1 Introduction

The study of nearly perfect sets started with the paper of Dunbar et al. (1995). A set S is called a nearly perfect set (NPS) if every vertex in $V - S$ is adjacent to at most one vertex in S. A set S is classified as 1-maximal nearly perfect set (1-max NPS) if, for every $i \in V \setminus S$, $S \cup \{i\}$ is not a NPS.

In the literature, only three studies have worked on nearly perfect sets, but they primarily concerned with finding the minimum cardinality of a 1-max NPS (denoted by $n_p(G)$) and/or the maximum cardinality of a 1-min NPS (denoted by $N_p(G)$). Dunbar et al. (1995) analyzed the complexity of computing $n_p(G)$ and $N_p(G)$. In their paper, they propose a linear time algorithm to find $n_p(G)$ for trees, and a linear time algorithm to find $N_p(G)$ for any graph. Furthermore, they show that the problem of finding $n_p(G)$ is NP-complete for general graphs (even when restricted to bipartite or chordal graphs). Kwa´snik and Perl (2004) and Perl (2007) computed $n_p(G)$ for the n-fold Cartesian product of graphs and in the n-fold strong product of graphs.

In this paper, we focus on the minimum weighted 1-max NPS problem, a generalization of the cardinality problem using polyhedral theory and integer programming. In the computational experiments, both trees and general graphs are investigated, as well as the effect of the proposed valid inequalities on these types of graphs.

The rest of the paper is organized as follows: First, the relevant definitions and preliminaries are presented in Section 2. Then, an IP formulation is presented in Section 3. In Section 4, we give several sets of valid inequalities to strenghten the formulation. In Section **??**, solution times and the effect of the proposed inequalities are analyzed in random trees and graphs.

2 Preliminaries and Problem Description

In this section, we first define the notation that will be used throughout the paper. We then introduce the problem that is the subject of this study; namely, the minimum weighted 1-max NPS problem.

Let $G = (V, E)$ be a simple undirected graph with vertex set V and edge set E. For any edge $\{i, j\} \in E$, i and j are said to be the endpoints of it. Two vertices $i, j \in V$ are said to be adjacent if $\{i, j\} \in E$. For a vertex $v \in V$, the open neighborhood of v, denoted by

$\delta(v)$, is the set of all vertices in G that are adjacent to v, i.e., $\delta(v) = \{j \in V \mid \{v, j\} \in E\}$. The closed neighborhood of v, denoted by $\Delta(v)$, is defined as $\delta(v) \cup \{v\}$. The degree of a vertex v, denoted by $deg(v)$, is defined as the number of vertices that are adjacent to v. A vertex v is said to be a pendant vertex if $deg(v) = 1$. Any vertex that is adjacent to a pendant vertex is called a support vertex. A vertex u is said to be universal if $deg(u) = |V| - 1$.

A graph is said to be connected if between every pair of distinct vertices, there exists a path joining them. Otherwise, the graph is said to be disconnected. In a disconnected graph G, the maximal connected subgraphs are called the connected components of G. An edge $e = \{i, j\}$ in a graph is said to be a bridge if its removal increases the number of connected components.

The two step neighborhood of a vertex i, denoted by $N_2(i)$, is defined as the set of vertices different from i that can be reached in exactly two steps from i. Formally,

$$N_2(i) = \{j \in V \setminus \{i\} \mid \exists m \in V, \{i, m\} \in E \text{ and } \{m, j\} \in E\}.$$

The narrowed two step neighborhood of a vertex i with respect to its adjacent vertex j, denoted by $NN_2(i, j)$, is defined as the set of vertices different from i that can be reached in exactly two steps from vertex i without using vertex j as the middle node. Mathematically, we have that

$$NN_2(i, j) = \{k \in V \setminus \{i\} \mid \exists m \in V \setminus \{j\}, \{i, m\} \in E \text{ and } \{m, k\} \in E\}.$$

1-maximal nearly perfect sets: A set $S \subseteq V$ is called a *nearly perfect set (NPS)*, if, for every vertex $i \in V \setminus S$, $|\delta(i) \cap S| \leq 1$. A S is said to be a *1-maximal nearly perfect set*, abbreviated as 1-max NPS, if S is a nearly perfect set and for every vertex $i \in V \setminus S$, $S \cup \{i\}$ is not a nearly perfect set.

To exemplify the definitions, we use the graph in Figure 1. Here, $\delta(1) = \{2, 3\}$, $\Delta(1) = \{1, 2, 3\}$, $N_2(1) = \{4, 5\}$, $NN_2(1, 2) = \{4\}$, and $NN_2(1, 3) = \{4, 5\}$. Consider the vertex sets $S_1 = \{2, 3\}$, $S_2 = \{2, 4\}$, and $S_3 = \{2, 4, 5\}$. Here S_1 is not a nearly perfect set as vertex 1 has two neighbors in S_1. On the other hand, S_2 is a nearly perfect set that is not 1-maximal as the set $S_2 \cup \{5\}$ is also a nearly perfect set. Finally, S_3 is both nearly perfect and 1-maximal (so it is a 1-max NPS) as addition of any vertex to S_3 results in a set that is not nearly perfect.

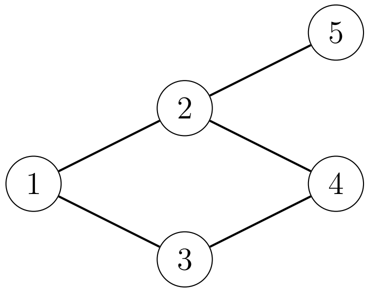

Figure 1: Example graph

Dunbar et al. (1995) provided the following alternative characterization for 1-max nearly perfect sets, which we use extensively in the paper:

Lemma 1. *For any graph $G = (V, E)$, a nearly perfect set S is 1-maximal if and only if every vertex i in $V \setminus S$ is adjacent to a vertex $j \in V \setminus S$ that is adjacent to exactly one vertex in S.*

Proof: The proof can be found in Dunbar et al. (1995).

In this paper, we study the *minimum weighted 1-max NPS (MW 1-max NPS) problem* which is defined next. We are given a graph $G = (V, E)$, and a weight w_i for every vertex $i \in V$. The MW 1-max NPS problem aims to find a 1-max NPS of minimum weight, where the weight of a set S, denoted by $W(S)$, is defined as

$$W(S) = \sum_{i \in S} w_i.$$

We assume, without loss of generality, that G is connected as for disconnected graphs, the MW 1-max NPS problem can be solved separately for each connected component.

This is the first study on the MW 1-max NPS problem. In the literature, only the unweighted version of the MW 1-max NPS problem has been studied which is the problem of finding a 1-max NPS of minimum cardinality. In addition, this study is the first attempt that aims to solve the MW 1-max NPS problem (and its unweighted version) using integer programming (IP) techniques. After developing an integer programming formulation for the MW 1-max NPS problem in the next section, we propose some classes of valid inequalities for the 1-max NPS polytope to strengthen our initial formulation which are presented in Section 4.

3 An Integer Programming Formulation for the MW 1-max NPS Problem

Given a graph $G = (V, E)$ and a weight $w(i)$ for each vertex $i \in V$, we propose an integer programming formulation for the MW 1-max NPS problem in this section.

Sets:
V: Set of nodes
$\delta(i)$: Open neighborhood of vertex $i \in V$
$N_2(i)$: Two-step neighborhood of vertex $i \in V$
$NN_2(i, j)$: Narrowed two-step neighborhood of vertex $i \in V$ with respect to its adjacent vertex $j \in V$

Parameters:
w_i: weight of vertex $i \in V$

Decision variables:
x_i: 1 if vertex $i \in V$ is selected in a 1-max NPS, 0 otherwise

Formulation (MW 1-Max NPS-IP):

$$\min \sum_{i \in V} w_i x_i \tag{1}$$

s.t.

$$x_i + (1 - x_{j_1}) + (1 - x_{j_2}) \geq 1 \qquad \forall i \in V, \forall j_1, j_2 \in \delta(i), j_1 \neq j_2 \tag{2}$$

$$x_i + \sum_{j \in N_2(i)} x_j \geq 1 \qquad \forall i \in V \tag{3}$$

$$x_i + (1 - x_j) + \sum_{k \in \mathrm{NN}_2(i,j)} x_k \geq 1 \qquad \forall i \in V, \forall j \in \delta(i) \tag{4}$$

$$x_i \in \{0, 1\} \qquad \forall i \in V \tag{5}$$

The objective function (1) aims to minimize the total weight of the selected vertices. Constraints (2) ensure that the selected vertices form an NPS. Constraints (3) and (4) make sure that the NPS is 1-maximal. Finally, constraints (5) imply that the variables take on binary values. The proposed model has $\min\{O(|V|^3), O(|E|^2)\}$ constraints and $O(|V|)$ variables.

Next, we will prove the correctness of the formulation (MW 1-Max NPS-IP). We will prove that for any feasible solution of (MW 1-Max NPS-IP), the selected vertices form a 1-max NPS. Moreover, for any 1-max NPS S, its incidence vector formed by giving x_i the value of 1 if and only if $i \in S$ is feasible to (MW 1-Max NPS-IP). Therefore the feasible solutions of (MW 1-Max NPS-IP) are in one-to-one correspondence with 1-max NPS's of G.

Proof of Correctness:

Let F denote the feasible region of (MW 1-Max NPS-IP) and $S \subseteq V$ be a 1-max NPS in G. We first show that the incidence vector of S denoted by $\bar{x}$ is in F, where $\bar{x}_i = 1$ if $i \in S$ and 0 if $i \notin S$, by showing that $\bar{x}_i$ satisfies (2), (3), and (4) for all $i \in V$. If $\bar{x}_i = 1$, then it clearly satisfies all the constraints that are written for i. Suppose now that $\bar{x}_i = 0$ for some $i \in V$ which means that $i \in V \setminus S$. From the definition of an NPS, every vertex in $V \setminus S$ is adjacent to at most one vertex in S, therefore for any two vertices $j_1, j_2 \in \delta(i)$, at least one of $\bar{x}_{j_1}$ or $\bar{x}_{j_2}$ is zero showing that Constraints (2) are satisfied. From Lemma 1, any $i \in V \setminus S$ has a neighbor in $V \setminus S$ which has a neighbor in S. This ensures that Constraints (3) are satisfied. To show that Constraints 4 are satisfied, consider a vertex $j \in \delta(i)$. If $\bar{x}_j = 0$, Constraints 4 are clearly satisfied. If $\bar{x}_j = 1$, then by Lemma 1, we know that there exists $k \in NN_2(i, j)$ with $\bar{x}_k = 1$ and hence Constraints (4) are satisfied.

To show that any point in F corresponds to a 1-max NPS, let $x^* \in F$. Construct S such that $i \in S$ if $x^*_i = 1$ and $i \notin S$ if $x^*_i = 0$. By Constraints (2), it is immediate that S is an NPS. We will show that S is a 1-max NPS by showing each $i \in V \setminus S$ satisfies Lemma 1.

<u>Case 1</u>: i has no neighbors in S. Then by Constraint (3) written for i, i satisfies the condition in Lemma 1.

<u>Case 2</u>: i has exactly one neighbor, say j, in S. Then, by Constraint (4) written for i and j, we know that $\exists m, k \in V$ such that $\{i, m\}, \{m, k\} \in E$ with $m \neq j$, $x^*_m = 0$ and $x^*_k = 1$. Therefore i satisfies the condition in Lemma 1. We have shown that there is a one-to-one correspondence between points in F and 1-max NPS's in $G = (V, E)$.

4 Valid Inequalities for the 1-Max NPS Polytope

Let F be the feasible region of (MW 1-Max NPS-IP) and $conv(F)$ denote its convex hull which is called the 1-Max NPS Polytope. To our knowledge, the 1-Max NPS Polytope has not been studied before in the literature. We take some initial steps in this direction and propose some valid inequalities for $conv(F)$. Adding valid inequalities to an Integer Programming formulation makes it stronger and may lead to solving the problem as a linear program in some classes of graphs. The methodology we use in the search for valid inequalities is as follows. First, we generate some instances, usually small ones, and solve the linear programming relaxation of (MW 1-Max NPS-IP) for them. If the resulting solution is fractional, we try to come up with a class of valid inequalities that cuts it off. After finding such inequalities, they are added to the formulation to make it stronger and the procedure is repeated.

For each proposed class of inequalities, a proof of validity as well as a fractional point that is cut off is provided. This fractional point is not cut off by any of the previous inequalities which shows that the new class of inequalities is not dominated by the previous ones. Below each inequality, a graph is presented where the numbers inside the nodes represent the labeling of the nodes and the values below the nodes are the x_i values in the fractional solution that is cut by that inequality.

Proposition 1. *Let $G = (V, E)$ be a graph and $\{i, j\} \in E$ be a bridge. Let V' and V'' be the vertex sets of the resulting connected components after the removal of $\{i, j\}$, where $i \in V'$ and $j \in V''$. If $V' \setminus \{i\} \neq \emptyset$, then inequality (6) is valid and similarly if $V'' \setminus \{j\} \neq \emptyset$ then inequality (7) is valid for $conv(F)$:*

$$\sum_{k \in V'} x_k \geq 1 \tag{6}$$

$$\sum_{k \in V''} x_k \geq 1 \tag{7}$$

Proof: Without loss of generality, assume that $V' \setminus \{i\} \neq \emptyset$.

Case 1: *Vertex $i \in S$*

The inequality is trivially satisfied.

Case 2: *Vertex $i \notin S$, Vertex $j \in S$*

We will show that there is a vertex $k \in V'$ which is in S. Consider vertex i. By Lemma 1, there must exist a vertex $m \in \delta(i)$ such that $m \notin S$ and m should have a neighbor $k \in S$. This means that $m \neq j$ and hence $m \in V' \setminus \{i\}$. As the neighbors of any vertex in $V' \setminus \{i\}$ are in V' (because $\{i, j\}$ is a bridge), we have that k is in V' and hence inequality (6) is satisfied.

<u>Case 3:</u> *Vertex $i \notin S$, Vertex $j \notin S$*

Let $m \neq i$ be a vertex in V'. If $m \in S$, then inequality (6) is satisfied. Otherwise, for m to satisfy Lemma 1, there must exist a vertex $k \in N_2(m)$ such that $k \in S$. Because $i, j \notin S$ and any vertex $p \in V'' \setminus \{j\}$ has a distance of at least three to m, k is necessarily in $V' \setminus \{i\}$ and hence inequality (6) is satisfied.

The proof of the validity of inequality (7) is the same assuming $V'' \setminus \{j\} \neq \emptyset$. □

Remark: Inequality 8 written for the edge $\{3, 4\}$ cuts off the following extreme point in Figure 4.

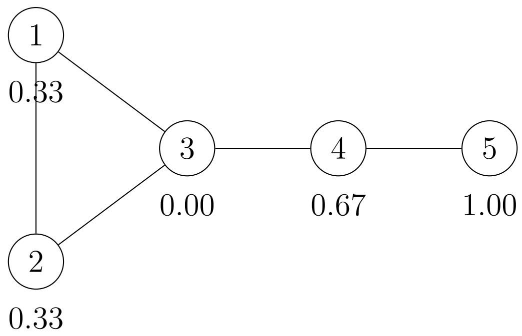

Figure 2: Extreme point cut by inequality 6

Remark: The inequalities (6) and (7) can be made stronger for trees, as shown in Proposition 2.

Proposition 2. *Let $T = (V, E)$ be a tree and $\{i, j\} \in E$. Let V' and V'' be the vertex sets of the resulting connected components after the removal of $\{i, j\}$, where $i \in V'$ and $j \in V''$. If $V' \setminus \{i\} \neq \emptyset$, then inequality (8) is valid and similarly if $V'' \setminus \{j\} \neq \emptyset$ then* *inequality (9) is valid for* $conv(F)$*:*

$$\sum_{k \in V' \setminus \{i\}} x_k \geq 1, \tag{8}$$

$$\sum_{k \in V'' \setminus \{j\}} x_k \geq 1. \tag{9}$$

.

Proof: Without loss of generality, assume that $V' \setminus \{i\} \neq \emptyset$. Let S be a 1-max NPS in T.

Case 1: *Vertex* $i \in S$

As $V' \setminus \{i\} \neq \emptyset$, we know that there exists $k \in V' \setminus \{i\}$. If $k \in S$, the inequality is valid so assume $k \notin S$. By Lemma 1, there must exist a vertex $m \in V' \setminus \{i\}$ where $m \in \delta(k)$, $m \notin S$ such that there exists a vertex $n \in \delta(m)$ where $n \in S$. $\{i, j\}$ being a bridge implies that $m, n \in V'$. If $n \neq i$, then inequality (8) is valid. Assume $n = i$. With a similar argument for the vertex m, by Lemma 1, there must exist a vertex $p \in \delta(m)$, where $p \notin S$ such that there exists a vertex $r \in \delta(p)$ such that $r \in S$. Furthermore, we have that r is a different vertex than i, as $r = i$ would imply that m, p, r form a triangle, contradicting that T is a tree.

Case 2: Vertex $i \notin S$

If $j \in S$, by Lemma 1, there must exist a vertex $v \in V' \setminus \{i\}$ such that $v \in S$. Otherwise, assume $j \notin S$. As $V' \setminus \{i\} \neq \emptyset$, we know that there exists $a \in V' \setminus \{i\}$. If $a \in S$, the inequality is valid so further assume $a \notin S$. Then, by Lemma 1, a needs to have a vertex $b \in N_2(a)$ where $b \in S$. Because $j \notin S$ and, b is necessarily in $V' \setminus \{i\}$.

These two cases prove the validity of inequality (8). The proof for inequality (9) is the same.

Remark: The stronger version of the inequality is only valid for trees. A counterexample is provided in Figure 3, where we take the bridge to be {3,4}, it is possible to construct the incidence vector of a 1-max NPS in S that violates 8.

Remark: Inequality 8 written for the bridge $\{2, 3\}$ cuts off the following extreme point in Figure 4.

Proposition 3. *Let* $G = (V, E)$ *be a graph having two bridges* $\{i_1, j_1\}, \{i_2, j_2\} \in E$*. The removal of these bridges divides* V *into three disjoint sets,* V'*,* V'' *and* V''' *where* $i_1 \in V'$*,*

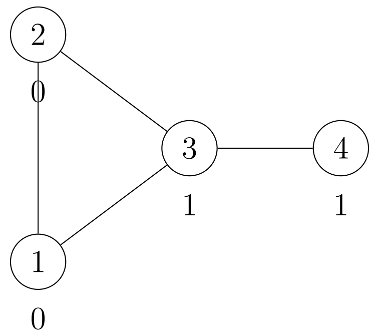

Figure 3: Counterexample for bridge inequalities in general graphs

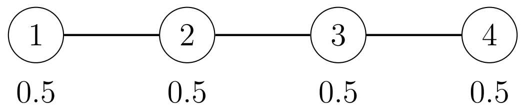

Figure 4: Example of an extreme point cut by inequality 8

$j_1, i_2 \in V''$ and $j_2 \in V'''$. If $\{j_1, i_2\} \notin E$, then the following inequality is valid for conv(S):

$$\sum_{j \in V''} x_j \geq 1 \tag{10}$$

Proof: Let S be a NPS in G. Because $\{j_1, i_2\} \notin E$, and j_1 and i_2 are in the same connected component, there must exist another vertex $k \in V''$. By Lemma 1, either $k \in S$, or there exists a vertex $m \in \delta(k)$ and $m \in V \setminus S$ such that m havs a neighbor $n \in S$. If $n \in V''$, inequality (10) is valid. Otherwise, either $n = j_2$ or $n = i_1$. This is because $\{i_1, j_1\}, \{i_2, j_2\}$ are bridges, hence any vertex in $V'' \setminus \{i_2, j_1\}$ must have distance at least 2 to i_1 and j_2. Without loss of generality, assume $n = i_1$. By Lemma 1 for vertex j_1, there must exist a vertex $p \in \delta(j_1)$ and $p \in V \setminus S$ such that p has a neighbor $q \in S$. Because $p \in V \setminus S$, we have that $p \neq i_1$, and hence q must be in V''. $\square$

Proposition 4. *Let $G = (V, E)$ be a graph. Suppose vertex $i \in V$ is a support vertex. Let $N_p(i) = \{j \in \delta(i) \mid j$ is pendant$\}$. If i has exactly one non-pendant neighbor, the following equality is valid for conv(F):*

$$\sum_{j \in N_p(i)} x_j - 1 = (|N_p(i)| - 1) \cdot x_i \tag{11}$$

Proof: Let S be a 1-max NPS in G and $i \in V$ be a support vertex with a single nonpendant neighbor $np \in \delta(i)$. i is a support vertex implies $N_p(i) \neq \emptyset$. Let $p \in \delta(i)$ be a pendant neighbor of i.

Case 1: *Vertex $i \in S$*
We will show that all pendant neighbors of i are in S, which will prove the equality for $x_i = 1$. The proof is by contradiction. Assume $p \notin S$. By Lemma 1, p must have a neighbor in $V \setminus S$. However, the only neighbor of p is i and $i \in S$, which is a contradiction. Therefore, we have that all vertices $p \in N_p(i)$ are in S.
Case 2: *Vertex $i \notin S$*
When $i \notin S$, the inequality reduces to

$$\sum_{j \in N_p(i)} x_j = 1$$

We will show that if $i \notin S$, exactly one of the vertices $p \in N_p(i)$ is in S. By Lemma 1 for vertex i, there must exist a vertex $m \in \delta(i)$, where $m \in V \setminus S$ and m must have a neighbor $k \in S$. We have that m has at least two vertices, namely i and k, therefore m cannot be a pendant vertex. As m is nonpendant, we have that $m = np$, and hence $np \notin S$.

Due to the NPS condition for vertex i, $i \notin S$ implies at most one of the vertices $v \in \delta(i)$ can be in S. By Lemma 1 for vertex p, we have that either $p \in S$, or there must exist a vertex $r \in N_2(p)$ such that $r \in S$. Because p is pendant, we have that $\{p\} \cup N_2(p) = \delta(i)$, therefore we have that there exists at least one vertex $v \in \delta(i)$, (either p or r) such that $v \in S$. Finally, knowing that there exists at least one vertex $v \in \delta(i)$, and at most one of the vertices $v \in \delta(i)$ can be in S; we can conclude there exists exactly one vertex $v \in \delta(i)$ such that $v \in S$. Such a vertex v is necessarily pendant as we have that $np \notin S$ which completes the proof. □
Remark: The following extreme point in Figure 5 is cut by inequality 11 written for vertex 1.

Proposition 5. *Let $G = (V, E)$ be a graph. If $u \in V$ is a universal vertex and $p \in V$ is a pendant vertex, the following inequality is valid for $conv(F)$:*

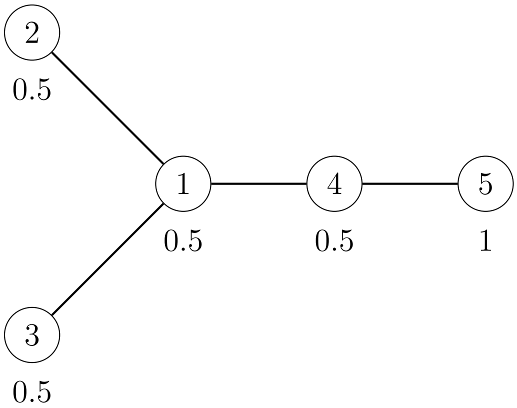

Figure 5: Extreme point cut by inequality 11

$$x_u = x_p \tag{12}$$

Proof:
Let S be a NPS in G and $u \in V$ be a universal vertex and $p \in V$ be a pendant vertex. The inequality implies either $u, p \in S$ or $u, p \notin S$.
Case 1: $u \in S$
Proof is by contradiction. Assume $p \notin S$. By Lemma 1.
, there must exist $v \in V$ such that $v \notin S$. However, the only neighbor of p is u, and $u \in S$. This is a contradiction, therefore we have that $p \in S$.
Case 2: $u \notin S$
By Lemma 1 , $\exists k \in N_2(u)$ such that $k \in S$. u and k are adjacent because u is a universal vertex. To satisfy the NPS condition for vertex u, p cannot be in S, and we have that $p \notin S$.

□

Remark: The following extreme point in Figure 6 is cut by inequality 12 written for the universal vertex 1 and the pendant vertex 2.

Proposition 6. *Let $G = (V, E)$ be a graph and let $u \in V$ be a universal vertex. Let $V_1, V_2, \ldots, V_k$ be the vertex sets of the resulting connected components after the removal of u. If $i, j \in V_i$ for some $i \in \{1, 2, \ldots, k\}$, the following inequalities are valid for conv(F):*

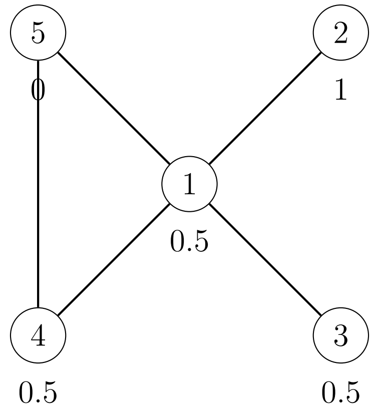

Figure 6: Star graph with 5 nodes

$$(1 - x_u) + (1 - x_j) + x_i \geq 1 \tag{13}$$
$$(1 - x_u) + (1 - x_i) + x_j \geq 1 \tag{14}$$

Proof: Let S be a 1-max NPS in G, $u \in V$ be a universal vertex and $V_1, V_2, \ldots, V_k$ be the vertex sets of the resulting connected components after the removal of u. Without loss of generality, assume $i, j \in V_1$. The inequalities 13 and 14 imply that if $u \in S$, then i, j are either both in S or both in $V \setminus S$. Assume that u is in S.

Assume $i \in S$. Then, any vertex $w \in \delta(u)$ must also be in S because it has two neighbors in S, namely i and u. For any $p \in \delta(w)$, all its neighbors must also be in S because it has two neighbors in S, namely w and u. Continuing like this, all the vertices in the same connected component as i must also be in S.

Similarly, assume $i \in V \setminus S$. Then no vertex $v \in \delta(i)$ can be in S to ensure S is a NPS because u is in S and is a neighbor of i. Continuing like this, none of the vertices in the same connected component as q can be in S.

Remark: Inequalities 13 and 14 cut off the extreme point in Figure 7 written for the universal vertex 3 and the vertices 1 and 2.

Proposition 7. *Let $G = (V, E)$ be a graph. For any $v \in V$, if there exists $i, j \in V$ such that for all $k \in \delta(v)$, if $i, j \in \delta(k)$, then the following inequality is valid for conv(F):*

$$x_i + x_j \leq 1 + x_v \tag{15}$$

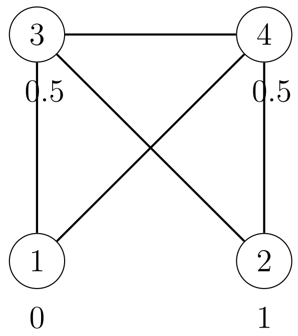

Figure 7: Example extreme point cut by inequalities 13 and 14

Proof: Let S be an 1-max NPS. Let there be vertices $v, i, j \in V$ such that for all $k \in \delta(v)$, $i, j \in \delta(k)$. The inequality implies that if $i, j \in S$, then $v \in S$.

Assume that i and j are in S. Then, any vertex that is in both $\delta(i)$ and $\delta(j)$ are necessarily in S. Because all neighbors of v are neighbors of both i and j, any $k \in \delta(v)$ is also in S. This implies that v is also in S, which completes the proof. □

Remark: Inequality 15 cuts off the following extreme point in Figure 8 written for the vertices $v = 1, i = 3, j = 4$.

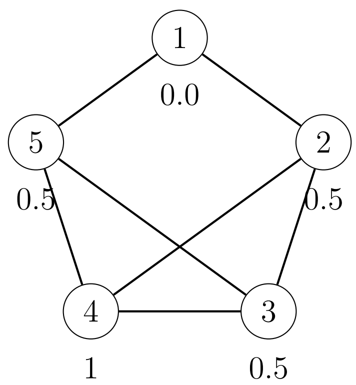

Figure 8: Example extreme point cut by inequality 15

Remark: If i and j are both neighbors of v, then the inequality is redundant as it is the same as the constraint 2

Proposition 8. *Let $G = (V, E)$ be a graph. Let the vertices $v_1, v_2, ..., v_{2k+1} \in V$ induce the subgraph K_{2k+1}, the complete graph of size $2k + 1$. Then, the following inequality is valid for conv(S):*

$$1 + \sum_{i=1}^{k} v_i \geq \sum_{i=k+1}^{2k+1} v_i \tag{16}$$

Proof: Let $S \subseteq V$ be a 1-max NPS. Enumerating all the NPS's in G, it is either the case that S is a singleton set, or $S = V$. In either case, the inequality is clearly valid.
Remark: For cliques of size 3, the inequality is the same as constraint2 from the initial formulation.
Remark: Inequality 16 cuts off the following extreme point in Figure 9.

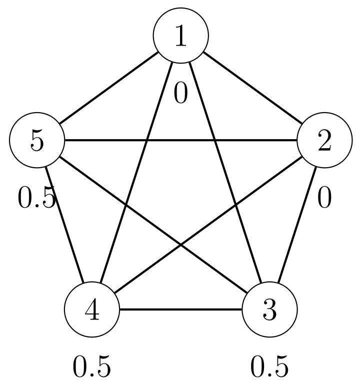

Figure 9: Example extreme point cut by inequality 16

Proposition 9. *Let $G = (V, E)$ be a graph. Let the vertices $v_1, v_2, ..., v_k \in V$ induce the subgraph K_k, the complete graph of size k. Then, the following inequality is valid for $conv(S)$:*

$$1 + (k-2) \cdot x_1 \geq x_2 + x_3 + \cdots + x_k \tag{17}$$

Proof: Let $S \subseteq V$ be a 1-max NPS. Enumerating all the NPS's in G, it is either the case that S is a singleton set, or $S = V$. In either case, the inequality is clearly valid.
Remark: Inequality 17 cuts off the following extreme point in Figure 9.

Theorem 1. *Let P' denote the polyhedron defined by the constraints (2), (3), (4), (17) as well as $0 \leq x_i \leq 1$ for all i. Then $P' = conv(S)$ for cliques.*

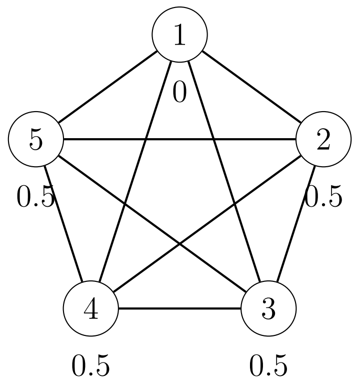

Figure 10: Example extreme point cut by inequality 17

Proof: Let $G = (V, E)$ be a complete graph of order n. Let S be the set of incidence vectors of all 1-max NPS of G. $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n, \mathbf{1}\}$ where $\mathbf{e}_i$ is the unit vector with i^{th} component 1; and $\mathbf{1}$ is a vector of all ones. Let $\bar{x}$ be a fractional point in P'. We will show that for any $\bar{x}$, it can be written as a convex combination of elements of S.

Let $\bar{\mathbf{x}} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix}$. Without loss of generality, assume $\bar{x_1} \leq \bar{x_2} \leq \cdots \leq \bar{x_n}$.Then, we can write $\bar{x}$ as follows:

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix} = a \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + (\bar{x}_1 - a)\mathbf{e}_1 + (\bar{x}_2 - a)\mathbf{e}_2 + \cdots + (\bar{x}_n - a)\mathbf{e}_n$$

where we choose $a = \frac{x_1 + x_2 + \cdots + x_n - 1}{n-2}$ so that the coefficients add up to 1. To show that this is a convex combination, we need to show that all the coefficients are non-negative. a is clearly nonnegative due to the inequality 3. From inequality 17, it follows that $(\bar{x_1} - a)$ is nonnegative. Because $\bar{x_1}$ is the smallest, all the other coefficients are nonnegative as well. Therefore, we conclude that for any $\bar{x}$, it is possible to write it as a convex combination of points in S which completes the proof.

Proposition 10. *Let $G = (V, E)$ be a graph. Let $k \geq 2$ be an integer, and the vertices $v_1, v_2, ..., v_{2k+1} \in V$. If $deg(v_2) = deg(v_3) = ... = deg(v_{2k}) = 2$ and the induced subgraph of*

$v_1, v_2, ..., v_{2k+1}$ *is either a path graph or a cycle of size* $2k+1$*, then the following inequality is valid for conv*(S)*:*

$$1 + x_{v_2} + x_{v_4} + \cdots + x_{v_{2k}} \geq x_{v_1} + x_{v_3} + x_{v_5} + \cdots + x_{v_{2k+1}} \tag{18}$$

Equivalently, the same inequality can be written as:

$$1 + \sum_{i=1}^{k} x_{v_{2i}} \geq \sum_{i=0}^{k} x_{v_{2i+1}} \tag{19}$$

Proof: Let S be a 1-max NPS and let there be $v_1, v_2, ..., v_{2k+1} \in V$ such that $deg(v_2) = deg(v_3) = ... = deg(v_{2k}) = 2$ and the induced subgraph of $v_1, v_2, ..., v_{2k+1}$ is either a path graph or a cycle of size $2k+1$. Consider any even indexed v_{2i}. First, we show that if $v_{2i} \notin S$,then exactly one of v_{2i-1} or v_{2i+1} is in S. The proof is by contradiction. Both v_{2i-1} and v_{2i+1} cannot be in S due to the NPS condition of v_{2i}. Assume both v_{2i-1}, $v_{2i-1} \notin S$. Then, due to Lemma 1, v_{2i} must have a neighbor in $V \setminus S$ which has a neighbor in S. Without loss of generality, call it v_{2i-2}, the neighbor of v_{2i-1}. However, this means that v_{2i-1} cannot satisfy Lemma 1 , therefore S is not maximal.

This implies that for any $x_{v_{2i}}$=0 on the left hand side, there is another vertex on the right hand side with the value zero, i.e. there are at least as many 0-valued variables on the right hand side as the left hand side, completing the proof.

Remark: Inequality 18 cuts off the following extreme point in Figure 11:

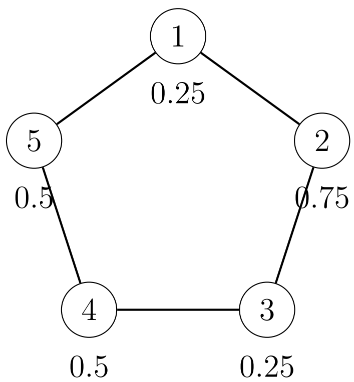

Figure 11: Example extreme point cut by inequality 18 and 19

Proposition 11. *Let $G = (V, E)$ be a graph and let $i, j \in V$ be two-step neighbors with some common neighbor $m \in V$, where $m \in \delta(i)$ and $m \in \delta(j)$. If $N_2(j) \subseteq \delta(i)$ and $m \notin N_2(j)$, then the following inequality is valid for conv(S):*

$$x_m \leq x_i + x_j \tag{20}$$

Proof: Let $G = (V, E)$ be a graph and let $i, j \in V$ be two-step neighbors with a common neighbor $m \in V$. Let S be a 1-max NPS. The inequality implies that if $i, j \notin S$, then $m \notin S$. Proof is by contradiction. Assume $i, j \notin S$ and $m \in S$ .Then, none of the neighbors of i or j except m can be in S due to the nearly perfect set condition. Because $N_2(j) \subseteq \delta(i)$ and $m \notin N_2(i)$, $S \cap N_2(i) = \emptyset$ This contradicts Lemma 1.
Remark: Inequality 20 cuts off the extreme point in 12

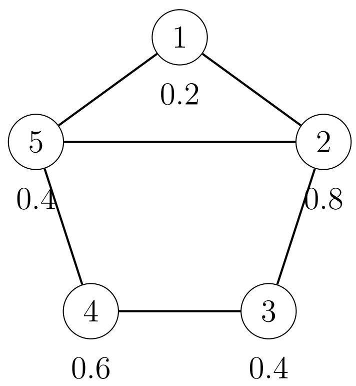

Figure 12: Example extreme point cut by inequality 20

Definition 1. *Let $G = (V, E)$ be a graph. Its two-step graph is defined as the graph $G' = (V, E')$ where $\{i, j\} \in E'$ if $j \in N_2(i)$ in G.*

Theorem 2. *Let $G = (V, E)$ be a graph. Let $G' = (V, E')$ be the two-step graph associated with G. The minimum cardinality dominating set of G', known as $\gamma(G')$, gives a lower bound on the minimum cardinality of a 1-max NPS ($n_p(G)$).*

Proof: Let $G = (V, E)$ be a graph, S be a 1-max nearly perfect set on G and Let G' be the two-step graph of G. Then, by Lemma 1 , any $v \in V$ is either in V, or has a two-step neighbor in V. This implies that S is a dominating set in G'. The converse is clearly not true.

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