The Mathematics of Complex Systems: Theory and Applications

Day 2: Euler's Method for Ordinary Differential Equations

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January 20, 2026

Motivation

- Many differential equations cannot be solved explicitly.
- Even when an exact solution exists, it may be complicated.
- Numerical methods allow us to **approximate** solutions.
- Euler's method is the simplest and most intuitive numerical method.

Initial Value Problems (IVPs)

An **initial value problem** has the form:

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = y_0 \end{cases}$$

- $y(t)$ is the unknown function
- $f(t, y)$ gives the slope of the solution
- The initial condition fixes a unique solution

Geometric Interpretation

- The equation $y' = f(t, y)$ defines a slope at each point (t, y).
- The exact solution follows these slopes continuously.
- Euler's method follows the slope using straight line segments.

Key idea: Replace the curve by small linear steps.

Derivation of Euler's Method

From calculus:

$$y(t_{n+1}) \approx y(t_n) + y'(t_n)(t_{n+1} - t_n)$$

Using $y'(t_n) = f(t_n, y_n)$ and step size h:

$$t_{n+1} = t_n + h$$

$$\boxed{y_{n+1} = y_n + h\, f(t_n, y_n)}$$

Euler's Method Algorithm

Given:

- Initial condition (t_0, y_0)
- Step size h

Repeat for $n = 0, 1, 2, \dots$:

$$t_{n+1} = t_n + h$$
$$y_{n+1} = y_n + hf(t_n, y_n)$$

This produces a sequence of approximate values $\{y_n\}$.

Worked Example

Solve approximately:

$$y' = y, \quad y(0) = 1$$

Exact solution:

$$y(t) = e^t$$

Euler's method with step size h:

$$y_{n+1} = y_n + hy_n = (1 + h)y_n$$

Numerical Approximation

Starting from $y_0 = 1$:

$$y_1 = 1 + h$$
$$y_2 = (1 + h)^2$$
$$y_n = (1 + h)^n$$

- Smaller h gives better approximation
- Larger h accumulates more error

Local and Global Error

- **Local truncation error:**
  - Error made in one step
  - Order $O(h^2)$
- **Global error:**
  - Error after many steps
  - Order $O(h)$

Euler's method is a **first-order method**.

Main Flaws of Euler's Method

1. Low accuracy (slow convergence)
2. Errors accumulate quickly
3. Requires very small step sizes
4. Can become unstable for some equations

Stability Issues

Consider:

$$y' = -\lambda y, \quad \lambda > 0$$

Euler update:

$$y_{n+1} = (1 - h\lambda)y_n$$

- If $|1 - h\lambda| > 1$, solution grows artificially
- Even when the exact solution decays

This shows **numerical instability**.

Example of Failure

For:

$$y' = -10y, \quad y(0) = 1$$

- Exact solution decays rapidly
- Large step sizes cause oscillations or divergence
- Euler method gives incorrect qualitative behavior

When is Euler's Method Useful?

- Conceptual understanding of numerical methods
- Simple problems
- Very small time intervals
- As a building block for better methods

Beyond Euler's Method

More accurate methods include:

- Improved Euler (Heun's method)
- Runge–Kutta methods
- Adaptive step-size methods

Euler's method is the foundation for all of them.

Summary

- Euler's method approximates solutions of IVPs
- Uses tangent lines to advance step by step
- Simple but not very accurate
- Important for understanding numerical ODE solvers

Questions?