Numerical Methods for ODEs: 4th-Order Runge-Kutta (RK4)

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- Ordinary Differential Equations (ODEs) often cannot be solved analytically.
- Numerical methods approximate the solution at discrete points.
- Euler's method: simple but low accuracy.
- Runge-Kutta methods improve accuracy without higher derivatives.
- RK4: widely used due to high accuracy and stability.

Fourth-Order Runge-Kutta (RK4) Method

ODE Problem

Consider the initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

RK4 Formula

$$\begin{aligned}
k_1 &= f(t_n, y_n) \\
k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) \\
k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right) \\
k_4 &= f(t_n + h, y_n + hk_3) \\
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$

Comparison with Euler's Method

Euler's Method:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

- First-order accuracy
- Easy to implement
- Can be unstable for large h

RK4 Method:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

- Fourth-order accuracy
- Very stable for moderate h
- Requires 4 evaluations of f per step

Example: Simple ODE

Solve

$$\frac{dy}{dt} = -2y, \quad y(0) = 1$$

- Exact solution: $y(t) = e^{-2t}$
- Step size: $h = 0.1$

RK4 Steps:

$$\begin{aligned}
k_1 &= -2y_n \\
k_2 &= -2\left(y_n + \frac{h}{2}k_1\right) \\
k_3 &= -2\left(y_n + \frac{h}{2}k_2\right) \\
k_4 &= -2\left(y_n + hk_3\right) \\
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$

Numerical Comparison

t	Exact $y(t)$	Euler y_n	RK4 y_n
0	1	1	1
0.1	0.8187	0.8	0.8187
0.2	0.6703	0.64	0.6703
0.3	0.5488	0.512	0.5488
0.4	0.4493	0.4096	0.4493

Limitations of RK4

- Accuracy still depends on step size h
- Not suitable for very stiff ODEs
- Computationally heavier than Euler (4 evaluations per step)
- Adaptive step-size methods can improve efficiency

Summary

- RK4 is a robust method for solving non-stiff ODEs numerically.
- Offers fourth-order accuracy, much better than Euler or RK2.
- Easy to implement and widely used in engineering and physics.
- For stiff problems or extremely high accuracy, other methods may be required.