# A practical construction method for Polar Codes in AWGN channels

Huijun Li and Jinhong Yuan
School of Electrical Engineering and Telecommunications
The University of New South Wales, Sydney, NSW 2052, Australia
Email: huijun.li@student.unsw.edu.au, jinhong@ee.unsw.edu.au

***Abstract*—Polar coding, introduced by Arikan, is the first code construction method that could be proved to construct capacity-achieving codes for any symmetric binary-input discrete memoryless channels (B-DMCs). However, this construction method is not explicit or efficient if the channel is not binary erasure channel (BEC). Density evolution is proposed to help solve this problem for any B-DMCs but the implementation is not tractable and requires a high computation complexity. Here, we introduce an efficient method to calculate Bhattacharyya parameter and construct polar codes based on Gaussian approximation. Then we evaluate the code performance and compare it with the existing methods. It is shown that the constructed code using Gaussian approximation can have a comparable error performance with a low computation complexity.**

## I. Introduction

In recent years, polar codes have attracted much attention since they are the first error correcting codes that are theoretically capacity achieving for any symmetric B-DMCs with low encoding and decoding complexity [1]. To date, there are many high performance codes like turbo and LDPC codes which could be empirically shown to achieve rates very close to channel capacity. However, they cannot be proved in theory rigorously for any channels other than the BEC [2]. Thus, the attribute of provably capacity-achieving in theory makes polar codes distinct with other traditional codes and makes it possible to be applied to help prove many theoretical problems.

Polar codes have been applied in wireless communication comprehensively. Not only for channel coding, Korada *et al*. also demonstrate that Shannon's rate-distortion bound could be achieved under proper construction of polar codes [3] [4]. In addition, much work has been done to extend binary input polar codes to polar codes with a non-binary input alphabets for channel and source coding [5] [6]. In [6], Reed-Solomon codes are investigated and algebraic geometry core is selected as the new kernel of polar codes. Abbe *et al*. use matroids to prove that polar coding schemes can also achieve channel capacity in multiple access channels [7]. In the field of wireless physical layer security, polar codes are widelyed utilized to help obtain the secrecy capacity of wiretap channels and to make sure the information is transmitted both reliably and securely between the transmitter and the legal receiver [8] [9].

However, in [1], no exact code design rule for general B-DMCs but the BEC is raised for polar codes. Instead, Arikan presents a heuristic manner to construct polar codes in [10]. Later, Mori *et al*. use density evolution tool to improve the code construction for any B-DMCs in [11] [12]. Requiring large memory and high computation complexity increasing with the code length, the implementation of density evolution is not tractable.

In this paper, we use the idea of Gaussian Approximation to propose a simple and efficient construction method for polar codes under AWGN channel and compare its performance with Arikan's heuristic method and density evolution, respectively. In section II, the construction and decoding process of polar codes proposed in [1] and some basic notations are briefly presented. In section III, the method of using Gaussian approximation to calculate Bhattacharyya parameter and to construct polar codes is described. In section IV, the performances of the three methods are compared and analyzed through simulation result. Finally, the conclusion of this paper is give in section V.

## II. Preliminaries

In this section, we briefly describe the encoding and decoding process of polar codes based on the work of [1], lay the foundation, and set the notation for the rest of this paper.

### A. Construction of Polar codes

Let $W : \mathcal{X}\ \{0,1\} \rightarrow \mathcal{Y}$ denote a B-DMC with input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$, and transition probability $p(y|x), x \in \mathcal{X}, y \in \mathcal{Y}$. Polar codes are constructed on the basis of channel polarization defined in [1]. Let $G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and apply the transform $G_2^{\otimes n}$ (where "$\otimes$" denotes Kronecker product) to a block of $N = 2^n$ bits and transmit the output through independent copies of $W$. As the block length $N$ increases and approaches infinity, the channels seen by individual bit get polarized and become either a noiseless channel or a very noisy one. We denote these polarized channels with bit channels $W_N^{(i)}$. Channel polarization suggests us use the "perfect" channel to transmit information bits and fix the symbols transmitted under noisy ones to a value published to both the receiver and the transmitter. The generator matrix can be defined recursively as

$$G_N = R_N G_2^{\otimes n}$$

where $R_N$ is a shuffle matrix and re-orders the input bits. In the following, we denote the input bits, coded bits, and received bits with $u_1^N = (u_1, ..., u_N)$, $x_1^N = (x_1, ..., x_N)$, and

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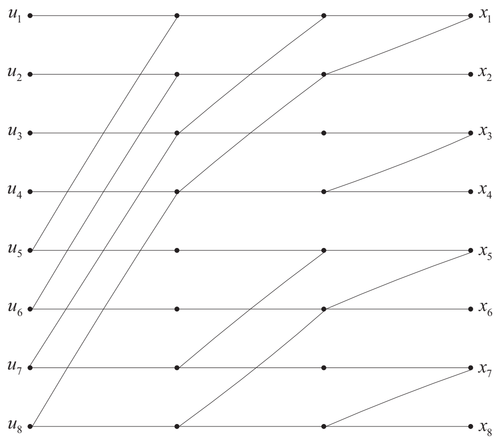


Fig. 1. Realization of the encoding process for polar codes with code length $N = 8$. The message vector $u_1^N$ has been reversed after the operation of $R_N$. At each node, the values are added modulo-2.

$y_1^N = (y_1, ..., y_N)$ respectively. The encoding process is shown in Fig. 1 and it is easy to see that polar codes have a linear encoding complexity $O(N \log N)$.

### *B. Decoding of Polar codes*

For polar codes, a successive cancelation (SC) decoding scheme is used in [1] with low decoding complexity. Since we focus on AWGN channels here, the channel is represented in the log-likelihood domain. Let $llr_N^{(i)}$ denote the log-likelihood ratio (LLR) $\log(p(y|0)/p(y|1))$ of the $i$th bit channel $W_N^{(i)}$. Since the symbols under pure-noise channels are fixed and known to both the transmitter and the receiver, then only the information bits are needed to be estimated. The information bits are decoded in sequence and each bit is determined under maximum-likelihood (ML) decoding. The $i$th message bit is decoded as

$$\hat{u}_N^{(i)} = \begin{cases} 0, & \text{if } llr_N^{(i)}(y_1^N, \hat{u}_1^{i-1}|u_i) > 0 \\ 1, & \text{if } llr_N^{(i)}(y_1^N, \hat{u}_1^{i-1}|u_i) < 0 \end{cases} \tag{1}$$

If $llr_N^{(i)}$ equals to 0, $u_i$ is determined to 0 or 1 with equal probability.

## III. Polar codes construction using Gaussian Approximation under AWGN channel

In this section we propose to use Gaussian approximation method to calculate Bhattacharyya parameter and then construct polar codes under the AWGN channel. Two existing methods of constructing polar codes: heuristic method and density evolution are briefly mentioned at first.

### *A. Two existing methods of constructing polar codes*

In order to construct polar codes, we need to select the noiseless channels to transmit information bits. Bhattacharyya parameter $Z(W)$ can be used to measure the error performance of a channel. It is the upper bound of error probability under (ML) decoding when transmitting over $W$ and is defined as

$$Z(W) = \sum_{y \in \mathcal{Y}} \sqrt{p(y|0)p(y|1)}$$

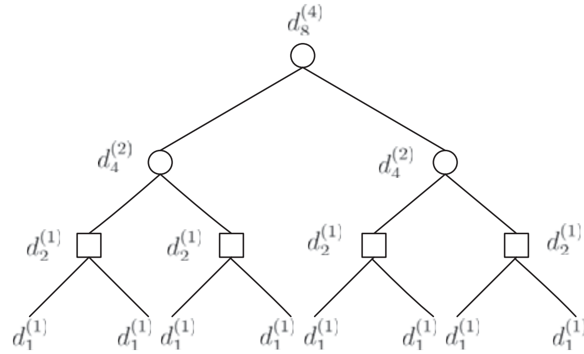


Fig. 2. Tree-like calculation path of $d_8^{(4)}$ for polar codes with code length $N = 8$. The squares and circles denote the the convolution on check nodes and variable nodes, respectively.

Let $Z_N^{(i)},\ i = 1, ..., N$ denote the Bhattacharyya parameter of the $i$th bit channel $W_N^{(i)}$. By selecting $Z_N^{(i)}$ with smaller value, we can choose the noiseless channels. However, Bhattacharyya parameter can only be exactly calculated under the BEC as [1]

$$\begin{aligned} Z_N^{(2j-1)} &= 2Z_{N/2}^{(j)} - (Z_{N/2}^{(j)})^2 \\ Z_N^{(2j)} &= (Z_{N/2}^{(j)})^2 \end{aligned} \tag{2}$$

where $Z_1^{(1)}$ is the erasure probability of the channel $W$. For other channels, Arikan proposed a heuristic method in [10] by treating other channel as an equivalent BEC with the same channel capacity.

The decoding error probability $P_e^{(i)}$ of the $i$th bit channel $W_N(i)$ under belief propagation (BP) decoding via density evolution is used to construct polar codes instead of Bhattacharyya parameter in [6]. In this paper, we assume that the channel $W$ is symmetric and all-zero information is transmitted. Let $d_N^{(i)}$ denote the probability density function (pdf) of $llr_N^{(i)}$ and it is evolved as [6]

$$d_N^{(2j-1)} = d_{N/2}^{(j)} \boxast d_{N/2}^{(j)},\ d_N^{(2j)} = d_{N/2}^{(j)} \circledast d_N^{(j)} \tag{3}$$

where $\boxast$ and $\circledast$ denote the convolutions of LLR density function (defined in [14]), corresponding to check nodes and variables, respectively. The evolution path of $d_N^{(i)}$ for polar codes with code length $N = 8$ is shown in Fig. 2. After obtaining $d_N^{(i)}$, $P_e^{(i)}$ can be calculated by integration as $P_e^{(i)} = \lim_{\epsilon \to 0}(\int_{-\infty}^{-\epsilon} d_N^{(i)}(x)dx + \frac{1}{2}\int_{-\epsilon}^{+\epsilon} d_N^{(i)}(x)dx)$. The rule of selecting good channels using density evolution is similar to Arikan's heuristic method by selecting the channels with smaller $P_e^{(i)}$ to transmit information.

### *B. Polar codes construction using Gaussian approximation*

Density evolution is exacter to evaluate the performance of subchannels $W_N^{(i)}$ compared with the heuristic method. But it is very complex to implement the convolution operation. Here, we propose to use Gaussian approximation to simplify density evolution.

Let $\sigma^2$ denote the additive Gaussian white noise variance of the original channel $W$. Then the LLR message from channel

$W$ is Gaussian distributed with mean $\frac{2}{\sigma^2}$ and variance $\frac{4}{\sigma^2}$. We use $m_N^{(i)}$ to denote the mean of the LLR density function of the $i$th bit.

As the pdf of $llr_N^{(i)}$ is obtained recursively through equation (3), the corresponding mean $m_N^{(i)}$ can also be computed recursively as

$$m_N^{(2j-1)} = f_1(m_{N/2}^{(j)}),\ m_N^{(2j)} = f_2(m_{N/2}^{(j)}) \tag{4}$$

where $f_1(\ )$ and $f_2(\ )$ denote the functions on the variable $m_{N/2}^{(j)}$, corresponding to check nodes and variable nodes, respectively.

Chung *et al*. have derived the formulas for calculating the mean of LLR message on check nodes and variable nodes using Gaussian approximation for LDPC codes in [13]. We apply the formulas for regular LDPC codes in [13] and the $m_N^{(i)}$ of polar codes can be calculated as

$$m_N^{(2j)} = 2m_{N/2}^{(j)} \tag{5}$$

$$m_N^{(2j-1)} = f^{-1}(1-(1-f(m_{N/2}^{(j)}))^2) \tag{6}$$

where $m_1^{(1)} = \frac{2}{\sigma^2}$.

The function $f(x)$ is approximated as [13]

$$f(x) = \begin{cases} e^{\alpha x^{\gamma}+\beta}, & x < 10 \\ \frac{1}{2}(\sqrt{\frac{\pi}{x}}e^{-\frac{x}{4}}(1-\frac{3}{x}) + \sqrt{\frac{\pi}{x}}e^{-\frac{x}{4}}(1+\frac{1}{7x})), & x > 10 \end{cases}$$

where $\alpha = -0.4527$, $\beta = 0.0218$, and $\gamma = 0.86$.

Let $(\sigma_N^{(i)})^2$ denote the noise variance of each bit channel $W_N^{(i)}$. Knowing the value of $m_N^{(i)}$, $(\sigma_N^{(i)})^2$ is set to $2/m_N^{(i)}$. Over AWGN channels, Bhattacharyya parameter $Z(W)$ can be computed as

$$\begin{aligned} Z(W) &= \int_{-\infty}^{\infty} \sqrt{p(y|0)p(y|1)}dy \\ &= \int_{-\infty}^{\infty} \sqrt{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-1)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y+1)^2}{2\sigma^2}}}dy \\ &= e^{-\frac{1}{2\sigma^2}} \end{aligned} \tag{7}$$

For a BEC, $Z(W) = \epsilon$, where $\epsilon$ is the erasure probability. For a BSC, $Z(W)$ is $2\sqrt{p(1-p)}$, where $p$ is the crossover probability. Then, with the knowledge of $\sigma_N^{(i)}$, the Bhattacharyya parameter $Z_N^{(i)}$ can be calculated directly as $e^{-1/(2(\sigma_N^{(i)})^2)}$. After obtaining $Z_N^{(i)}$ of each polarized bit channel, the encoding construction is realized by ordering the $N$ channels and choose the noiseless ones to convey message. Knowing $Z_N^{(i)}$, the upper and lower bounds of block error probability can be calculated as

$$\begin{aligned} P_B^U(e) &= \sum_{i\in\mathcal{A}} Z_N^{(i)} \\ P_B^L(e) &= max_{i\in\mathcal{A}} Z_N^{(i)} \end{aligned} \tag{8}$$

where $\mathcal{A}$ is the information set with the Bhattacharyya parameter lower than a given threshold.

In Gaussian approximation, only one parameter $m_N^{(i)}$ needs to be tracked and calculated and the complexity of computing $m_N^{(i)}$ is still $O(N\log N)$. Gaussian approximation construction possesses the same complexity with Arikan's heuristic method but its performance is improved. Compared with density evolution, the computation complexity is reduced. In density evolution, quantization and discrete convolution are implemented in practice to track the LLR density function which would require more memory and computation time.

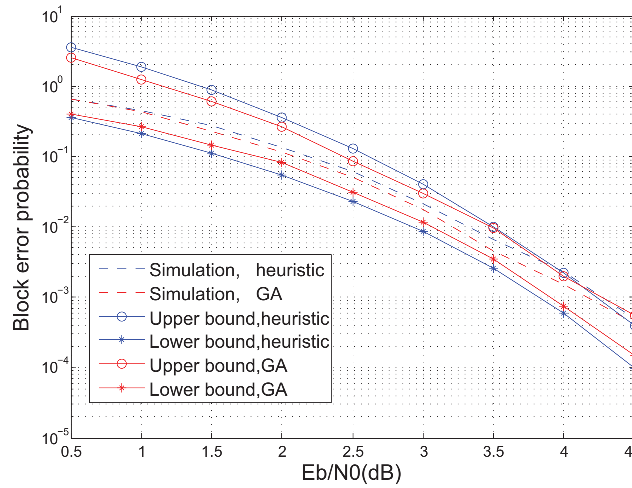


Fig. 3. Calculation results of block error probability bounds using Arikan's heuristic method and Gaussian approximation over AWGN channels. The code length is 128 and code rate is 0.5.

## IV. Simulation result and performance analysis

In this section, we compare the code performances in simulation using Arikan's heuristic method, density evolution (DE), and Gaussian approximation (GA) to construct polar codes. The block error probability bounds calculated through Bhattacharyya parameter are also compared between Arikan's method and Gaussian approximation.

In Fig. 3, we compare the upper and lower bounds of the block error probability defined in (8) between Arikan's heuristic method and Gaussian approximation. The result shows that using Gaussian approximation to calculate Bhattacharyya parameter could obtain a tighter bound than using the heuristic method.

Fig. 4 and Fig. 5 show the simulation results of the block error probability of polar codes over AWGN channels with code length 128 and 256, respectively. We can see that using Gaussian approximation and density evolution makes an improvement of performance compared with Arikan's heuristic method. This is because considering the distribution of output is also important for code construction. Fig. 4 and Fig. 5 also show that using Gaussian approximation could obtain nearly the same error performance for polar codes compared with density evolution.

Next, we investigate the average error probability of each bit for the polar code between three methods. Let us define the corresponding bit channels with the highest average error probabilities as the least reliable bit channels (LRBC). In the following, we will evaluate whether the LRBC computed

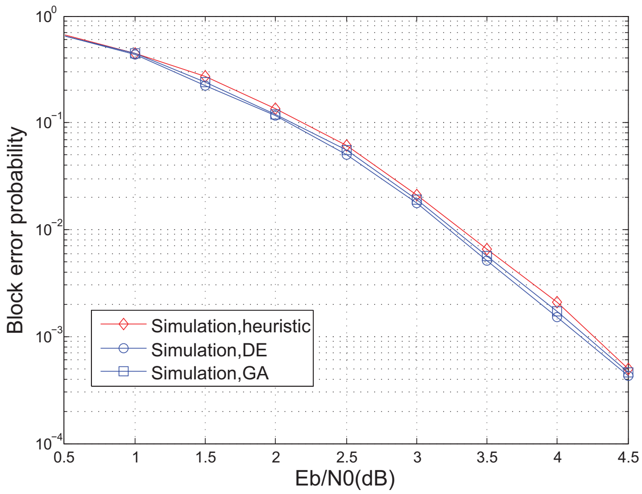


Fig. 4. Comparison of error performance for polar codes under three construction methods over AWGN channels. The code length is 128 and code rate is 0.5.

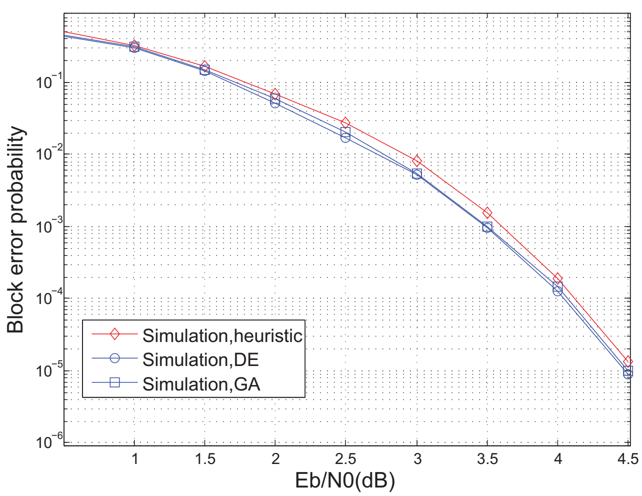


Fig. 5. Comparison of error performance for polar codes under three construction methods over AWGN channels. The code length is 256 and code rate is 0.5.

based on Bhattacharyya parameter (Arikan's heuristic method and Gaussian approximation) or decoding error probability (Density evolution) would match the simulation result.

TABLE I
COMPARISON BETWEEN ANALYSIS AND SIMULATION FOR LRBC. THE CODE LENGTH IS 128, RATE $\frac{1}{2}$, AND SNR 1DB

| M | Heuristic method | Density evolution | Gaussian approximation |
|---|---|---|---|
| 10 | 2 | 6 | 6 |
| 20 | 6 | 15 | 14 |
| 30 | 15 | 24 | 23 |

In Table I, we show that among the $M = 10$, 20, and 30 LRBC computed based on Bhattacharyya parameter or decoding error probability, the number of bit channels belong to the M LRBC through simulated bit error probability results. From the table, we can see that the LRBC based on density evolution and Gaussian approximation have a much better closeness to the simulation result, compared with Arikan's heuristic method. In addition, we see that Gaussian approximation is comparable with density evolution in code performance.

## V. CONCLUSION AND FUTURE WORKS

In this paper, we propose to use Gaussian approximation method to simplify the implementation of density evolution for the construction of polar codes and find a new way to calculate the Bhattacharyya parameter in AWGN channels. The simulation results show that using Gaussian approximation could obtain a tighter block error probability bound a better code performance compared with Arikan's heuristic method. The results also demonstrate that the performance under Gaussian approximation is comparable with that using density evolution with a lower computation complexity.

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