dp_qlbs_oneset_m3_ex2_v3

October 28, 2018

0.1 The QLBS model for a European option

Welcome to your 2nd assignment in Reinforcement Learning in Finance. In this exercise you will arrive to an option price and the hedging portfolio via standard toolkit of Dynamic Pogramming (DP). QLBS model learns both the optimal option price and optimal hedge directly from trading data.

Instructions: - You will be using Python 3. - Avoid using for-loops and while-loops, unless you are explicitly told to do so. - Do not modify the (# GRADED FUNCTION [function name]) comment in some cells. Your work would not be graded if you change this. Each cell containing that comment should only contain one function. - After coding your function, run the cell right below it to check if your result is correct. - When encountering `# dummy code - remove` please replace this code with your own

After this assignment you will: - Re-formulate option pricing and hedging method using the language of Markov Decision Processes (MDP) - Setup foward simulation using Monte Carlo - Expand optimal action (hedge) $a_t^\star(X_t)$ and optimal Q-function $Q_t^\star(X_t, a_t^\star)$ in basis functions with time-dependend coefficients

Let's get started!

0.2 About iPython Notebooks

iPython Notebooks are interactive coding environments embedded in a webpage. You will be using iPython notebooks in this class. You only need to write code between the ### START CODE HERE ### and ### END CODE HERE ### comments. After writing your code, you can run the cell by either pressing "SHIFT"+"ENTER" or by clicking on "Run Cell" (denoted by a play symbol) in the upper bar of the notebook.

We will often specify "(X lines of code)" in the comments to tell you about how much code you need to write. It is just a rough estimate, so don't feel bad if your code is longer or shorter.

```
In [1]: #import warnings
        #warnings.filterwarnings("ignore")

        import numpy as np
        import pandas as pd
        from scipy.stats import norm
        import random
        import time
        import matplotlib.pyplot as plt
        import sys
```

```
sys.path.append("..")
import grading
```

```
In [2]: ### ONLY FOR GRADING. DO NOT EDIT ###
        submissions=dict()
        assignment_key="wLtf3SoiEeieSRL7rCBNJA"
        all_parts=["15mYc", "h1P6Y", "q9QW7","s7MpJ","Pa177"]
        ### ONLY FOR GRADING. DO NOT EDIT ###
```

```
In [18]: COURSERA_TOKEN = 'gF094cwtidz2YQpP' # the key provided to the Student under his/her e
         COURSERA_EMAIL = 'cilsya@yahoo.com' # the email
```

0.3 Parameters for MC simulation of stock prices

```
In [4]: S0 = 100       # initial stock price
        mu = 0.05      # drift
        sigma = 0.15   # volatility
        r = 0.03       # risk-free rate
        M = 1          # maturity

        T = 24         # number of time steps
        N_MC = 10000   # number of paths

        delta_t = M / T                # time interval
        gamma = np.exp(- r * delta_t)  # discount factor
```

0.3.1 Black-Sholes Simulation

Simulate N_{MC} stock price sample paths with T steps by the classical Black-Sholes formula.

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad S_{t+1} = S_t e^{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t} Z}$$

where Z is a standard normal random variable.
Based on simulated stock price S_t paths, compute state variable X_t by the following relation.

$$X_t = -\left(\mu - \frac{1}{2}\sigma^2\right) t\Delta t + \log S_t$$

Also compute

$$\Delta S_t = S_{t+1} - e^{r\Delta t} S_t \qquad \Delta \hat{S}_t = \Delta S_t - \Delta \bar{S}_t \qquad t = 0, ..., T-1$$

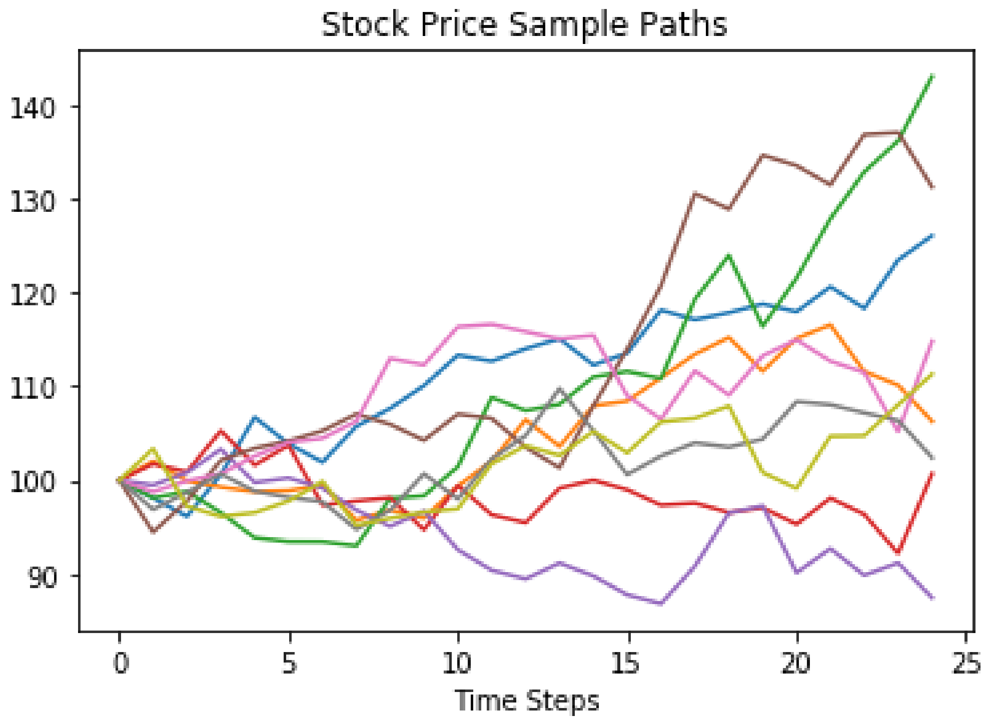
where $\Delta \bar{S}_t$ is the sample mean of all values of ΔS_t.
Plots of 5 stock price S_t and state variable X_t paths are shown below.

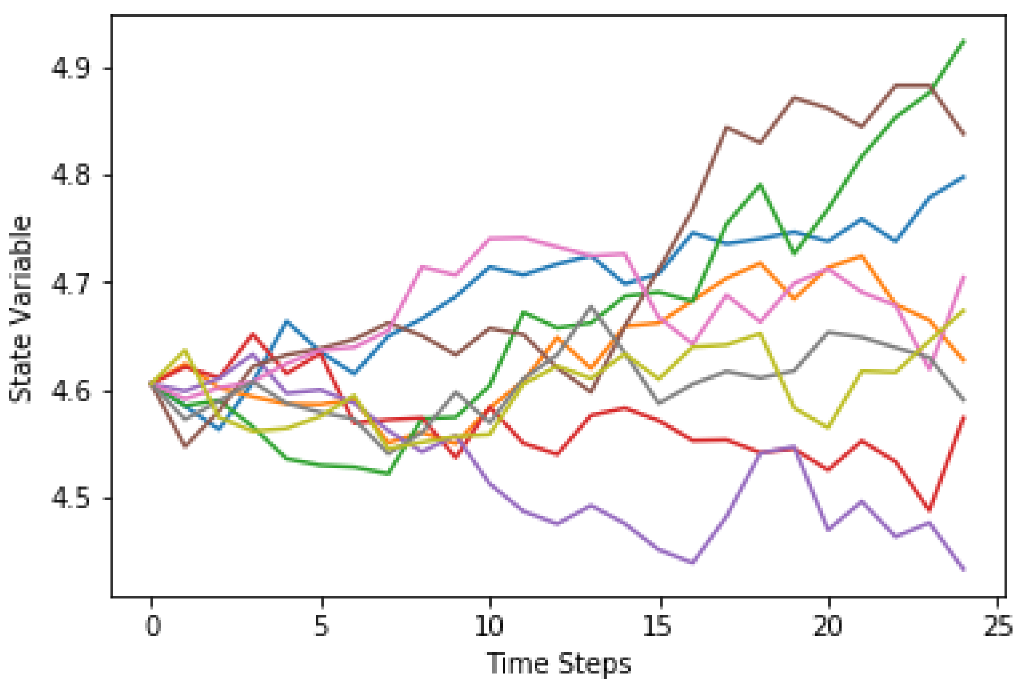
```
In [5]: # make a dataset
        starttime = time.time()
        np.random.seed(42)

        # stock price
```

```
S = pd.DataFrame([], index=range(1, N_MC+1), columns=range(T+1))
S.loc[:,0] = S0

# standard normal random numbers
RN = pd.DataFrame(np.random.randn(N_MC,T), index=range(1, N_MC+1), columns=range(1, T+

for t in range(1, T+1):
    S.loc[:,t] = S.loc[:,t-1] * np.exp((mu - 1/2 * sigma**2) * delta_t + sigma * np.sq

delta_S = S.loc[:,1:T].values - np.exp(r * delta_t) * S.loc[:,0:T-1]
delta_S_hat = delta_S.apply(lambda x: x - np.mean(x), axis=0)

# state variable
X = - (mu - 1/2 * sigma**2) * np.arange(T+1) * delta_t + np.log(S)   # delta_t here is

endtime = time.time()
print('\nTime Cost:', endtime - starttime, 'seconds')
```

```
Time Cost: 0.20280027389526367 seconds
```

In [6]:
```
# plot 10 paths
step_size = N_MC // 10
idx_plot = np.arange(step_size, N_MC, step_size)

plt.plot(S.T.iloc[:,idx_plot])
plt.xlabel('Time Steps')
plt.title('Stock Price Sample Paths')
plt.show()

plt.plot(X.T.iloc[:,idx_plot])
plt.xlabel('Time Steps')
plt.ylabel('State Variable')
plt.show()
```





Define function *terminal_payoff* to compute the terminal payoff of a European put option.

$$H_T(S_T) = \max(K - S_T, 0)$$

```
In [7]: def terminal_payoff(ST, K):
            # ST   final stock price
            # K    strike
            payoff = max(K - ST, 0)
            return payoff
```

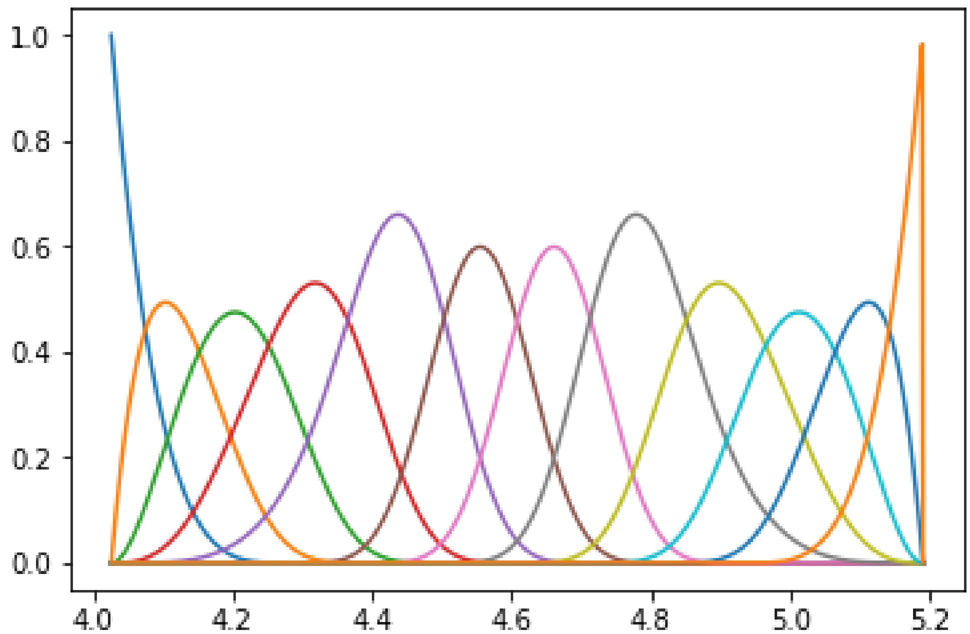
```
In [8]: type(delta_S)
```

```
Out[8]: pandas.core.frame.DataFrame
```

0.4 Define spline basis functions

```
In [9]: import bspline
        import bspline.splinelab as splinelab

        X_min = np.min(np.min(X))
        X_max = np.max(np.max(X))
        print('X.shape = ', X.shape)
        print('X_min, X_max = ', X_min, X_max)

        p = 4              # order of spline (as-is; 3 = cubic, 4: B-spline?)
        ncolloc = 12

        tau = np.linspace(X_min,X_max,ncolloc)  # These are the sites to which we would like t

        # k is a knot vector that adds endpoints repeats as appropriate for a spline of order p
        # To get meaninful results, one should have ncolloc >= p+1
        k = splinelab.aptknt(tau, p)

        # Spline basis of order p on knots k
        basis = bspline.Bspline(k, p)

        f = plt.figure()
        # B   = bspline.Bspline(k, p)     # Spline basis functions
        print('Number of points k = ', len(k))
        basis.plot()

        plt.savefig('Basis_functions.png', dpi=600)
```

```
X.shape =  (10000, 25)
X_min, X_max =  4.024923524903037 5.190802775129617
Number of points k =  17
```



<Figure size 432x288 with 0 Axes>

```
In [10]: type(basis)

Out[10]: bspline.bspline.Bspline

In [11]: X.values.shape

Out[11]: (10000, 25)
```

0.4.1 Make data matrices with feature values

"Features" here are the values of basis functions at data points The outputs are 3D arrays of dimensions num_tSteps x num_MC x num_basis

```
In [12]: num_t_steps = T + 1
         num_basis =  ncolloc # len(k) #

         data_mat_t = np.zeros((num_t_steps, N_MC,num_basis ))
         print('num_basis = ', num_basis)
         print('dim data_mat_t = ', data_mat_t.shape)

         t_0 = time.time()
         # fill it
         for i in np.arange(num_t_steps):
```

```
            x = X.values[:,i]
            data_mat_t[i,:,:] = np.array([ basis(el) for el in x ])

        t_end = time.time()
        print('Computational time:', t_end - t_0, 'seconds')
```

```
num_basis =  12
dim data_mat_t =  (25, 10000, 12)
Computational time: 55.5485999584198 seconds
```

In [13]:
```
# save these data matrices for future re-use
np.save('data_mat_m=r_A_%d' % N_MC, data_mat_t)
```

In [14]:
```
print(data_mat_t.shape)  # shape num_steps x N_MC x num_basis
print(len(k))
```

```
(25, 10000, 12)
17
```

0.5 Dynamic Programming solution for QLBS

The MDP problem in this case is to solve the following Bellman optimality equation for the action-value function.

$$Q_t^\star\left(x,a\right) = \mathbb{E}_t\left[R_t\left(X_t,a_t,X_{t+1}\right) + \gamma\max_{a_{t+1}\in\mathcal{A}}Q_{t+1}^\star\left(X_{t+1},a_{t+1}\right)|X_t = x, a_t = a\right], t = 0,...,T-1, \quad \gamma = e^{-r\Delta t}$$

where $R_t\left(X_t,a_t,X_{t+1}\right)$ is the one-step time-dependent random reward and $a_t\left(X_t\right)$ is the action (hedge).

Detailed steps of solving this equation by Dynamic Programming are illustrated below.

With this set of basis functions $\left\{\Phi_n\left(X_t^k\right)\right\}_{n=1}^N$, expand the optimal action (hedge) $a_t^\star\left(X_t\right)$ and optimal Q-function $Q_t^\star\left(X_t,a_t^\star\right)$ in basis functions with time-dependent coefficients.

$$a_t^\star\left(X_t\right) = \sum_n^N\phi_{nt}\Phi_n\left(X_t\right) \qquad Q_t^\star\left(X_t,a_t^\star\right) = \sum_n^N\omega_{nt}\Phi_n\left(X_t\right)$$

Coefficients ϕ_{nt} and ω_{nt} are computed recursively backward in time for $t = T1,...,0$.

Coefficients for expansions of the optimal action $a_t^\star\left(X_t\right)$ are solved by

$$\phi_t = \mathbf{A}_t^{-1}\mathbf{B}_t$$

where $\mathbf{A}_t$ and $\mathbf{B}_t$ are matrix and vector respectively with elements given by

$$A_{nm}^{(t)} = \sum_{k=1}^{N_{MC}}\Phi_n\left(X_t^k\right)\Phi_m\left(X_t^k\right)\left(\Delta\hat{S}_t^k\right)^2 \qquad B_n^{(t)} = \sum_{k=1}^{N_{MC}}\Phi_n\left(X_t^k\right)\left[\hat{\Pi}_{t+1}^k\Delta\hat{S}_t^k + \frac{1}{2\gamma\lambda}\Delta S_t^k\right]$$

$$\Delta S_t = S_{t+1} - e^{-r\Delta t}S_t \quad t = T-1,...,0$$

where $\Delta\hat{S}_t$ is the sample mean of all values of ΔS_t.

Define function *function_A* and *function_B* to compute the value of matrix $\mathbf{A}_t$ and vector $\mathbf{B}_t$.

0.6 Define the option strike and risk aversion parameter

In [15]:
```
risk_lambda = 0.001 # risk aversion
K = 100             # option stike

# Note that we set coef=0 below in function function_B_vec. This correspond to a pure
```

0.6.1 Part 1 Calculate coefficients ϕ_{nt} of the optimal action $a_t^\star(X_t)$

Instructions: - implement function_A_vec() which computes $A_{nm}^{(t)}$ matrix - implement function_B_vec() which computes $B_n^{(t)}$ column vector

In [16]:
```
# functions to compute optimal hedges
def function_A_vec(t, delta_S_hat, data_mat, reg_param):
    """
    function_A_vec - compute the matrix A_{nm} from Eq. (52) (with a regularization!)
    Eq. (52) in QLBS Q-Learner in the Black-Scholes-Merton article

    Arguments:
    t - time index, a scalar, an index into time axis of data_mat
    delta_S_hat - pandas.DataFrame of dimension N_MC x T
    data_mat - pandas.DataFrame of dimension T x N_MC x num_basis
    reg_param - a scalar, regularization parameter

    Return:
    - np.array, i.e. matrix A_{nm} of dimension num_basis x num_basis
    """

    ### START CODE HERE ### ( 5-6 lines of code)
    # store result in A_mat for grading

#     # The cell above shows the equations we need
#     #  Eq. (53) in QLBS Q-Learner in the Black-Scholes-Merton article we are trying
#     #  Phi* = (At^-1)(Bt)
#     #
#     # This function solves for the A coeffecient, which is shown in the cell above,
#     # Eq. (52) in QLBS Q-Learner in the Black-Scholes-Merton article
#     #
#     # The article is located here
#     # https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3087076

#     # Get the data matrix at this specific time index
#     Xt = data_mat[t,:,:]

#     # As shown in the description of the arguments in this function
#     # data_mat - pandas.DataFrame of dimension T x N_MC x num_basis
#     #
#     # We got Xt at a certain t time index, so
```

```
#     # Xt pandas.DataFrame of dimension N_MC x num_basis
#     #
#     # Therefore...
#     num_basis = Xt.shape[1]

#     # Now we need Delta S hat at this time index for the
#     # 'A' coefficient from the
#     # Eq. (52) in QLBS Q-Learner in the Black-Scholes-Merton article
#     #
#     # We are feed the parameter delta_S_hat into this function
#     # and
#     # delta_S_hat - pandas.DataFrame of dimension N_MC x T
#     #
#     # We what the delta_S_hat at this time index
#     #
#     # Therefore...
#     current_delta_S_hat = delta_S_hat.loc[:, t]

#     # The last term in the A coefficient calculation in the
#     # Eq. (52) in QLBS Q-Learner in the Black-Scholes-Merton article
#     # is delta_S_hat squared
#     #
#     # NOTE: There is .reshape(-1,1) which means that 1 for the columns
#     #       MUST be respected, but the -1 for the rows means that whatever
#     #       elements are left, fill it up to be whatever number.
#     current_delta_S_hat_squared = np.square(current_delta_S_hat).reshape( -1, 1)

#     # Now we have the terms to make up the equation.
#     # Eq. (52) in QLBS Q-Learner in the Black-Scholes-Merton article
#     # NOTE: The summation is not done in this function.
#     # NOTE: You do not see it in the equation
#     #       Eq. (52) in QLBS Q-Learner in the Black-Scholes-Merton article
#     #       but regularization is a technique used in Machine Learning.
#     #       You add the term.
#     #       np.eye() creates an identity matrix of size you specify.
#     #
#     # NOTE: When doing dot products, might have to transpose so the dimensions
#     #       align.
#     A_mat = ( np.dot( Xt.T, Xt*current_delta_S_hat_squared )
#               +
#               reg_param * np.eye(num_basis) )

    X_mat = data_mat[t, :, :]
    num_basis_funcs = X_mat.shape[1]
    this_dS = delta_S_hat.loc[:, t]
    hat_dS2 = (this_dS ** 2).reshape(-1, 1)
    A_mat = np.dot(X_mat.T, X_mat * hat_dS2) + reg_param * np.eye(num_basis_funcs)
```

```
    ### END CODE HERE ###
    return A_mat


def function_B_vec(t,
                   Pi_hat,
                   delta_S_hat=delta_S_hat,
                   S=S,
                   data_mat=data_mat_t,
                   gamma=gamma,
                   risk_lambda=risk_lambda):
    """
    function_B_vec - compute vector B_{n} from Eq. (52) QLBS Q-Learner in the Black-S

    Arguments:
    t - time index, a scalar, an index into time axis of delta_S_hat
    Pi_hat - pandas.DataFrame of dimension N_MC x T of portfolio values
    delta_S_hat - pandas.DataFrame of dimension N_MC x T
    S - pandas.DataFrame of simulated stock prices of dimension N_MC x T
    data_mat - pandas.DataFrame of dimension T x N_MC x num_basis
    gamma - one time-step discount factor $exp(-r \delta t)$
    risk_lambda - risk aversion coefficient, a small positive number
    Return:
    np.array() of dimension num_basis x 1
    """
    # coef = 1.0/(2 * gamma * risk_lambda)
    # override it by zero to have pure risk hedge

    ### START CODE HERE ### ( 5-6 lines of code)
    # store result in B_vec for grading

#     # Get the data matrix at this specific time index
#     Xt = data_mat[t,:,:]

#     # Computer the first term in the brackets.
#     first_term = Pi_hat[ :, t+1 ] * delta_S_hat.loc[:, t]

#     # NOTE: for the last term in the equation
#     # Eq. (52) QLBS Q-Learner in the Black-Scholes-Merton article
#     #
#     # would be
#     # last_term =  1.0/(2 * gamma * risk_lambda) * S.loc[:, t]
#     last_coefficient = 1.0/(2 * gamma * risk_lambda)
#     #
#     # But the instructions say make it equal override it by zero to have pure risk 
#     last_coefficient = 0
#     last_term =  last_coefficient * S.loc[:, t]
```

```
#     # Compute
#     second_factor = first_term + last_term

#     # Compute the equation
#     # NOTE: When doing dot products, might have to transpose so the dimensions
#     #       align.
#     B_vec = np.dot(Xt.T, second_factor)

    tmp = Pi_hat.loc[:,t+1] * delta_S_hat.loc[:, t]
    X_mat = data_mat[t, :, :]  # matrix of dimension N_MC x num_basis
    B_vec = np.dot(X_mat.T, tmp)

    ### END CODE HERE ###
    return B_vec
```

In [19]:
```
### GRADED PART (DO NOT EDIT) ###
reg_param = 1e-3
np.random.seed(42)

A_mat = function_A_vec(T-1, delta_S_hat, data_mat_t, reg_param)
idx_row = np.random.randint(low=0, high=A_mat.shape[0], size=50)

np.random.seed(42)
idx_col = np.random.randint(low=0, high=A_mat.shape[1], size=50)


part_1 = list(A_mat[idx_row, idx_col])
try:
    part1 = " ".join(map(repr, part_1))
except TypeError:
    part1 = repr(part_1)


submissions[all_parts[0]]=part1
grading.submit(COURSERA_EMAIL, COURSERA_TOKEN, assignment_key,all_parts[:1],all_parts
A_mat[idx_row, idx_col]
### GRADED PART (DO NOT EDIT) ###
```

```
D:\application\Anaconda3\envs\pyalgo\lib\site-packages\ipykernel_launcher.py:82: FutureWarning


Submission successful, please check on the coursera grader page for the status
```

Out[19]:
```
array([12261.42554869,  1259.28492179,   176.92982137, 11481.78830269,
        6579.62177219, 12261.42554869,   628.29798339,   189.70711815,
       12261.42554869,   176.92982137,   176.92982137, 11481.78830269,
        6579.62177219,  1259.28492179, 11481.78830269, 11481.78830269,
         189.70711815, 10408.62274335,  6579.62177219,    18.31727282,
```

```
11481.78830269,    32.94988345, 10408.62274335,    18.31727282,
   32.94988345,  6579.62177219,    16.09789819,    32.94988345,
  628.29798339, 10408.62274335,    32.94988345,  3275.69869791,
   16.09789819,   176.92982137,   176.92982137,   628.29798339,
   32.94988345,    32.94988345,   189.70711815,    32.94988345,
12261.42554869,  1259.28492179,  3275.69869791,   189.70711815,
 6579.62177219,   189.70711815, 12261.42554869,  6579.62177219,
 3275.69869791, 12261.42554869])
```

In [20]:
```python
### GRADED PART (DO NOT EDIT) ###
np.random.seed(42)
risk_lambda = 0.001
Pi = pd.DataFrame([], index=range(1, N_MC+1), columns=range(T+1))
Pi.iloc[:,-1] = S.iloc[:,-1].apply(lambda x: terminal_payoff(x, K))

Pi_hat = pd.DataFrame([], index=range(1, N_MC+1), columns=range(T+1))
Pi_hat.iloc[:,-1] = Pi.iloc[:,-1] - np.mean(Pi.iloc[:,-1])
B_vec = function_B_vec(T-1, Pi_hat, delta_S_hat, S, data_mat_t, gamma, risk_lambda)

part_2 = list(B_vec)
try:
    part2 = " ".join(map(repr, part_2))
except TypeError:
    part2 = repr(part_2)


submissions[all_parts[1]]=part2
grading.submit(COURSERA_EMAIL, COURSERA_TOKEN, assignment_key,all_parts[:2],all_parts

B_vec
### GRADED PART (DO NOT EDIT) ###
```

```
Submission successful, please check on the coursera grader page for the status
```

Out[20]:
```
array([ 3.29073713e+01, -2.95729027e+02, -8.73272905e+02, -3.31856654e+03,
       -1.25928899e+04, -1.14032852e+04, -2.91636810e+03, -3.38216415e+00,
       -1.33830723e+02, -1.36875328e+02, -6.60942460e+01, -3.07904971e+01])
```

0.7 Compute optimal hedge and portfolio value

Call *function_A* and *function_B* for $t = T-1, ..., 0$ together with basis function $\Phi_n(X_t)$ to compute optimal action $a_t^\star(X_t) = \sum_n^N \phi_{nt}\Phi_n(X_t)$ backward recursively with terminal condition $a_T^\star(X_T) = 0$.

Once the optimal hedge $a_t^\star(X_t)$ is computed, the portfolio value Π_t could also be computed backward recursively by

$$\Pi_t = \gamma\left[\Pi_{t+1} - a_t^\star \Delta S_t\right] \quad t = T-1, ..., 0$$

together with the terminal condition $\Pi_T = H_T(S_T) = \max(K - S_T, 0)$ for a European put option.

Also compute $\hat{\Pi}_t = \Pi_t - \bar{\Pi}_t$, where $\bar{\Pi}_t$ is the sample mean of all values of Π_t.

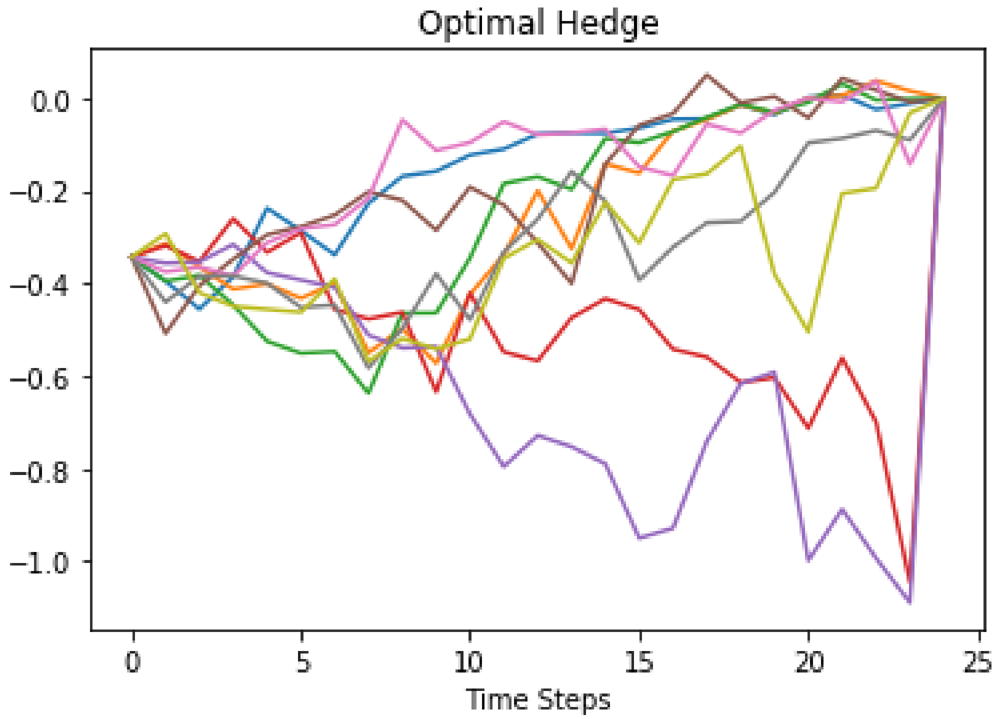
Plots of 5 optimal hedge $a_t^\star$ and portfolio value Π_t paths are shown below.

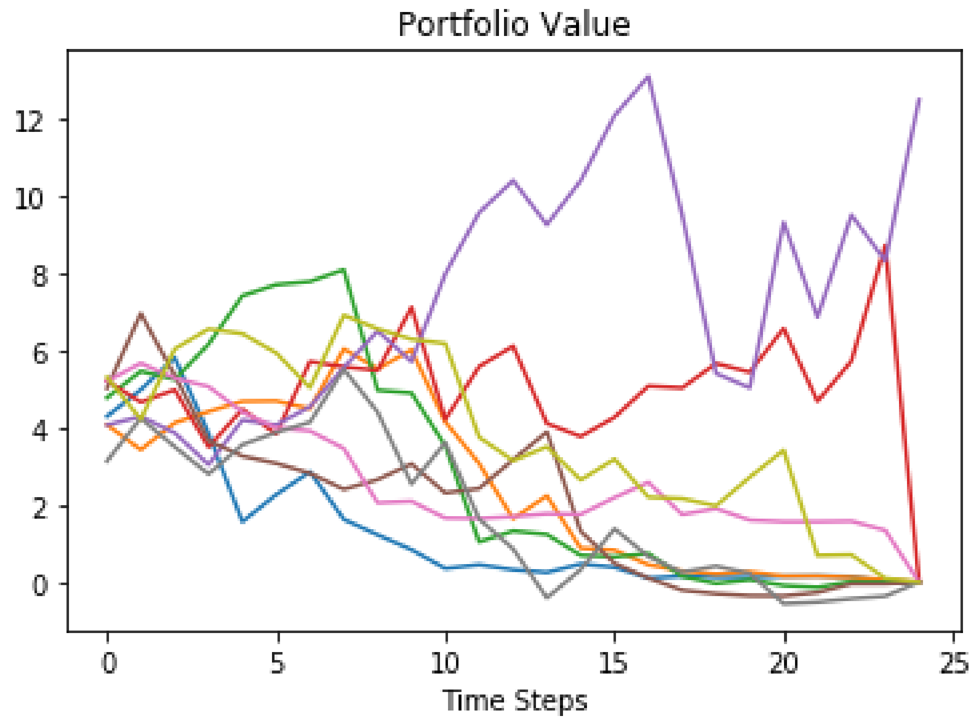
```
In [21]: starttime = time.time()

         # portfolio value
         Pi = pd.DataFrame([], index=range(1, N_MC+1), columns=range(T+1))
         Pi.iloc[:,-1] = S.iloc[:,-1].apply(lambda x: terminal_payoff(x, K))

         Pi_hat = pd.DataFrame([], index=range(1, N_MC+1), columns=range(T+1))
         Pi_hat.iloc[:,-1] = Pi.iloc[:,-1] - np.mean(Pi.iloc[:,-1])

         # optimal hedge
         a = pd.DataFrame([], index=range(1, N_MC+1), columns=range(T+1))
         a.iloc[:,-1] = 0

         reg_param = 1e-3 # free parameter
         for t in range(T-1, -1, -1):
             A_mat = function_A_vec(t, delta_S_hat, data_mat_t, reg_param)
             B_vec = function_B_vec(t, Pi_hat, delta_S_hat, S, data_mat_t, gamma, risk_lambda)
             # print ('t =  A_mat.shape = B_vec.shape = ', t, A_mat.shape, B_vec.shape)

             # coefficients for expansions of the optimal action
             phi = np.dot(np.linalg.inv(A_mat), B_vec)

             a.loc[:,t] = np.dot(data_mat_t[t,:,:],phi)
             Pi.loc[:,t] = gamma * (Pi.loc[:,t+1] - a.loc[:,t] * delta_S.loc[:,t])
             Pi_hat.loc[:,t] = Pi.loc[:,t] - np.mean(Pi.loc[:,t])

         a = a.astype('float')
         Pi = Pi.astype('float')
         Pi_hat = Pi_hat.astype('float')

         endtime = time.time()
         print('Computational time:', endtime - starttime, 'seconds')
```

```
D:\application\Anaconda3\envs\pyalgo\lib\site-packages\ipykernel_launcher.py:82: FutureWarning


Computational time: 0.6784000396728516 seconds
```

```
In [22]: # plot 10 paths
         plt.plot(a.T.iloc[:,idx_plot])
         plt.xlabel('Time Steps')
         plt.title('Optimal Hedge')
         plt.show()
```

```
plt.plot(Pi.T.iloc[:,idx_plot])
plt.xlabel('Time Steps')
plt.title('Portfolio Value')
plt.show()
```





0.8 Compute rewards for all paths

Once the optimal hedge $a_t^\star$ and portfolio value Π_t are all computed, the reward function $R_t\left(X_t, a_t, X_{t+1}\right)$ could then be computed by

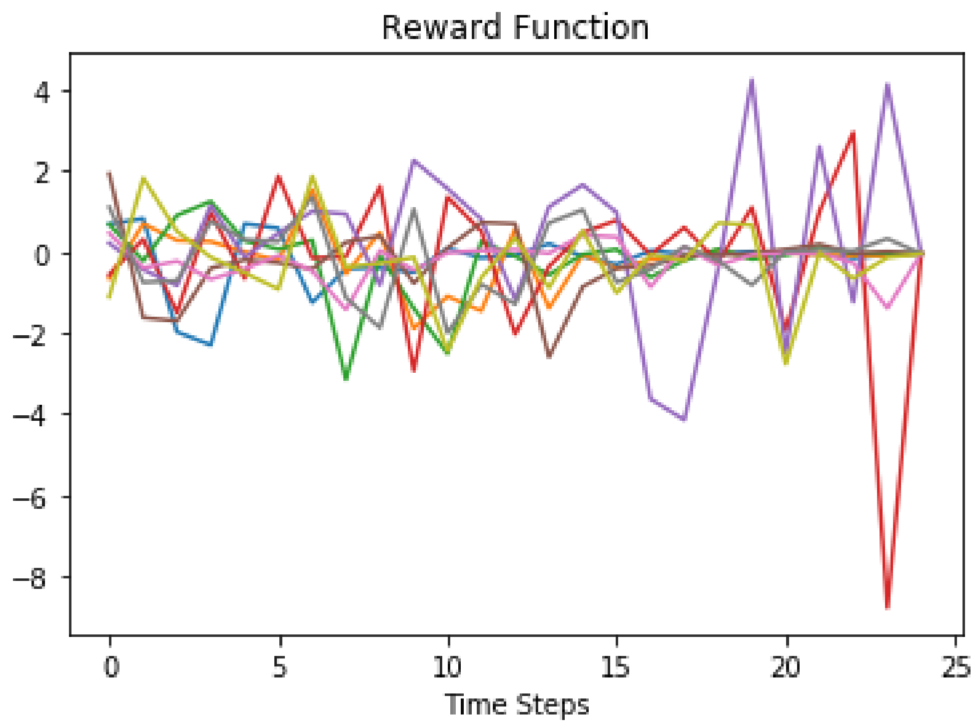
$$R_t\left(X_t, a_t, X_{t+1}\right) = \gamma a_t \Delta S_t - \lambda Var\left[\Pi_t | \mathcal{F}_t\right] \quad t = 0, ..., T-1$$

with terminal condition $R_T = -\lambda Var\left[\Pi_T\right]$.
Plot of 5 reward function R_t paths is shown below.

In [23]:
```
# Compute rewards for all paths
starttime = time.time()
# reward function
R = pd.DataFrame([], index=range(1, N_MC+1), columns=range(T+1))
R.iloc[:,-1] = - risk_lambda * np.var(Pi.iloc[:,-1])

for t in range(T):
    R.loc[1:,t] = gamma * a.loc[1:,t] * delta_S.loc[1:,t] - risk_lambda * np.var(Pi.lo

endtime = time.time()
print('\nTime Cost:', endtime - starttime, 'seconds')

# plot 10 paths
plt.plot(R.T.iloc[:, idx_plot])
```

```
plt.xlabel('Time Steps')
plt.title('Reward Function')
plt.show()
```

```
Time Cost: 0.1530001163482666 seconds
```



0.9 Part 2: Compute the optimal Q-function with the DP approach

Coefficients for expansions of the optimal Q-function $Q_t^\star\left(X_t, a_t^\star\right)$ are solved by

$$\omega_t = \mathbf{C}_t^{-1}\mathbf{D}_t$$

where $\mathbf{C}_t$ and $\mathbf{D}_t$ are matrix and vector respectively with elements given by

$$C_{nm}^{(t)} = \sum_{k=1}^{N_{MC}} \Phi_n\left(X_t^k\right)\Phi_m\left(X_t^k\right) \qquad D_n^{(t)} = \sum_{k=1}^{N_{MC}} \Phi_n\left(X_t^k\right)\left(R_t\left(X_t, a_t^\star, X_{t+1}\right) + \gamma \max_{a_{t+1}\in\mathcal{A}} Q_{t+1}^\star\left(X_{t+1}, a_{t+1}\right)\right)$$

Define function *function_C* and *function_D* to compute the value of matrix $\mathbf{C}_t$ and vector $\mathbf{D}_t$.

Instructions: - implement function_C_vec() which computes $C_{nm}^{(t)}$ matrix - implement function_D_vec() which computes $D_n^{(t)}$ column vector

In [24]:
```
def function_C_vec(t, data_mat, reg_param):
    """
    function_C_vec - calculate C_{nm} matrix from Eq. (56) (with a regularization!)
    Eq. (56) in QLBS Q-Learner in the Black-Scholes-Merton article

    Arguments:
    t - time index, a scalar, an index into time axis of data_mat
    data_mat - pandas.DataFrame of values of basis functions of dimension T x N_MC x 
    reg_param - regularization parameter, a scalar

    Return:
    C_mat - np.array of dimension num_basis x num_basis
    """
    ### START CODE HERE ### ( 5-6 lines of code)
    # your code here ....
    # C_mat = your code here ...
    X_mat = data_mat[t, :, :]
    num_basis_funcs = X_mat.shape[1]
    C_mat = np.dot(X_mat.T, X_mat) + reg_param * np.eye(num_basis_funcs)
    ### END CODE HERE ###
    return C_mat

def function_D_vec(t, Q, R, data_mat, gamma=gamma):
    """
    function_D_vec - calculate D_{nm} vector from Eq. (56) (with a regularization!)
    Eq. (56) in QLBS Q-Learner in the Black-Scholes-Merton article

    Arguments:
    t - time index, a scalar, an index into time axis of data_mat
    Q - pandas.DataFrame of Q-function values of dimension N_MC x T
    R - pandas.DataFrame of rewards of dimension N_MC x T
    data_mat - pandas.DataFrame of values of basis functions of dimension T x N_MC x 
    gamma - one time-step discount factor $exp(-r \delta t)$

    Return:
    D_vec - np.array of dimension num_basis x 1
    """

    ### START CODE HERE ### ( 5-6 lines of code)
    # your code here ....
    # D_vec = your code here ...
    X_mat = data_mat[t, :, :]
    D_vec = np.dot(X_mat.T, R.loc[:,t] + gamma * Q.loc[:, t+1])
    ### END CODE HERE ###
    return D_vec
```

In [25]:
```
### GRADED PART (DO NOT EDIT) ###
C_mat = function_C_vec(T-1, data_mat_t, reg_param)
```

```
np.random.seed(42)
idx_row = np.random.randint(low=0, high=C_mat.shape[0], size=50)

np.random.seed(42)
idx_col = np.random.randint(low=0, high=C_mat.shape[1], size=50)

part_3 = list(C_mat[idx_row, idx_col])
try:
    part3 = " ".join(map(repr, part_3))
except TypeError:
    part3 = repr(part_3)


submissions[all_parts[2]]=part3
grading.submit(COURSERA_EMAIL, COURSERA_TOKEN, assignment_key,all_parts[:3],all_parts

C_mat[idx_row, idx_col]
### GRADED PART (DO NOT EDIT) ###
```

```
Submission successful, please check on the coursera grader page for the status
```

```
Out[25]: array([1.09774699e+03, 2.10343651e+02, 4.56877655e+00, 8.41911156e+02,
                8.69395802e+02, 1.09774699e+03, 3.30191775e+01, 3.53203253e+01,
                1.09774699e+03, 4.56877655e+00, 4.56877655e+00, 8.41911156e+02,
                8.69395802e+02, 2.10343651e+02, 8.41911156e+02, 8.41911156e+02,
                3.53203253e+01, 1.10328718e+03, 8.69395802e+02, 4.35986560e+00,
                8.41911156e+02, 1.04949534e+00, 1.10328718e+03, 4.35986560e+00,
                1.04949534e+00, 8.69395802e+02, 2.34165631e+00, 1.04949534e+00,
                3.30191775e+01, 1.10328718e+03, 1.04949534e+00, 1.99059232e+02,
                2.34165631e+00, 4.56877655e+00, 4.56877655e+00, 3.30191775e+01,
                1.04949534e+00, 1.04949534e+00, 3.53203253e+01, 1.04949534e+00,
                1.09774699e+03, 2.10343651e+02, 1.99059232e+02, 3.53203253e+01,
                8.69395802e+02, 3.53203253e+01, 1.09774699e+03, 8.69395802e+02,
                1.99059232e+02, 1.09774699e+03])
```

```
In [26]: ### GRADED PART (DO NOT EDIT) ###
Q = pd.DataFrame([], index=range(1, N_MC+1), columns=range(T+1))
Q.iloc[:,-1] = - Pi.iloc[:,-1] - risk_lambda * np.var(Pi.iloc[:,-1])
D_vec = function_D_vec(T-1, Q, R, data_mat_t,gamma)


part_4 = list(D_vec)
try:
    part4 = " ".join(map(repr, part_4))
except TypeError:
    part4 = repr(part_4)
```

```
submissions[all_parts[3]]=part4
grading.submit(COURSERA_EMAIL, COURSERA_TOKEN, assignment_key,all_parts[:4],all_parts

D_vec
### GRADED PART (DO NOT EDIT) ###
```

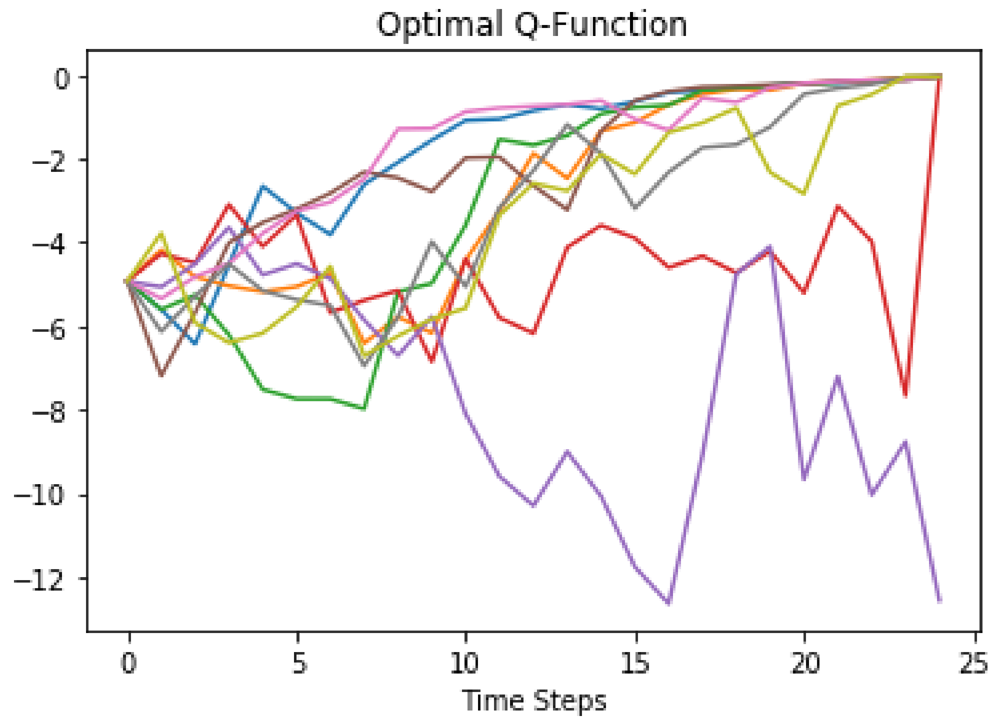
```
Submission successful, please check on the coursera grader page for the status
```

Out[26]:
```
array([-1.33721037e+02, -5.99514760e+02, -3.18661973e+03, -1.02120353e+04,
       -1.76323018e+04, -7.20169691e+03, -1.13250111e+03, -1.66673355e+02,
       -5.20254025e+01, -1.55950276e+01, -5.86197625e+00, -4.96858215e+00])
```

Call *function_C* and *function_D* for $t = T-1, ..., 0$ together with basis function $\Phi_n(X_t)$ to compute optimal action Q-function $Q_t^\star(X_t, a_t^\star) = \sum_n^N \omega_{nt}\Phi_n(X_t)$ backward recursively with terminal condition $Q_T^\star(X_T, a_T = 0) = -\Pi_T(X_T) - \lambda Var[\Pi_T(X_T)]$.

In [27]:
```
starttime = time.time()

# Q function
Q = pd.DataFrame([], index=range(1, N_MC+1), columns=range(T+1))
Q.iloc[:,-1] = - Pi.iloc[:,-1] - risk_lambda * np.var(Pi.iloc[:,-1])

reg_param = 1e-3
for t in range(T-1, -1, -1):
    ######################
    C_mat = function_C_vec(t,data_mat_t,reg_param)
    D_vec = function_D_vec(t, Q,R,data_mat_t,gamma)
    omega = np.dot(np.linalg.inv(C_mat), D_vec)

    Q.loc[:,t] = np.dot(data_mat_t[t,:,:], omega)

Q = Q.astype('float')
endtime = time.time()
print('\nTime Cost:', endtime - starttime, 'seconds')

# plot 10 paths
plt.plot(Q.T.iloc[:, idx_plot])
plt.xlabel('Time Steps')
plt.title('Optimal Q-Function')
plt.show()
```

```
Time Cost: 0.16299986839294434 seconds
```



The QLBS option price is given by $C_t^{(QLBS)}\left(S_t, ask\right) = -Q_t\left(S_t, a_t^\star\right)$

0.10 Summary of the QLBS pricing and comparison with the BSM pricing

Compare the QLBS price to European put price given by Black-Sholes formula.

$$C_t^{(BS)} = Ke^{-r(T-t)}\mathcal{N}\left(-d_2\right) - S_t\mathcal{N}\left(-d_1\right)$$

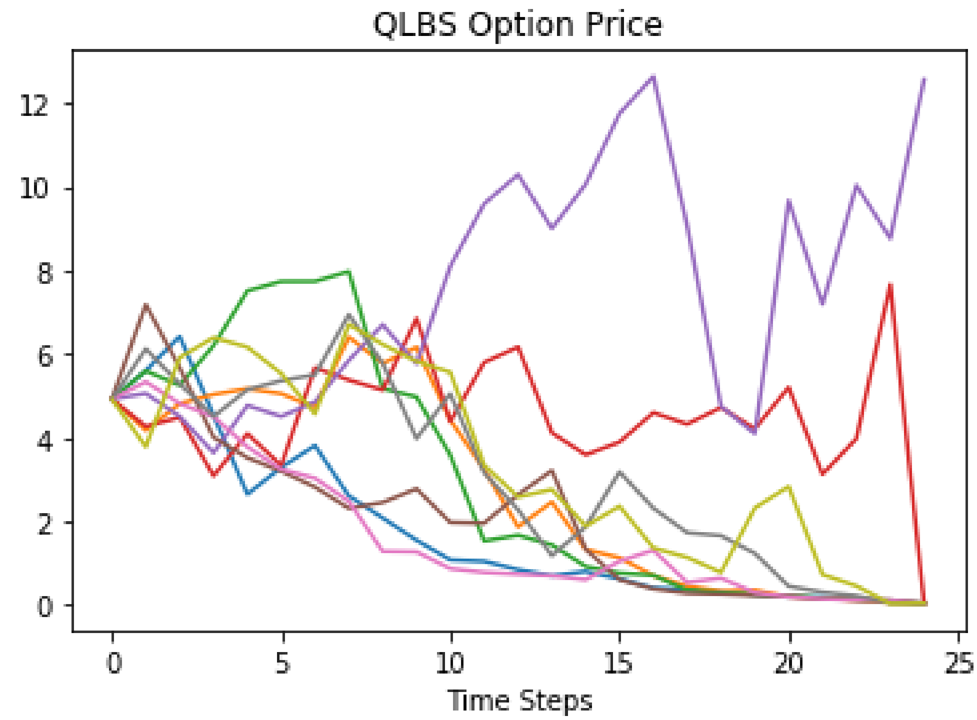
```
In [28]: # The Black-Scholes prices
         def bs_put(t, S0=S0, K=K, r=r, sigma=sigma, T=M):
             d1 = (np.log(S0/K) + (r + 1/2 * sigma**2) * (T-t)) / sigma / np.sqrt(T-t)
             d2 = (np.log(S0/K) + (r - 1/2 * sigma**2) * (T-t)) / sigma / np.sqrt(T-t)
             price = K * np.exp(-r * (T-t)) * norm.cdf(-d2) - S0 * norm.cdf(-d1)
             return price

         def bs_call(t, S0=S0, K=K, r=r, sigma=sigma, T=M):
             d1 = (np.log(S0/K) + (r + 1/2 * sigma**2) * (T-t)) / sigma / np.sqrt(T-t)
             d2 = (np.log(S0/K) + (r - 1/2 * sigma**2) * (T-t)) / sigma / np.sqrt(T-t)
             price = S0 * norm.cdf(d1) - K * np.exp(-r * (T-t)) * norm.cdf(d2)
             return price
```

0.11 The DP solution for QLBS

```
In [29]: # QLBS option price
         C_QLBS = - Q.copy()
```

```python
print('---------------------------------------------')
print('       QLBS Option Pricing (DP solution)      ')
print('---------------------------------------------\n')
print('%-25s' % ('Initial Stock Price:'), S0)
print('%-25s' % ('Drift of Stock:'), mu)
print('%-25s' % ('Volatility of Stock:'), sigma)
print('%-25s' % ('Risk-free Rate:'), r)
print('%-25s' % ('Risk aversion parameter: '), risk_lambda)
print('%-25s' % ('Strike:'), K)
print('%-25s' % ('Maturity:'), M)
print('%-26s %.4f' % ('\nQLBS Put Price: ', C_QLBS.iloc[0,0]))
print('%-26s %.4f' % ('\nBlack-Sholes Put Price:', bs_put(0)))
print('\n')

# plot 10 paths
plt.plot(C_QLBS.T.iloc[:,idx_plot])
plt.xlabel('Time Steps')
plt.title('QLBS Option Price')
plt.show()
```

```
---------------------------------------------
       QLBS Option Pricing (DP solution)
---------------------------------------------

Initial Stock Price:      100
Drift of Stock:           0.05
Volatility of Stock:      0.15
Risk-free Rate:           0.03
Risk aversion parameter:  0.001
Strike:                   100
Maturity:                 1

QLBS Put Price:           4.9261

Black-Sholes Put Price:   4.5296
```




```
In [30]: ### GRADED PART (DO NOT EDIT) ###

         part5 = str(C_QLBS.iloc[0,0])
         submissions[all_parts[4]]=part5
         grading.submit(COURSERA_EMAIL, COURSERA_TOKEN, assignment_key,all_parts[:5],all_parts

         C_QLBS.iloc[0,0]
         ### GRADED PART (DO NOT EDIT) ###
```

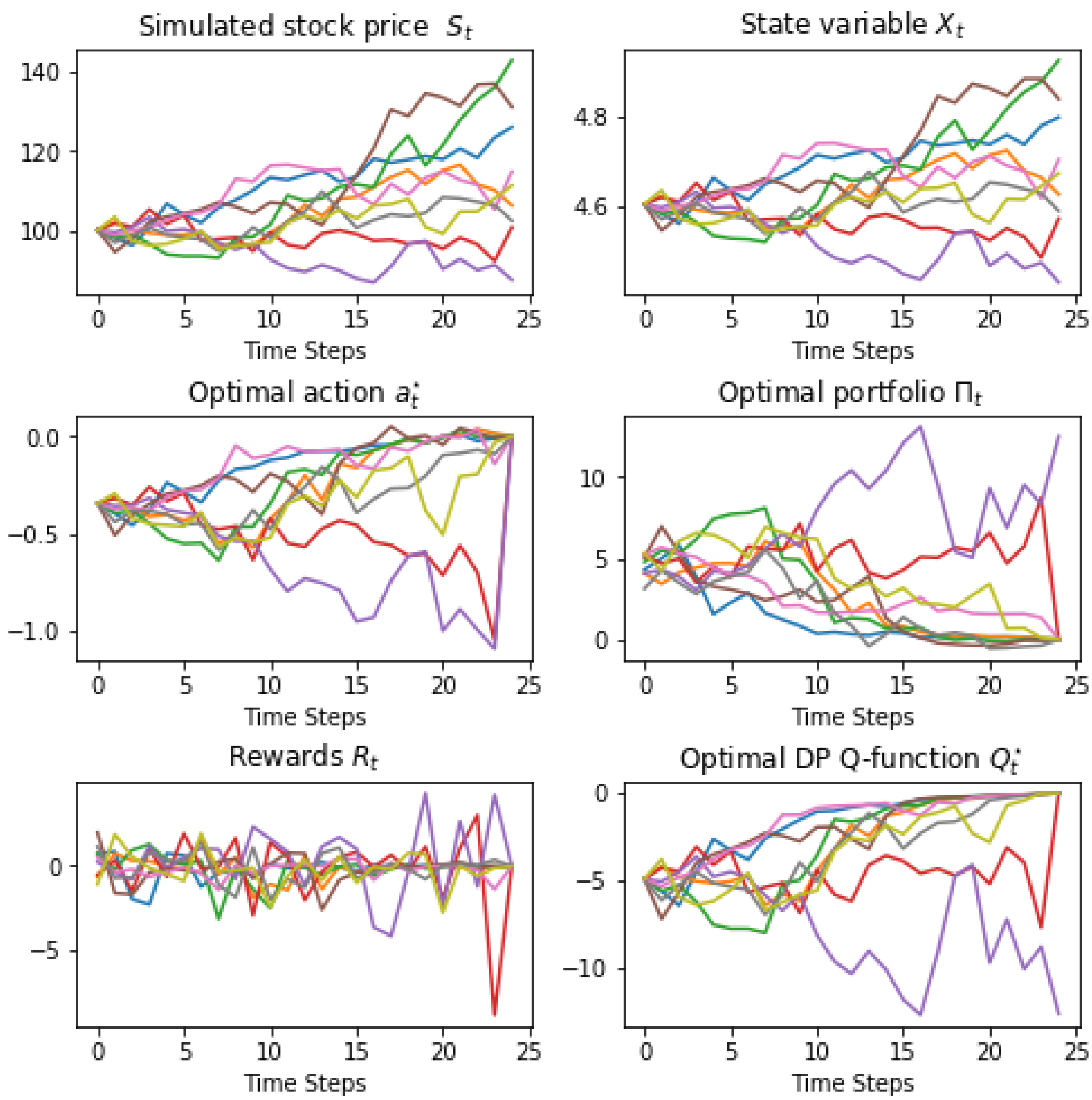
```
Submission successful, please check on the coursera grader page for the status
```

```
Out[30]: 4.926125422447431
```

0.11.1 make a summary picture

```
In [33]: # plot: Simulated S_t and X_t values
         # optimal hedge and portfolio values
         # rewards and optimal Q-function

         f, axarr = plt.subplots(3, 2)
         f.subplots_adjust(hspace=.5)
         f.set_figheight(8.0)
         f.set_figwidth(8.0)
```

```
axarr[0, 0].plot(S.T.iloc[:,idx_plot])
axarr[0, 0].set_xlabel('Time Steps')
axarr[0, 0].set_title(r'Simulated stock price  $S_t$')

axarr[0, 1].plot(X.T.iloc[:,idx_plot])
axarr[0, 1].set_xlabel('Time Steps')
axarr[0, 1].set_title(r'State variable $X_t$')

axarr[1, 0].plot(a.T.iloc[:,idx_plot])
axarr[1, 0].set_xlabel('Time Steps')
axarr[1, 0].set_title(r'Optimal action $a_t^{\star}$')

axarr[1, 1].plot(Pi.T.iloc[:,idx_plot])
axarr[1, 1].set_xlabel('Time Steps')
axarr[1, 1].set_title(r'Optimal portfolio $\Pi_t$')

axarr[2, 0].plot(R.T.iloc[:,idx_plot])
axarr[2, 0].set_xlabel('Time Steps')
axarr[2, 0].set_title(r'Rewards $R_t$')

axarr[2, 1].plot(Q.T.iloc[:,idx_plot])
axarr[2, 1].set_xlabel('Time Steps')
axarr[2, 1].set_title(r'Optimal DP Q-function $Q_t^{\star}$')


# plt.savefig('QLBS_DP_summary_graphs_ATM_option_mu=r.png', dpi=600)
# plt.savefig('QLBS_DP_summary_graphs_ATM_option_mu>r.png', dpi=600)
#plt.savefig('QLBS_DP_summary_graphs_ATM_option_mu>r.png', dpi=600)
plt.savefig('r.png', dpi=600)

plt.show()
```



In [34]:
```
# plot convergence to the Black-Scholes values

# lam = 0.0001, Q = 4.1989 +/- 0.3612 # 4.378
# lam = 0.001: Q = 4.9004 +/- 0.1206  # Q=6.283
# lam = 0.005: Q = 8.0184 +/- 0.9484 # Q = 14.7489
# lam = 0.01: Q = 11.9158 +/- 2.2846 # Q = 25.33

lam_vals = np.array([0.0001, 0.001, 0.005, 0.01])
# Q_vals =  np.array([3.77, 3.81, 4.57, 7.967,12.2051])
Q_vals =  np.array([4.1989, 4.9004, 8.0184, 11.9158])
Q_std =  np.array([0.3612,0.1206, 0.9484, 2.2846])

BS_price = bs_put(0)
```

```
# f, axarr = plt.subplots(1, 1)
fig, ax = plt.subplots(1, 1)

f.subplots_adjust(hspace=.5)
f.set_figheight(4.0)
f.set_figwidth(4.0)

# ax.plot(lam_vals,Q_vals)
ax.errorbar(lam_vals, Q_vals, yerr=Q_std, fmt='o')

ax.set_xlabel('Risk aversion')
ax.set_ylabel('Optimal option price')
ax.set_title(r'Optimal option price vs risk aversion')
ax.axhline(y=BS_price,linewidth=2, color='r')
textstr = 'BS price = %2.2f'% (BS_price)
props = dict(boxstyle='round', facecolor='wheat', alpha=0.5)
# place a text box in upper left in axes coords
ax.text(0.05, 0.95, textstr, fontsize=11,transform=ax.transAxes, verticalalignment='to
plt.savefig('Opt_price_vs_lambda_Markowitz.png')
plt.show()
```