discrete_black_scholes_m3_ex1_v3

October 25, 2018

0.1 Discrete-Time Black Scholes

Welcome to your 1st assignment in Reinforcement Learning in Finance. This exercise will introduce Black-Scholes model as viewed through the lens of pricing an option as discrete-time replicating portfolio of stock and bond.

Instructions: - You will be using Python 3. - Avoid using for-loops and while-loops, unless you are explicitly told to do so. - Do not modify the (# GRADED FUNCTION [function name]) comment in some cells. Your work would not be graded if you change this. Each cell containing that comment should only contain one function. - After coding your function, run the cell right below it to check if your result is correct.

Let's get started!

0.2 About iPython Notebooks

iPython Notebooks are interactive coding environments embedded in a webpage. You will be using iPython notebooks in this class. You only need to write code between the ### START CODE HERE ### and ### END CODE HERE ### comments. After writing your code, you can run the cell by either pressing "SHIFT"+"ENTER" or by clicking on "Run Cell" (denoted by a play symbol) in the upper bar of the notebook.

We will often specify "(X lines of code)" in the comments to tell you about how much code you need to write. It is just a rough estimate, so don't feel bad if your code is longer or shorter.

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        %matplotlib inline

        from numpy.random import standard_normal, seed

        import scipy.stats as stats
        from scipy.stats import norm

        import sys

        sys.path.append("..")
        import grading

        import datetime
        import time
```

```
import bspline
import bspline.splinelab as splinelab
```

In [2]:
```
### ONLY FOR GRADING. DO NOT EDIT ###
submissions=dict()
assignment_key="J_L65CoiEeiwfQ53m1Mlug"
all_parts=["9jLRK","YoMns","Wc3NN","fcl3r"]
### ONLY FOR GRADING. DO NOT EDIT ###
```

In [3]:
```
COURSERA_TOKEN = 'Ky2vvzIxTBraMmfM' # the key provided to the Student under his/her emai
COURSERA_EMAIL = 'cilsya@yahoo.com' # the email
```

In [4]:
```
# The Black-Scholes prices
def bs_put(t, S0, K, r, sigma, T):
    d1 = (np.log(S0/K) + (r + 1/2 * sigma**2) * (T-t)) / sigma / np.sqrt(T-t)
    d2 = (np.log(S0/K) + (r - 1/2 * sigma**2) * (T-t)) / sigma / np.sqrt(T-t)
    price = K * np.exp(-r * (T-t)) * norm.cdf(-d2) - S0 * norm.cdf(-d1)
    return price

def bs_call(t, S0, K, r, sigma, T):
    d1 = (np.log(S0/K) + (r + 1/2 * sigma**2) * (T-t)) / sigma / np.sqrt(T-t)
    d2 = (np.log(S0/K) + (r - 1/2 * sigma**2) * (T-t)) / sigma / np.sqrt(T-t)
    price = S0 * norm.cdf(d1) - K * np.exp(-r * (T-t)) * norm.cdf(d2)
    return price

def d1(S0, K, r, sigma, T):
    return (np.log(S0/K) + (r + sigma**2 / 2) * T)/(sigma * np.sqrt(T))

def d2(S0, K, r, sigma, T):
    return (np.log(S0 / K) + (r - sigma**2 / 2) * T) / (sigma * np.sqrt(T))
```

Simulate N_{MC} stock price sample paths with T steps by the classical Black-Sholes formula.

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad S_{t+1} = S_t e^{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t} Z}$$

where Z is a standard normal random variable.
MC paths are simulated by GeneratePaths() method of DiscreteBlackScholes class.

0.2.1 Part 1

Class DiscreteBlackScholes implements the above calculations with class variables to math symbols mapping of:

$$\Delta S_t = S_{t+1} - e^{-r\Delta t} S_t \quad t = T-1, ..., 0$$

Instructions: Some portions of code in DiscreteBlackScholes have bee taken out. You are to implement the missing portions of code in DiscreteBlackScholes class.

$$\Pi_t = e^{-r\Delta t} \left[\Pi_{t+1} - u_t \Delta S_t\right] \quad t = T-1, ..., 0$$

- implement DiscreteBlackScholes.function_A_vec() method

$$A_{nm}^{(t)} = \sum_{k=1}^{N_{MC}} \Phi_n\left(X_t^k\right) \Phi_m\left(X_t^k\right) \left(\Delta \hat{S}_t^k\right)^2$$

- implement DiscreteBlackScholes.function_B_vec() method

$$B_n^{(t)} = \sum_{k=1}^{N_{MC}} \Phi_n\left(X_t^k\right) \left[\hat{\Pi}_{t+1}^k \Delta \hat{S}_t^k + \frac{1}{2\gamma\lambda} \Delta S_t^k\right]$$

- implement DiscreteBlackScholes.gen_paths() method using the following relation:

$$S_{t+1} = S_t e^{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t} Z}$$

where $Z \sim N(0,1)$

- implement parts of DiscreteBlackScholes.roll_backward()
  - DiscreteBlackScholes.bVals corresponds to B_t and is computed as

$$B_t = e^{-r\Delta t}\left[B_{t+1} + (u_{t+1} - u_t) S_{t+1}\right] \quad t = T-1, ..., 0$$

DiscreteBlackScholes.opt_hedge corresponds to ϕ_t and is computed as

$$\phi_t = \mathbf{A}_t^{-1} \mathbf{B}_t$$

```
In [5]: class DiscreteBlackScholes:
            """
            Class implementing discrete Black Scholes
            DiscreteBlackScholes is class for pricing and hedging under
            the real-world measure for a one-dimensional Black-Scholes setting
            """

            def __init__(self,
                         s0,
                         strike,
                         vol,
                         T,
                         r,
                         mu,
                         numSteps,
                         numPaths):
                """
                :param s0: initial price of the underlying
                :param strike: option strike
                :param vol: volatility
                :param T: time to maturity, in years
                :param r: risk-free rate,
                :param mu: real drift, asset drift
```

```python
        :param numSteps: number of time steps
        :param numPaths: number of Monte Carlo paths
        """
        self.s0 = s0
        self.strike = strike
        self.vol = vol
        self.T = T
        self.r = r
        self.mu = mu
        self.numSteps = numSteps
        self.numPaths = numPaths

        self.dt = self.T / self.numSteps  # time step
        self.gamma = np.exp(-r * self.dt)  # discount factor for one time step, i.e. gam

        self.sVals = np.zeros((self.numPaths, self.numSteps + 1), 'float')  # matrix of

        # initialize half of the paths with stock price values ranging from 0.5 to 1.5 o
        # the other half of the paths start with s0
        half_paths = int(numPaths / 2)

        if False:
            # Grau (2010) "Applications of Least-Squares Regressions to Pricing and Hedg
            self.sVals[:, 0] = (np.hstack((np.linspace(0.5 * s0, 1.5 * s0, half_paths),
                                           s0 * np.ones(half_paths, 'float')))).T

        self.sVals[:, 0] = s0 * np.ones(numPaths, 'float')
        self.optionVals = np.zeros((self.numPaths, self.numSteps + 1), 'float')  # matri
        self.intrinsicVals = np.zeros((self.numPaths, self.numSteps + 1), 'float')

        self.bVals = np.zeros((self.numPaths, self.numSteps + 1), 'float')  # matrix of
        self.opt_hedge = np.zeros((self.numPaths, self.numSteps + 1),
                              'float')  # matrix of optimal hedges calculated from cross
        self.X = None
        self.data = None  # matrix of features, i.e. self.X as sum of basis functions
        self.delta_S_hat = None

        # coef = 1.0/(2 * gamma * risk_lambda)
        # override it by zero to have pure risk hedge
        self.coef = 0.

    def gen_paths(self):
        """
        A simplest path generator
        """
        np.random.seed(42)
        # Spline basis of order p on knots k
```

```
### START CODE HERE ### ( 3-4 lines of code)
# self.sVals = your code goes here ...
# for-loop or while loop is allowed heres

# https://docs.scipy.org/doc/numpy-1.15.0/reference/generated/numpy.random.norma
# NOTE: Given in the instructions above
#       Z ~ N(0,1)
# NOTE: Z must match the size of the matrix of stock values, hence why we define
Z = np.random.normal( 0,
                      1,
                      size = (self.numSteps + 1, self.numPaths))

# Cycle through each time step (column) to simulate.
# The rows are all the stock values at the time step.
# Going to be implementing the equation given above
# St+1 = Ste*e^(mu - 1/2*(sigma^2))*dt + sigma*sqrt(dt)*Z
for t in range(self.numSteps):

    # For an entire current column of the self.sVals matrix of stock values,
    # sVals matrix rows should represent each stock, the columns represent the t
    # are the stock values of the stocker ticker (row) at that time (columnn)
    # set the value from the relation equation given.
    # NOTE: : means whatever amount of rows.
    #         t+1 because it is index base zero
    #         NOTE: we are using numpy broadcasting here. All of the tickers (ro
    #               that time (column) will be updated.
    #
    # Using the member variables supplied by the class to implement this equatio
    # St+1 = Ste*e^(mu - 1/2*(sigma^2))*dt + sigma*sqrt(dt)*Z
    #
    # NOTE: The member variables were commented in the class constructor __init_
    #       It may seem cryptic but it is just plugging in the values but dealin
    #       so transpose may get thrown in the mix.
    self.sVals[:, t+1] = self.sVals[:, t] * np.exp( (self.mu - 0.5*self.vol**2)*
                                                    +
                                                    (self.vol*np.sqrt(self.dt) *
### END CODE HERE ###

# like in QLBS
delta_S = self.sVals[:, 1:] - np.exp(self.r * self.dt) * self.sVals[:, :self.num
self.delta_S_hat = np.apply_along_axis(lambda x: x - np.mean(x), axis=0, arr=del

# state variable
# delta_t here is due to their conventions
self.X = - (self.mu - 0.5 * self.vol ** 2) * np.arange(self.numSteps + 1) * self

X_min = np.min(np.min(self.X))
X_max = np.max(np.max(self.X))
```

```python
        print('X.shape = ', self.X.shape)
        print('X_min, X_max = ', X_min, X_max)

        p = 4  # order of spline (as-is; 3 = cubic, 4: B-spline?)
        ncolloc = 12
        tau = np.linspace(X_min, X_max, ncolloc)  # These are the sites to which we woul

        # k is a knot vector that adds endpoints repeats as appropriate for a spline of
        # To get meaningful results, one should have ncolloc >= p+1
        k = splinelab.aptknt(tau, p)
        basis = bspline.Bspline(k, p)

        num_basis = ncolloc  # len(k) #
        self.data = np.zeros((self.numSteps + 1, self.numPaths, num_basis))

        print('num_basis = ', num_basis)
        print('dim self.data = ', self.data.shape)

        # fill it, expand function in finite dimensional space
        # in neural network the basis is the neural network itself
        t_0 = time.time()
        for ix in np.arange(self.numSteps + 1):
            x = self.X[:, ix]
            self.data[ix, :, :] = np.array([basis(el) for el in x])
        t_end = time.time()
        print('\nTime Cost of basis expansion:', t_end - t_0, 'seconds')

    def function_A_vec(self, t, reg_param=1e-3):
        """
        function_A_vec - compute the matrix A_{nm} from Eq. (52) (with a regularization!
        Eq. (52) in QLBS Q-Learner in the Black-Scholes-Merton article

        Arguments:
        t - time index, a scalar, an index into time axis of data_mat
        reg_param - a scalar, regularization parameter

        Return:
        - np.array, i.e. matrix A_{nm} of dimension num_basis x num_basis
        """
        X_mat = self.data[t, :, :]
        num_basis_funcs = X_mat.shape[1]
        this_dS = self.delta_S_hat[:, t]
        hat_dS2 = (this_dS ** 2).reshape(-1, 1)
        A_mat = np.dot(X_mat.T, X_mat * hat_dS2) + reg_param * np.eye(num_basis_funcs)
        return A_mat

    def function_B_vec(self, t, Pi_hat):
```

```python
        """
        function_B_vec - compute vector B_{n} from Eq. (52) QLBS Q-Learner in the Black-
        
        Arguments:
        t - time index, a scalar, an index into time axis of delta_S_hat
        Pi_hat - pandas.DataFrame of dimension N_MC x T of portfolio values
        Return:
        B_vec - np.array() of dimension num_basis x 1
        """
        tmp = Pi_hat * self.delta_S_hat[:, t] + self.coef * (np.exp((self.mu - self.r) *
        X_mat = self.data[t, :, :]  # matrix of dimension N_MC x num_basis

        B_vec = np.dot(X_mat.T, tmp)
        return B_vec

    def seed_intrinsic(self, strike=None, cp='P'):
        """
        initilaize option value and intrinsic value for each node
        """
        if strike is not None:
            self.strike = strike

        if cp == 'P':
            # payoff function at maturity T: max(K - S(T),0) for all paths
            self.optionVals = np.maximum(self.strike - self.sVals[:, -1], 0).copy()
            # payoff function for all paths, at all time slices
            self.intrinsicVals = np.maximum(self.strike - self.sVals, 0).copy()
        elif cp == 'C':
            # payoff function at maturity T: max(S(T) -K,0) for all paths
            self.optionVals = np.maximum(self.sVals[:, -1] - self.strike, 0).copy()
            # payoff function for all paths, at all time slices
            self.intrinsicVals = np.maximum(self.sVals - self.strike, 0).copy()
        else:
            raise Exception('Invalid parameter: %s'% cp)

        self.bVals[:, -1] = self.intrinsicVals[:, -1]

    def roll_backward(self):
        """
        Roll the price and optimal hedge back in time starting from maturity
        """

        for t in range(self.numSteps - 1, -1, -1):

            # determine the expected portfolio value at the next time node
            piNext = self.bVals[:, t+1] + self.opt_hedge[:, t+1] * self.sVals[:, t+1]
            pi_hat = piNext - np.mean(piNext)
```

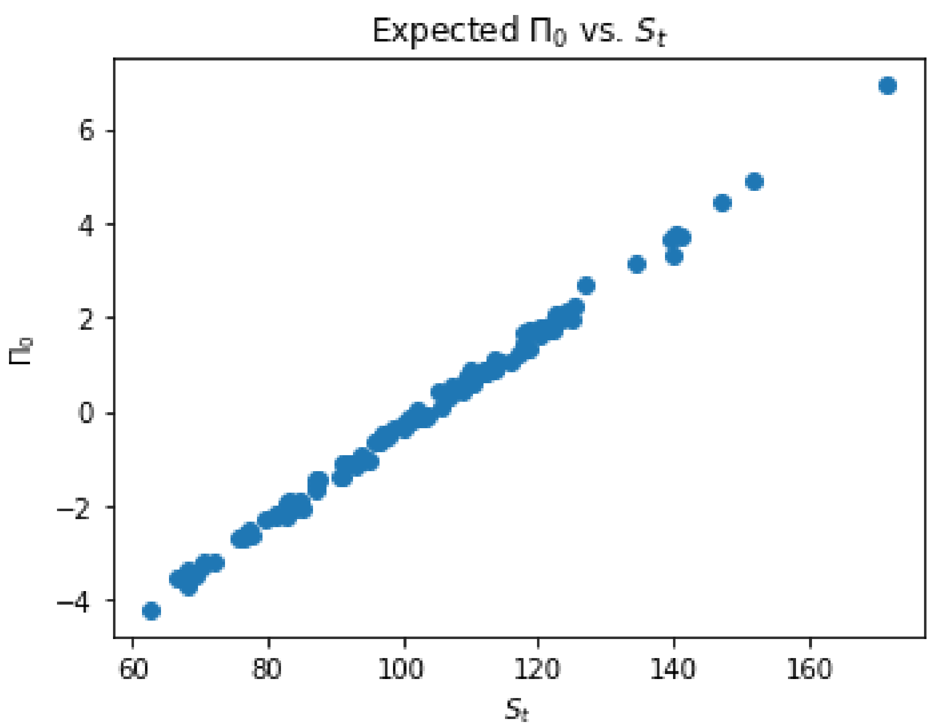
```
                A_mat = self.function_A_vec(t)
                B_vec = self.function_B_vec(t, pi_hat)
                phi = np.dot(np.linalg.inv(A_mat), B_vec)
                self.opt_hedge[:, t] = np.dot(self.data[t, :, :], phi)

                ### START CODE HERE ### ( 1-2 lines of code)
                # implement code to update self.bVals
                # self.bVals[:,t] = your code goes here ....

                # Implementing the equation provided above.
                # Again, the variables are supplied above in the constructor .__i
                # NOTE: opt_hedge corresponds to phi at time t.
                self.bVals[:,t] = np.exp( -self.r * self.dt) * (self.bVals[:,t+1]
                                          +
                                          (self.opt_hedge[:, t+1] - self.opt_hedge[:,t]) * s


                ### END CODE HERE ###

            # calculate the initial portfolio value
            initPortfolioVal = self.bVals[:, 0] + self.opt_hedge[:, 0] * self.sVals[:, 0]

            # use only the second half of the paths generated with paths starting from S0
            optionVal = np.mean(initPortfolioVal)
            optionValVar = np.std(initPortfolioVal)
            delta = np.mean(self.opt_hedge[:, 0])

            return optionVal, delta, optionValVar
```

```
In [6]: np.random.seed(42)
        strike_k = 95
        test_vol = 0.2
        test_mu = 0.03
        dt = 0.01
        rfr = 0.05
        num_paths = 100
        num_periods = 252

        hMC = DiscreteBlackScholes(100, strike_k, test_vol, 1., rfr, test_mu, num_periods, num_p
        hMC.gen_paths()

        t = hMC.numSteps - 1
        piNext = hMC.bVals[:, t+1] + 0.1 * hMC.sVals[:, t+1]
        pi_hat = piNext - np.mean(piNext)

        A_mat = hMC.function_A_vec(t)
        B_vec = hMC.function_B_vec(t, pi_hat)
        phi = np.dot(np.linalg.inv(A_mat), B_vec)
```

```
        opt_hedge = np.dot(hMC.data[t, :, :], phi)

        # plot the results
        fig = plt.figure(figsize=(12,4))
        ax1 = fig.add_subplot(121)

        ax1.scatter(hMC.sVals[:,t], pi_hat)
        ax1.set_title(r'Expected $\Pi_0$ vs. $S_t$')
        ax1.set_xlabel(r'$S_t$')
        ax1.set_ylabel(r'$\Pi_0$')
```

```
X.shape =  (100, 253)
X_min, X_max =  4.10743882917 5.16553756345
num_basis =  12
dim self.data =  (253, 100, 12)

Time Cost of basis expansion: 12.204232454299927 seconds
```

Out[6]: <matplotlib.text.Text at 0x7f0f48c59f98>




```
In [ ]: ### GRADED PART (DO NOT EDIT) ###
```

```
part_1 = list(pi_hat)
try:
    part1 = " ".join(map(repr, part_1))
except TypeError:
    part1 = repr(part_1)
submissions[all_parts[0]]=part1
grading.submit(COURSERA_EMAIL, COURSERA_TOKEN, assignment_key,all_parts[:1],all_parts,su
pi_hat
### GRADED PART (DO NOT EDIT) ###
```

```
Submission successful, please check on the coursera grader page for the status
```

Out[ ]:

```
array([ 0.81274895, -3.49043554,  0.69994334,  1.61239986, -0.25153316,
       -3.19082265,  0.8848621 , -2.0380868 ,  0.45033564,  3.74872863,
       -0.6568227 ,  1.74148929,  0.94314331, -4.19716113,  1.72135256,
       -0.66188482,  6.95675041, -2.20512677, -0.14942482,  0.30067272,
        3.33419402,  0.68536713,  1.65097153,  2.69898611,  1.22528159,
        1.47188744, -2.48129898, -0.37360224,  0.81064666, -1.05269459,
        0.02476551, -1.88267258,  0.11748169, -0.9038195 ,  0.69753811,
       -0.54805029,  1.97594593, -0.44331403,  0.62134931, -1.86191032,
       -3.21226413,  2.24508097, -2.23451292, -0.13488281,  3.64364848,
       -0.11270281, -1.15582237, -3.30169455,  1.74454841, -1.10425448,
        2.10192819,  1.80570507, -1.68587001, -1.42113397, -2.70292006,
        0.79454199, -2.05396827,  3.13973887, -1.08786662,  0.42347686,
        1.32787012,  0.55924965, -3.54140814, -3.70258632,  2.14853641,
        1.11495458,  3.69639676,  0.62864736, -2.62282995, -0.05315552,
        1.05789698,  1.8023196 , -3.35217374, -2.30436466, -2.68609519,
        0.95284884, -1.35963013, -0.56273408, -0.08311276,  0.79044269,
        0.46247485, -1.04921463, -2.18122285,  1.82920128,  1.05635272,
        0.90161346, -1.93870347, -0.37549305, -1.96383274,  1.9772888 ,
       -1.37386984,  0.95230068,  0.88842589, -1.42214528, -2.60256696,
       -1.53509699,  4.47491253,  4.87735375, -0.19068803, -1.08711941])
```

In [ ]:

```
# input parameters
s0 = 100.0
strike = 100.0
r = 0.05
mu = 0.07 # 0.05
vol = 0.4
T = 1.0

# Simulation Parameters
numPaths = 50000  # number of Monte Carlo trials
numSteps = 6

# create the class object
hMC = DiscreteBlackScholes(s0, strike, vol, T, r, mu, numSteps, numPaths)
```

```
# calculation
hMC.gen_paths()
hMC.seed_intrinsic()
option_val, delta, option_val_variance = hMC.roll_backward()
bs_call_value = bs_put(0, s0, K=strike, r=r, sigma=vol, T=T)
print('Option value = ', option_val)
print('Option value variance = ', option_val_variance)
print('Option delta = ', delta)
print('BS value', bs_call_value)
```

```
X.shape =  (50000, 7)
X_min, X_max =  2.96880459823 6.37164911461
num_basis =  12
dim self.data =  (7, 50000, 12)

Time Cost of basis expansion: 147.85048985481262 seconds
Option value =  13.1083499076
Option value variance =  5.17079676287
Option delta =  -0.356133722777
BS value 13.1458939003
```

In [ ]:

```
### GRADED PART (DO NOT EDIT) ###
part2 = str(option_val)
submissions[all_parts[1]]=part2
grading.submit(COURSERA_EMAIL, COURSERA_TOKEN, assignment_key,all_parts[:2],all_parts,su
option_val
### GRADED PART (DO NOT EDIT) ###
```

```
Submission successful, please check on the coursera grader page for the status
```

Out[ ]: 13.108349907565021

In [ ]:

```
strikes = np.linspace(85, 110, 6)
results = [None] * len(strikes)
bs_prices = np.zeros(len(strikes))
bs_deltas = np.zeros(len(strikes))
numPaths = 50000
hMC = DiscreteBlackScholes(s0, strike, vol, T, r, mu, numSteps, numPaths)
hMC.gen_paths()
for ix, k_strike in enumerate(strikes):
    hMC.seed_intrinsic(k_strike)
    results[ix] = hMC.roll_backward()
    bs_prices[ix] = bs_put(0, s0, K=k_strike, r=r, sigma=vol, T=T)
    bs_deltas[ix] = norm.cdf(d1(s0, K=k_strike, r=r, sigma=vol, T=T)) - 1
bs_prices
```

```
X.shape =  (50000, 7)
X_min, X_max =  2.96880459823 6.37164911461
num_basis =  12
dim self.data =  (7, 50000, 12)

Time Cost of basis expansion: 148.89949584007263 seconds
```

```
Out[ ]: array([  6.70326307,   8.59543726,  10.74614496,  13.1458939 ,
                15.78197485,  18.63949388])
```

```
In [ ]: mc_prices = np.array([x[0] for x in results])
        mc_deltas = np.array([x[1] for x in results])
        price_variances = np.array([x[-1] for x in results])
        prices_diff = mc_prices - bs_prices
        deltas_diff = mc_deltas - bs_deltas
        # price_variances
```

```
In [ ]: ### GRADED PART (DO NOT EDIT) ###

        part_3 = list(prices_diff)
        try:
            part3 = " ".join(map(repr, part_3))
        except TypeError:
            part3 = repr(part_3)
        submissions[all_parts[2]]=part3
        grading.submit(COURSERA_EMAIL, COURSERA_TOKEN, assignment_key,all_parts[:3],all_parts,su
        prices_diff
        ### GRADED PART (DO NOT EDIT) ###
```

```
Submission successful, please check on the coursera grader page for the status
```

```
Out[ ]: array([-0.03641514, -0.04034142, -0.039966  , -0.03754399, -0.03240012,
               -0.02997066])
```

```
In [ ]: ### GRADED PART (DO NOT EDIT) ###
        part_4 = list(deltas_diff)
        try:
            part4 = " ".join(map(repr, part_4))
        except TypeError:
            part4= repr(part_4)
        submissions[all_parts[3]]=part4
        grading.submit(COURSERA_EMAIL, COURSERA_TOKEN, assignment_key,all_parts[:4],all_parts,su
        deltas_diff
        ### GRADED PART (DO NOT EDIT) ###
```

```
Submission successful, please check on the coursera grader page for the status
```

Out[ ]:
```
array([ 0.01279806,  0.01416029,  0.01532709,  0.01645681,  0.01715352,
        0.01780661])
```