CIMPA School Recife 2026
Applications of Topological and Geometric Approaches in Epidemiology

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- Motivation: Topological Data Analysis (TDA) to investigate epidemic behavior trough an exploratory perspective;
- Results across many fields, e.g, Neuroscience and Stock markets.
- Dengue disease represents a huge health concern for the world scenario;
- In parallel, the COVID-19 pandemic scenario is also challenging science;
- Here, we compare geometric and topological approaches applied to epidemics.

- Inpiration:

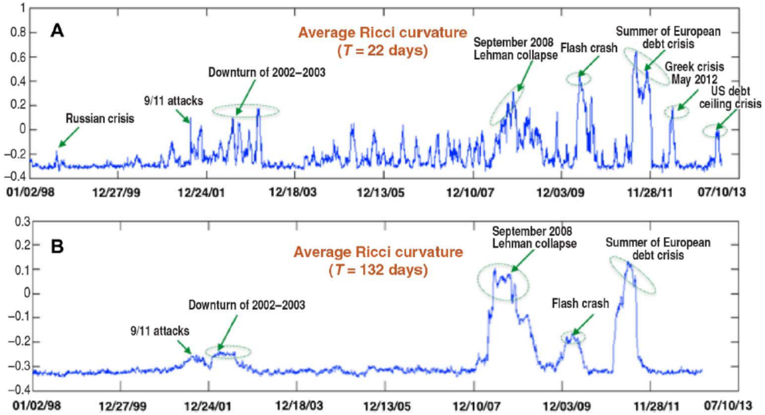

Figure: Ricci curvature: An economic indicator for market fragility and systemic risk

- Motivation: discrete versions of Ricci curvatures and Euler characteristic on epidemic networks.

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Networks

Definition (Simple graph)

$G = (V, E)$ is an underected simple graph if V is a set of finite elemens and $E = \{e = (x, y) | x, y \in V, x \neq y\}$. E is said the set of edges V is said the set of nodes.

Definition (Subgraph)

We say that $H = (V', E') \subseteq G$ is a subgraph of G if $V' \subseteq V$ and $E' \subset E$.

Remark

Here in this work, we are going to refer to simple underected graphs as graphs or networks.

Neighborhood of a node

Let $x \in V$. We say that y is a neighbor of x when there is an edge that connects x to y, and we denote by $x \sim y$.

$$\mathrm{neigh(x)} = \{y \in V, x \sim y\}. \tag{1}$$

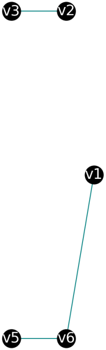
Degree of a node

Let $x \in V$.

$$\mathrm{deg(x)} = \#\mathrm{neigh(x)}. \tag{2}$$

Networks

Figure: Example of a simple graph.



Node	Degree
v1	1
v2	1
v3	1
v4	0
v5	1
v6	2

Networks

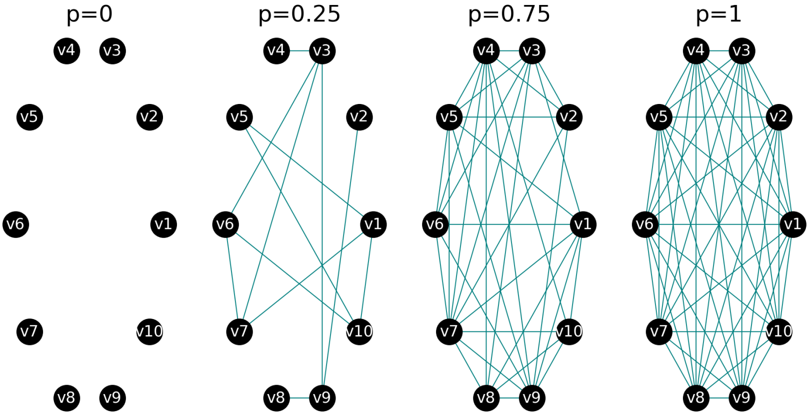


Figure: Examples of Erdos-Renyi graphs, for different values of probability p.

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Graph cells

d-Cells

Let $G = (V, E)$ a graph and $d \in \mathbb{N} \bigcup \{0\}$. The structures equivalent to the d-dimensional open disks in the continuous approach are called d-cells ,or d-faces, or also d-simplexes. They are also equivalent to a $(d+1)$-vertex clique.

CW-complex

The space constructed from the graph G is called a Cell Complex or a CW-Complex

Figure: Examples of $d - cells$ of different dimensions.

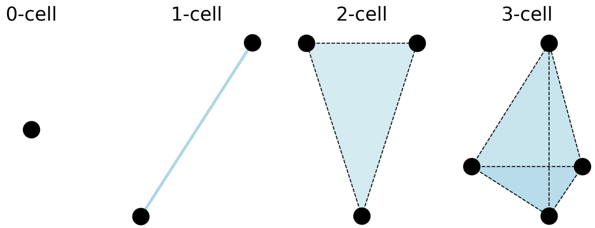
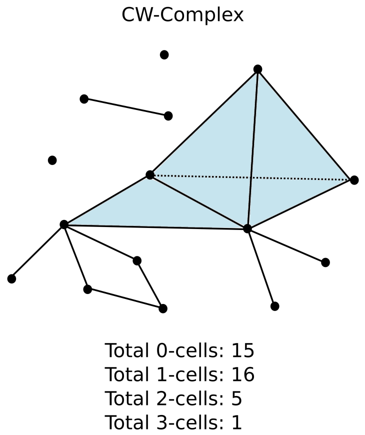


Figure: Example of a CW-Complex.



Topological approaches

Boundary Operator

We also define the boundary operator, $\partial_d : C_d \rightarrow C_{d-1}$, as follows:

$$\partial_d([v_1, \cdots, v_{d+1}]) = \sum_{i=1}^{d+1} (-1)^{i-1} [v_1, \cdots, \hat{v}_i, \cdots, v_{d+1}], \qquad (3)$$

where $\hat{v}_i$ denotes the omitted variable.

Betti number

The *d*-th boundary space of X is denoted by $B_d = B_d(X)$ and is defined as the $\mathrm{im}\partial_{d+1}$, and the *d*-th cycle space, $Z_d = Z_d(X)$ is $\ker \partial_d$. The d-dimentional homology group:

$$H_d = \frac{Z_d}{B_d}. \tag{4}$$

Finally, the d-th Betti number of the simplicial complex X is defined as

$$\beta_d(X) = \dim\ \ker \partial_d - \dim \mathrm{im}\partial_{d+1}. \tag{5}$$

Topological Approaches

Euler Characteristics (via Betti numbers)

$$\chi(X) = \sum_{i=1}^{\infty} (-1)^i \beta_i(X). \tag{6}$$

Alternative version:

Euler Characteristics (Clique version)

$$\chi(G) = \sum_{d=0}^{d_{\max}} (-1)^{(d+1)} k_d, \tag{7}$$

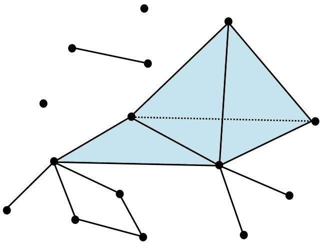
where k_d is the number of d-cells ($d-1$-cliques) in G.

Topological Approaches

Figure: Example: $\chi(X) = \beta_1 - \beta_2 + \beta_3 = 4 - 1 + 0 = 3.$

Also, $\chi(G) = 15 - 16 + 5 - 1 = 3.$

CW-Complex



Total 0-cells: 15
Total 1-cells: 16
Total 2-cells: 5
Total 3-cells: 1

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Geometric Approaches

- Ollivier-Ricci curvature
- Forman-Ricci curvatures

Ollivier-Ricci curvature

- Ollivier-Ricci curvature

Ollivier-Ricci curvature

Let $G = (V, E)$ be a graph and $d : V \times V \to \mathbb{R}^+$ the shortest path length function. Let $\alpha \in [0, 1]$ and $x \in V$. Let $\alpha \in [0, 1]$ and $x \in V$.
We define m_x^α, a probability measure over the set of nodes as

$$m_x^\alpha(y) = \begin{cases} \alpha & \text{if } y = x \\ (1-\alpha)/\deg(x) & \text{if } y \in \text{neigh}(x) \\ 0 & \text{otherwise} \end{cases} . \tag{8}$$

Ollivier-Ricci curvature

Ollivier-Ricci curvature

Let $e = (x, y) \in E$,

$$\kappa(x, y) = 1 - \frac{W(m_x^\alpha, m_y^\alpha)}{d(x, y)}, \tag{9}$$

where W is the discrete Wasserstein distance given by

$$W(m_x^\alpha, m_y^\alpha) \quad = \quad \inf_{\mu \in \Pi(m_x^\alpha, m_y^\alpha)} \sum_{x', y' \in V} d(x', y') \mu(x', y'), \tag{10}$$

where $\Pi(m_x^\alpha, m_y^\alpha)$ denotes the set of all probability measures $\mu : V \times V \to \mathbb{R}^+$ that satisfy

$$\sum_{y' \in V} \mu(x', y') = m_x^\alpha(x'), \quad \sum_{x' \in V} \mu(x', y') = m_y^\alpha(y'). \tag{11}$$

Forman-Ricci curvatures

Let $e = (x, y) \in E$.

Weighted Forman-Ricci curvature

$$\mathrm{F}(x,y) = w_x + w_y - w_{xy}\left(\sum_{z \in \pi_x,\, z \neq y} \frac{w_x}{\sqrt{w_{xy} w_{xz}}} + \sum_{s \in \pi_y,\, s \neq x} \frac{w_y}{\sqrt{w_{xy} w_{sy}}} \right),$$

where w_x, w_y and w_{xy} are the weights of vertices x, y and edge (x, y), respectively.

Forman-Ricci curvature

$$\mathrm{F}(x,y) = 4 - \deg(x) - \deg(y). \tag{12}$$

Forman-Ricci curvatures

- d-th Forman-Ricci curvature

Forman-Ricci curvatures

Given two d-cells α_1 and α_2, we denote that α_1 is contained in the boundary of α_2 by $\alpha_1 < \alpha_2$.

Neighborhood of a $d-$cell

1. There is a $(d+1)$-cell β such that $\alpha_1 < \beta$ and $\alpha_2 < \beta$;
2. There is a $(d-1)$-cell γ such that $\gamma < \alpha_1$ and $\gamma < \alpha_2$.

Paralell and Tranverse Neighbors

We say that that α_1 and α_2 are parallel neighbors if conditions (1) and (2) is true but not simultaneously. In case of both (1) and (2) are true, α_1 and α_2 are said transverse neighbors.

Forman-Ricci curvatures

$d-$th Forman-Ricci curvature

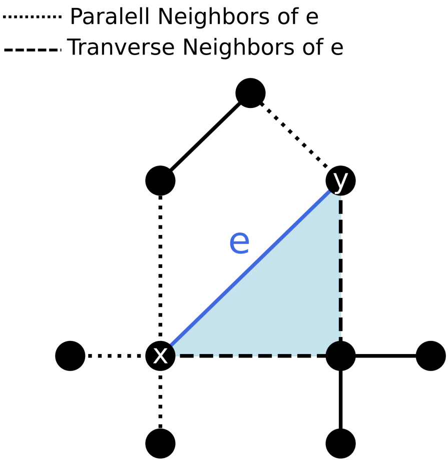
$$\mathrm{F}_d(\alpha) = \begin{aligned}&\#\{(d+1)\text{-cells } \beta > \alpha\}\\ &+\#\{(d-1)\text{-cells } \gamma < \alpha\}\\ &-\#\{\text{parallel neighbors of } \alpha\}.\end{aligned} \tag{13}$$

Augmented Forman-Ricci curvature (d=1)

$$\mathrm{F}(e) = \begin{aligned}&\#\{2\text{-cells } \beta > e\} + 2\\ &-\#\{\text{parallel neighbors of } e\}.\end{aligned} \tag{14}$$

Forman-Ricci curvatures

Figure: Example ($d = 1$): $F(e) = 1 + 2 - 4 = -1$.



Forman-Ricci curvatures

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Epidemic Networks

Introducing "Epidemic Network":

- An evolving graph generated from (pieces of) epidemic time-series;
- Each time-series represents are generated by a region (nodes);
- The level of connection between epidemic time-series are mesuared by (weighted) edges.

Epidemic Networks

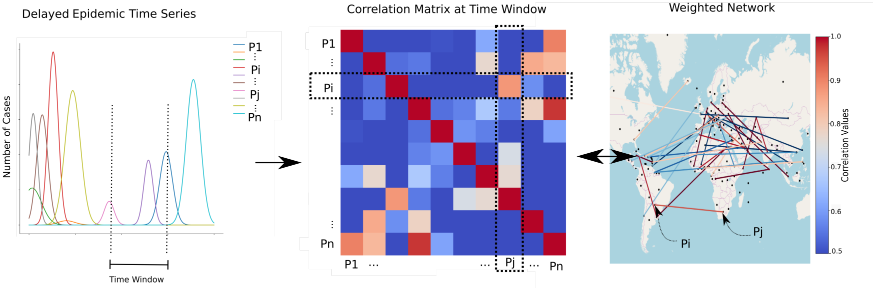
The network construction is made as follows:

- A moving window on data is selected;
- The resulting data will provide the time-series on each place to be analysed;
- The place names are provided to the nodes, while the connection between epidemic time-series is given by edges between nodes via Pearson correlation coefficient, given by

$$\rho = \frac{\mathrm{cov}(X, Y)}{\mathrm{var}(X)\mathrm{var}(Y)}; \tag{15}$$

- The process is repeated for the next time period, until the whole analysed data be fully covered.

Epidemic Networks



Epidemic Networks

Possible issues:

- The resultant networks might be very complex;
- Time processing can be compromised;
- The analysis might be damaged by the excess of spurious information.

Solution: Network filtration.

- Let $G = (V, E)$ a weighted graph. A subgraph $G_\epsilon = (V, E_\epsilon) \subset G$ will be constructed;
- The set of edges will be filtered by its weights as follows:

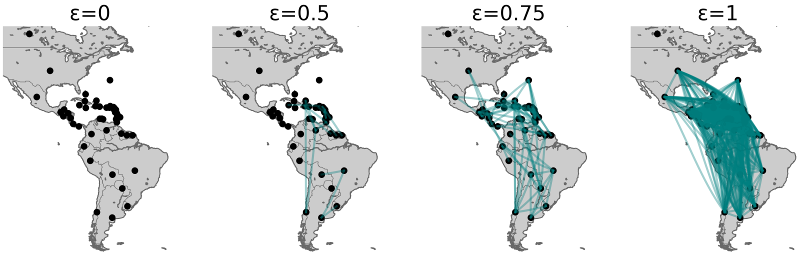
$$E_\epsilon = \{e \in E; \; w_e \geq 1 - \epsilon\}, \tag{16}$$

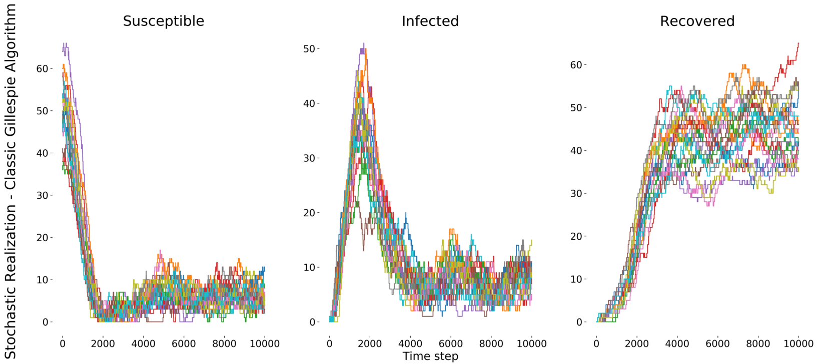
  for some $\epsilon \in [0, 2]$.

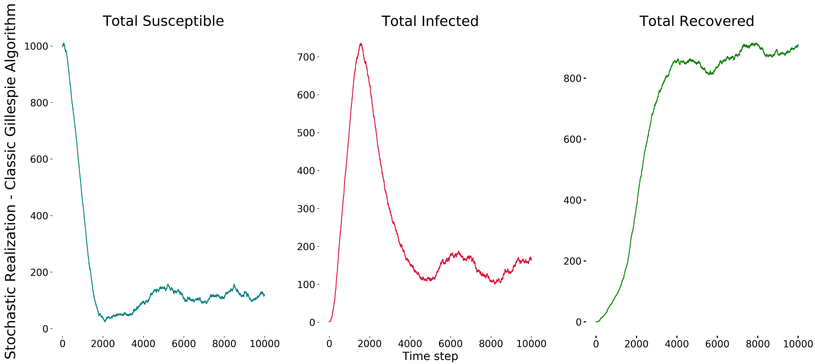
- An appropriate value of ϵ will be find so that the both original and filtered graph has the same number of connected components:

$$\epsilon_c = \inf\{\epsilon \in [0, 2] \; ; \; |G_\epsilon| = |G|\}. \tag{17}$$

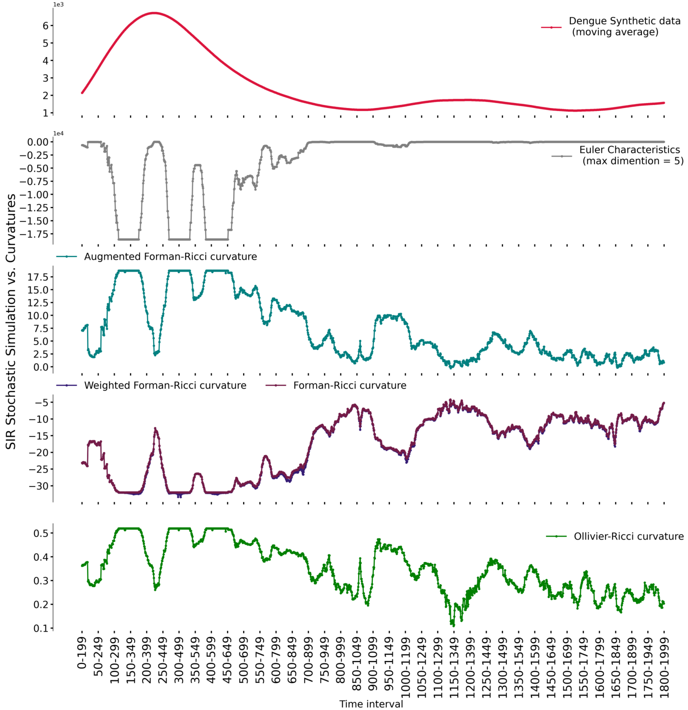
Epidemic Networks




Stochastic Realization - Classic Gillespie Algorithm
Susceptible
Infected
Recovered
Time step
0
2000
4000
6000
8000
10000
0
10
20
30
40
50
60


Stochastic Realization - Classic Gillespie Algorithm
Total Susceptible
Total Infected
Total Recovered
1000
800
600
400
200
0
700
500
300
100
0
2000
4000
6000
8000
10000
Time step

- Comparison between Synthetic data of Dengue Disease and curvatures:



Dengue Synthetic data (moving average)
1e3
Euler Characteristics (max dimention = 5)
1e4
Augmented Forman-Ricci curvature
Weighted Forman-Ricci curvature
Forman-Ricci curvature
Ollivier-Ricci curvature
SIR Stochastic Simulation vs. Curvatures
Time interval

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Results

Real data application:

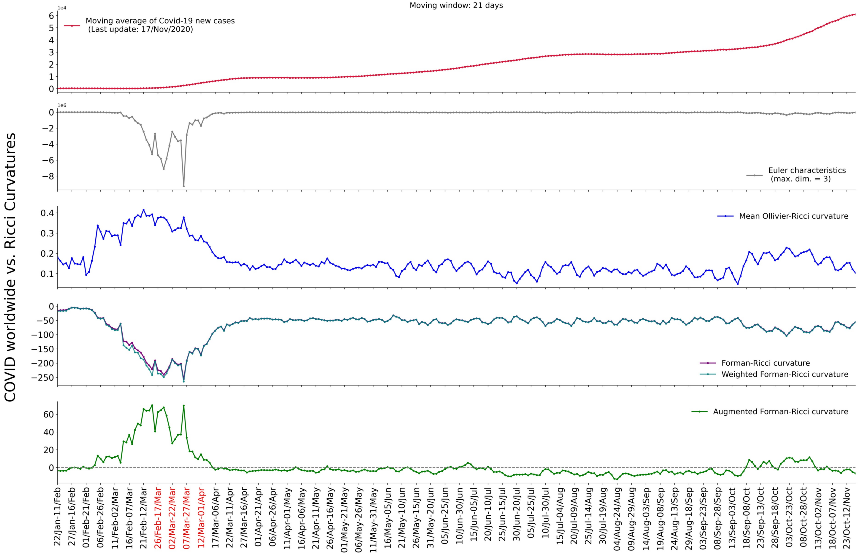
- World Health Organization - COVID-19 data (John Hopkings)

- COVID-19:

Results

Moving window: 7 days
Moving average of Covid-19 new cases
(Last update: 17/Nov/2020)
Euler characteristics
(max. dim. = 3)
Mean Ollivier-Ricci curvature
Forman-Ricci curvature
Weighted Forman-Ricci curvature
Augmented Forman-Ricci curvature
COVID worldwide vs. Ricci Curvatures
1e4
1e6

Results



Moving window: 21 days
Moving average of Covid-19 new cases
(Last update: 17/Nov/2020)
Euler characteristics
(max. dim. = 3)
Mean Ollivier-Ricci curvature
Forman-Ricci curvature
Weighted Forman-Ricci curvature
Augmented Forman-Ricci curvature
COVID worldwide vs. Ricci Curvatures

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Conclusion

- We believe that our work could pave the way for parameter-free, topological and geometric approaches to epidemic networks and open the possibility for studying epidemics from a geometric and topological perspective.

Conclusion

	Time processing	Computational complexity	Information	Noise
Forman-Ricci curvatures	Low	Low	Med	Med
Ollivier-Ricci curvature	High	Med	High	High
Euler caracteristics	High	High	High	Low

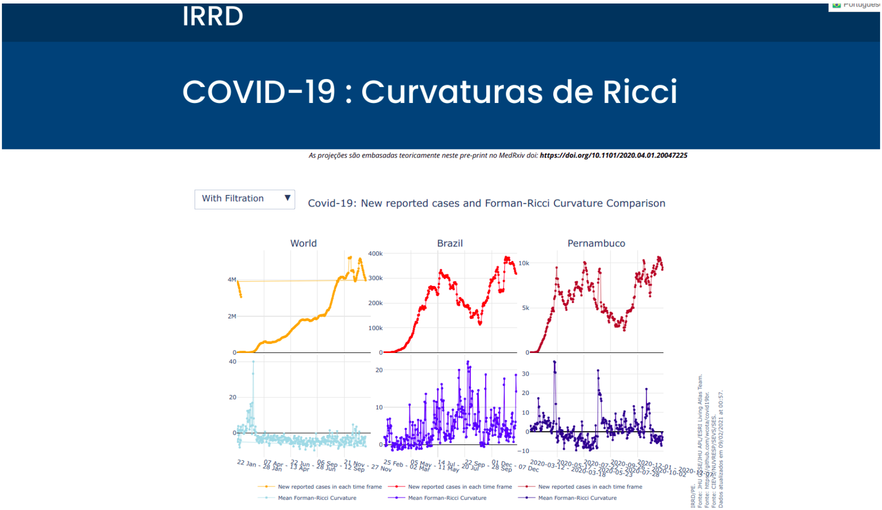
Conclusion

Publications

- Using discrete Ricci curvatures to infer COVID-19 epidemic network fragility – Journal of Statistical Mechanics: Theory and Experiment (JSTAT)
- Topological and Geometric approaches to Dengue outbreaks

Parallel work

Figure: http://www.irrd.org



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Obrigado!