation are zero, so the Lyapunov equation is singular, and owing to k is arbitrary, so the solution is not unique.

3.33

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} a_1 a_1 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & a_2 a_2 & a_2 a_3 \\ a_3 a_1 & a_3 a_2 & a_3 a_3 \end{bmatrix}$$

For A : the leading principle minor

$$\det \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} < 0$$

so the matrix A is neither positive definite nor semidefinite.

For B :

$$\det \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} < 0$$

so the matrix A is neither positive definite nor semidefinite.

For C : all the principle minors of C are greater than or equal to zero, so the matrix is semidefinite.

3.34

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$AA^* = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

The eigenvalues of AA^* are 1, 6, so the singular values of A are $1, \sqrt{6}$.

For B , B is a Hermite matrix, so the singular values of B are absolute values of B 's eigenvalues, the eigenvalues of B are $\frac{3+\sqrt{41}}{2}, \frac{3-\sqrt{41}}{2}$, so the singular values of B is $\frac{3+\sqrt{41}}{2}, \frac{\sqrt{41}-3}{2}$

3.35

The singular values is just the absolute values of its eigenvalues.

3.36

from the formula:

$$\det(I_m + AB) = \det(I_n + BA)$$

then:

$$\det(I_n + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} [b_1 \quad b_2 \cdots b_n]) = \det(I_1 + [b_1 \quad b_2 \cdots b_n] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}) = \det(1 + \sum_{m=1}^n a_m b_m) = 1 + \sum_{m=1}^n a_m b_m$$

3.37

$$N = \begin{bmatrix} \sqrt{s}I_m & A \\ 0 & \sqrt{s}I_n \end{bmatrix} \quad Q = \begin{bmatrix} \sqrt{s}I_m & 0 \\ B & \sqrt{s}I_n \end{bmatrix} \quad P = \begin{bmatrix} \sqrt{s}I_m & -A \\ -B & \sqrt{s}I_n \end{bmatrix}$$

then we have

$$NP = \begin{bmatrix} sI_m - AB & 0 \\ \sqrt{s}B & sI_n \end{bmatrix} \quad QP = \begin{bmatrix} sI_m & -\sqrt{s}A \\ 0 & sI_n - BA \end{bmatrix}$$

$$\det(N) = s^{\frac{m+n}{2}} = \det(Q)$$

then:

$$\det(NP) = \det(QP)$$

$$s^n \det(sI_m - AB) = s^m \det(sI_n - BA)$$

3.38

i

if $n > m$, then $A' A$ is singular, thus $(A' A)^{-1}$ is not exist, so the $(A' A)^{-1} A' y$ is not a solution.

ii

if $n = m$, then matrix A is reversible, so $(A' A)^{-1} A' y$ is just the $A^{-1} y$, it is a solution of $Ax = y$.