Solving the Minimum Weighted Paired-Dominating Set Problem in Trees Using Linear Programming

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Abstract

This is a draft shared for PhD admission purposes. It is incomplete and may contain errors.

We study the minimum weighted paired-dominating set problem (MWPDSP) in trees using integer programming techniques. We first formulate the MW-PDSP as an integer programming problem whose linear programming relaxation does not always represent the convex hull of the incidence vectors of paired-dominating sets in trees. We then strengthen the formulation and show that even though the constraint matrix of the strengthened formulation is not always totally unimodular, its linear programming relaxation always defines the convex hull of the incidence vectors of paired-dominating sets in trees and thus solves the MWPDSP in trees.

Keywords: graph theory, domination, paired-domination, paired-dominating set, integer programming, linear programming

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1. Introduction

Domination in graph theory is an extensively studied area of research with various applications in fields such as location theory, social networks, and computer communication networks [1]. In this paper, we study a variant of domination, namely paired-domination, in trees using integer programming.

Let $G = (V, E)$ be a simple undirected graph with vertex set V and edge set E. Two vertices i and j are said to be adjacent if $\{i, j\} \in E$. A vertex $i \in V$ and an edge $e \in E$ are said to be incident if $i \in e$. Two edges $e, f \in E, e \neq f$ are said to be adjacent if $e \cap f \neq \emptyset$. For a vertex $v \in V$, the open neighborhood of v, denoted by $\delta(v)$, is defined to be the set of all vertices in V that are adjacent to v, i.e., $\delta(v) = \{j \in V | \{v, j\} \in E\}$. The closed neighborhood of a vertex $v \in V$, denoted by $\Delta(v)$, is defined as $\delta(v) \cup \{v\}$. The degree of a vertex $v \in V$, denoted by $deg(v)$, is equal to the number of vertices in the open neighborhood of v, i.e., $deg(v) = |\delta(i)|$. A vertex $v \in V$ is called a pendant vertex if $deg(v) = 1$. Any vertex that is adjacent to a pendant vertex is called a support vertex.

For a subset S of V, $G[S]$ represents the subgraph of G induced by the vertices in S. A matching M in G is a subset of E such that no two edges in M are adjacent. A matching M in $G = (V, E)$ is said to be perfect if every vertex $v \in V$ is incident to an edge of M.

A subset S of V is a dominating set of $G = (V, E)$ if every vertex in $V \setminus S$ has at least one neighbor in S. A vertex v in a dominating set S is said to dominate all vertices in $\delta(v)$. The domination number of a graph G is the minimum cardinality of a dominating set of G. Given a graph $G = (V, E)$, let w_i be the weight associated with vertex $i \in V$. The minimum weighted dominating set problem (MW-DSP) is the problem of finding a dominating set of G of minimum weight, where the weight of a dominating set S is defined to be $\sum_{i \in S} w_i$.

A subset S of V is a paired-dominating set if S is a dominating set and $G[S]$ has a perfect matching. Every graph without isolated vertices has a paired-dominating set. Motivated by the following application, paired-domination was introduced by Haynes and Slater [2]. Assume that each region of a museum is to be protected by a guard in the region or a guard in a neighboring region. If all the regions are protected, then the regions with the guards form a dominating set in the graph which is obtained by taking the regions as vertices and adding an edge between two vertices if the corresponding regions are neighbors. If, in addition, it is required that each guard is paired up with an adjacent guard so that they can back each other up, then the regions with the guards form a paired-dominating set. For another application of paired-domination, see [3]. The paired-domination number of a graph G is the minimum cardinality of a paired-dominating set of G. Let w_i be the weight associated with vertex $i \in V$

and α_e the weight associated with edge $e \in E$. If $e = \{i, j\}$, then we also use the notation α_{ij} for α_e. The minimum weighted paired-dominating set problem (MW-PDSP) is the problem of finding a dominating set S with

$$\sum_{i \in S} w_i + \sum_{e \in E(G[S])} \alpha_e \tag{1}$$

the minimum, where $E(G[S])$ represents the set of edges of $G[S]$. Note that most of the studies in the literature on paired-domination focus on the unweighted version of the MW-PDSP, i.e., $w_i = 1, \forall i \in V$ and $\alpha_e = 0, \forall e \in E$ and study the paired domination number for general or specific graphs, see e.g., [4, 5]. The studies dealing with the weighted version consider only weights on the vertices, i.e., $\alpha_e = 0, \forall e \in E$. In this study, we will study the MW-PDSP in trees where there are weights both on the vertices and the edges. Our results will continue to hold if there are no weights on the edges.

There are a number of studies in the literature investigating paired-domination in trees. The authors in [6] present a linear time algorithm to compute the paired-domination number in trees and characterize trees with equal domination and paired-domination numbers. The study in [7] characterizes the set of vertices of a tree that are contained in every, or in no, paired-dominating set of minimum cardinality. The authors in [8] study the MW-PDSP in trees where there are weights only on the vertices and propose a linear time dynamic programming style algorithm for its solution. Some other studies in the literature on paired-domination in trees that are less related to this study are [9, 10].

In this paper, we study the minimum weighted paired-dominating set problem in trees using integer programming techniques. To the knowledge of the author, this is the first study proposing an integer programming formulation for the MW-PDSP. After proposing an initial integer programming formulation, we propose a stronger formulation and show that the linear programming relaxation of the latter formulation solves the MW-PDSP in trees. Moreover, to the knowledge of the author, this is the first study in the literature that considers edge weights in paired-domination.

2. Integer Programming and Paired-domination in Trees

Given a graph $G = (V, E)$ and a weight w_i associated with each vertex $i \in V$, an integer programming formulation for the MW-DSP is given as follows [11]

$$\text{(IP-D)} \qquad \begin{array}{lll} \text{minimize} & \sum_{i \in V} w_i x_i & \\ \text{s.t.} & \sum_{j \in \Delta(i)} x_j \geq 1, \forall i \in V & (1) \\ & x_i \in \{0, 1\}, \forall i \in V. & (2) \end{array}$$

Here, x_i is a binary variable that takes the value 1 if and only if vertex i is included (or selected) in the dominating set. The objective function minimizes the weight of the dominating set. The first set of constraints make sure that every vertex $i \in V$ is either included in the dominating set or is dominated by a vertex in the dominating set.

The formulation (IP-D) can be extended to obtain an integer programming formulation for the MW-PDSP. Keeping the x_i's in the formulation, we introduce for each edge $\{i, j\} \in E$ a binary variable y_{ij} which takes the value 1 if and only the edge $\{i, j\}$ is included in the perfect matching of $G[S]$, where $S = \{i \in V | x_i = 1\}$. An integer programming formulation for the MW-PDSP is given as follows

$$\text{(IP-PD-1)} \qquad \begin{array}{lll} \text{minimize} & \sum_{i \in V} w_i x_i + \sum_{\{i,j\} \in E} \alpha_{ij} y_{ij} & \\ \text{s.t.} & \sum_{j \in \Delta(i)} x_j \geq 1, \forall i \in V & (3) \\ & x_i = \sum_{j \in \delta(i)} y_{ij}, \forall i \in V & (4) \\ & x_i \in \{0, 1\}, \forall i \in V & (5) \\ & y_{ij} \in \{0, 1\}, \forall \{i, j\} \in E. & (6) \end{array}$$

In (IP-PD-1), the constraints in (3) ensure that the selected vertices form a dominating set. Constraints in (4) make sure that x_i is equal to 1 if and only if some y_{ij} takes the value 1 for a vertex $j \in \delta(i)$ which forces x_j to take the value 1 as well. Therefore it is ensured that, each vertex selected in the dominating set is paired up with an adjacent selected vertex. By (4), it is also guaranteed that a selected vertex cannot be paired up with more than one adjacent selected vertex and therefore implying that the selected vertices form a paired-dominating set.

Let $P_1(G)$ be the polytope described by the linear programming (LP) relaxation of (IP-PD-1) for a given graph G. Consider the tree T_1 in Figure 1 which is a star graph of order four with internal vertex 0 and three outer vertices $1, 2$, and 3.

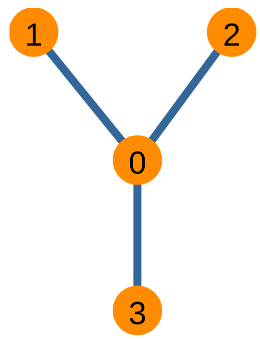

Figure 1: Tree T_1 with $P_1(T_1)$ non-integral

It can be shown that the following non-integral point (x', y') is an extreme point of $P_1(T_1)$ implying that $P_1(T_1)$ is not integral. Hence the linear programming relaxation of (IP-PD-1) does not always solve the MW-PDSP in trees.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \\ y'_{01} \\ y'_{02} \\ y'_{03} \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}$$

Consider a paired-dominating set S of $G = (V, E)$. Note that any vertex not in S is dominated by a vertex in S as S is a dominating set and any vertex in S is dominated by some other vertex in S as S is a paired-dominating set. Therefore it can be concluded that if S is a paired-dominating set, then every vertex of G is dominated by some vertex in S. With this observation, we now provide a stronger formulation for the MW-PDSP.

$$
\text{(IP-PD-2)}\quad
\begin{aligned}
\text{minimize} \quad & \textstyle\sum_{i\in V} w_i x_i + \sum_{\{i,j\}\in E} \alpha_{ij} y_{ij} && \\
\text{s.t.} \quad & \textstyle\sum_{j\in\delta(i)} x_j \geq 1, \forall i \in V && (7)\\
& x_i = \textstyle\sum_{j\in\delta(i)} y_{ij}, \forall i \in V && (8)\\
& x_i \in \{0,1\}, \forall i \in V && (9)\\
& y_{ij} \in \{0,1\}, \forall \{i,j\} \in E. && (10)
\end{aligned}
$$

The only difference of (IP-PD-2) from (IP-PD-1) is the constraint set (7). As the incidence vector of any paired-dominating set satisfies all the inequalities in (7), the formulation (IP-PD-2) is still valid for the MW-PDSP. Note that the non-integral vector (x', y') does not satisfy the inequality (7) when it is written for the internal vertex 0 of T_1.

Let $P_2(G)$ be the polytope described by the LP relaxation of (IP-PD-2) for a given graph G. In the LP relaxation, the constraints in (9) and (10) are replaced with

$$0 \leq x_i \leq 1, \forall i \in V, \tag{11}$$

$$0 \leq y_{ij} \leq 1, \forall \{i,j\} \in E, \tag{12}$$

respectively.

We first prove that for the support vertices, the corresponding x variables can be fixed.

Proposition 2.1. *Let $i \in V$ be a support vertex in $G = (V, E)$. Then for any $(\overline{x}, \overline{y}) \in P_2(G)$, we have that $\overline{x_i} = 1$.*

Proof. Let $j \in V$ be a pendant vertex that is adjacent to i. We have that $\overline{x_i} \geq 1$ by (7) and $\overline{x_i} \leq 1$ by (11) resulting in $\overline{x_i} = 1$. □

Note that Proposition 2.1 is stronger than stating that $\overline{x_i} = 1$ for every feasible vector $(\overline{x}, \overline{y})$ of (IP-PD-2) as $P_2(G)$ contains the feasible region of (IP-PD-2).

Does the LP relaxation of the formulation (IP-PD-2) solve the MW-PDSP in trees? In the search for an answer to this question, one can check whether the coefficient matrix of the LP relaxation of (IP-PD-2) is totally unimodular or not. When total unimodularity is checked, it can be seen that the coefficient matrix of the LP relaxation of (IP-PD-2) is totally unimodular for any tree of order less than or equal to four. For the following tree T_2 which is of order five, the corresponding coefficient matrix, however, is not totally unimodular.

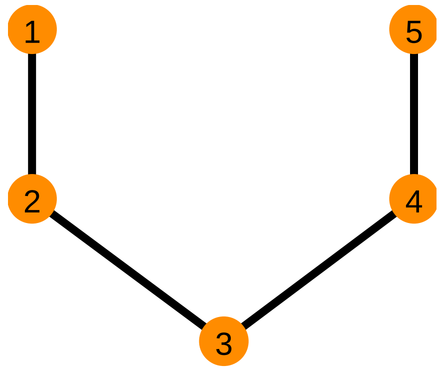

Figure 2: A tree for which the coefficient matrix of the LP relaxation of (IP-PD-2) is not totally unimodular

A part of the coefficient matrix of the LP relaxation of (IP-PD-2) for T_2 is

$$\left[\begin{array}{ccccc|cccc}
0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\
\hline
\mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-1} & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{-1} & \mathbf{-1} & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{-1} & \mathbf{-1} & 0 \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{-1} & \mathbf{-1} \\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{-1}
\end{array}\right]$$

where the first five rows correspond to the inequalities (7), the last five rows to the inequalities (8), the first five columns to the x variables, and the last four columns to the y variables. The following submatrix obtained by taking rows $2, 3, 4, 6, 7, 9, 10$ and columns $1, 2, 3, 4, 5, 6, 9$ has determinant 2.

$$\left[\begin{array}{ccccc|cc}
\mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\
\hline
\mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-1} & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{-1} & 0 \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{-1} \\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{-1}
\end{array}\right]$$

Even though the coefficient matrix of the LP relaxation of (IP-PD-2) is not always totally unimodular for trees, we will show that $P_2(T)$ is always integral for every tree T implying that the LP relaxation of (IP-PD-2) solves the MW-PDSP in trees. Before proving the main theorem, we will prove that for any star graph S, $P_2(S)$ is integral.

Proposition 2.2. *Let $S = (V, E)$ be a star graph with internal vertex 0 and outer vertices $1, 2, \ldots, n$. We have that $P_2(S)$ is integral.*

Proof. There are n paired-dominating sets in S which are obtained by selecting the internal vertex and any of the outer vertices. We will show that any $(\overline{x}, \overline{y}) \in P_2(S)$ can be written as a convex combination of the incidence vectors of these n paired-dominating sets. As 0 is a support vertex, by Proposition 2.1, we have that $\overline{x_0} = 1$. By (8), we have that $\overline{x_0} = \overline{y_{01}} + \overline{y_{02}} + \cdots + \overline{y_{0n}} = 1$. We have that $\overline{x_i} = \overline{y_{0i}}$ for any $i \in \{1, 2, \ldots, n\}$ again by (8) which implies that $\overline{x_1} + \overline{x_2} + \cdots + \overline{x_n} = 1$. We can now write $(\overline{x}, \overline{y})$ as a convex combination of the incidence vectors of the paired-dominating sets of S as

$$\begin{pmatrix}\overline{x}\\ \overline{y}\end{pmatrix} = \begin{pmatrix}\overline{x_0}\\ \overline{x_1}\\ \overline{x_2}\\ \vdots\\ \overline{x_n}\\ \overline{y_{01}}\\ \overline{y_{02}}\\ \vdots\\ \overline{y_{0n}}\end{pmatrix} = \overline{x_1}\begin{pmatrix}1\\1\\0\\\vdots\\0\\1\\0\\\vdots\\0\end{pmatrix} + \overline{x_2}\begin{pmatrix}1\\0\\1\\\vdots\\0\\0\\1\\\vdots\\0\end{pmatrix} + \cdots + \overline{x_n}\begin{pmatrix}1\\0\\0\\\vdots\\1\\0\\0\\\vdots\\1\end{pmatrix}.$$

□

We will now state the main theorem.

Theorem 2.3. *For any tree T, $P_2(T)$ is integral.*

We will prove Theorem 2.3 after stating and proving some claims. Any tree of order two or three is a star graph, and by Proposition 2.2, for such trees the polytope described by the LP relaxation of (IP-PD-2) is integral. Assume that Theorem 2.3 is not true. That means that there exists a smallest counterexample $T' = (V', E')$, i.e., a tree T' with $|E(T')|$ the smallest such that $P_2(T')$ is not integral. As $P_2(T')$ is not integral, it has a non-integral extreme point $(\overline{x}, \overline{y})$. We will study some properties of the point $(\overline{x}, \overline{y})$ and then show that it's existence results in a contradiction.

Claim 2.4. *We have that $\overline{x_i} > 0$ for all $i \in V'$.*

Proof of Claim 2.4 Assume to the contrary that $\overline{x_i} = 0$ for some $i \in V'$. If i is a pendant vertex of T', then i has exactly one adjacent vertex, say j, and by Proposition 2.1, we have that $\overline{x_j} = 1$. Removing the vertex i and the edge $\{i, j\}$ from T', we obtain a new tree, say T'', such that the subvector of $(\overline{x}, \overline{y})$ obtained by removing $\overline{x_i}$ and $\overline{y_{ij}}$, call it $(\overline{x'}, \overline{y'})$, belongs to $P_2(T'')$. As T' is the smallest tree with $P_2(T')$ non-integral, we have that $P_2(T'')$ is integral.

Therefore $(\overline{x'}, \overline{y'})$ can be written as a convex combination of the incidence vectors of paired-dominating sets of T'' as

$$(\overline{x'}, \overline{y'}) = \sum_{t=1}^{k} \alpha_t (\overline{x^t}, \overline{y^t}),$$

where $0 \leq \alpha_t \leq 1$ for each $t \in \{1, 2, \dots, k\}$, $\sum_{t=1}^{k} \alpha_t = 1$, and $(\overline{x^t}, \overline{y^t})$ is the incidence vector of some paired-dominating set of T'' for each $t \in \{1, 2, \dots, k\}$. Note that as $\overline{x_j} = \overline{x'_j} = 1$, it should be that $\overline{x^t_j} = 1$ for each $t \in \{1, 2, \dots, n\}$. This means that $(\overline{x'}, \overline{y'})$ is a convex combination of incidence vectors of paired-dominating sets of T'' such that in each paired-dominating set the vertex j is selected. Note that any paired-dominating set of T'' in which vertex j is selected is also a paired-dominating set of T' as the vertex i is dominated by j. Therefore adding two components with zero values (for $\overline{x_i}$ and $\overline{y_{ij}}$) to each vector $(\overline{x^t}, \overline{y^t})$, we obtain an incidence vector of a paired-dominating set of T', call it (x^t, y^t). We then have

$$(\overline{x}, \overline{y}) = \sum_{t=1}^{k} \alpha_t (x^t, y^t),$$

showing that $(\overline{x}, \overline{y})$ can be written as a convex combination of incidence vectors of paired-dominating sets of T'. This contradicts to the fact that $(\overline{x}, \overline{y})$ is an extreme point of $P_2(T')$.

On the other hand, if i is not a pendant vertex, then it is clearly not a support vertex either as $\overline{x_i} = 0$. By (7), we have that $\sum_{j \in \delta(i)} \overline{x_j} \geq 1$. As $(\overline{x}, \overline{y})$ is an extreme point of $P_2(T')$, there exist vectors c and d such that $(\overline{x}, \overline{y})$ is the unique optimal solution of the minimization of $c^T x + d^T y$ subject to $(x, y) \in P_2(T')$. Now we remove the vertex i and all the edges in T' that are incident to it and obtain $r = |\delta(i)|$ many trees, call them $T_1, T_2, \dots, T_r$, each of an order lower than the order of T'. Denoting by $(\overline{x}, \overline{y})_{T_\ell}$ the subvector of $(\overline{x}, \overline{y})$ corresponding to the vertices and edges of T_ℓ, it can be seen that $(\overline{x}, \overline{y})_{T_\ell}$ belongs to $P_2(T_\ell)$ for every $\ell \in \{1, 2, \dots, r\}$. We have that $P_2(T_\ell)$ is integral for each $\ell \in \{1, 2, \dots, r\}$. Consider a vertex $j \in \delta(i)$ such that $\overline{x_j} > 0$ (such a vertex exists by (7)) and assume that it belongs to some T_ℓ. If we minimize $c_\ell^T x + d_\ell^T y$ over $P_2(T_\ell)$, where c_ℓ and d_ℓ are the subvectors of c and d, respectively, corresponding to the vertices and edges of T_ℓ, then as $P_2(T_\ell)$ is integral, there exists an optimal solution which is the incidence vector of some paired-dominating set of T_ℓ in which the vertex i is selected (call this paired-dominating set of T_ℓ as S'_ℓ). If we minimize $c_\ell^T x + d_\ell^T y$ over $P_2(T_\ell)$ for each $\ell \in \{1, 2, \ldots, r\}$, we know that there exists an optimal solution (incidence vector of some paired-dominating set of T_ℓ) that is integral. Observe that if $S_1, S_2, \ldots, S_r$ are paired-dominating sets of $T_1, T_2, \ldots, T_r$, respectively, such that at least one of the vertices in $\delta(i)$ is selected in $S_1 \cup S_2 \cup \cdots \cup S_r$, then $S_1 \cup S_2 \cup \cdots \cup S_r$ is a paired-dominating set of T'. With this observations, we take the union of the paired-dominating sets of $T_1, T_2, \ldots, T_r$ (one of which is S'_ℓ) whose incidence vectors are optimal solutions of the problem of the minimization of $c_\ell^T x + d_\ell^T y$ over $P_2(T_\ell)$. The incidence vector of the union which is integral is an alternative optimal solution to the minimization of $c^T x + d^T y$ subject to $(x, y) \in P_2(T')$ contradicting to the fact that $(\overline{x}, \overline{y})$ is the unique optimal solution.

Claim 2.5. *We have that $\overline{y_{ij}} > 0$ for all $\{i, j\} \in E'$.*

Proof of Claim 2.5 Assume that $\overline{y_{ij}} = 0$. If i or j is a pendant vertex, WLOG assume that i is pendant, then we have by (8) that $\overline{x_i} = \overline{y_{ij}} = 0$ contradicting Claim 2.4.

If both i and j are support vertices, then we have by Proposition 2.1 that $\overline{x_i} = \overline{x_j} = 1$. Removal of the edge $\{i, j\}$ in this case creates a smaller counter example than T'.

If none of i and j is a support or a pendant vertex, then as T' is a tree, we have that $\{i, j\}$ is a bridge. Let the removal of $\{i, j\}$ result in two connected components, $T_I = (V_I, E_I)$ and $T_J = (V_J, E_J)$, where $i \in V_1$ and $j \in V_2$. We can represent $(\bar{x}, \bar{y})$ as $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \bar{x}_I \\ \bar{y}_I \\ \bar{x}_J \\ \bar{y}_J \\ \bar{y}_{ij} \end{pmatrix}$, where x_I, y_I and x_J, y_J refers to the x and y variables in T_I and T_J respectively.

Using T_I and T_J, we construct two new trees, $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ where $V_1 = V_I \cup \{k_1, k_2\}$, $E_1 = E_I \cup \{i, k_1\} \cup \{k_1, k_2\}$; and similarly $V_2 = V_J \cup \{k_3, k_4\}$, $E_2 = E_J \cup \{j, k_3\} \cup \{k_3, k_4\}$. In addition, we construct two new vectors, $\begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \end{pmatrix} = \begin{pmatrix} \bar{x}_I \\ \bar{y}_I \\ \bar{y}_{i,k_1} = 0 \\ \bar{y}_{k_1,k_2} = 1 \\ \bar{x}_{k_1} = 1 \\ \bar{x}_{k_2} = 1 \end{pmatrix}$ and $\begin{pmatrix} \bar{x}_2 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} \bar{x}_J \\ \bar{y}_J \\ \bar{y}_{j,k_3} = 0 \\ \bar{y}_{k_3,k_4} = 1 \\ \bar{x}_{k_3} = 1 \\ \bar{x}_{k_4} = 1 \end{pmatrix}$.

Lemma 1. *$\begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \end{pmatrix}$ is in $P_2(T_1)$, and similarly, $\begin{pmatrix} \bar{x}_2 \\ \bar{y}_2 \end{pmatrix}$ is in $P_2(T_2)$.*

Proof of Lemma 1: We prove that $\mathbf{v} = \begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \end{pmatrix}$ is in $P_2(T_1)$ by showing that it satisfies all the inequalities defining $P_2(T_1)$. For the vertices $v \in V_1 \setminus \{i, k_1, k_2\}$, no x or y variable containing v is changed. $\mathbf{v} = \begin{pmatrix} \bar{x}_I \\ \bar{y}_I \end{pmatrix}$ being a feasible solution to $P_2(T_I)$ implies that vertices $v \in V_1 \setminus \{i, k_1, k_2\}$ satisfy the constraints written for them.

For vertex i, the only change in variables containing i from x_I, y_I is that instead of $\bar{y}_{i,j} = 0$, we have that $\bar{y}_{i,k_1} = 0$, which does not change the constraint (2) written for i. As $\bar{x}_{k_1} = 1$ and $k_1 \in \delta(i)$, constraint (3) written for i is also satisfied. Lastly, as $\bar{y}_{k_1,k_2} = 1$ and $bary_{i,k_1} = 0$, the constraints written for k_1 and k_2 are satisfied.

This shows that all the constraints are satisfied, so we conclude that $\begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \end{pmatrix}$ is in $P_2(T_1)$.

The proof for $\begin{pmatrix} \bar{x}_2 \\ \bar{y}_2 \end{pmatrix} \in P_2(T_2)$ is similar and omitted.

$\square$

Let $ext(T_1) = \{v_1, v_2, \dots v_n\}$ and $ext(T_2) = \{w_1, w_2, \dots w_m\}$ be the set of all extreme points of $P_2(T_1)$ and $P_2(T_2)$ respectively. We have that T_1 and T_2 are smaller than T', therefore by the initial assumption they have integral extreme points.

As $\begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \end{pmatrix}$ is in $P_2(T_1)$, we can write it as a convex combination of elements in $ext(T_1)$, that is; there exists $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}_{\geq 0}$ such that

$$\begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \end{pmatrix} = \sum_{p=1}^{n} \alpha_p \mathbf{v}_p, \quad \text{where} \quad \sum_{p=1}^{n} \alpha_p = 1. \tag{2}$$

and similarly, as $\begin{pmatrix} \bar{x}_2 \\ \bar{y}_2 \end{pmatrix}$ is in $P_2(T_2)$, we can write it as a convex combination of elements in $ext(T_2)$ there exists $\beta_1, \beta_2, \dots, \beta_m \in \mathbb{R}_{\geq 0}$

$$\begin{pmatrix} \bar{x}_2 \\ \bar{y}_2 \end{pmatrix} = \sum_{p=1}^{m} \beta_p \mathbf{w}_p, \quad \text{where} \quad \sum_{p=1}^{m} \beta_p = 1. \tag{3}$$

Let (α_p, v_p) and (β_r, w_r) be tuples of weights from the convex combination and the extreme points respectively.

We create a partition of $ext(T_1)$ into three sets A, B, C such that

$A = \{e \in ext(T_1) \mid x_i = 1\}$

$B = \{e \in ext(T_1) \mid x_i = 0, \exists q \in \delta(i) \setminus \{k_1\} \text{ such that } x_q = 1\}$

$C = \{e \in ext(T_1) \mid x_i = 0, \forall q \in \delta(i) \setminus \{k_1\}, x_q = 0\}$

and similarly, create a partition of $ext(T_2)$ into three sets D, E, F, such that

$D = \{e \in ext(T_2) \mid x_j = 1\}$

$E = \{e \in ext(T_2) \mid x_j = 0, \exists q \in \delta(j) \setminus \{k_3\} \text{ such that } x_q = 1\}$

$F = \{e \in ext(T_2) \mid x_j = 0, \forall q \in \delta(j) \setminus \{k_3\}, x_q = 0\}$

and define the set of tuples, $A_T = \{(\alpha_v, v) \mid v \in A\}$, $B_T = \{(\alpha_v, v) \mid v \in B\}$, $C_T = \{(\alpha_v, v) \mid v \in C\}$; $D_T = \{(\beta_w, w) \mid w \in D\}$, $E_T = \{(\beta_w, w) \mid w \in E\}$,

$F_T = \{(\beta_w, w) \mid w \in F\}$ where for each tuple, α_v is the same as in 2 for extreme point v, and β_v is the same is in 3.

Lemma 2. *Let α_i and β_i be the coefficients used to write the convex combinations in the equations 2 and 3. The following inequalities are true about the coefficients:*

$$\sum_{p \in D} \alpha_p \geq \sum_{r \in C} \beta_r \tag{4}$$

$$\sum_{r \in A} \beta_r \geq \sum_{p \in F} \alpha_p \tag{5}$$

Proof of Lemma 2: Consider the variable x_i. We need to have the x_i component of $\sum_{p \in ext(T_1)} \alpha_p v_p$ to be $\bar{x}_i$. It is only the extreme points in D that have nonzero x_i component, therefore we can write $\sum_{p \in D} \alpha_p = \bar{x}_i$.

By constraint (2) in T_1 written for vertex j, we have that $\bar{x}_i + \sum_{k \in \delta(i)} \bar{x}_k \geq 1$. Subtracting $\bar{x}_i$ from both sides of the inequality, we obtain $\sum_{k \in \delta(j) \setminus \{i\}} x_k >= 1 - \bar{x}_i$.

As the x values in $\sum_{k \in \delta(j) \setminus \{i\}} x_k >= 1 - \bar{x}_i$ do not change from $(\bar{x}, \bar{y})$ to $(\bar{x}_1, \bar{y}_1)$, it is true for both of them.

Consider the convex combination from 2 to write $(\bar{x}_1, \bar{y}_1)$. A vector $v_p \in ext(T_1)$ contributes to the convex combination of the component $\bar{x}_{1_i}$ if $\alpha_p v_{p_i} \neq 0$. Such contributing vectors for the vertices $k \in \delta(j) \setminus \{i\}$ can only be found in the set $A \cup B$, as the vectors in the set C have $x_k = 0$ for all $k \in \delta(j) \setminus \{i\}$. Therefore, we can write $\sum_{p \in A \cup B} \alpha_p \geq 1 - \bar{x}_i$.

Finally, as (2) is a convex combination, we have $\sum_{p \in ext(T_1)} \alpha_p = 1$. Subtracting the inequality $\sum_{p \in A \cup B} \alpha_p \geq 1 - \bar{x}_i$, we are left with $\sum_{p \in C} \alpha_p \leq \bar{x}_i$. Comparing this with the inequality from the first paragraph, namely $\sum_{p \in D} \alpha_p = \bar{x}_i$, we conclude that the proposed inequality (4) is true.

The proof for inequality (5) is similar and omitted.

$\square$

Algorithm 1: Concat

Input: Two extreme points, $v \in ext(T_1)$,$w \in ext(T_2)$

Output: A vector with the same dimension as $\bar{x}, \bar{y}$

Concatenate v and w into a vector vw;

Remove dimensions from vw where y_{ij} has $i = k_1, k_2, k_3, k_4$ or $j = k_1, k_2, k_3, k_4$;

Remove dimensions from vw where x_i has $i = k_1, k_2, k_3, k_4$;

Add the dimension $y_{ij} = 0$ to vw;

return *The resulting vector* vw;

Lemma 3. *Taking a vector from the set C, picking any vector from the set D and using the Algorithm 1 will result in an extreme point of $P_2(T')$. Similarly, taking a vector from the set F, picking any vector from the set A and using the Algorithm 1 will result in an extreme point of $P_2(T')$.*

Proof of Lemma 3: As the inputs $v \in C, w \in ext(T_2)$ to Algorithm 1 are integral vectors, the output is also integral and has the same dimension as $\{\bar{x}, \bar{y}\}$. We will show that the output $vw \in P_2(T')$.

As $w \in D$, it satisfies the inequalities defining T_2. Similarly, $v \in C$ implies v satisfies the inequalities defining T_1. Furthermore, when we use Algorithm 1, we drop the vertices k_1, k_2, k_3, k_4 and the related x and y variables. As the only neighbors to k variables are i and j, it is sufficient to show that constraints written for i and j are satisfied.

For all $w \in D$, j has a pair from the set $p \in \delta(j) \setminus \{k_1\}$. Therefore, $y_{jp} = 1$ and $x_j = 1$ and the constraints written for j is satisfied. For all $v \in C$, $x_i = 0$ and $y_{ik} = 0$ for all $k \in \delta(i)$. As $x_j = 1$, we have that vertex i is dominated and the constraints written for vertex i is satisfied. $\square$

Lemma 4. *Taking a vector $v \in A \cup B$, picking any vector $w \in D \cup E$ and using the Algorithm 1 will result in an extreme point of $P_2(T')$.*

Proof of Lemma 4: The proof is similar to Lemma 3 and is omitted. $\square$

Algorithm 2: Matching

Input: $A_T, B_T, C_T, D_T, E_T, F_T$

Output: Set S containing weight and extreme point tuples of a convex combination of $\bar{x}, \bar{y}$

Initialize $S \leftarrow \emptyset$;

while *there exists a vector* $v \in C_T$ *with* $\alpha_v \geq 0$ **do**

  Pick the vector $w \in D_T$ with the largest β_w;

  $vw \leftarrow \text{Concat}(v, w)$;

  Add $\big(\min\{\alpha_v, \beta_w\}, vw\big)$ to S;

  Update (α_v, v) in C_T: $\alpha_v \leftarrow \alpha_v - \min\{\alpha_v, \beta_w\}$;

  Update (β_w, w) in D_T: $\beta_w \leftarrow \beta_w - \min\{\alpha_v, \beta_w\}$;

end

while *there exists a vector* $w \in F_T$ *with* $\beta_w \neq 0$ **do**

  Pick the vector $v \in A_T$ with the largest α_v;

  $vw \leftarrow \text{Concat}(v, w)$;

  Add $\big(\min\{\alpha_v, \beta_w\}, vw\big)$ to S;

  Update (α_v, v) in A_T: $\alpha_v \leftarrow \alpha_v - \min\{\alpha_v, \beta_w\}$;

  Update (β_w, w) in F_T: $\beta_w \leftarrow \beta_w - \min\{\alpha_v, \beta_w\}$;

end

`// At this point, there should be no more elements with nonzero` α `or` β `in` C_T `and` F_T

while *there exists a vector* $v \in A_T \cup B_T$ *with* $\alpha_v \geq 0$ **do**

  Pick the vector $w \in D_T \cup E_T$ with the largest β_w;

  $vw \leftarrow \text{Concat}(v, w)$;

  Add $\big(\min\{\alpha_v, \beta_w\}, vw\big)$ to S;

  Update (α_v, v) in $A_T \cup B_T$: $\alpha_v \leftarrow \alpha_v - \min\{\alpha_v, \beta_w\}$;

  Update (β_w, w) in $D_T \cup E_T$: $\beta_w \leftarrow \beta_w - \min\{\alpha_v, \beta_w\}$;

end

return S;

Let S be the output of Algorithm 2. Using elements of S, we can write a convex combination of $(\bar{x}, \bar{y})$ in the following way:

$$\sum_{(\gamma_{vw}, vw) \in S} \gamma_{vw} \cdot \mathbf{vw} = (\bar{x}, \bar{y}), \quad \text{where} \quad \sum_{v,w \in S} \gamma_{vw} = 1.$$

.

To prove this, By Lemma 2, Equation 4, we know that

$$\sum_{p \in D} \alpha_p \geq \sum_{r \in C} \beta_r$$

In each iteration, $\min\{\alpha_v, \beta_w\}$ is subtracted from both sides of the inequality. The right hand side will reach zero first due to Lemma 2, which ensures that the while loop always finds an element with nonzero β in D_T if there is an element with nonzero α in C_T. Similarly, the second while loop always finds an element with nonzero α in A_T if there is an element with nonzero β in F_T due to Lemma 2. Moreover, all the concatenations in the first two while loops result in extreme points of $P_2(T_2)$ by Lemma 3.

After the first two while loops terminate, the sum of α in C_T and β in F_T is zero and the sum of α in $A_T \cup B_T$ is equal to the sum of β in $C_T \cup D_T$. In each iteration, $\min\{\alpha_v, \beta_w\}$ is removed from both sides, which means that both sides reach zero at the same time. In the third while loop, all concatenations result in extreme points of $P_2(T')$ by Lemma 4.

When the third while loop ends, in all the sets, α and β values are zero, and all elements of S are extreme points of $P_2(T_2)$. For any $v \in ext(T_1)$, the sum of α's that have v as a subvector is unchanged. Therefore, it is possible to write $(\bar{x}, \bar{y})$ as a convex combination of extreme points in $P_2(T')$, which contradicts that $(\bar{x}, \bar{y})$ is an extreme point. □

Claim 2.6. *We have that $\overline{y_{ij}} < 1$ for all $\{i, j\} \in E'$.*

Proof of Claim 2.6 Assume that $\overline{y_{ij}} = 1$. As T' is not a star graph, at least one of i and j is not a pendant vertex. This means that one of these two vertices, say i, is adjacent to at least one other vertex, say k. This implies by (8) that $\overline{y_{ik}} = 0$ contradicting Claim 2.5.

Claim 2.7. *If i is neither a pendant nor a support vertex and $\sum_{j\in\delta(i)} \overline{x_j} = 1$, then we have that $\overline{x_i} < 1$.*

Proof of Claim 2.7

We know by Claim 2.5 that $\overline{y_{ij}} > 0$ for all $\{i,j\} \in E'$. Assume to the contrary that i is an internal vertex, $\sum_{j\in\delta(i)} \overline{x_j} = 1$, and $\overline{x_i} = 1$. By (8), we have that $\overline{x_j} \geq \overline{y_{ij}}$ for every $j \in \delta(i)$ and the inequality is strict if j is not a pendant vertex. This implies that $\sum_{j\in\delta(i)} \overline{x_j} \geq \sum_{j\in\delta(i)} \overline{y_{ij}}$ and the inequality is strict if at least one vertex in $\delta(i)$ is not a pendant vertex. As $1 = \overline{x_i} = \sum_{j\in\delta(i)} \overline{y_{ij}}$ and $\sum_{j\in\delta(i)} \overline{x_j} = 1$, we have that $\sum_{j\in\delta(i)} \overline{x_i} = \sum_{j\in\delta(i)} \overline{y_{ij}}$ which implies that all the vertices in $\delta(i)$ are pendant and hence T' is a star graph. This contradicts to the fact that T' cannot be a star graph.

Claim 2.8. *If i is a support vertex, then we have that $\sum_{j\in\delta(i)} x_j > 1$.*

Proof of Claim 2.8 Assume i is a support vertex and $\sum_{j\in\delta(i)} x_j = 1$. By Proposition 2.2, T' cannot be a star graph. This implies that $\exists m, v \in V'$ such that $\{i,m\}, \{m,v\} \in E'$.

Proof of Theorem 2.3

We are now ready to prove Theorem 2.3. Let P' and S' be the set of pendant and support vertices of T', respectively. Let us denote by I' the set $V' \setminus (P' \cup S')$. We rewrite the inequalities describing $P_2(T')$ as

$$\sum_{j\in\delta(i)} x_j \geq 1, \forall i \in S' \qquad (7a)$$

$$\sum_{j\in\delta(i)} x_j \geq 1, \forall i \in I' \qquad (7b)$$

$$x_i = \sum_{j\in\delta(i)} y_{ij}, \forall i \in V' \qquad (8)$$

$$x_i \geq 0, \forall i \in V' \qquad (11a)$$

$$x_i \leq 1, \forall i \in P' \qquad (11b)$$

$$x_i = 1, \forall i \in S' \qquad (11c)$$

$$x_i \leq 1, \forall i \in I' \qquad (11d)$$

$$y_{ij} \geq 0, \forall \{i,j\} \in E' \qquad (12a)$$

$$y_{ij} \geq 0, \forall \{i,j\} \in E' \qquad (12b)$$

Note that the inequality (7) written for a pendant vertex is implied by (11c) and is therefore not needed. As $(\overline{x}, \overline{y})$ is an extreme point of $P_2(T')$, it has to satisfy $|V'| + |E'|$ many linearly independent inequalities among (7),(8),(11), and (12) at equality.

Let us now count the maximum number of inequalities that $(\overline{x}, \overline{y})$ may satisfy. By Proposition 2.8, $(\overline{x}, \overline{y})$ does not satisfy any of the inequalities in (7a) at equality. Assume that $(\overline{x}, \overline{y})$ satisfies k inequalities of (7b) at equality. This implies by Proposition 2.7 that $(\overline{x}, \overline{y})$ satisfies at most $|I'|-k$ many inequalities of (11d) at equality. $(\overline{x}, \overline{y})$ satisfies all the ($|V'|$ many) equalities in (8). $(\overline{x}, \overline{y})$ does not satisfy any of the inequalities in (11a), (11b), (12a), and (12b). Finally $(\overline{x}, \overline{y})$ satisfies $|S'|$ many inequalities due to (11c). In total, the number of linearly independent inequalities satisfied by $(\overline{x}, \overline{y})$ at equality is at most $k + |I'| - k + |V'| + |S'| = |V'| + |V'| - |P'|$. As each tree has at least two pendant vertices, this number is at most $2|V'| - 2$ which is strictly less than $|V'| + |E'|$. This shows that such a non-integral extreme point $(\overline{x}, \overline{y})$ cannot exist and completes the proof.

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